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ORDERING OF MINIMAL ENERGIES IN UNICYCLIC SIGNED GRAPHS

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Abstract. Let $S = (G, \sigma)$ be a signed graph of order n and size m and let $t_1, t_2, \ldots, t_n$ be the eigenvalues of S. The energy of S is defined as $E(S) = \sum_{j=1}^{n} |t_j|$. A connected signed graph is said to be unicyclic if its order and size are the same. In this paper we characterize, up to switching, the unicyclic signed graphs with first 11 minimal energies for all $n \geq 11$. For $3 \leq n \leq 7$, we provide complete orderings of unicyclic signed graphs with respect to energy. For $8 \leq n \leq 10$, we determine unicyclic signed graphs with first 13 minimal energies.

1. Introduction

Let $S = (G, \sigma)$ be a signed graph of order n, where $G = (V, E)$ is its underlying graph and $\sigma : E \to \{-1, 1\}$ is its signature. Let A be the adjacency matrix of S. In a signed graph, a cycle is said to be positive if it contains an even number of negative edges, and negative otherwise. A signed graph is said to be balanced if all its cycles are positive. For undefined terms related to signed graphs, we refer the reader to [1]. The characteristic polynomial $P(S, x)$ of S is the characteristic polynomial of its adjacency matrix A and is given by

$$P(S, t) = \det(tI - A) = \sum_{r=0}^{n} a_r(S) t^{n-r},$$

with

$$a_r(S) = \sum_{l \in \mathscr{L}_r} (-1)^{k(l)} 2^{|c(l)|} \prod_{X \in c(l)} Z(X), \tag{1.1}$$

where $\mathscr{L}_r$ denotes the set of all linear signed subgraphs (also known as basic figures) on r vertices, $k(l)$ denotes the number of components in l, $c(l)$ denotes the set of cycles in l, and $Z(X) = \prod_{e \in X} \sigma(e)$ is the sign of X. Let S be a signed graph with

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vertex set V. For any $X \subseteq V$, let S^X denote the signed graph obtained from S by reversing the signs of the edges between X and $V - X$. Then, we say S^X is switching equivalent to S. Here we note that switching is an equivalence relation and preserves the eigenvalues including their multiplicities. We use a single signed graph as representative of a switching class. Germina et al. [2] defined the energy of a signed graph S with eigenvalues $t_1, t_2, \dots, t_n$ as $E(S) = \sum_{j=1}^{n} |t_j|$. Note that the definition of the energy of a signed graph is transferred from the domain of unsigned graphs. Signed graphs significantly enrich algebraic and geometric properties compared to unsigned graphs [7].

It is well known that even and odd coefficients of the characteristic polynomial of a unicyclic signed graph alternate in sign [1, Lemma 2.7]. Putting $c_j(S) = |a_j(S)|$, we have the following integral representation for the energy of a unicyclic signed graph S:

$$E(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{t^2} \log \left[\left(\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} c_{2j}(S) t^{2j} \right)^2 + \left(\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} c_{2j+1}(S) t^{2j+1} \right)^2 \right] dt. \tag{1.2}$$

From the above integral formula, we see that the energy of a unicyclic signed graph is a monotonic increasing function of the coefficients c_j, where $j = 0, 1, \dots, n$. For signed graphs S_1 and S_2 of the same order, say n, if $c_j(S_1) \leq c_j(S_2)$ for all j, then we write $S_1 \preceq S_2$. Moreover, if $S_1 \preceq S_2$ and there is a strict inequality in $c_j(S_1) \leq c_j(S_2)$ for some $j = 1, 2, \dots, n$, then we write $S_1 \prec S_2$. Hence, if $S_1 \preceq S_2$, then $E(S_1) \leq E(S_2)$ and if $S_1 \prec S_2$, then $E(S_1) < E(S_2)$. Also, if $c_j(S_1) = c_j(S_2)$ for all j, then we write $S_1 \sim S_2$. Hence, if $S_1 \sim S_2$, then $E(S_1) = E(S_2)$.

Let $S_{n,l}$ denote the set of unicyclic signed graphs with n vertices and a cycle of length $l \leq n$. Let $e = uv$ be a pendant edge of a signed graph $S \in S_{n,l}$ with v as the pendant vertex. Then the following relation holds [1, Lemma 3.2] for c_j's of a signed graph S and its vertex-deleted signed subgraphs:

$$c_j(S) = c_j(S - v) + c_{j-2}(S - u - v). \tag{1.3}$$

2. Unicyclic signed graphs of order n with the first eleven minimal energies

Let C_r^σ $(r = 3, 4)$ be signed cycles on 3 and 4 vertices respectively, and k be a nonnegative integer. Let $S_{n,n}^k$ be a unicyclic signed graph obtained from C_3^σ by connecting k pendant vertices to any vertex and the remaining $n - k - 3$ pendant vertices to any other vertex of C_3^σ. Also, let $B_{n,n}^k$ be a unicyclic signed graph obtained from C_4^σ by connecting k pendant vertices to any vertex and the remaining $n - k - 4$ pendant vertices to any other vertex of C_4^σ which is at a distance of 2 from this vertex. There are two switching classes in $S_{n,n}^k$ and two in $B_{n,n}^k$. We use $S_{n,n}^{k,1}$, $S_{n,n}^{k,2}$ and $B_{n,n}^{k,1}$, $B_{n,n}^{k,2}$, respectively, as the representatives for these two switching classes as shown in Figure 1 (here a positive edge is denoted by a plain line and a negative edge by a dotted line). Note that $S_{n,n}^{k,1}$ and $B_{n,n}^{k,1}$ are balanced, while $S_{n,n}^{k,2}$ and $B_{n,n}^{k,2}$ are unbalanced. With these notations, we have the following result.

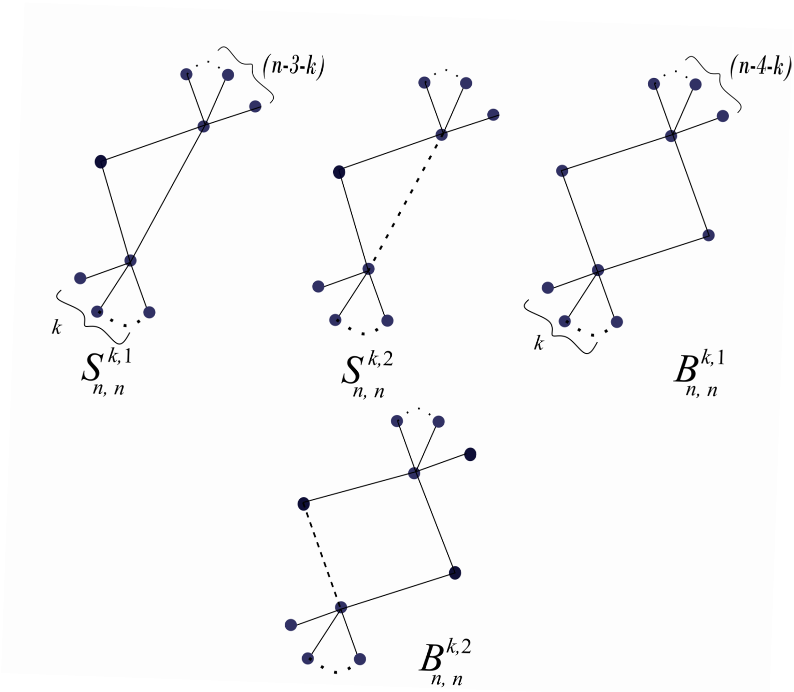

FIGURE 1. Switching classes corresponding to unicyclic signed graphs $S^k_{n,n}$ and $B^k_{n,n}$.

Lemma 2.1.

(i) *For all* $n \geq 11$ *and* $0 \leq k < n-5$, $E(S^{k,1}_{n,n}) = E(S^{k,2}_{n,n}) < E(B^{k,1}_{n,n})$.

