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BROWNIAN MOTION ON INVOLUTIVE BRAIDED SPACES

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Abstract. We study (quantum) stochastic processes with independent and stationary increments (i.e., Lévy processes), and in particular Brownian motions in braided monoidal categories. The notion of increments is based on a bialgebra or Hopf algebra structure, and positivity is taken w.r.t. an involution. We show that involutive bialgebras and Hopf algebras in the Yetter–Drinfeld categories of a quasi- or coquasi-triangular $*$-bialgebra admit a symmetrization (or bosonization) and that their Lévy processes are in one-to-one correspondence with a certain class of Lévy processes on their symmetrization. We classify Lévy processes with quadratic generators, i.e., Brownian motions, on several braided Hopf-$*$-algebras that are generated by their primitive elements (also called braided $*$-spaces), and on the braided SU(2)-quantum groups.

Introduction

The study of random variables and stochastic processes with values in algebraic structures has a long and rich history; see, e.g., the monograph [9]. In quantum probability [24], the commutative algebras of functions on the underlying probability space and the state space of a stochastic process are replaced by possibly noncommutative algebras, typically realized as algebras of operators on a Hilbert space. Quantum (or noncommutative) probability was initially motivated by the wish to have a common framework for classical probability and quantum physics, but it also opened the stage for many new interactions between probability and algebraic structures. In [1, 27], Lévy processes on involutive bialgebras were introduced and studied as a common generalization of Lévy processes with values in groups and semigroups, and of factorizable representations of groups and Lie algebras.

In this paper we will study quantum stochastic processes and in particular Brownian motions in braided categories. In quantum probability, Brownian motion is

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defined as a Lévy process with quadratic generator, cf. [27, Section 5.1]. This generalizes Brownian motions in Euclidean spaces, Lie groups, and Riemannian manifolds, which are fundamental examples of stochastic processes, and which have been the guiding examples in the development of the stochastic calculus and potential theory. The generator of the Markov semigroup of classical Brownian motion is a Laplace or Laplace–Beltrami operator and contains valuable information about the geometry of the underlying space.

In this paper we will develop the theory of Brownian motions and Lévy processes on braided involutive bialgebras, i.e., involutive bialgebras which are objects of a braided monoidal category, cf. [19, 10]. Our main examples will be such algebras, also called braided spaces, which are generated by a finite number of primitive elements satisfying certain quadratic commutative relations coming from an R-matrix. This article builds on ideas and results from [16, 17, 2, 4, 6].

Our paper introduces several interesting new classes of quantum stochastic processes. In dimension one (i.e., with one self-adjoint primitive generator) there is only one R-matrix which leads to a braided involutive space in our setting, the 1×1 matrix $R = (q)$ with $q \in \mathbb{R}\backslash\{0\}$. In this case the braiding can be defined via a group, and we obtain, e.g., the Azéma process, whose surprising properties have been studied in [3, 23, 26]. But in higher dimension there are many possibilities to choose an R-matrix with a compatible involution, and most of them cannot be obtained from groups. As an example, we will classify the Brownian motions (i.e., Lévy processes with quadratic generator) on the braided spaces associated to the sl_2- and sl_3-R-matrix in Section 6.

Let us outline the structure of this paper.

In Section 1, we give the basic definitions and several fundamental results needed to define and study braided Lévy processes.

In Section 2, we show how braided categories can be constructed from a Hopf algebra, a coquasi-triangular or a quasi-triangular bialgebra. Our particular setting has been chosen since it will be convenient when we consider Lévy processes on braided $*$-bialgebras in these categories. We consider three cases. In the first case we have a Hopf-$*$-algebra $\mathcal{A}$ and the objects of the category are Yetter–Drinfeld modules carrying an involution, which has to satisfy some compatibility conditions. In the second case $\mathcal{A}$ is a coquasi-triangular bialgebra equipped with an involution. We show that if the r-form satisfies a compatibility condition with respect to the involution, then we can build a braided category from the comodules. In the last case, $\mathcal{A}$ is a quasi-triangular bialgebra with an involution. If the R-matrix satisfies a certain compatibility condition w.r.t. the involution, then we can again build a braided category, this time from the modules.

In Section 3 we show that for every braided $*$-bialgebra $\mathcal{B}$ in either of the three types of category we can construct a symmetrization (or bosonization), i.e., we can embed it into a bigger $*$-bialgebra $\mathcal{H}$ as an algebra in a way that allows us to "lift" Lévy processes on $\mathcal{B}$ to Lévy processes on $\mathcal{H}$. This generalizes the symmetrization in [27, Ch. 3], where the braiding was defined via actions and coactions of a group. For more general braidings this kind of construction was introduced by Majid (see

[19, Section 9.4]) but without the involution, which is crucial for defining and studying quantum stochastic processes. We show that the symmetrization reduces the construction and classification of Lévy processes on braided $*$-bialgebras to usual involutive bialgebras.

In Section 4, we explicitly construct a family of braided $*$-spaces and their symmetrization. The construction starts from a bi-invertible R-matrix of real type I and it yields a braided $*$-space in the category of comodules of the quasi-triangular $*$-bialgebra associated to the R-matrix by the Faddeev–Reshetikhin–Takhtajan construction (cf. [25]). Furthermore, these braided $*$-spaces always come with a canonical quadratic generator that gives rise to a standard Brownian motion on these spaces.

In Section 5, we prove the existence of these invariant, conditionally positive, quadratic functionals on our braided $*$-spaces and study the associated Brownian motions. These processes can be considered as multi-dimensional analogues of the Azéma martingale.

Section 6 contains the explicit quantum stochastic differential equations defining the standard Brownian motion on the braided line and the braided plane as well as the classification of all quadratic generators for the braided $*$-spaces associated to the 1×1 R-matrix (q), and the standard sl_2- and sl_3-R-matrices, and for the braided SU(2)-quantum groups.

1. Preliminaries

1.1. **Lévy processes on braided Hopf-$*$-algebras.** In this section we introduce braided Hopf-$*$-algebras, braided $*$-bialgebras, and Lévy processes on braided $*$-bialgebras in order to recall some of their elementary theory, in particular the one-to-one correspondence between these processes and their generators. See [19, 10] and the references therein for information on braided tensor categories and Hopf algebras, and [27, 7] for the theory of quantum Lévy processes.

A *tensor category* or *monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying certain conditions (see, for example, [15]). A *braiding* Ψ in a tensor category is a natural isomorphism between the functors $\otimes : (A, B) \mapsto A \otimes B$ and $\otimes \circ \tau : (A, B) \mapsto B \otimes A$ satisfying the so-called Hexagon Axioms. It is called a *symmetry* if it is involutive, i.e., if $\Psi_{B,A} \circ \Psi_{A,B} = \mathrm{id}_{A\otimes B}$ for all objects A, B of $\mathcal{C}$. A *braided tensor category* or *braided monoidal category* is a pair $(\mathcal{C}, \Psi)$ consisting of a tensor category $\mathcal{C}$ and a braiding Ψ of $\mathcal{C}$. It is called *symmetric tensor category* if the braiding is a symmetry (cf. [15]).

In this paper we always assume that the objects of our braided tensor categories $(\mathcal{C}, \Psi)$ are complex vector spaces and that the morphisms are linear maps. We call a linear map $L : V \to W$ Ψ*-invariant* if $(\mathrm{id}_X \otimes L) \circ \Psi_{V,X} = \Psi_{W,X} \circ (L \otimes \mathrm{id}_X)$ holds for any X. Note that due to the naturality of Ψ, morphisms of the category $(\mathcal{C}, \Psi)$ always have to satisfy both conditions $(\mathrm{id}_X \otimes L) \circ \Psi_{X,V} = \Psi_{X,W} \circ (L \otimes \mathrm{id}_X)$ and $(L \otimes \mathrm{id}_X) \circ \Psi_{X,V} = \Psi_{X,W} \circ (\mathrm{id}_X \otimes L)$, i.e., they are Ψ-invariant and Ψ^{-1}-invariant. The notions of bialgebras and Hopf algebras can also be defined in a braided tensor category, this leads to *braided bialgebras* and *braided Hopf algebras*. The product, coproduct, unit, counit and antipode now have to be morphisms of the braided tensor category and satisfy similar axioms as in the usual case, cf. [19]. Bialgebras and Hopf algebras are special cases of braided bialgebras and braided Hopf algebras where the flip automorphism $\tau : A \otimes B \to B \otimes A$, $\tau(u \otimes v) = v \otimes u$ is the braiding.

Since the axioms proposed by Majid in [18, 19] do not guarantee that the braided coproduct is a $*$-algebra homomorphism, we follow the ones in [5].

We want the $*$-structure $*_{\mathcal{B}\otimes\mathcal{B}}$ on $\mathcal{B} \otimes \mathcal{B}$ to be an anti-homomorphism and the canonical inclusions $\mathcal{B} \xrightarrow{\iota_1} \mathcal{B} \otimes \mathcal{B} \xleftarrow{\iota_2} \mathcal{B}$ to be $*$-homomorphisms. Thus we get for $a, b \in \mathcal{B}$

$$\begin{aligned}
*_{\mathcal{B}\otimes\mathcal{B}}(a \otimes b) &= (*_{\mathcal{B}\otimes\mathcal{B}} \circ m_{\mathcal{B}\otimes\mathcal{B}})(a \otimes \mathbb{1} \otimes \mathbb{1} \otimes b) \\
&= (m_{\mathcal{B}\otimes\mathcal{B}} \circ (*_{\mathcal{B}\otimes\mathcal{B}} \otimes *_{\mathcal{B}\otimes\mathcal{B}}) \circ \tau_{\mathcal{B}\otimes\mathcal{B},\mathcal{B}\otimes\mathcal{B}})(a \otimes \mathbb{1} \otimes \mathbb{1} \otimes b) \\
&= ((m \otimes m) \circ (\mathrm{id} \otimes \Psi \otimes \mathrm{id}))(\mathbb{1} \otimes b^* \otimes a^* \otimes \mathbb{1}) \\
&= \big(((m \circ (\mathbb{1} \otimes \mathrm{id})) \otimes (m \circ (\mathrm{id} \otimes \mathbb{1}))) \circ \Psi\big)(b^* \otimes a^*) \\
&= (\Psi \circ (* \otimes *) \circ \tau)(a \otimes b).
\end{aligned}$$

Hence we get the following definition.

Definition 1.1. A *braided $*$-bialgebra* is a braided bialgebra $(\mathcal{B}, m, \mathbb{1}, \Delta, \varepsilon, \Psi)$ over $\mathbb{C}$ with an anti-linear map $* : \mathcal{B} \to \mathcal{B}$, such that $(\mathcal{B}, m, \mathbb{1}, *)$ is a $*$-algebra and $*_{\mathcal{B}\otimes\mathcal{B}} := \Psi \circ (* \otimes *) \circ \tau$ is a self-inverse map turning the coproduct Δ into a $*$-algebra homomorphism for $*_{\mathcal{B}\otimes\mathcal{B}}$.

Remark 1.2.

- This definition is equivalent to the one given in [5, Def. 3.8.2].
- One can show that $*_{\mathcal{B}\otimes\mathcal{B}}$ is indeed an anti-homomorphism and that the canonical inclusions are $*$-algebra-homomorphisms.
- In general, the convolution of two positive functionals on a braided $*$-bialgebra is not again positive. But if we have two positive functionals ϕ and θ on a braided $*$-bialgebra $\mathcal{A}$ such that ϕ is Ψ^{-1}-invariant or θ is Ψ-invariant, then the convolution $\phi \star \theta = (\phi \otimes \theta) \circ \Delta$ is positive (cf. [5, Lemma 4.2.2]).

We briefly recall the definition of Lévy processes on braided $*$-bialgebras (see also [5, Ch. 4]). A *quantum probability space* is a pair $(\mathcal{A}, \Phi)$ consisting of a $*$-algebra and a state (i.e., a normalized positive linear functional) Φ on $\mathcal{A}$. A *quantum random variable* j over a quantum probability space $(\mathcal{A}, \Phi)$ on a $*$-algebra $\mathcal{B}$ is a $*$-algebra homomorphism $j : \mathcal{B} \to \mathcal{A}$. A *quantum stochastic process* is a family of quantum random variables over the same quantum probability space, indexed by some set, and defined on the same algebra. Two quantum stochastic processes $(j_t)_{t\in I}$ and $(k_t)_{t\in I}$, indexed by the same set I, on the same $*$-algebra $\mathcal{B}$ over the quantum probability spaces $(\mathcal{A}_j, \Phi_j)$ and $(\mathcal{A}_k, \Phi_k)$ are called *equivalent* if all their finite-dimensional distributions agree, i.e., if

$$\Phi_j\big(j_{t_1}(b_1) \dots j_{t_n}(b_n)\big) = \Phi_k\big(k_{t_1}(b_1) \dots k_{t_n}(b_n)\big)$$

for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$, $b_1, \dots, b_n \in \mathcal{B}$.

Definition 1.3 ([5, Def. 4.2.1]). Let $(\mathcal{A}, \Phi)$ be a quantum probability space, $\mathcal{B}$ a $*$-algebra, and $\Psi : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ a linear map. An n-tuple $(j_1, \dots, j_n)$ of quantum random variables $j_i : \mathcal{B} \to \mathcal{A}$, $i = 1, \dots, n$, over $(\mathcal{A}, \Phi)$ on $\mathcal{B}$ is called Ψ*-independent* or *braided independent* if

(i) $\Phi\Big(j_{\sigma(1)}(b_1) \dots j_{\sigma(n)}(b_n)\Big) = \Phi(j_{\sigma(1)}(b_1)) \dots \Phi(j_{\sigma(n)}(b_n))$ for all permutations $\sigma \in \mathcal{S}(n)$ and all $b_1, \dots, b_n \in \mathcal{B}$, and
(ii) $m_{\mathcal{A}} \circ (j_l \otimes j_k) = m_{\mathcal{A}} \circ (j_k \otimes j_l) \circ \Psi$ for all $1 \leq k < l \leq n$.

Definition 1.4 ([5, Def. 4.3.1]). Let $\mathcal{B}$ be a braided $*$-bialgebra. A quantum stochastic process $(j_{st})_{0 \leq s \leq t}$ on $\mathcal{B}$ over some quantum probability space $(\mathcal{A}, \Phi)$ is called a *Lévy process* if the following conditions are satisfied.

(1) (Increment property)
$$j_{rs} \star j_{st} = j_{rt} \quad \text{for all } 0 \leq r \leq s \leq t,$$
$$j_{tt} = \mathbb{1} \circ \varepsilon \quad \text{for all } 0 \leq t,$$
where $j_{rs} \star j_{st} = m_{\mathcal{A}} \circ (j_{rs} \otimes j_{st}) \circ \Delta_{\mathcal{B}}$ denotes the convolution of j_{rs} and j_{st}.
(2) (Independence of increments) The family $(j_{st})_{0 \leq s \leq t}$ is Ψ-independent, i.e., $(j_{s_1 t_1}, \dots, j_{s_n t_n})$ is Ψ-independent for all $n \in \mathbb{N}$ and all $0 \leq s_1 \leq t_1 \leq s_2 \leq \dots \leq t_n$.
(3) (Stationarity of increments) The distribution $\varphi_{st} = \Phi \circ j_{st}$ of j_{st} depends only on the difference $t - s$,
(4) (Weak continuity) j_{st} converges to j_{ss} $(= \mathbb{1} \circ \varepsilon)$ in distribution for $t \searrow s$, i.e., we have $\lim_{t \searrow s} \Phi\big(j_{st}(b)\big) = \varepsilon(b)$ for all $b \in \mathcal{B}$.

Let (j_{st}) be a Lévy process on some $*$-bialgebra. The states on $\mathcal{B}$ defined by $\varphi_t = \Phi \circ j_{0t}$ are called *marginal distributions* of (j_{st}) and uniquely determine a Lévy process (up to equivalence). The marginal distributions form a convolution semigroup. Using the fundamental theorem of coalgebras one can show that there exists a unique, conditionally positive (i.e., positive on the kernel of ε) hermitian, linear functional $L : \mathcal{B} \to \mathbb{C}$ with $L(\mathbb{1}) = 0$ such that $\varphi_t = \exp_\star tL$. The functional L is called the *generator of the process* (j_{st}). Using Schoenberg correspondence, we see that the convolution semigroup generated by a conditionally positive, hermitian linear functional L with $L(\mathbb{1}) = 0$ consists of states, i.e., normalized positive functionals. Due to Remark 1.2 and the braided version of the Schoenberg correspondence for Ψ-invariant functionals (see [8]), these results remain true for Lévy processes on braided $*$-bialgebras. This is summarized in the following proposition.

Proposition 1.5 ([5, Prop. 4.3.2]). *Let $\mathcal{B}$ be a $*$-bialgebra over $\mathbb{C}$ in a braided category $(\mathcal{C}, \Psi)$. Then there is a one-to-one correspondence between (equivalence classes of) Lévy processes (j_{st}), the set of convolution semigroups $(\varphi_t)_t$ of Ψ-invariant states on $\mathcal{B}$, and the set of Ψ-invariant, hermitian, conditionally positive linear functionals $L : \mathcal{B} \to \mathbb{C}$.*

1.2. **Quasi-triangular and coquasi-triangular bialgebras.** We will now recall some definitions (see, e.g., [14]) which allow us later on to "symmetrize" braided

$*$-bialgebras. Let $\mathcal{B}$ be a bialgebra and $R \in \mathcal{B} \otimes \mathcal{B}$. For $R = \sum_i x_i \otimes y_i$, define R_{12}, R_{13} and $R_{23} \in \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$ by

$$R_{12} \coloneqq \sum_i x_i \otimes y_i \otimes 1, \quad R_{13} \coloneqq \sum_i x_i \otimes 1 \otimes y_i, \quad R_{23} \coloneqq \sum_i 1 \otimes x_i \otimes y_i.$$

Definition 1.6. A *quasi-triangular* bialgebra $\mathcal{B}$ is a bialgebra $\mathcal{B}$ equipped with an invertible element $R \in \mathcal{B} \otimes \mathcal{B}$, called *universal R-matrix*, such that the equations

$$\begin{aligned} \Delta^{op}(a) &= R \cdot \Delta(a) \cdot R^{-1}, \\ (\Delta \otimes \mathrm{id})(R) &= R_{13} R_{23}, \\ (\mathrm{id} \otimes \Delta)(R) &= R_{13} R_{12} \end{aligned} \tag{1.1}$$

hold.

Note that the universal R-matrix of a quasi-triangular bialgebra satisfies the quantum Yang–Baxter equation (QYBE)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}; \tag{1.2}$$

see, e.g., [14, Ch. 8.1.1, Prop. 2], [13, Theorem VIII.2.4].

A *coquasi-triangular bialgebra* is a bialgebra $\mathcal{A}$ equipped with a *universal r-form* on $\mathcal{A}$, i.e., a linear functional $\boldsymbol{r} : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$ that is invertible w.r.t. the convolution product (i.e., there exists another functional $\bar{\boldsymbol{r}} : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$ such that $\boldsymbol{r} \star \bar{\boldsymbol{r}} = \bar{\boldsymbol{r}} \star \boldsymbol{r} = \varepsilon \otimes \varepsilon$) that satisfies

$$\begin{aligned} m^{op} &= \boldsymbol{r} \star m \star \bar{\boldsymbol{r}}, \\ \boldsymbol{r_{13}} \star \boldsymbol{r_{23}} &= \boldsymbol{r} \circ (m \otimes \mathrm{id}), \\ \boldsymbol{r_{13}} \star \boldsymbol{r_{12}} &= \boldsymbol{r} \circ (\mathrm{id} \otimes m), \end{aligned} \tag{1.3}$$

where $\boldsymbol{r_{12}}, \boldsymbol{r_{23}}, \boldsymbol{r_{13}} : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$ are defined by $\boldsymbol{r_{12}} \coloneqq \boldsymbol{r} \otimes \varepsilon$, $\boldsymbol{r_{23}} \coloneqq \varepsilon \otimes \boldsymbol{r}$ and $\boldsymbol{r_{13}} \coloneqq (\boldsymbol{r} \otimes \varepsilon) \circ (\mathrm{id} \otimes \tau)$. Furthermore we have

$$\boldsymbol{r} \circ (\mathbb{1} \otimes \mathrm{id}) = \varepsilon = \boldsymbol{r} \circ (\mathrm{id} \otimes \mathbb{1}) \tag{1.4}$$

and, if $\mathcal{A}$ has an antipode S, then the antipode is invertible and satisfies

$$\boldsymbol{r} \circ (S \otimes \mathrm{id}) = \bar{\boldsymbol{r}} = \boldsymbol{r} \circ (\mathrm{id} \otimes S^{-1}).$$