(ii) *For all* $n \geq 11$, $E(B^{k,1}_{n,n}) < E(S^{k+1,1}_{n,n}) = E(S^{k+1,2}_{n,n})$ *for* $k = 0, 1$, *and* $E(S^{k+1,1}_{n,n}) = E(S^{k+1,2}_{n,n}) < E(B^{k,1}_{n,n})$ *for all* $2 \leq k \leq n-4$.

(iii) *For all* $n \geq 11$ *and* $0 \leq k \leq n-4$, $E(S^{k+1,1}_{n,n}) = E(S^{k+1,2}_{n,n}) < E(B^{k,2}_{n,n})$.

(iv) *For all* $n > 2k+9$ *and* $k \geq 0$, $E(B^{k,2}_{n,n}) < E(B^{k+1,1}_{n,n})$.

(v) *For all* $n \geq 11$ *and* $k \geq 0$, $E(B^{k,1}_{n,n}) < E(B^{k,2}_{n,n})$.

Proof. (i) By (1.1), we have

$$\begin{aligned} p(B^{k,1}_{n,n}, t) &= t^{n-4}\{t^4 - nt^2 + [(k+2)(n-k-4) + 2k]\}, \\ p(S^{k,1}_{n,n}, t) &= t^{n-4}\{t^4 - nt^2 - 2t + [(k+1)(n-k-3) + k]\}, \\ \text{and} \quad p(S^{k,2}_{n,n}, t) &= t^{n-4}\{t^4 - nt^2 + 2t + [(k+1)(n-k-3) + k]\}. \end{aligned}$$

It is clear that $S^{k,1}_{n,n} \sim S^{k,2}_{n,n}$ and so $E(S^{k,1}_{n,n}) = E(S^{k,2}_{n,n})$. Therefore, to compare the energy of $S^{k,r}_{n,n}$ for $r = 1, 2$ and $B^{k,1}_{n,n}$, it is enough to compare the energy of $S^{k,1}_{n,n}$ and $B^{k,1}_{n,n}$. Clearly, $S^{k,1}_{n,n}$ and $B^{k,1}_{n,n}$ are not quasi-order comparable. We use the integral formula (1.2) in this case, and we have

$$E(B_{n,n}^{k,1}) - E(S_{n,n}^{k,1}) = \frac{1}{\pi}\int_0^\infty \ln \frac{\{1+nt^2+[(k+2)(n-k-4)+2k]t^4\}^2}{\{1+nt^2+[(k+1)(n-k-3)+k]t^4\}^2+4t^6}\,dt.$$

Put

$$f_1(t) = \{1+nt^2+[(k+2)(n-k-4)+2k]t^4\}^2$$
$$\text{and}\quad g_1(t) = \{1+nt^2+[(k+1)(n-k-3)+k]t^4\}^2+4t^6.$$

Since $n > k+5$, we get

$$\begin{aligned} f_1(t)-g_1(t) &= 2(n-k-5)t^4 + 2[n(n-k-5)-2]t^6 \\ &\quad + (n-k-5)[n-k-5+2(k+1)(n-k-3)+2k]t^8 > 0 \end{aligned}$$

for $n \geq 11$ and $t > 0$, and thus $E(B_{n,n}^{k,1}) > E(S_{n,n}^{k,1})$.

(ii) The characteristic polynomials of $B_{n,n}^{k,1}$ and $S_{n,n}^{k+1,r}$ for $r = 1,2$ are given by

$$\begin{aligned} p(B_{n,n}^{k,1},t) &= t^{n-4}\{t^4 - nt^2 + [(k+2)(n-k-4)+2k]\}, \\ p(S_{n,n}^{k+1,1},t) &= t^{n-4}\{t^4 - nt^2 - 2t + [(k+2)(n-k-4)+k+1]\}, \\ \text{and}\quad p(S_{n,n}^{k+1,2},t) &= t^{n-4}\{t^4 - nt^2 + 2t + [(k+2)(n-k-4)+k+1]\}. \end{aligned}$$

To prove the result, it is enough to show that $E(B_{n,n}^{k,1}) < E(S_{n,n}^{k+1,1})$ for $k = 0,1$ and $E(B_{n,n}^{k,1}) > E(S_{n,n}^{k+1,1})$ for all $2 \leq k \leq n-4$. Clearly, $B_{n,n}^{k,1} \prec S_{n,n}^{k+1,1}$ for $k = 0,1$, and therefore $E(B_{n,n}^{k,1}) < E(S_{n,n}^{k+1,1})$ for $k = 0,1$ and for all $n \geq 11$. To compare the energy of $S_{n,n}^{k+1,1}$ and $B_{n,n}^{k,1}$ for all $2 \leq k \leq n-4$, it is enough to compare the energy of $S_{n,n}^{k+1,1}$ and $B_{n,n}^{k,1}$. Clearly, $S_{n,n}^{k+1,1}$ and $B_{n,n}^{k,1}$ are not quasi-order comparable for $2 \leq k \leq n-4$. By (1.2), we have

$$E(B_{n,n}^{k,1}) - E(S_{n,n}^{k+1,1}) = \frac{1}{\pi}\int_0^\infty \ln \frac{\{1+nt^2+[(k+2)(n-k-4)+2k]t^4\}^2}{\{1+nt^2+[(k+2)(n-k-4)+k+1]t^4\}^2+4t^6}\,dt.$$

Put

$$f_2(t) = \{1+nt^2+[(k+2)(n-k-4)+2k]t^4\}^2$$
$$\text{and}\quad g_2(t) = \{1+nt^2+[(k+2)(n-k-4)+k+1]t^4\}^2+4t^6.$$

Since $n \geq k+4$, we get

$$\begin{aligned} f_2(t)-g_2(t) &= 2(k-1)t^4 + 2[n(k-1)-2]t^6 \\ &\quad + [2(k+2)(n-k-4)(k-1)+3k^2-2k-1]t^8 > 0 \end{aligned}$$

for all $2 \leq k \leq n-4$, $n \geq 11$ and $t > 0$, and therefore $E(B_{n,n}^{k,1}) > E(S_{n,n}^{k+1,1})$ for all $2 \leq k \leq n-4$ and $n \geq 11$.

(iii) The characteristic polynomial of $B_{n,n}^{k,2}$ is given by

$$p(B_{n,n}^{k,2},t) = t^{n-4}\{t^4 - nt^2 + [(k+2)(n-k-4)+2k+4]\}.$$

We show that $E(B^{k,2}_{n,n}) > E(S^{k+1,1}_{n,n})$ for all $0 \le k \le n-4$. Clearly, $S^{k+1,1}_{n,n}$ and $B^{k,2}_{n,n}$ are not quasi-order comparable. Proceeding similarly to part (ii), we can prove that $E(S^{k+1,1}_{n,n}) = E(S^{k+1,2}_{n,n}) < E(B^{k,2}_{n,n})$ for all $0 \le k \le n-4$ and $n \ge 11$.

(iv) We have

$$p(B^{k+1,1}_{n,n}, t) = t^{n-4}\{t^4 - nt^2 + [(k+3)(n-k-5) + 2k + 2]\}.$$

Clearly, $B^{k+1,1}_{n,n} \succ B^{k,2}_{n,n}$ for all $n > 2k+7$, and therefore $E(B^{k,2}_{n,n}) < E(B^{k+1,1}_{n,n})$ for all $n > 2k+9$.

(v) This follows by [1, Theorem 2.10 (i)]. □

Let $F^1_{n,n}$ be a unicyclic graph as shown in Figure 2. There are two switching classes on the signings of $F^1_{n,n}$. Let $F^1_{n,n}$ and $F^2_{n,n}$ be the representatives for these two switching classes, where $F^1_{n,n}$ contains a positive cycle of length 4 and $F^2_{n,n}$ contains a negative cycle of length 4. With these notations, we have the following lemma.