The inverse $\bar{\boldsymbol{r}}$ satisfies similar conditions, i.e.,

$$\begin{aligned} \bar{\boldsymbol{r}}_{\mathbf{12}} \star \bar{\boldsymbol{r}}_{\mathbf{13}} \star \bar{\boldsymbol{r}}_{\mathbf{23}} &= \bar{\boldsymbol{r}}_{\mathbf{23}} \star \bar{\boldsymbol{r}}_{\mathbf{13}} \star \bar{\boldsymbol{r}}_{\mathbf{12}}, \\ \bar{\boldsymbol{r}}_{\mathbf{23}} \star \bar{\boldsymbol{r}}_{\mathbf{13}} &= \bar{\boldsymbol{r}} \circ (m \otimes \mathrm{id}), \\ \bar{\boldsymbol{r}}_{\mathbf{12}} \star \bar{\boldsymbol{r}}_{\mathbf{13}} &= \bar{\boldsymbol{r}} \circ (\mathrm{id} \otimes m), \\ \bar{\boldsymbol{r}} \circ (\mathbb{1} \otimes \mathrm{id}) &= \bar{\boldsymbol{r}} \circ (\mathrm{id} \otimes \mathbb{1}) = \varepsilon. \end{aligned} \tag{1.5}$$

2. Construction of braided categories

We will now study the special case of tensor categories whose objects are Yetter–Drinfeld modules of some given bialgebra $\mathcal{A}$. Our main goal is to construct such categories with objects that are equipped with an involution which is compatible with the braiding in the sense of Definition 1.1. Let $\mathcal{A}$ be a bialgebra. A $\mathbb{C}$-vector space V is called *(left) Yetter–Drinfeld module (over $\mathcal{A}$)* if it is both a left $\mathcal{A}$-module with action $\alpha : \mathcal{A}\otimes V \to V$ and a left $\mathcal{A}$-comodule with left coaction $\gamma : V \to \mathcal{A}\otimes V$, such that the (left) Yetter–Drinfeld equation

$$\sum a_{(1)}v^{(1)} \otimes a_{(2)}.v^{(2)} = \sum (a_{(1)}.v)^{(1)}a_{(2)} \otimes (a_{(1)}.v)^{(2)},$$

or equivalently

$$\begin{aligned}(m \otimes \alpha)\circ&(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \gamma)\\ &= (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\gamma \otimes \mathrm{id}) \circ (\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\Delta \otimes \mathrm{id}),\end{aligned} \tag{2.1}$$

is satisfied for all $a \in \mathcal{A}$, $v \in V$, where $a.b \coloneqq \alpha(a \otimes b)$ and $\gamma(v) = v^{(1)} \otimes v^{(2)}$. The category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}$ of (left) Yetter–Drinfeld modules is well studied (see for example [29] or [22, Ch. 10]). It is a braided tensor category with braiding $\Psi = (\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\gamma \otimes \mathrm{id})$. For actions α_V, α_W on a bialgebra $\mathcal{A}$, a linear map $\Phi : V \to W$ is called *(left) module map*, if $\Phi \circ \alpha_V = \alpha_W \circ (\mathrm{id} \otimes \Phi)$ and *(left) comodule map* for coactions γ_V, γ_W, if $\gamma_W \circ \Phi = (\mathrm{id} \otimes \Phi) \circ \gamma_V$.

2.1. **The category $({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*, \Psi)$ of Yetter–Drinfeld modules with an involution.** Now we want to equip the objects in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}$ with an involution and construct a new category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*$.

Lemma 2.1. *Let $\mathcal{A}$ be a Hopf-$*$-algebra with antipode S and $V \in \mathrm{Obj}({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD})$ with action α and coaction γ. Suppose that there exists a $*$-structure on V such that α and γ satisfy*

$$* \circ \alpha = \alpha \circ (* \otimes *) \circ (S \otimes \mathrm{id}), \tag{2.2}$$

$$\gamma \circ * = (* \otimes *) \circ \gamma. \tag{2.3}$$

Then $*_{V\otimes V} = (\alpha \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma) \circ (* \otimes *)$ is an involution on $V \otimes V$.

Proof. Recall that the inverse of $\Psi = (\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\gamma \otimes \mathrm{id})$ is given by

$$\Psi^{-1} = (\mathrm{id} \otimes \alpha) \circ (\tau \otimes \mathrm{id}) \circ (S^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\gamma \otimes \mathrm{id}) \circ \tau$$

(see [29, Theorem 7.2]). We have

$$\begin{aligned}(* \otimes *) \circ \Psi \circ (* \otimes *) &\overset{(2.2)\text{ and }(2.3)}{=} (\alpha \otimes \mathrm{id}) \circ (* \otimes * \otimes *) \circ (S \otimes \tau)\\ &\qquad\qquad \circ (* \otimes * \otimes *) \circ (\gamma \otimes \mathrm{id})\\ &= \tau \circ \Psi^{-1} \circ \tau,\end{aligned}$$

since $* \circ S \circ * = S^{-1}$. This proves that $*_{V\otimes V} = \Psi \circ (* \otimes *) \circ \tau$ is its own inverse. ☐

Theorem 2.2. *Let $\mathcal{A}$ be a Hopf-$*$-algebra. Then we can define a braided category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*$ as follows. The objects $(V, \alpha, \gamma, *)$ are Yetter–Drinfeld modules equipped with an involution $*$, such that equations* (2.2) *and* (2.3) *are satisfied. The morphisms are the linear maps that are module and comodule maps. The tensor product of objects is given by*

$$(V, \alpha_V, \gamma_V, *_V) \otimes (W, \alpha_W, \gamma_W, *_W) \coloneqq (V \otimes W, \alpha_{V\otimes W}, \gamma_{V\otimes W}, *_{V\otimes W}),$$

where

$$\alpha_{V\otimes W} \coloneqq (\alpha_V \otimes \alpha_W) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id} \otimes \mathrm{id}),$$

$$\gamma_{V\otimes W} \coloneqq (m\otimes \mathrm{id}\otimes \mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\gamma_V\otimes\gamma_W),$$
$$*_{V\otimes W} \coloneqq \Psi_{W,V}\circ(*_W\otimes *_V)\circ\tau_{V,W}.$$

The braiding is again given by $\Psi = (\alpha\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\gamma\otimes\mathrm{id})$.

Proof. To show that this is again a braided category it only remains to show that $(V\otimes W, \alpha_{V\otimes W}, \gamma_{V\otimes W}, *_{V\otimes W})$ is again an object in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*$. First we show that $*_{V\otimes W}$ satisfies equation (2.2). We have

$$\begin{aligned}
_{V\otimes W}\circ\alpha_{V\otimes W} &= \Psi\circ(\otimes *)\circ\tau\circ(\alpha\otimes\alpha)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\\
&\overset{(2.2)}{=} \Psi\circ(\alpha\otimes\alpha)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(*\otimes *\otimes *)\\
&\qquad\circ(S\otimes\mathrm{id}\otimes\mathrm{id})\\
&= (\alpha\otimes\alpha)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Psi)\circ(*\otimes *\otimes *)\circ(S\otimes\tau)\\
&= \alpha_{V\otimes W}\circ(*_{\mathcal{A}}\otimes *_{V\otimes W})\circ(S\otimes\mathrm{id}),
\end{aligned}$$

because Ψ is a module map, since it has to be a morphism in the category. Now we show that it also satisfies equation (2.3):

$$\begin{aligned}
&(*_{\mathcal{A}}\otimes *_{V\otimes W})\circ\gamma_{V\otimes W}\\
&\quad= (\mathrm{id}\otimes\Psi)\circ(m\otimes\mathrm{id})\circ(\tau\otimes\tau)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(*\otimes *\otimes *\otimes *)\circ(\gamma\otimes\gamma)\\
&\quad= \gamma_{V\otimes W}\circ *_{V\otimes W}.
\end{aligned}$$
□

2.2. **The category $({}^{\mathcal{A}}\mathrm{C}_*, \Psi)$ of comodules over a coquasi-triangular $*$-bialgebra.** We start by considering the category ${}^{\mathcal{A}}\mathrm{C}$ of $\mathcal{A}$-comodules over a coquasi-triangular bialgebra $\mathcal{A}$.

Lemma 2.3. *Let $\mathcal{A}$ be a coquasi-triangular bialgebra. If γ is a coaction of $\mathcal{A}$ on V, then*

$$\alpha_\gamma = (\bar{\boldsymbol{r}}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\gamma_V)$$

defines an action of $\mathcal{A}$ on V and $(V, \alpha_\gamma, \gamma)$ is a Yetter–Drinfeld module. Furthermore, if a linear map $f\colon V\to W$ between two comodules V, W is γ-invariant (i.e., a comodule map), then it is also α_γ-invariant.