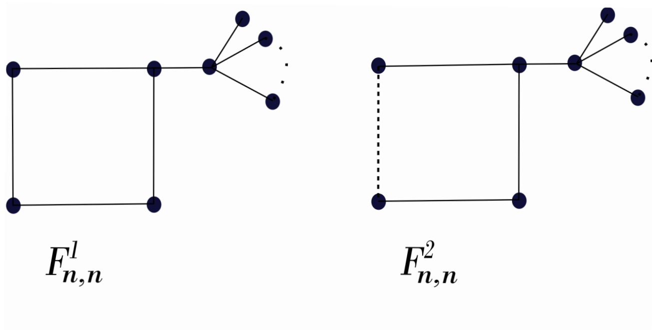

FIGURE 2. Unicyclic signed graphs $F^1_{n,n}$ and $F^2_{n,n}$.

Lemma 2.2.

(i) *For all $n \ge 6$, we have $E(B^{2,1}_{n,n}) < E(F^1_{n,n}) < E(B^{2,2}_{n,n}) < E(F^2_{n,n})$.*

(ii) *For all $n \ge 9$, $n = 2k+9$, $E(B^{k,2}_{n,n}) = E(B^{k+1,1}_{n,n})$.*

Proof. (i) The characteristic polynomials of $B^{2,r}_{n,n}$ for $r = 1, 2$, $F^1_{n,n}$, and $F^2_{n,n}$ are given by

$$\begin{aligned} p(B^{2,2}_{n,n}, t) &= t^{n-4}\{t^4 - nt^2 + (4n-16)\}, \\ p(B^{2,1}_{n,n}, t) &= t^{n-4}\{t^4 - nt^2 + (4n-20)\}, \\ p(F^1_{n,n}, t) &= t^{n-4}\{t^4 - nt^2 + (4n-18)\}, \\ \text{and} \quad p(F^2_{n,n}, t) &= t^{n-6}\{t^6 - nt^4 + (4n-14)t^2 - 4(n-5)\}. \end{aligned}$$

Clearly, $F^2_{n,n} \succ B^{2,2}_{n,n} \succ F^1_{n,n} \succ B^{2,1}_{n,n}$, and therefore $E(B^{2,1}_{n,n}) < E(F^1_{n,n}) < E(B^{2,2}_{n,n}) < E(F^2_{n,n})$ for all $n \ge 6$.

(ii) We have

$$p(B^{k+1,1}_{n,n},t)=t^{n-4}\{t^4-nt^2+[(k+3)(n-k-5)+2k+2]\}$$
$$\text{and}\quad p(B^{k+1,1}_{n,n},t)=t^{n-4}\{t^4-nt^2+[(k+2)(n-k-4)+2k+4]\}.$$

Clearly, $B^{k+1,1}_{n,n}\sim B^{k,2}_{n,n}$ for $n=2k+7$, and therefore $E(B^{k,2}_{n,n})=E(B^{k+1,1}_{n,n})$ for all $n=2k+9$. ☐

Combining Lemma 2.1 and Lemma 2.2, we have the following result.

Theorem 2.3.

(i) *For $n=11$, we have $E(S^{0,1}_{11,11})=E(S^{0,2}_{11,11})<E(B^{0,1}_{11,11})<E(S^{1,1}_{11,11})=E(S^{1,2}_{11,11})<E(B^{0,2}_{11,11})<E(B^{1,1}_{11,11})<E(S^{2,1}_{11,11})=E(S^{2,2}_{11,11})<E(S^{3,1}_{11,11})=E(S^{3,2}_{11,11})<E(B^{1,2}_{11,11})=E(B^{2,1}_{11,11})<E(F^{1}_{11,11})<E(B^{2,2}_{11,11})$.*

(ii) *For all $n\geq 12$, we have $E(S^{0,1}_{n,n})=E(S^{0,2}_{n,n})<E(B^{0,1}_{n,n})<E(S^{1,1}_{n,n})=E(S^{1,2}_{n,n})<E(B^{0,2}_{n,n})<E(B^{1,1}_{n,n})<E(S^{2,1}_{n,n})=E(S^{2,2}_{n,n})<E(B^{1,2}_{n,n})<E(S^{3,1}_{n,n})=E(S^{3,2}_{n,n})<E(B^{2,1}_{n,n})<E(F^{1}_{n,n})<E(B^{2,2}_{n,n})$.*

Let $S^{l\sigma}_n$ denote the signed graph obtained by identifying the center of the signed star S_{n-l+1} with a vertex of C^{σ}_l. The following theorem shows that among all unicyclic signed graphs with cycle length greater than 5, S^{6-}_n has the minimal energy.

Theorem 2.4. *Let $S\in\mathcal{S}_{n,l}$, where $S\neq S^{6-}_n$, $n\geq l$, $n\geq 7$, and $l\geq 6$. Then $S\succ S^{6-}_n$ and $E(S)>E(S^{6-}_n)$.*

Proof. By (1.1), we have

$$p(S^{6-}_n,t)=t^{n-6}\{t^6-nt^4+(4n-15)t^2-(3n-18)\}.$$

In view of integral formula (1.2), it suffices to prove that $c_i(S^{6-}_n)\leq c_i(S)$ for all $i=4,6$, with strict inequality for at least one i. Here, we need to consider two cases.

Case 1. Let $S\in\mathcal{S}_{n,l}$ be unbalanced, where $n\geq l$, $n\geq 7$, and $l\geq 6$. Then, by [1, Theorem 3.3], it suffices to show that $c_i(S^{6-}_n)\leq c_i(S^{l-}_n)$ for all $i=4,6$, with strict inequality for at least one i. We use induction on $n-l$ for $n\geq l$, where $n\geq 7$ and $l\geq 6$.

If $n-l=0$, then $S^{l-}_n=C^-_n$. We have $c_4(C^-_n)=\frac{n(n-3)}{2}$, $c_4(S^{6-}_n)=4n-15$, $c_6(C^-_n)=\frac{n(n-4)(n-5)}{6}$ and $c_6(S^{6-}_n)=3n-18$. Clearly, $c_i(S^{6-}_n)<c_i(C^-_n)$ for $i=4,6$ and $n\geq 7$. By (1.3), for $i=4,6$, we have

$$c_i(S^{l-}_n)=c_i(S^{l-}_{n-1})+c_{i-2}(P_{l-1})$$

and

$$c_i(S^{6-}_n)=c_i(S^{6-}_{n-1})+c_{i-2}(P_5).$$

By induction, $S^{l-}_{n-1}\succ S^{6-}_{n-1}$. Since $l\geq 6$, P_{l-1} has P_5 as a subgraph, and hence $c_i(S^{6-}_n)\leq c_i(S^{l-}_n)$ for all $i=4,6$, with strict inequality for at least one i.

Case 2. This is similar to Case 1. ☐

Lemma 2.5. *For all $n \geq 6$, we have*
(i) $E(S_n^{6-}) > E(B_{n,n}^{2,2})$,
(ii) $E(S_n^{5+}) > E(B_{n,n}^{2,2})$.

Proof. (i) The characteristic polynomials of S_n^{6-} and $B_{n,n}^{2,2}$ are respectively given by

$$p(S_n^{6-}, t) = t^{n-6}\{t^6 - nt^4 + (4n-15)t^2 - (3n-18)\}$$

and

$$p(B_{n,n}^{2,2}, t) = t^{n-6}\{t^6 - nt^4 + (4n-16)t^2\}.$$

Clearly, $S_n^{6-} \succ B_{n,n}^{2,2}$ for all $n \geq 6$, and therefore $E(S_n^{6-}) > E(B_{n,n}^{2,2})$ for all $n \geq 6$.