Proof. First we show that α_γ is an action. We have

$$\begin{aligned}
\alpha_\gamma\circ(m\otimes\mathrm{id}) &= (\bar{\boldsymbol{r}}\otimes\mathrm{id})\circ(m\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\gamma)\\
&\overset{(1.5)}{=} \big((\bar{\boldsymbol{r}}_{\mathbf{23}}\star\bar{\boldsymbol{r}}_{\mathbf{13}})\otimes\mathrm{id}\big)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\gamma)\\
&= \Big(\Big(\big((\varepsilon\otimes\bar{\boldsymbol{r}})\otimes\big((\bar{\boldsymbol{r}}\otimes\varepsilon)\circ(\mathrm{id}\otimes\tau)\big)\big)\circ\Delta_{\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}}\Big)\otimes\mathrm{id}\Big)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\gamma)\\
&= (\bar{\boldsymbol{r}}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\gamma)\circ\Big(\mathrm{id}\otimes\big((\bar{\boldsymbol{r}}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\gamma)\big)\Big) = \alpha_\gamma\circ(\mathrm{id}\otimes\alpha_\gamma)
\end{aligned}$$

as well as

$$\alpha_\gamma\circ(\mathbb{1}\otimes\mathrm{id}) = \big((\bar{\boldsymbol{r}}\circ(\mathbb{1}\otimes\mathrm{id}))\otimes\mathrm{id}\big)\circ\gamma \overset{(1.4)}{=} (\varepsilon\otimes\mathrm{id})\circ\gamma = \mathrm{id}.$$

Now we want to show that γ and α_γ satisfies the Yetter–Drinfeld equation (2.1):

$$
\begin{aligned}
&(m\otimes\alpha_\gamma)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\gamma)\\
&\quad=(m\otimes\bar{\boldsymbol{r}}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta\otimes\Delta\otimes\mathrm{id})\circ(\mathrm{id}\otimes\gamma)\\
&\quad=\big((m\star\bar{\boldsymbol{r}})\otimes\mathrm{id}\big)\circ(\mathrm{id}\otimes\gamma)\\
&\quad\overset{(1.3)}{=}\big((\bar{\boldsymbol{r}}\star m^{op})\otimes\mathrm{id}\big)\circ(\mathrm{id}\otimes\gamma)\\
&\quad=(\bar{\boldsymbol{r}}\otimes m\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\tau)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\gamma\otimes\mathrm{id})\circ(\mathrm{id}\otimes\gamma\otimes\mathrm{id})\\
&\qquad\quad\circ(\mathrm{id}\otimes\tau)\circ(\Delta\otimes\mathrm{id})\\
&\quad=(m\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\gamma\otimes\mathrm{id})\circ(\alpha_\gamma\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\Delta\otimes\mathrm{id}).
\end{aligned}
$$

It remains to show that α_γ is a module map. We have

$$\alpha_W\circ(\mathrm{id}\otimes f)=(\bar{\boldsymbol{r}}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes f)\circ(\mathrm{id}\otimes\gamma_W). \qquad \square$$

Note that, for the braiding $\Psi=(\alpha_\gamma\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\gamma\otimes\mathrm{id})$ defined in this way between two $\mathcal{A}$-comodules V and W, we have

$$\Psi(v\otimes w)=\bar{\boldsymbol{r}}(v^{(1)}\otimes w^{(1)})w^{(2)}\otimes v^{(2)}$$

for $v\in V$, $w\in W$, and where $\gamma_V(v)=v^{(1)}\otimes v^{(2)}$, $\gamma(w)=w^{(1)}\otimes w^{(2)}$. Up to a conjugation by the flip and the use of $\bar{\boldsymbol{r}}$ instead of $\boldsymbol{r}$, this is the same as the definition of the braiding associated to a universal r-form in [13, Equation (VIII.5.9)].

One can show as in [13] that the category of $\mathcal{A}$-comodules becomes a braided category $({}^{\mathcal{A}}\mathrm{C},\Psi)$ in this way. Unlike for the construction for Yetter–Drinfeld modules in Theorem 2.2, this does not require $\mathcal{A}$ to be a Hopf algebra, because the invertibility of Ψ follows from that of the universal r-form.

Now we want to extend the objects in the category ${}^{\mathcal{A}}\mathrm{C}$ by an involution $*$ and define the category ${}^{\mathcal{A}}\mathrm{C}_*$.

A *coquasi-triangular ∗-bialgebra* (resp., *Hopf-∗-algebra*) is a coquasi-triangular bialgebra (resp., Hopf-algebra) which is also a ∗-bialgebra (resp., Hopf-∗-algebra) such that the universal r-form $\boldsymbol{r}$ satisfies the equation

$$\bar{\ }\circ\boldsymbol{r}=\bar{\boldsymbol{r}}\circ(*\otimes*), \tag{2.4}$$

where $\bar{\ }$ denotes the complex conjugation on $\mathbb{C}$. A universal r-form that satisfies equation (2.4) is called an *involutive r-form.*

Theorem 2.4. *Let $\mathcal{A}$ be a coquasi-triangular ∗-bialgebra. Then we can construct a braided category $({}^{\mathcal{A}}\mathrm{C}_*,\Psi)$ as follows. The objects are triples $(V,\gamma_V,*_V)$ consisting of an $\mathcal{A}$-comodule V with coaction γ and an involution $*$ such that equation* (2.3) *is satisfied. The morphisms between two objects are the comodule maps. The tensor product of objects is given by*

$$(V,\gamma_V,*_V)\otimes(W,\gamma_W,*_W)\coloneqq(V\otimes W,\gamma_{V\otimes W},*_{V\otimes W}),$$

where

$$
\begin{aligned}
\gamma_{V\otimes W}&\coloneqq(m\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\gamma_V\otimes\gamma_W)\\
*_{V\otimes W}&\coloneqq\Psi_{W,V}\circ(*_W\otimes*_V)\circ\tau_{V,W}.
\end{aligned}
$$

The braiding Ψ *is given by*

$$\begin{aligned}\Psi_{V,W} &= (\alpha_\gamma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\gamma \otimes \mathrm{id}) \\ &= \tau \circ (\bar{\boldsymbol{r}} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\gamma_V \otimes \gamma_W).\end{aligned}$$

Proof. Let $(V, \gamma_V, *_V)$, $(W, \gamma_W, *_W)$ be two objects of $({}^{\mathcal{A}}\mathrm{C}_*, \Psi)$, i.e., $\mathcal{A}$-comodules with a coaction and an involution that satisfy (2.3). We have to check that $*_{V\otimes W}$ is an involution and that $V \otimes W$ is again an object of $({}^{\mathcal{A}}\mathrm{C}_*, \Psi)$. For $v \in V$, $w \in W$ with $\gamma_V(v) = v^{(1)} \otimes v^{(2)}$, $\gamma(w) = w^{(1)} \otimes w^{(2)}$ we have

$$\begin{aligned}(v \otimes w)^* = \Psi(w^* \otimes v^*) &= \bar{\boldsymbol{r}}\left((w^{(1)})^* \otimes (v^{(1)})^*\right)\left(v^{(2)}\right)^* \otimes \left(w^{(2)}\right)^* \\ &= \overline{\boldsymbol{r}\left(w^{(1)} \otimes v^{(1)}\right)}\left(v^{(2)}\right)^* \otimes \left(w^{(2)}\right)^*\end{aligned}$$

and

$$\begin{aligned}((v \otimes w)^*)^* &= \boldsymbol{r}\left(v^{(1)} \otimes w^{(1)}\right)\left(\left(v^{(2)}\right)^* \otimes \left(w^{(2)}\right)^*\right)^* \\ &= \boldsymbol{r}\left(w^{(1)} \otimes v^{(1)}\right)\Psi\left(w^{(2)} \otimes v^{(2)}\right) \\ &= \boldsymbol{r}\left(w^{(1)} \otimes v^{(1)}\right)\bar{\boldsymbol{r}}\left(w^{(2)} \otimes v^{(2)}\right) v^{(3)} \otimes w^{(3)} \\ &= v \otimes w,\end{aligned}$$

since $r \star \bar{r} = \varepsilon \otimes \varepsilon$. Furthermore,

$$\begin{aligned}\gamma_{V\otimes W} \circ *_{V\otimes W}(v \otimes w) &= \gamma_{V\otimes W}\left(\overline{\boldsymbol{r}\left(w^{(1)} \otimes v^{(1)}\right)}\left(v^{(2)}\right)^* \otimes \left(w^{(2)}\right)^*\right) \\ &= \left(\boldsymbol{r}\left(w^{(1)} \otimes v^{(1)}\right) w^{(2)} v^{(2)}\right)^* \left(v^{(3)}\right)^* \otimes \left(w^{(3)}\right)^* \\ &= \left(v^{(1)} w^{(1)} \boldsymbol{r}\left(w^{(2)} \otimes v^{(2)}\right)\right)^* \left(v^{(3)}\right)^* \otimes \left(w^{(3)}\right)^* \\ &= \left(v^{(1)} w^{(1)}\right)^* \otimes \left(\overline{\boldsymbol{r}\left(w^{(2)} \otimes v^{(2)}\right)}\left(v^{(3)}\right)^* \otimes \left(w^{(3)}\right)^*\right) \\ &= \left(\gamma_{V\otimes W}(v \otimes w)\right)^{*\otimes *_{V\otimes W}},\end{aligned}$$

since $\boldsymbol{r} \star m = m^{\mathrm{op}} \star \boldsymbol{r}$. I.e., $\gamma_{V\otimes W}$ satisfies (2.3), and therefore $(V \otimes W, \gamma_{V\otimes W}, *_{V\otimes W})$ is an object of $({}^{\mathcal{A}}\mathrm{C}_*, \Psi)$. □

Remark 2.5. In general, $\mathcal{A}$ does not have an antipode. Thus equation (2.2) is not satisfied and $(V, \alpha_\gamma, \gamma, *)$ is not an object in $({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*, \Psi)$, and therefore $({}^{\mathcal{A}}\mathrm{C}_*, \Psi)$ cannot be interpreted as a subcategory of $({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*, \Psi)$. But if $\mathcal{A}$ is a coquasi-triangular Hopf-$*$-algebra, then the antipode S is automatically invertible and we have $\bar{\boldsymbol{r}} = \boldsymbol{r} \circ (S \otimes \mathrm{id})$, which, combined with equation (2.4), implies that α_γ satisfies equation (2.2). So if $\mathcal{A}$ is a coquasi-triangular Hopf-$*$-algebra, then because of Lemma 2.3, the category ${}^{\mathcal{A}}\mathrm{C}_*$ can be viewed as a subcategory of ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*$.

2.3. **The category $({}_{\mathcal{A}}\mathrm{C}_*, \Psi)$ of modules over a quasi-triangular $*$-bialgebra.** We start with the category ${}_{\mathcal{A}}\mathrm{C}$ of $\mathcal{A}$-modules over a quasi-triangular bialgebra $\mathcal{A}$, and then introduce the category ${}_{\mathcal{A}}\mathrm{C}_*$, if $\mathcal{A}$ has an involution that is compatible with the R-matrix, see below. Let $\mathcal{A}$ be a quasi-triangular bialgebra and define $\widetilde{R} : \mathbb{C} \to \mathcal{A} \otimes \mathcal{A}$ by $\widetilde{R}(c) = cR$ for all $c \in \mathbb{C}$.