(ii) We have

$$p(S_n^{5+}, t) = t^{n-6}\{t^6 - nt^4 + (3n-10)t^2 - 2t - (n-5)\}.$$

The signed graphs S_n^{5+} and $B_{n,n}^{2,2}$ are not quasi-order comparable. Consider the functions $f_3(t) = t^6 - nt^4 + (3n-10)t^2 - 2t - (n-5)$ and $g_3(t) = t^4 - nt^2 + (4n-16)$. It is easy to see that $f_3(\frac{3}{5}) < 0$, $f_3(1) > 0$, $f_3(\frac{7}{5}) > 0$, $f_3(2) < 0$, $f_3(\sqrt{n-3}) < 0$, and $f_3(\sqrt{n-2}) > 0$ for all $n \geq 10$. Also, $g_3(2) = 0$ and $g_3(\sqrt{n-4}) = 0$. By Descartes's rule of signs, $f_3(t)$ has three positive and three negative zeros and $g_3(t)$ has two positive and two negative zeros. As the energy of a signed graph is twice the sum of its positive eigenvalues, we have

$$E(S_n^{5+}) > 2(2 + \sqrt{n-3}) > 2(2 + \sqrt{n-4}) = E(B_{n,n}^{2,2})$$

for all $n \geq 10$. We have verified the result directly for $n = 6, 7, 8, 9$.

Let $Q_{n,n}^{r,1}$ $(r = 1, 2, 3, 4)$ be the graphs shown in Figure 3. It is easy to see that there are two switching classes on the signings of $Q_{n,n}^{r,1}$ for all $r = 1, 2, 3, 4$. Let $Q_{n,n}^{r,1}$ and $Q_{n,n}^{r,2}$ $(r = 1, 2, 3, 4)$ be the representatives for these two switching classes, where $Q_{n,n}^{r,1}$ $(r = 1, 2, 3, 4)$ contains a positive cycle and $Q_{n,n}^{r,2}$ $(r = 1, 2, 3, 4)$ contains a negative cycle. We have the following lemma, the proof of which is similar to that of Lemma 2.5, and so we skip it here. □

Lemma 2.6. *For all $n \geq 6$, we have*
(i) $E(B_{n,n}^{2,2}) < E(Q_{n,n}^{1,1}) < E(Q_{n,n}^{1,2})$,
(ii) $E(Q_{n,n}^{1,1}) < E(Q_{n,n}^{2,1}) < E(Q_{n,n}^{2,2})$,
(iii) $E(B_{n,n}^{2,2}) < E(Q_{n,n}^{3,1}) = E(Q_{n,n}^{3,2})$,
(iv) $E(B_{n,n}^{2,2}) < E(Q_{n,n}^{4,1}) = E(Q_{n,n}^{4,2})$.

A unicyclic signed graph can be obtained by attaching rooted signed trees to the vertices of the cycle C_l^σ. Thus, if $T_1, T_2, \ldots, T_l$ are l rooted signed trees, then we denote by $U(T_1, T_2, \ldots, T_l, \sigma)$ the signed graph obtained by attaching the rooted signed trees T_i to the vertices v_i of the cycle $C_l^\sigma = v_1 v_2 \ldots v_l v_1$. When T_i is a rooted signed star K_{1,n_i} with the center of the star as its root, we write $U(n_1, n_2, \ldots, n_l, \sigma)$ instead of $U(T_1, T_2, \ldots, T_l, \sigma)$. For example, $B_{n,n}^{2,1} = U(n-6, 0, 2, 0, +)$ and $B_{n,n}^{2,2} = U(n-6, 0, 2, 0, -)$.

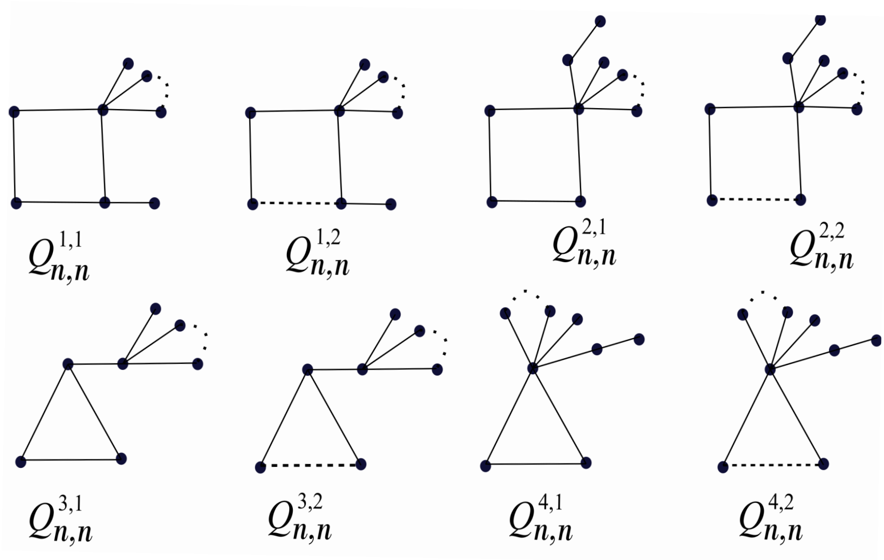

FIGURE 3. Unicyclic signed graphs $Q_{n,n}^{r,s}$ ($r = 1, 2, 3, 4$ and $s = 1, 2$).

Also, when T_i is a rooted signed star K_{1,n_i}, with a pendant vertex of the star as its root, we simplify the notation $U(T_1, T_2, \ldots, T_l, \sigma)$ by replacing T_i by the pair $(n_i - 1, 1)$. For example, $F_{n,n}^1 = U((n-5,1), 0, 0, 0, +)$ and $F_{n,n}^2 = U((n-5,1), 0, 0, 0, -)$. Let $T(m-2,2)$ be the rooted signed tree obtained by identifying the end vertex of a path of length 2 with the center of the star $K_{1,m-2}$, and let the vertex of degree $m-1$ be the root. Clearly, $Q_{n,n}^{2,1} = U(T(n-6,2), 0, 0, 0, +)$ and $Q_{n,n}^{2,2} = U(T(n-6,2), 0, 0, 0, -)$.

Let $S(n)$ be the set of all unicyclic signed graphs of order n. Let $S \in S(n)$ and u be a vertex of S. Let T be a rooted signed tree and $S_u(T)$ be the signed graph obtained by attaching T to S such that the root of T is u. When T is a signed path P_{m+1} with one endpoint as the root, we write $S_u(T)$ as $S_u(m)$. When T is a star $K_{1,m}$ with the center as its root, we write $S_u(T)$ as $S_u^*(m)$. When T is a star $K_{1,m}$ with a pendant vertex as its root, we write $S_u(T)$ as $S_u^*(m-1,1)$. For example, if $S = C_3^-$, then $S_u^*(n-3) = S_{n,n}^{0,2}$ and $S_u^*(n-4,1) = Q_{n,n}^{3,2}$. With these notations, we have following lemmas.

Lemma 2.7 ([6]). *Let $S \in S(n)$ be balanced and u be a vertex of S. Let T be a tree of order $m+1$ rooted at u. Then we have:*

(1) if $S_u(T) \neq S_u(m)$, then $S_u(T) \prec S_u(m)$;
(2) if $S_u(T) \neq S_u^*(m)$, then $S_u(T) \succ S_u^*(m)$.