Lemma 2.6. *If α is an action of $\mathcal{A}$ on V, then*

$$\gamma_\alpha = (\mathrm{id} \otimes \alpha) \circ (\tau \otimes \mathrm{id}) \circ (\widetilde{R} \otimes \mathrm{id})$$

defines a coaction of $\mathcal{A}$ on V and $(V, \alpha, \gamma_\alpha)$ is a Yetter–Drinfeld module. Furthermore, if a linear map $f \colon V \to W$ between two modules V, W is α-invariant (i.e., a module map), then it is also γ_α-invariant.

Proof. First we show that γ_α is a coaction. We have

$$\begin{aligned}
&(\Delta \otimes \mathrm{id}) \circ \gamma_\alpha \\
&\quad = (\mathrm{id} \otimes \mathrm{id} \otimes \alpha) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta \otimes \mathrm{id}) \circ (\widetilde{R} \otimes \mathrm{id}) \\
&\quad \overset{(1.1)}{=} (\mathrm{id} \otimes \mathrm{id} \otimes \alpha) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (m \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}) \\
&\qquad\quad \circ (\mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\widetilde{R} \otimes \widetilde{R} \otimes \mathrm{id}) \\
&\quad = (\mathrm{id} \otimes \gamma_\alpha) \circ \gamma_\alpha,
\end{aligned}$$

as well as

$$(\varepsilon \otimes \mathrm{id}) \circ \gamma_\alpha = \alpha \circ (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}) \circ (\widetilde{R} \otimes \mathrm{id}) = \alpha \circ (\mathbb{1} \otimes \mathrm{id}) = \mathrm{id}.$$

Now we show that α and γ_α satisfy the Yetter–Drinfeld equation (2.1):

$$\begin{aligned}
&(m \otimes \alpha) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \gamma_\alpha) \\
&\quad = (m \otimes \alpha) \circ (\mathrm{id} \otimes \tau \otimes \alpha) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \widetilde{R} \otimes \mathrm{id}) \\
&\quad = (\mathrm{id} \otimes \alpha) \circ (m \otimes \mathrm{id} \otimes \alpha) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\tau \otimes \tau \otimes \mathrm{id}) \circ (\widetilde{R} \otimes \Delta \otimes \mathrm{id}) \\
&\quad = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\gamma_\alpha \otimes \mathrm{id}) \circ (\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\Delta \otimes \mathrm{id}).
\end{aligned}$$

That γ_α is a comodule map follows directly. ☐

Define a *quasi-triangular $*$-bialgebra* as a quasi-triangular bialgebra which is also a $*$-bialgebra such that $(* \otimes *)(R) = R^{-1}$.

Theorem 2.7. *Let $\mathcal{A}$ be a quasi-triangular Hopf-$*$-algebra with antipode S. Then we can construct a braided category $({}_{\mathcal{A}}\mathrm{C}_*, \Psi)$ as follows. The objects are triples $(V, \alpha, *)$ consisting of an $\mathcal{A}$-module V with action α and an involution $*$ such that equation* (2.2) *is satisfied. The morphisms between two objects are the module maps. The tensor product of objects is given by*

$$(V, \alpha_V, *_V) \otimes (W, \alpha_W, *_W) \coloneqq (V \otimes W, \alpha_{V\otimes W}, *_{V \otimes W}),$$

where

$$\begin{aligned}
\alpha_{V \otimes W} &\coloneqq (\alpha_V \otimes \alpha_W) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id} \otimes \mathrm{id}), \\
*_{V \otimes W} &\coloneqq \Psi_{W,V} \circ (*_W \otimes *_V) \circ \tau_{V,W}.
\end{aligned}$$

The braiding Ψ is given by

$$\begin{aligned}\Psi_{V,W} &= (\alpha\otimes \mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\gamma_\alpha\otimes\mathrm{id})\\ &= (\alpha\otimes\alpha)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\tau\otimes\tau)\circ(\widetilde{R}\otimes\mathrm{id}\otimes\mathrm{id}).\end{aligned}$$

Proof. This follows directly from Lemma 2.6 and Theorem 2.2. ☐

Lemma 2.8. *Let $\mathcal{A}$ be a quasi-triangular Hopf-$*$-algebra and let $(V,\alpha,*)$ be an object in $({}_{\mathcal{A}}\mathrm{C}_*,\Psi)$. Then γ_α satisfies equation* (2.3)*.*

Proof. Since $(S\otimes\mathrm{id})\circ\widetilde{R}=\widetilde{R^{-1}}$, we have

$$\begin{aligned}(*\otimes *)\circ\gamma_\alpha &= (*\otimes *)\circ(\mathrm{id}\otimes\alpha)\otimes(\tau\otimes\mathrm{id})\circ(\widetilde{R}\otimes\mathrm{id})\\ &\overset{(2.2)}{=} (\mathrm{id}\otimes\alpha)\circ(*\otimes *\otimes *)\circ(\mathrm{id}\otimes S\otimes\mathrm{id})\circ(\tau\otimes\mathrm{id})\circ(\widetilde{R}\otimes\mathrm{id})\\ &= (\mathrm{id}\otimes\alpha)\circ(\tau\otimes\mathrm{id})\circ(*\otimes *\otimes *)\circ(\widetilde{R^{-1}}\otimes\mathrm{id})\\ &= (\mathrm{id}\otimes\alpha)\circ(\widetilde{R}\otimes *)=\gamma_\alpha\circ *.\end{aligned}$$ ☐

Remark 2.9. It follows from Lemma 2.8 and Lemma 2.6 that $(V,\alpha,\gamma_\alpha,*)\in\mathrm{Obj}({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD})$ and hence $({}_{\mathcal{A}}\mathrm{C}_*,\Psi)$ is a subcategory of $\left({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*,\Psi\right)$.

2.3.1. *Cocommutative bialgebras.* If the bialgebra $\mathcal{A}$ is cocommutative (i.e., $\tau\circ\Delta=\Delta$), then $1\otimes 1$ defines an R-matrix. The corresponding braiding is simply the flip τ.

We show now that the construction in Theorem 2.2 includes as a special case the construction given in [27]. Schürmann's construction has as input the group algebra $\mathcal{A}=\mathbb{C}\Gamma$ of some group Γ, and actions and coactions of $\mathcal{A}$ on the objects V that satisfy the compatibility condition

$$\gamma\circ\alpha=(\mathrm{ad}\otimes\alpha)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\gamma), \tag{2.5}$$

(see [27, p. 14]), where $\mathrm{ad}=m\circ(\mathrm{id}\otimes m)\circ(\mathrm{id}\otimes\mathrm{id}\otimes S)\circ(\mathrm{id}\otimes\tau)\circ(\Delta\otimes\mathrm{id})$ is the adjoint action of $\mathcal{A}$ on itself. The following lemma shows that our construction is a generalisation of the one presented there.

Lemma 2.10. *Let $\mathcal{A}$ be a cocommutative bialgebra. Let α and γ be an action and a coaction of $\mathcal{A}$ on some vector space V. If α and γ satisfy* (2.5)*, then (V,α,γ) is a Yetter–Drinfeld module.*

Proof. Substituting (2.5) into the right side of (2.1), we get after some simplifications

$$(m^{(4)}\otimes\alpha)\circ(\mathrm{id}\otimes\tau\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau_{\mathcal{A}\otimes\mathcal{A},\mathcal{A}}\circ\mathrm{id})\circ(\mathrm{id}_{\mathcal{A}}\otimes S\otimes\mathrm{id})\circ(\Delta^{(4)}\otimes\gamma),$$

where $m^{(4)}=m\circ(m\otimes\mathrm{id})\circ(m\otimes\mathrm{id}\otimes\mathrm{id})$, $\Delta^{(4)}=(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id})\circ\Delta$. Using the cocommutativity, we can produce a term of the form $m\circ(S\otimes\mathrm{id})\circ\Delta$, to which we can apply the antipode axiom. Using the unit and the counit axiom to clean up the resulting expression, we get the desired result. ☐

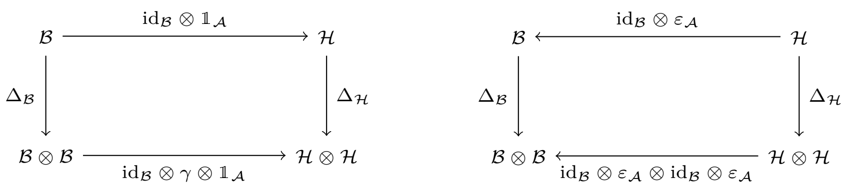
3. Symmetrization of braided $*$-bialgebras and their Lévy processes

3.1. **Symmetrizing braided $*$-bialgebras.** Now we will present a construction that will allow us in Subsection 3.2 to associate with every Lévy process on a braided $*$-bialgebra $\mathcal{B}$ a Lévy process on a usual (i.e., *symmetric* or τ-braided) $*$-bialgebra. The idea is to construct a bigger (symmetric) bialgebra $\mathcal{H}$ that contains the braided bialgebra $\mathcal{B}$ as a subalgebra and whose coproduct is related to that of $\mathcal{B}$ in a "nice way". For the case where the braiding is defined through the action and coaction of a group, this construction can be found in [27, Ch. 3]. For the general case a similar construction, called bosonization, was introduced by Majid (see [19, Section 9.4] and the references therein). But the role of the involution is not studied there. In this section we study the left symmetrization of braided $*$-bialgebras. Note that the whole theory works analogously for categories consisting of right modules and comodules.