Lemma 2.8 ([8]). *Let $S \in S(n)$ be balanced, u be a vertex of S, and T be a tree of order $m+1$ ($m \geq 3$) rooted at u. If $S_u(T) \neq S_u(T(m-2,2)), S_u^*(m-1,1), S_u^*(m)$, then $S_u(T) \succ S_u(T(m-2,2))$.*

For a signed graph S, let $d_S(v)$ denote the degree of a vertex v. Recall that $U(n_1, n_2, \ldots, n_l, \sigma)$ denotes the unicyclic signed graph obtained by attaching the rooted signed star K_{1,n_i}, with the center of the star as its root, to the vertices v_i for $i = 1, 2, \ldots, l$ of the cycle $C_l^\sigma = v_1 v_2 \ldots v_l v_1$. We denote by $S_{v_1,v_i}(m, n) = U(m, 0, 0, 0, \ldots, n, n_{i+1}, n_{i+2}, \ldots, n_l, \sigma)$ $(2 \leq i \leq l)$ the signed graph obtained by attaching m pendant edges and n pendant edges to the vertices v_1 and v_i of the signed graph $U(0, 0, 0, 0, \ldots, 0, n_{i+1}, n_{i+2}, \ldots, n_l, \sigma)$ $(2 \leq i \leq l)$, respectively, as shown in Figure 4. Then $S_{v_1,v_i}(m+1, n-1)$ is the signed graph obtained from the signed graph $S_{v_1,v_i}(m, n)$ by deleting a pendant edge which is adjacent to v_i and adding a pendant edge to v_1, also called edge grafting π. Proceeding in an exactly similar way to [8, Theorem 2.2], we obtain the following lemma.

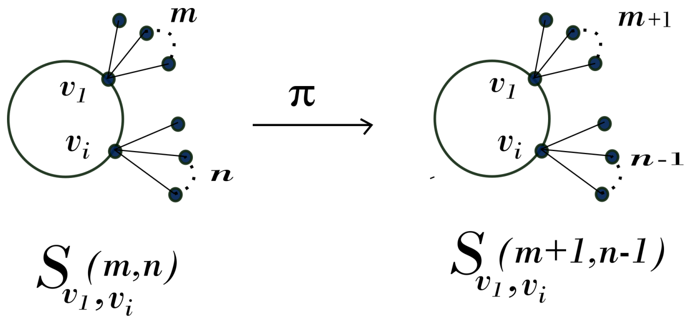

FIGURE 4. The edge-grafting π.

Lemma 2.9. *Let m and n be positive integers. If $m \geq n$, then $S_{v_1,v_i}(m, n) \succ S_{v_1,v_i}(m + 1, n - 1)$.*

Let $\mathscr{U}_n = \{S \mid S \in S(n),\, S \neq S_{n,n}^{k,r}\ (k = 0, 1, 2, 3,\ r = 1, 2),\, S \neq B_{n,n}^{k,r}\ (k = 0, 1, 2,\ r = 1, 2), \text{ and } S \neq F_{n,n}^1\}$. Also, let $C_l^\sigma = v_1 v_2 \ldots v_l v_1$ be the unique cycle of the unicyclic signed graph S and $N(S) = \{v_i \mid d(v_i) > 2,\, v_i \in V(C_l^\sigma)\}$.

Theorem 2.10. *Let $S \in \mathscr{U}_n$. If $n \geq 11$, then $E(S) > E(B_{n,n}^{2,2})$.*

Proof. Let $C_l^\sigma = v_1 v_2 \ldots v_l v_1$ be the unique cycle of the unicyclic signed graph S and $N(S) = \{v_i \mid d(v_i) > 2,\, v_i \in V(C_l^\sigma)\}$. Then the following cases arise:

Case 1. If $l \geq 5$, then the following two subcases arise.
Subcase 1.1. If $l \geq 6$, then the result follows by Theorem 2.4 and Lemma 2.5.
Subcase 1.2. If $l = 5$, then the result follows by [1, Theorem 2.9], [4, Theorem 4] and Lemma 2.5.

Case 2. Let $l = 4$. If $S = F_{n,n}^2$, then the result follows by Lemma 2.2. Also, by [1, Theorem 2.10 (i)], it is enough to show that $E(S) > E(B_{n,n}^{2,2})$, where S is balanced. Therefore, the following subcases arise.
Subcase 2.1. If $|N(S)| = 1$, then by Lemma 2.8, $S \succ Q_{n,n}^{2,1}$, and therefore $E(S) > E(Q_{n,n}^{2,1})$. Hence the result follows by Lemma 2.6.

Subcase 2.2. If $|N(S)| = 2$, then by Lemma 2.7, $S \succeq U(n-4-r, r, 0, 0, +)$ $(r \geq 1)$ or $S \succeq U(n-4-r, 0, r, 0, +)$ $(r \geq 3)$. By Lemma 2.9, $U(n-4-r, r, 0, 0, +) \succeq Q^{1,1}_{n,n}$ $(r \geq 1)$ or $U(n-4-r, 0, r, 0, +) \succeq B^{3,1}_{n,n} \succ B^{2,2}_{n,n}$ $(r \geq 3)$, and therefore $E(S) \geq E(Q^{1,1}_{n,n})$ or $E(S) > E(B^{2,2}_{n,n})$. Hence the result follows by Lemma 2.6.

Subcase 2.3. If $|N(S)| = 3$, then by Lemma 2.7, $S \succeq U(n-4-r-s, r, s, 0, +)$ $(r \geq 1,\ s \geq 1)$ or $S \succeq U(n-4-r-s, r, 0, s, +)$ $(r \geq 1,\ s \geq 1)$. By Lemma 2.9, $U(n-4-r-s, r, s, 0, +) \succ U(n-4-s, 0, s, 0, +)$ or $U(n-4-r-s, r, 0, s, +) \succ U(n-4-s, 0, 0, s, +)$. Again by Lemma 2.9, $U(n-4-s, 0, s, 0, +) \succeq B^{3,1}_{n,n} \succ B^{2,2}_{n,n}$ $(s \geq 3)$ or $U(n-4-s, 0, 0, s, +) \succeq Q^{1,1}_{n,n}$ $(s \geq 1)$, and therefore $E(S) > E(Q^{1,1}_{n,n})$ or $E(S) > E(B^{2,2}_{n,n})$. Hence the result follows by Lemma 2.6.

Subcase 2.4. If $|N(S)| = 4$, then by Lemma 2.7, $S \succeq U(n-4-r_1-r_2-r_3, r_1, r_2, r_3, +)$ $(r_i \geq 1,\ i = 1, 2, 3)$. Now applying Lemma 2.9 repeatedly, we have $U(n-4-r_1-r_2-r_3, r_1, r_2, r_3, +) \succ U(n-4-r_2-r_3, 0, r_2, r_3, +) \succ U(n-4-r_3, 0, 0, r_3, +) \succeq Q^{1,1}_{n,n}$ $(r_3 \geq 1,)$, and therefore $E(S) > E(Q^{1,1}_{n,n})$. Hence the result follows by Lemma 2.6.

Case 3. Let $l = 3$. By [1, Theorem 2.9], it is enough to show that $E(S) > E(B^{2,2}_{n,n})$, where S is balanced. Therefore the following subcases arise.

Subcase 3.1. If $|N(S)| = 1$, we have by Lemma 2.8 that, if $S \neq Q^{r,1}_{n,n}$ $(r = 3, 4)$, then $S \succ Q^{4,1}_{n,n}$, and therefore $E(S) > E(Q^{4,1}_{n,n})$. Hence the result follows by Lemma 2.6.

Subcase 3.2. If $|N(S)| = 2$, then by Lemma 2.7, we have $S \succeq U(n-3-r, r, 0, +)$ $(r \geq 1)$. By Lemma 2.9, $U(n-3-r, r, 0, +) \succeq U(n-7, 4, 0, +)$ $(r \geq 4)$. By Sach's theorem, we have

$$p(U(n-7, 4, 0, +), t) = t^{n-4}\{t^4 - nt^2 - 2t + (5n-31)\}.$$

Clearly, $U(n-7, 4, 0, +) \succ B^{2,2}_{n,n}$, and therefore $E(U(n-7, 4, 0, +)) > E(B^{2,2}_{n,n})$. Hence the result follows in this subcase.