Theorem 3.1. *Let $\mathcal{A}$ be a Hopf-$*$-algebra and let $\mathcal{B}$ be a braided $*$-bialgebra in $\big({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*, \Psi\big)$. Then $\mathcal{H} = \mathcal{B} \otimes \mathcal{A}$ (as a vector space) becomes a $*$-bialgebra with*

$$
\begin{aligned}
m_{\mathcal{H}} &= (m_{\mathcal{B}} \otimes m_{\mathcal{A}}) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \mathrm{id}),\\
\mathbb{1}_{\mathcal{H}} &= \mathbb{1}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{A}},\\
\Delta_{\mathcal{H}} &= (\mathrm{id} \otimes m \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta_{\mathcal{B}} \otimes \Delta_{\mathcal{A}}),\\
\varepsilon_{\mathcal{H}} &= \varepsilon_{\mathcal{B}} \otimes \varepsilon_{\mathcal{A}},\\
*_{\mathcal{H}} &= (\alpha \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ (*_{\mathcal{B}} \otimes *_{\mathcal{A}}).
\end{aligned}
$$

The map $\mathrm{id}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{A}} \colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A} \cong \mathcal{H}$ defines an embedding, i.e., an injective $*$-algebra homomorphism. Furthermore we have the following commutative diagrams:



Proof. For the proof that $\mathcal{H}$ is a bialgebra, see [19, Section 9.4]. To show that it is even a $*$-bialgebra we first have to verify that $*_{\mathcal{H}}$ is its own inverse. Using the facts that Δ is a $*$-algebra homomorphism and $*$ is self-inverse, as well as equation (2.2) and the relation $* \circ S \circ * = S^{-1}$, we get

$$
\begin{aligned}
*_{\mathcal{H}} \circ *_{\mathcal{H}} = (\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})\\
\circ (\mathrm{id} \otimes \mathrm{id} \otimes \Delta) \circ (S^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta).
\end{aligned}
$$

Applying the module equation and the coassociativity, this expression transforms into

$$
(\alpha \otimes \mathrm{id}) \circ \big((m \circ (\mathrm{id} \otimes S^{-1}) \circ \tau \circ \Delta) \otimes \mathrm{id} \otimes \mathrm{id}\big) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta),
$$

and, using the antipode axiom, into

$$(\alpha\otimes\mathrm{id})\circ((\mathbb{1}\circ\varepsilon)\otimes\tau)\circ(\Delta\otimes\mathrm{id})\circ\tau=\tau\circ\tau=\mathrm{id}\otimes\mathrm{id},$$

which shows that $*_{\mathcal{H}}$ is its own inverse. Next we want to show that $*_{\mathcal{H}}$ is an algebra antihomomorphism. After applying some basic transformations, as well as equation (2.2) and the bialgebra equation

$$\Delta\circ m=(m\otimes m)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\Delta)$$

twice, we get the expression

$$\begin{aligned}*_{\mathcal{H}}\circ m_{\mathcal{H}}=&(m\otimes\mathrm{id})\circ(\alpha\otimes\alpha\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\tau)\circ(m\otimes m\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\\&\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\Delta\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\\&\circ(\mathrm{id}\otimes\mathrm{id}\otimes m\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\\&\circ(\Delta\otimes\Delta\otimes\mathrm{id}\otimes\mathrm{id})\circ\tau_{\mathcal{H}\otimes\mathcal{H}}\circ(\tau\otimes\tau)\circ(\mathrm{id}\otimes\alpha\otimes\mathrm{id}\otimes\mathrm{id})\\&\circ(\mathrm{id}\otimes S^{-1}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\mathrm{id})\circ(*\otimes*\otimes*\otimes*).\end{aligned}$$

Again after using some basic transformations and reordering, we get the expression

$$\begin{aligned}(m\otimes m)&\circ(\mathrm{id}\otimes\alpha\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\mathrm{id})\\&\circ(\alpha\otimes\mathrm{id}\otimes\alpha\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\tau)\circ(\Delta\otimes\mathrm{id}\otimes\Delta\otimes\mathrm{id})\circ(\tau\otimes\tau)\\&\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\tau\otimes\tau)\circ(\mathrm{id}\otimes\alpha\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes m\otimes\tau\otimes\mathrm{id})\\&\circ(\mathrm{id}\otimes S\otimes\mathrm{id}\otimes S\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\\&\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\mathrm{id})\\&\circ(\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\circ(*\otimes*\otimes*\otimes*).\end{aligned}$$

After applying the antipode axiom and some simplifications, this becomes

$$\begin{aligned}(m\otimes m)&\circ(\mathrm{id}\otimes\alpha\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\mathrm{id})\\&\circ(\alpha\otimes\mathrm{id}\otimes\alpha\otimes\mathrm{id})\circ(\tau\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\Delta)\circ(*\otimes*\otimes*\otimes*)\\&\circ(\tau\otimes\tau)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})=m_{\mathcal{H}}\circ(*_{\mathcal{H}}\otimes*_{\mathcal{H}})\circ\tau_{\mathcal{H}\otimes\mathcal{H}}.\end{aligned}$$

Thus we have shown that $*_{\mathcal{H}}$ is an algebra homomorphism.

Now we want to show that Δ is a $*$-algebra homomorphism. After using the coassociativity twice and some reordering, we get the expression

$$\begin{aligned}&\Delta_{\mathcal{H}}\circ*_{\mathcal{H}}\\&\quad=\big(\mathrm{id}\otimes\big((m\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\gamma\otimes\mathrm{id})\circ(\alpha\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\Delta\otimes\mathrm{id})\big)\otimes\mathrm{id}\big)\\&\qquad\circ\big(\big((\alpha\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\Delta\otimes\mathrm{id})\big)\otimes\mathrm{id}\otimes\mathrm{id}\big)\\&\qquad\circ(\tau\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\Delta)\circ(*\otimes*).\end{aligned}$$

Using the Yetter–Drinfeld equation as well as the equations (2.2) and (2.3), this transforms into

$$\begin{aligned}(*\otimes*\otimes*\otimes*)&\circ\big(\mathrm{id}\otimes\big((m\otimes\alpha)\circ(\tau\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\big)\otimes\mathrm{id}\big)\\&\circ\big(\big((\alpha\otimes\mathrm{id})\circ(S^{-1}\otimes\tau)\circ(\Delta\otimes\mathrm{id})\circ\tau\big)\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\big)\circ(\alpha\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id})\\&\circ(S^{-1}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\tau\otimes\gamma\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\gamma\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta\otimes\Delta).\end{aligned}$$

After a few more basic transformations, this becomes

$$\begin{aligned}(*\otimes *\otimes *\otimes *)\circ\big(((\alpha\circ(S^{-1}\otimes \mathrm{id})\circ\tau)\otimes \mathrm{id}\otimes(\alpha\circ(S^{-1}\otimes\mathrm{id})\circ\tau)\otimes\mathrm{id}\big)\\ \circ\big(\mathrm{id}\otimes\big((m\otimes m)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\Delta)\big)\otimes\mathrm{id}\otimes\mathrm{id}\big)\\ \circ(\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\gamma\otimes\mathrm{id}\otimes\Delta)\circ(\Delta\otimes\Delta).\end{aligned}$$

Using the bialgebra equation, we get

$$\begin{aligned}\big(((\alpha\otimes\mathrm{id})\circ(\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)\circ(*\otimes *))\\ \otimes\big((\alpha\otimes\mathrm{id})\circ(\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)\circ(*\otimes *)\big)\big)\\ \circ\big(\mathrm{id}\otimes\big((m\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\gamma\otimes\mathrm{id})\big)\otimes\mathrm{id}\big)\circ(\Delta\otimes\Delta),\end{aligned}$$

and thus we have shown that Δ is a $*$-algebra homomorphism. It remains to show that the inclusions are $*$-algebra homomorphisms and that both diagrams commute. This can easily be verified by straightforward calculations. □

Due to the connection between the categories we get the following corollaries for the categories $\big({}^{\mathcal{A}}\mathrm{C}_*,\Psi\big)$ and $\big({}_{\mathcal{A}}\mathrm{C}_*,\Psi\big)$.

Corollary 3.2. *Let $\mathcal{A}$ be a coquasi-triangular $*$-bialgebra and $\mathcal{B}$ a braided $*$-bialgebra in $\big({}^{\mathcal{A}}\mathrm{C}_*,\Psi\big)$. Then $\mathcal{H}=\mathcal{B}\otimes\mathcal{A}$ (as a vector space) becomes a $*$-bialgebra with $\mathbb{1}_{\mathcal{H}}$, $\Delta_{\mathcal{H}}$ and $\varepsilon_{\mathcal{H}}$ as in Theorem 3.1 and*

$$\begin{aligned}m_{\mathcal{H}}&=(m_{\mathcal{B}}\otimes m_{\mathcal{A}})\circ(\mathrm{id}\otimes\alpha_\gamma\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\mathrm{id})\\ &=(m_{\mathcal{B}}\otimes m_{\mathcal{A}})\circ(\mathrm{id}\otimes\bar{\boldsymbol{r}}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta\otimes\gamma\otimes\mathrm{id}),\\ *_{\mathcal{H}}&=(\alpha_\gamma\otimes\mathrm{id})\circ(\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)\circ(*_{\mathcal{B}}\otimes *_{\mathcal{A}})\\ &=(\bar{\boldsymbol{r}}\otimes\mathrm{id}\otimes\mathrm{id})\circ(\tau\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\gamma\otimes\Delta)\circ(*_{\mathcal{B}}\otimes *_{\mathcal{A}}).\end{aligned}$$

Corollary 3.3. *Let $\mathcal{A}$ be a quasi-triangular $*$-bialgebra and $\mathcal{B}$ a braided $*$-bialgebra in $({}_{\mathcal{A}}\mathrm{C}_*,\Psi)$. Then $\mathcal{H}=\mathcal{B}\otimes\mathcal{A}$ (as a vector space) becomes a $*$-bialgebra with $m_{\mathcal{H}}$, $\mathbb{1}_{\mathcal{H}}$, $\varepsilon_{\mathcal{H}}$ and $*_{\mathcal{H}}$ as in Theorem 3.1 and*

$$\begin{aligned}\Delta_{\mathcal{H}}&=(\mathrm{id}\otimes m\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\gamma_\alpha\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta_{\mathcal{B}}\otimes\Delta_{\mathcal{A}})\\ &=(\mathrm{id}\otimes m\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\alpha\otimes\mathrm{id}\otimes\mathrm{id})\\ &\qquad\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\widetilde{R}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta\otimes\Delta).\end{aligned}$$

3.2. **Symmetrizing braided Lévy processes.** The following proposition is important for symmetrizing Lévy processes, i.e., for constructing a Lévy process on $\mathcal{H}$ for a given Lévy process on $\mathcal{B}$. In Theorem 3.5 below we shall show that the process we construct on $\mathcal{H}$ allows us to recover a process on $\mathcal{B}$ which is equivalent to the original process.