Subcase 3.3. If $|N(S)| = 3$, then by Lemma 2.7, we have $S \succeq U(n-3-r_1-r_2, r_1, r_2, +)$ $(r_i \geq 1,\ i = 1, 2)$. Now applying Lemma 2.9 repeatedly, we have $U(n-3-r_1-r_2, r_1, r_2, +) \succ U(n-3-r_2, 0, r_2, +) \succeq U(n-7, 0, 4, +)$ $(r_2 \geq 4)$. By Sach's theorem, we get

$$p(U(n-7, 0, 4, +), t) = t^{n-4}\{t^4 - nt^2 - 2t + (5n-31)\}.$$

Clearly, $U(n-7, 0, 4, +) \succ B^{2,2}_{n,n}$, and therefore $E(U(n-7, 0, 4, +)) > E(B^{2,2}_{n,n})$. Hence the result follows. This completes the proof. □

By direct calculations via computer simulation, we have verified that the signed graphs $Q^{3,1}_{11,11}$, $Q^{3,2}_{11,11}$ have 11^{th} minimal energy, $Q^{4,1}_{11,11}$, $Q^{4,2}_{11,11}$ have 12^{th} minimal energy, and $Q^{1,1}_{11,11}$ has 13^{th} minimal energy for $n = 11$. The following theorem is the main result of our paper.

Theorem 2.11. (i) *Among all unicyclic signed graphs with $n = 11$ vertices, $Q^{1,1}_{11,11}$ is the signed graph with 13^{th} minimal energy. Also, we have an ordering of energies in ascending order as follows: $E(S^{0,1}_{11,11}) = E(S^{0,2}_{11,11}) < E(B^{0,1}_{11,11}) < E(S^{1,1}_{11,11}) = E(S^{1,2}_{11,11}) < E(B^{0,2}_{11,11}) < E(B^{1,1}_{11,11}) < E(S^{2,1}_{11,11}) =$*

$E(S^{2,2}_{11,11}) < E(S^{3,1}_{11,11}) = E(S^{3,2}_{11,11}) < E(B^{1,2}_{11,11}) = E(B^{2,1}_{11,11}) < E(F^{1}_{11,11}) < E(B^{2,2}_{11,11}) < E(Q^{3,1}_{11,11}) = E(Q^{3,2}_{11,11}) < E(Q^{4,1}_{11,11}) = E(Q^{4,2}_{11,11}) < E(Q^{1,1}_{11,11})$.

(ii) *Among all unicyclic signed graphs with $n \geq 12$ vertices, $B^{2,2}_{n,n}$ is the signed graph with 11^{th} minimal energy for all $n \geq 12$. Also, we have an ordering of energies in ascending order as follows:* $E(S^{0,1}_{n,n}) = E(S^{0,2}_{n,n}) < E(B^{0,1}_{n,n}) < E(S^{1,1}_{n,n}) = E(S^{1,2}_{n,n}) < E(B^{0,2}_{n,n}) < E(B^{1,1}_{n,n}) < E(S^{2,1}_{n,n}) = E(S^{2,2}_{n,n}) < E(B^{1,2}_{n,n}) < E(S^{3,1}_{n,n}) = E(S^{3,2}_{n,n}) < E(B^{2,1}_{n,n}) < E(F^{1}_{n,n}) < E(B^{2,2}_{n,n})$.

Proof. The result follows by Theorems 2.3 and 2.10. □

Finally, we consider the partial ordering by minimal energies of unicyclic signed graphs with at most 10 vertices. There do not exist any unicyclic signed graphs on one and two vertices.

For $n = 3$, there is a unique unicyclic unsigned graph, which gives two unicyclic signed graphs with the same number of vertices, up to switching. Let C_3^+ and C_3^- be balanced and unbalanced cycles on 3 vertices, respectively. By the coefficient theorem, we have $p(C_3^+, t) = t^3 - 3t - 2$ and $p(C_3^-, t) = t^3 - 3t + 2$. Therefore, by the integral formula (1.2), we get $E(C_3^+) = E(C_3^-)$.

For $n = 4$, there are 4 unicyclic signed graphs, up to switching. Let $S^{0,1}_{4,4}$, $S^{0,2}_{4,4}$, C_4^+, and C_4^- be 4 unicyclic signed graphs, up to switching, on 4 vertices. Their characteristic polynomials are respectively given as $p(S^{0,1}_{4,4}, t) = t^4 - 4t^2 - 2t + 1$, $p(S^{0,2}_{4,4}, t) = t^4 - 4t^2 + 2t + 1$, $p(C_4^+, t) = t^4 - 4t^2$, and $p(C_4^-, t) = t^4 - 4t^2 + 4$.

By direct calculations, we have $E(S^{0,1}_{4,4}) = 4.9622 = E(S^{0,2}_{4,4})$, $E(C_4^+) = 4$, and $E(C_4^-) = 5.657$. Therefore, we obtain

$$E(C_4^+) < E(S^{0,1}_{4,4}) = E(S^{0,2}_{4,4}) < E(C_4^-).$$

For $n = 5$, there are 10 unicyclic signed graphs, up to switching. Let $S^{0,1}_{5,5}$, $S^{0,2}_{5,5}$, $S^{1,1}_{5,5}$, $S^{1,2}_{5,5}$, $Q^{3,1}_{5,5}$, $Q^{3,2}_{5,5}$, $B^{0,1}_{5,5}$, $B^{0,2}_{5,5}$, C_5^+, and C_5^- be 10 unicyclic signed graphs, up to switching, on 5 vertices. Their characteristic polynomials are respectively given as $p(S^{0,1}_{5,5}, t) = t\{t^4 - 5t^2 - 2t + 2\}$, $p(S^{0,2}_{5,5}, t) = t\{t^4 - 5t^2 + 2t + 2\}$, $p(S^{1,1}_{5,5}, t) = t\{t^4 - 5t^2 - 2t + 3\}$, $p(S^{1,2}_{5,5}, t) = t\{t^4 - 5t^2 + 2t + 3\}$, $p(Q^{3,1}_{5,5}, t) = t^5 - 5t^3 - 2t^2 + 4t + 2$, $p(Q^{3,2}_{5,5}, t) = t^5 - 5t^3 + 2t^2 + 4t - 2$, $p(B^{0,1}_{5,5}, t) = t\{t^4 - 5t^2 + 2\}$, $p(B^{0,2}_{5,5}, t) = t\{t^4 - 5t^2 + 6\}$, $p(C_5^+, t) = t^5 - 5t^3 + 5t - 2$, and $p(C_5^-, t) = t^5 - 5t^3 + 5t + 2$.

By direct calculations, it is easy to see that $E(S^{0,1}_{5,5}) = 5.6272 = E(S^{0,2}_{5,5})$, $E(S^{1,1}_{5,5}) = 5.8416 = E(S^{1,2}_{5,5})$, $E(Q^{3,1}_{5,5}) = 6.4286 = E(Q^{3,2}_{5,5})$, $E(B^{0,1}_{5,5}) = 5.596$, $E(B^{0,2}_{5,5}) = 6.2926$, and $E(C_5^+) = 6.472 = E(C_5^-)$. Therefore, we have $E(B^{0,1}_{5,5}) < E(S^{0,1}_{5,5}) = E(S^{0,2}_{5,5}) < E(S^{1,1}_{5,5}) = E(S^{1,2}_{5,5}) < E(B^{0,2}_{5,5}) < E(Q^{3,1}_{5,5}) = E(Q^{3,2}_{5,5}) < E(C_5^+) = E(C_5^-)$.

For $n = 6$, there are 26 unicyclic signed graphs, up to switching. Let $S^{0,1}_{6,6}$, $S^{0,2}_{6,6}$, $S^{1,1}_{6,6}$, $S^{1,2}_{6,6}$, $Q^{3,1}_{6,6}$, $Q^{3,2}_{6,6}$, $Q^{4,1}_{6,6}$, $Q^{4,2}_{6,6}$, $B^{0,1}_{6,6}$, $B^{0,2}_{6,6}$, $B^{1,1}_{6,6}$, $B^{1,2}_{6,6}$, $Q^{1,1}_{6,6}$, $Q^{1,2}_{6,6}$, $F^{1}_{6,6}$, $F^{2}_{6,6}$, $Q_6^{1,1}$, $Q_6^{1,2}$, $Q_6^{2,1}$, $Q_6^{2,2}$, $Q_6^{3,1}$, $Q_6^{3,2}$, $Q_6^{4,1}$, $Q_6^{4,2}$, C_6^+, and C_6^- be 26 unicyclic signed graphs, up to switching, on 6 vertices. Here, the signed graphs $Q_6^{1,1}$, $Q_6^{1,2}$, $Q_6^{2,1}$, $Q_6^{2,2}$, $Q_6^{3,1}$, $Q_6^{3,2}$, $Q_6^{4,1}$, and $Q_6^{4,2}$ are shown in Figure 5.

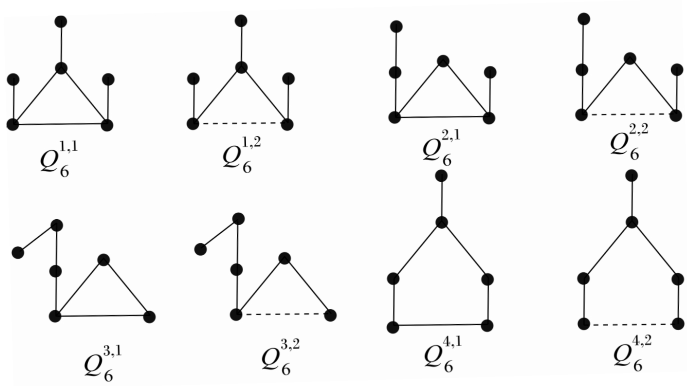

FIGURE 5. Unicyclic signed graphs $Q_6^{r,s}$, $r = 1, 2, 3, 4$ and $s = 1, 2$.

By direct calculations, it is easy to see that $E(S_{6,6}^{0,1}) = E(S_{6,6}^{0,2}) = 6.1722$, $E(S_{6,6}^{1,1}) = E(S_{6,6}^{1,2}) = 6.4852$, $E(Q_{6,6}^{3,1}) = E(Q_{6,6}^{3,2}) = 7.1916$, $E(Q_{6,6}^{4,1}) = E(Q_{6,6}^{4,2}) = 7.3426$, $E(B_{6,6}^{0,1}) = 6.3244$, $E(B_{6,6}^{0,2}) = 6.8284$, $E(B_{6,6}^{1,1}) = 6.4722$, $E(B_{6,6}^{1,2}) = 6.9284$, $E(Q_{6,6}^{1,1}) = 7.2078$, $E(Q_{6,6}^{1,2}) = 7.5176$, $E(F_{6,6}^{1}) = 6.6026$, $E(F_{6,6}^{2}) = 8.0548$, $E(Q_6^{1,1}) = E(Q_6^{1,2}) = 7.3004$, $E(Q_6^{2,1}) = E(Q_6^{2,2}) = 7.416$, $E(Q_6^{3,1}) = E(Q_6^{3,2}) = 7.5494$, $E(Q_6^{4,1}) = E(Q_6^{4,2}) = 7.4658, E(C_6^{+}) = 8$, and $E(C_6^{-}) = 6.9284$. Therefore, we have $E(S_{6,6}^{0,1}) = E(S_{6,6}^{0,2}) < E(B_{6,6}^{0,1}) < E(B_{6,6}^{1,1}) < E(S_{6,6}^{1,1}) = E(S_{6,6}^{1,2}) < E(F_{6,6}^{1}) < E(B_{6,6}^{0,2}) < E(B_{6,6}^{1,2}) = E(C_6^{-}) < E(Q_{6,6}^{3,1}) = E(Q_{6,6}^{3,2}) < E(Q_{6,6}^{1,1}) < E(Q_6^{1,1}) = E(Q_6^{1,2}) < E(Q_{6,6}^{4,1}) = E(Q_{6,6}^{4,2}) < E(Q_6^{2,1}) = E(Q_6^{2,2}) < E(Q_6^{4,1}) = E(Q_6^{4,2}) < E(Q_{6,6}^{1,2}) < E(Q_6^{3,1}) = E(Q_6^{3,2}) < E(C_6^{+}) < E(F_{6,6}^{2})$.

For $n = 7$, there are 33 unicyclic unsigned graphs on 7 vertices, which gives 66 unicyclic signed graphs with the same number of vertices, up to switching: $S_{7,7}^{r,s}$ ($r = 0, 1, 2$ and $s = 1, 2$), $B_{7,7}^{r,s}$ ($r = 0, 1$ and $s = 1, 2$), $Q_{7,7}^{r,s}$ ($r = 1, 2, 3, 4$ and $s = 1, 2$), $F_{7,7}^{r}$ ($r = 1, 2$), $Q_7^{r,s}$ ($r = 1, 2, \ldots, 22$ and $s = 1, 2$), C_7^{+}, and C_7^{-}.

By direct calculations, it is easy to see that $E(S_{7,7}^{0,1}) = E(S_{7,7}^{0,2}) = 6.6468 < E(B_{7,7}^{0,1}) = 6.899 < E(S_{7,7}^{1,1}) = E(S_{7,7}^{1,2}) = 7.0206 < E(B_{7,7}^{1,1}) = 7.1154 < E(S_{7,7}^{2,1}) = E(S_{7,7}^{2,2}) = 7.1232 < E(B_{7,7}^{0,2}) = E(F_{7,7}^{1}) = 7.3006 < E(B_{7,7}^{1,2}) = 7.4642 < E(Q_{7,7}^{3,1}) = E(Q_{7,7}^{3,2}) = 7.8102 < E(Q_{7,7}^{1,1}) = 7.9426 < E(Q_{7,7}^{4,1}) = E(Q_{7,7}^{4,2}) = 7.9688 < E(Q_{7,7}^{2,1}) = 8.004 < E(Q_7^{1,1}) = E(Q_7^{1,2}) = 8.0094 < E(Q_7^{16,1}) = 8.0628 < E(Q_7^{4,1}) = E(Q_7^{4,2}) = E(Q_7^{9,1}) = E(Q_7^{9,2}) = 8.0852 < E(Q_7^{5,1}) = E(Q_7^{5,2}) = 8.1178 < E(Q_7^{17,1}) = 8.12 < E(Q_7^{19,1}) = E(Q_7^{19,2}) = 8.1282 < E(Q_7^{15,1}) = 8.1528 < E(Q_7^{7,1}) = E(Q_7^{7,2}) = 8.171 < E(Q_{7,7}^{1,2}) = 8.175 < E(Q_7^{14,1}) = 8.2078 < E(Q_7^{13,1}) = E(Q_7^{13,2}) = 8.2618 < E(Q_7^{3,1}) = E(Q_7^{3,2}) = 8.3012 < E(Q_7^{21,1}) = E(Q_7^{21,2}) = 8.3184 < E(Q_7^{15,2}) =