Proposition 3.4. *The map $F:\mathcal{B}'\to\mathcal{H}'$, $\varphi\mapsto\varphi\otimes\varepsilon_{\mathcal{A}}$ is a unital injective algebra homomorphism w.r.t. the convolution product. Furthermore, it maps positive (resp., hermitian, conditionally positive) Ψ-invariant functionals $\varphi\in\mathcal{B}'$ to positive (resp., hermitian, conditionally positive) functionals $F(\varphi)\in\mathcal{H}'$.*

Proof. The injectivity of F is clear, because $\widetilde{F} : \mathcal{H}' \to \mathcal{B}' \cong (\mathcal{B} \otimes \mathbb{1})'$ defined by $\widetilde{F}(\psi) = \psi \circ (\mathrm{id} \otimes \mathbb{1})$ is a left inverse of F. It is unital, since $F(\varepsilon_{\mathcal{B}}) = \varepsilon_{\mathcal{B}} \otimes \varepsilon_{\mathcal{A}} = \varepsilon_{\mathcal{H}}$. Furthermore it preserves the convolution product, because

$$\begin{aligned}
F(\varphi_1) \star F(\varphi_2) &= (F(\varphi_1) \otimes F(\varphi_2)) \circ \Delta \\
&= (\varphi_1 \otimes \varepsilon \otimes \varphi_2 \otimes \varepsilon) \circ (\mathrm{id} \otimes m \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id}) \\
&\qquad \circ (\mathrm{id} \otimes \gamma \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \\
&= (\varphi_1 \otimes \varepsilon \otimes \varphi_2 \otimes \varepsilon \otimes \varepsilon) \circ (\mathrm{id} \otimes \gamma \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \\
&= ((\varphi_1 \otimes \varphi_2) \circ \Delta) \otimes \varepsilon \\
&= F((\varphi_1 \otimes \varphi_2) \circ \Delta) \\
&= F(\varphi_1 \star \varphi_2).
\end{aligned}$$

Assume now that φ is positive and Ψ-invariant. Let $c = \sum_k b_k \otimes a_k \in \mathcal{B} \otimes \mathcal{A} \cong \mathcal{H}$. We want to show that $F(\varphi)$ is again positive. Because $(b_k \otimes a_k)^* = \Psi(a_k^* \otimes b_k^*)$, we have

$$\begin{aligned}
c^*c = \Big(\sum_k b_k \otimes a_k\Big)^* \Big(\sum_k b_k \otimes a_k\Big) &= m_{\mathcal{H}}\Big(\sum_{k,l} (b_k \otimes a_k)^* \otimes (b_l \otimes a_l)\Big) \\
&= m_{\mathcal{H}}\Big(\sum_{k,l} \Psi(a_k^* \otimes b_k^*) \otimes b_l \otimes a_l\Big).
\end{aligned}$$

Since m is a morphism and thus Ψ-invariant, we have

$$(m \otimes m) \circ (\mathrm{id} \otimes \Psi \otimes \mathrm{id}) \circ (\Psi \otimes \mathrm{id} \otimes \mathrm{id}) = (\mathrm{id} \otimes m) \circ (\Psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m \otimes \mathrm{id}),$$

and therefore we get

$$\begin{aligned}
&\big(F(\varphi)\big)(c^*c) \\
&\quad = (\varphi \otimes \varepsilon) \circ m_{\mathcal{H}}\Big(\sum_{k,l} \Psi(a_k^* \otimes b_k^*) \otimes b_l \otimes a_l\Big) \\
&\quad = (\varphi \otimes \varepsilon \circ (m \otimes m) \circ (\mathrm{id} \otimes \Psi \otimes \mathrm{id}) \circ (\Psi \otimes \mathrm{id} \otimes \mathrm{id})\Big(\sum_{k,l} a_k^* \otimes b_k^* \otimes b_l \otimes a_l\Big) \\
&\quad = (\varphi \otimes \varepsilon) \circ (\mathrm{id} \otimes m) \circ (\Psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m \otimes \mathrm{id})\Big(\sum_{k,l} a_k^* \otimes b_k^* \otimes b_l \otimes a_l\Big) \\
&\quad = \varepsilon \circ m \circ (\Psi_{\mathcal{A},\mathbb{C}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m \otimes \mathrm{id})\Big(\sum_{k,l} a_k^* \otimes b_k^* \otimes b_l \otimes a_l\Big) \\
&\quad = (\varepsilon \circ m)\Big(\sum_{k,l} \varphi(b_k^* b_l) a_k^* \otimes a_l\Big) \\
&\quad = \sum_{k,l} \varphi(b_k^* b_l) \varepsilon(a_k^* a_l).
\end{aligned}$$

This is positive, since it is the Schur product of two positive definite matrices. Conditional positivity can be shown similarly and hermitianity is a straightforward calculation. □

Theorem 3.5. *Let $\mathcal{B}$ be a braided $*$-bialgebra in one of the categories $\left({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*\right)$, $\left({}^{\mathcal{A}}\mathrm{C}_*, \Psi\right)$ or $\left({}_{\mathcal{A}}\mathrm{C}_*, \Psi\right)$. Let $(j_{st})_{0\le s\le t}$ be a Lévy process on $\mathcal{B}$ with convolution semigroup $(\varphi_t)_{t\ge 0}$ and let $\left(j^{\mathcal{H}}_{st}\right)_{0\le s\le t}$ be a Lévy process on $\mathcal{H}\cong\mathcal{B}\otimes\mathcal{A}$ with convolution semigroup $(F(\varphi_t))_{t\ge0}$. Then $(\hat{\jmath}_{st})_{0\le s\le t}$ with*

$$\hat{\jmath}_{st} := m\circ\left(j^{\mathcal{H}}_{0s}\otimes j^{\mathcal{H}}_{st}\right)\circ\left(\mathbb{1}_{\mathcal{B}}\otimes\gamma\otimes\mathbb{1}_{\mathcal{A}}\right)$$

defines a Lévy process on $\mathcal{B}$. Furthermore $(\hat{\jmath}_{st})_{0\le s\le t}$ is equivalent to $(j_{st})_{0\le s\le t}$.

Remark 3.6. This theorem generalizes [27, Theorem 3.3.1].

Proof. We will use Sweedler's notation $\Delta_{\mathcal{B}}(b)=\sum b_{(1)}\otimes b_{(2)}\in\mathcal{B}\otimes\mathcal{B}$ and $\gamma(b)=\sum b^{(1)}\otimes b^{(2)}\in\mathcal{B}\otimes\mathcal{A}$ for the coproduct and coaction on an element $b\in\mathcal{B}$. We have

$$\hat{\jmath}_{st}(b)=m\circ\left(j^{\mathcal{H}}_{0s}\otimes j^{\mathcal{H}}_{st}\right)\circ\left(\mathbb{1}_{\mathcal{B}}\otimes\gamma\otimes\mathbb{1}_{\mathcal{A}}\right)(b)=j^{\mathcal{H}}_{0s}\left(\mathbb{1}\otimes b^{(1)}\right)\cdot j^{\mathcal{H}}_{st}\left(b^{(2)}\otimes\mathbb{1}\right)\tag{3.1}$$

as well as

$$\begin{aligned} j^{\mathcal{H}}_{0s}\left(1\otimes a\right)&=\left(j^{\mathcal{H}}_{0r}\star j^{\mathcal{H}}_{rs}\right)\left(\mathbb{1}\otimes a\right)=m\circ\left(j^{\mathcal{H}}_{0r}\otimes j^{\mathcal{H}}_{rs}\right)\circ\Delta_{\mathcal{H}}\left(\mathbb{1}\otimes a\right)\\ &=j^{\mathcal{H}}_{0r}\left(\mathbb{1}\otimes a_{(1)}\right)\cdot j^{\mathcal{H}}_{rs}\left(\mathbb{1}\otimes a_{(2)}\right)\end{aligned}\tag{3.2}$$

and

$$\Delta_{\mathcal{H}}\left(b^{(2)}\otimes\mathbb{1}\right)=b^{(2)}{}_{(1)}\otimes b^{(2)}{}_{(2)}{}^{(1)}\otimes b^{(2)}{}_{(2)}{}^{(2)}\otimes\mathbb{1}.\tag{3.3}$$

Let $0\le r\le s\le t$. Then we have

$$\begin{aligned}(\hat{\jmath}_{rs}\star\hat{\jmath}_{st})(b)&=m\circ(\hat{\jmath}_{rs}\otimes\hat{\jmath}_{st})\circ\Delta(b)\\ &=\hat{\jmath}_{rs}(b_{(1)})\cdot\hat{\jmath}_{st}(b_{(2)})\\ &\overset{(3.1)}{=}j^{\mathcal{H}}_{0r}(\mathbb{1}\otimes b_{(1)}{}^{(1)})\cdot j^{\mathcal{H}}_{rs}(b_{(1)}{}^{(2)}\otimes\mathbb{1})\cdot j^{\mathcal{H}}_{0s}(\mathbb{1}\otimes b_{(2)}{}^{(1)})\cdot j^{\mathcal{H}}_{st}(b_{(2)}{}^{(2)}\otimes\mathbb{1})\\ &\overset{(3.2)}{=}j^{\mathcal{H}}_{0r}(\mathbb{1}\otimes b_{(1)}{}^{(1)})\cdot j^{\mathcal{H}}_{0r}(\mathbb{1}\otimes b_{(2)}{}^{(1)}{}_{(1)})\cdot j^{\mathcal{H}}_{rs}(b_{(1)}{}^{(2)}\otimes\mathbb{1})\\ &\qquad\cdot j^{\mathcal{H}}_{rs}(\mathbb{1}\otimes b_{(2)}{}^{(1)}{}_{(2)})\cdot j^{\mathcal{H}}_{st}(b_{(2)}{}^{(2)}\otimes\mathbb{1})\\ &=j^{\mathcal{H}}_{0r}(\mathbb{1}\otimes b_{(1)}{}^{(1)}\cdot b_{(2)}{}^{(1)}{}_{(1)})\cdot j^{\mathcal{H}}_{rs}(b_{(1)}{}^{(2)}\otimes b_{(2)}{}^{(1)}{}_{(2)})\cdot j^{\mathcal{H}}_{st}(b_{(2)}{}^{(2)}\otimes\mathbb{1})\\ &=m\circ(m\otimes\mathrm{id})\circ(j^{\mathcal{H}}_{0r}\otimes j^{\mathcal{H}}_{rs}\otimes j^{\mathcal{H}}_{st})\\ &\qquad\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\gamma\otimes\mathrm{id})\circ(\mathrm{id}\otimes m\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\\ &\qquad\circ(\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\gamma\otimes\gamma\otimes\mathrm{id})\circ(\mathbb{1}_{\mathcal{B}}\otimes\Delta\otimes\mathbb{1}_{\mathcal{A}})(b)\\ &=j^{\mathcal{H}}_{0r}(1\otimes b^{(1)})\cdot j^{\mathcal{H}}_{rs}(b^{(2)}{}_{(1)}\otimes b^{(2)}{}_{(2)}{}^{(1)})\cdot j^{\mathcal{H}}_{st}(b^{(2)}{}_{(2)}{}^{(2)}\otimes\mathbb{1})\\ &\overset{(3.3)}{=}j^{\mathcal{H}}_{0r}(\mathbb{1}\otimes b^{(1)})\cdot\left(m\circ(j^{\mathcal{H}}_{rs}\otimes j^{\mathcal{H}}_{st})\circ\Delta_{\mathcal{H}}(b^{(2)}\otimes\mathbb{1})\right)\\ &=j^{\mathcal{H}}_{0r}(\mathbb{1}\otimes b^{(1)})\cdot(j^{\mathcal{H}}_{rs}\star j^{\mathcal{H}}_{st})(b^{(2)}\otimes\mathbb{1})=j^{\mathcal{H}}_{0r}(\mathbb{1}\otimes b^{(1)})\cdot j^{\mathcal{H}}_{rt}(b^{(2)}\otimes 1)\\ &=m\circ(j^{\mathcal{H}}_{0r}\otimes j^{\mathcal{H}}_{rt})\circ(\mathbb{1}_{\mathcal{B}}\otimes\gamma\otimes\mathbb{1}_{\mathcal{A}})(b)=\hat{\jmath}_{rt}(b).\end{aligned}$$