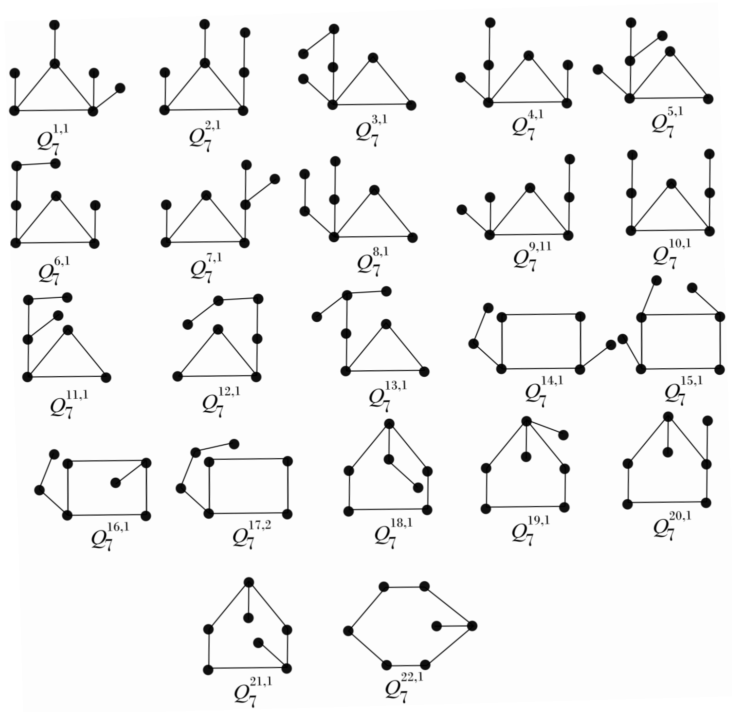

FIGURE 6. Unicyclic signed graphs $Q_7^{r,1}$, $r = 1, 2, 3, \dots, 22$.

$E(Q_7^{22,2}) = 8.3632 < E(Q_7^{2,1}) = E(Q_7^{2,2}) = 8.3898 < E(Q_7^{20,1}) = E(Q_7^{20,2}) = 8.4286 < E(Q_7^{6,1}) = E(Q_7^{6,2}) = 8.4556 < E(Q_{7,7}^{2,2}) = 8.647 < E(Q_7^{16,2}) = 8.6906 < E(Q_7^{22,1}) = 8.7266 < E(Q_7^{14,2}) = 8.7628 < E(Q_7^{8,1}) = E(Q_7^{2,2}) = E(F_{7,7}^2) = 8.8284 < E(Q_7^{11,1}) = E(Q_7^{11,2}) = 8.8696 < E(Q_7^{10,1}) = E(Q_7^{10,2}) = 8.8702 < E(Q_7^{18,1}) = E(Q_7^{18,2}) = 8.9172 < E(Q_7^{12,1}) = E(Q_7^{12,2}) = 8.9408 < E(C_7^+) = E(C_7^-) = 8.988 < E(Q_7^{17,2}) = 8.9838$, where unicyclic signed graphs $Q_7^{r,1}$, $r = 1, 2, 3, \dots, 22$, are shown in Figure 6. Also, $Q_7^{r,2}$ is an unbalanced unicyclic signed graph corresponding to the balanced signed graph $Q_7^{r,1}$.

Similarly, by direct calculations with the aid of MATLAB software, we obtain the following result.

Theorem 2.12. (i) *Among all* 178 *unicyclic signed graphs on* 8 *vertices,* $Q_8^{1,1}$ *and* $Q_8^{1,2}$ *are the signed graphs with* 13^{th} *minimal energy. Also, we have an ordering of energies in ascending order as follows:* $E(S_{8,8}^{0,1}) = E(S_{8,8}^{0,2}) < E(B_{8,8}^{0,1}) < E(S_{8,8}^{1,1}) = E(S_{8,8}^{1,2}) < E(B_{8,8}^{1,1}) < E(S_{8,8}^{2,1}) = E(S_{8,8}^{2,2}) < E(B_{8,8}^{0,2}) = E(B_{8,8}^{2,1}) < E(F_{8,8}^{1}) < E(B_{8,8}^{1,2}) < E(B_{8,8}^{2,2}) < E(Q_{8,8}^{3,1}) = E(Q_{8,8}^{3,2}) < E(Q_{8,8}^{2,1}) < E(Q_{8,8}^{1,1}) < E(Q_8^{1,1}) = E(Q_8^{1,2})$*, where* $Q_8^{1,r}$*,* $r = 1, 2$*, are shown in Figure 7.*

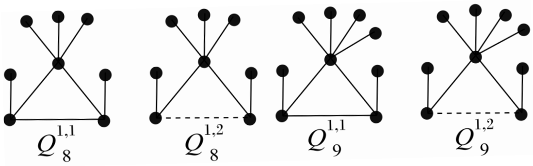

FIGURE 7. Unicyclic signed graphs $Q_8^{1,1}$, $Q_8^{1,2}$, $Q_9^{1,1}$, and $Q_9^{1,2}$.

(ii) *Among all* 480 *unicyclic signed graphs on* 9 *vertices,* $Q_{9,9}^{2,1}$ *is the signed graph with* 13^{th} *minimal energy. Also, we have an ordering of energies in ascending order as follows:* $E(S_{9,9}^{0,1}) = E(S_{9,9}^{0,2}) < E(B_{9,9}^{0,1}) < E(S_{9,9}^{1,1}) = E(S_{9,9}^{1,2}) < E(B_{9,9}^{0,2}) = E(B_{9,9}^{1,1}) < E(S_{9,9}^{2,1}) = E(S_{9,9}^{2,2}) < E(B_{9,9}^{2,1}) < E(F_{9,9}^{1}) = E(B_{9,9}^{1,2}) < E(B_{9,9}^{2,2}) < E(Q_{9,9}^{3,1}) = E(Q_{9,9}^{3,2}) < E(Q_{9,9}^{4,1}) = E(Q_{9,9}^{4,2}) < E(Q_{9,9}^{1,1}) < E(Q_9^{1,1}) = E(Q_9^{1,2}) < E(Q_{9,9}^{2,1})$*, where* $Q_9^{1,r}$*,* $r = 1, 2$*, are shown in Figure 7.*

(iii) *Among all* 667 *unicyclic unsigned graphs on* 10 *vertices, which gives* 1334 *unicyclic signed graphs, up to switching,* $Q_{10,10}^{4,1}$ *and* $Q_{10,10}^{4,2}$ *are the signed graphs with* 13^{th} *minimal energy. Also, we have an ordering of energies in ascending order as follows:* $E(S_{10,10}^{0,1}) = E(S_{10,10}^{0,2}) < E(B_{10,10}^{0,1}) < E(S_{10,10}^{1,1}) = E(S_{10,10}^{1,2}) < E(B_{10,10}^{0,2}) < E(B_{10,10}^{1,1}) < E(S_{10,10}^{2,1}) = E(S_{10,10}^{2,2}) < E(S_{10,10}^{3,1}) = E(S_{10,10}^{3,2}) < E(B_{10,10}^{2,1}) < E(B_{10,10}^{1,2}) < E(F_{10,10}^{1}) < E(B_{10,10}^{2,2}) < E(Q_{10,10}^{3,1}) = E(Q_{10,10}^{3,2}) < E(Q_{10,10}^{4,1}) = E(Q_{10,10}^{4,2})$*.*

Conclusion. In this paper, we were able to provide unicyclic signed graphs with first eleven minimal energies. After that the problem becomes difficult. For example, it is easy to see that the unicyclic signed graphs $Q_{n,n}^{3,1}$ and $Q_{n,n}^{4,1}$ have 12^{th} and 13^{th} minimal energy, respectively, for $n = 12$. But for $n = 1000$, $Q_{n,n}^{4,1}$ has 12^{th} minimal energy and $Q_{n,n}^{3,1}$ has 13^{th} minimal energy. It will be interesting to provide further orderings with respect to minimal energy. Energy ordering in other families of signed graphs, such as bipartite, k-cyclic ($k \geq 2$), complete signed graphs of fixed order, etc., remains a problem for future study. It will be useful to see the work on weighted graphs [3] and directed graphs [5].

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