Thus we have shown that $(\hat{\jmath}_{st})_{0\leq s\leq t}$ satisfies the increment property. Furthermore, because of the independence of increments, it follows that

$$\begin{aligned}\Phi\circ\hat{\jmath}_{st} &= \big((\Phi\circ j^{\mathcal{H}}_{0s})\otimes(\Phi\circ j^{\mathcal{H}}_{st})\big)\circ(\mathbb{1}_{\mathcal{B}}\otimes\gamma\otimes\mathbb{1}_{\mathcal{A}})\\ &= \big(F(\varphi_s)\otimes F(\varphi_{t-s})\big)\circ(\mathbb{1}_{\mathcal{B}}\otimes\gamma\otimes\mathbb{1}_{\mathcal{A}})\\ &= (\varphi_s\otimes\varepsilon_{\mathcal{A}}\otimes\varphi_{t-s}\otimes\varepsilon_{\mathcal{A}})\circ(\mathbb{1}_{\mathcal{B}}\otimes\gamma\otimes\mathbb{1}_{\mathcal{A}})\\ &= \varphi_{t-s},\end{aligned}$$

i.e., the processes $(j_{st})_{0\leq s\leq t}$ and $(\hat{\jmath}_{st})_{0\leq s\leq t}$ have the same marginal distributions. This implies the stationarity and the weak continuity of the increments of $(\hat{\jmath}_{st})_{0\leq s\leq t}$ and completes the proof that $(\hat{\jmath}_{st})_{0\leq s\leq t}$ is a Lévy process. Furthermore it establishes the equivalence of the two processes and completes the proof of the theorem. □

Lévy processes on symmetric $*$-bialgebras can be realized on Bose–Fock spaces using quantum stochastic differential calculus [11, 24, 21]. The necessary input is a triple (ρ,η,L), where ρ is a $*$-representation of $\mathcal{A}$ on some pre-Hilbert space P, $\eta:\mathcal{A}\to P$ is a ρ-cocycle (i.e.,

$$\eta(ab)=\rho(a)\eta(b)-\eta(a)\varepsilon(b)$$

for $a,b\in\mathcal{A}$), and $L:\mathcal{A}\to\mathbb{C}$ is a hermitian linear functional such that

$$\langle\eta(a^*),\eta(b)\rangle=\varepsilon(a)L(b)+L(ab)-L(a)\varepsilon(b)$$

for $a,b,\in\mathcal{A}$. Such triples are called *Schürmann triples* in [21, Ch. VII] and [7].

Let us recall the GNS-type construction of the triple (ρ,η,L) from the functional L (see [27, Section 2.3]). Let $\mathcal{B}$ be a $*$-algebra with a unital, hermitian character $\varepsilon:\mathcal{B}\to\mathbb{C}$ (i.e., $\varepsilon(\mathbb{1})=1$, $\varepsilon(b^*)=\overline{\varepsilon(b)}$, and $\varepsilon(ab)=\varepsilon(a)\varepsilon(b)$ for all a,b in $\mathcal{B}$) and let $L:\mathcal{B}\to\mathbb{C}$ be a generator. We define an inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}_0}:\mathcal{B}_0\times\mathcal{B}_0\to\mathbb{C}$ on $\mathcal{B}_0\coloneqq\ker\varepsilon$ by $(a,b)\mapsto L(a^*b)$. This inner product is positive semi-definite, since L is conditionally positive. We define the null space by $\mathcal{N}_0\coloneqq\{b\in\mathcal{B}_0\mid\langle b,b\rangle=0\}$. The quotient space $P\coloneqq\mathcal{B}_0/\mathcal{N}_0$ with inner product

$$\langle a+\mathcal{N}_0,b+\mathcal{N}_0\rangle_P\coloneqq\langle a,b\rangle_{\mathcal{B}_0}$$

becomes a pre-Hilbert space. The (left) action $\alpha:\mathcal{B}\times\mathcal{B}_0\to\mathcal{B}_0$ with $(a,b)\mapsto a\cdot b$ induces an action $\tilde{\alpha}$ on P, since $\alpha(\mathcal{N}_0)\subseteq\mathcal{N}_0$. Now we define $\rho(a)\in\mathcal{L}(P,P)$ by

$$\rho(a)(b+\mathcal{N}_0)\coloneqq\tilde{\alpha}(a,b+\mathcal{N}_0)$$

for $a\in\mathcal{B}$ and $b+\mathcal{N}_0\in P$, as well as

$$\begin{aligned}\eta:\mathcal{B}&\to P\\ b&\mapsto(b-\varepsilon(b)\cdot 1_{\mathcal{B}})+\mathcal{N}_0,\end{aligned}$$

where

$$\eta(b)=\begin{cases}b+\mathcal{N}_0 & \text{for all } b\in\mathcal{B}_0,\\ 0+\mathcal{N}_0 & \text{for all } b\in\langle 1_{\mathcal{B}}\rangle.\end{cases}$$

The equations

$$\eta(a\cdot b)=\rho(a)\eta(b)+\eta(a)\varepsilon(b)$$

and

$$\langle \eta(a), \eta(b)\rangle = L(a^*b) - \varepsilon(a^*)L(b) - L(a^*)\varepsilon(b)$$

hold. Thus (ρ, η, L) is a surjective triple, i.e., a triple whose cocycle η is surjective.

If we know the triple for a generator L on a braided $*$-bialgebra $\mathcal{B}$, then the following proposition tells us how to extend it to a triple for $L^{\mathcal{H}} = F(L)$.

Theorem 3.7. *Let $(j_{st})_{0\le s\le t}$ be a Lévy process on a braided $*$-bialgebra $\mathcal{B}$ in $\big({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}_*\big)$, $\big({}_{\mathcal{A}}\mathrm{C}_*, \Psi\big)$ or $\big({}^{\mathcal{A}}\mathrm{C}_*, \Psi\big)$ with α-invariant generator L and triple (ρ, η, L). Furthermore let $\big(j^{\mathcal{H}}_{st}\big)_{0\le s\le t}$ be the Lévy process on the symmetrization $\mathcal{H} \cong \mathcal{B} \otimes \mathcal{A}$ from Theorem 3.5 with triple $\big(\rho^{\mathcal{H}}, \eta^{\mathcal{H}}, L^{\mathcal{H}}\big)$. Then we have:*

- *The pre-Hilbert spaces P and $P^{\mathcal{H}}$ belonging to the triples are isometrically isomorphic, i.e., there exists a linear, bijective map $\check{T} : P^{\mathcal{H}} \to P$ such that*
$$\langle a, b\rangle_{P^{\mathcal{H}}} = \langle \check{T}(a), \check{T}(b)\rangle_P.$$
- *$\check{T} \circ \eta^{\mathcal{H}}(b \otimes a) = \varepsilon_{\mathcal{A}}(a)\eta(b)$, and $\check{T} \circ \eta^{\mathcal{H}}$ vanishes on $\mathbb{1}_{\mathcal{B}} \otimes \mathcal{A}$.*
- *$\rho^{\mathcal{H}}$ is determined by*
$$\check{T}\big(\rho^{\mathcal{H}}(\mathbb{1}_{\mathcal{B}} \otimes a)\eta^{\mathcal{H}}(b'_0 \otimes \mathbb{1}_{\mathcal{A}})\big) = \eta(\alpha(a \otimes b'_0))$$
$$\check{T}\big(\rho^{\mathcal{H}}(b \otimes \mathbb{1}_{\mathcal{A}})\eta^{\mathcal{H}}(b'_0 \otimes \mathbb{1}_{\mathcal{A}})\big) = \eta(bb'_0)$$
for $a \in \mathcal{A}$, $b \in \mathcal{B}$, $b'_0 \in \mathcal{B}_0$.

Proof. Using the construction above we know that the triple (ρ, η, L) belonging to $(j_{st})_{0\le s\le t}$ is defined on the pre-Hilbert-space $P := \mathcal{B}_0/\mathcal{N}_0$ with inner product

$$\langle a + \mathcal{N}_0, b + \mathcal{N}_0\rangle_P := \langle a, b\rangle_{\mathcal{B}_0} = L(a^*b),$$

where $\mathcal{N}_0 = \{G \in \mathcal{B}_0 | \langle G, G\rangle_{\mathcal{B}_0} = 0\}$. The representation ρ is given by $\rho(a)(b + \mathcal{N}_0) := (a \cdot b + \mathcal{N}_0)$ and the ρ-cocycle by

$$\eta(a) = (a - \varepsilon(a) \cdot \mathbb{1}_{\mathcal{B}}) + \mathcal{N}_0.$$

Analogously, the triple $\big(\rho^{\mathcal{H}}, \eta^{\mathcal{H}}, L^{\mathcal{H}}\big)$ of the process $\big(j^{\mathcal{H}}_{st}\big)_{0\le s\le t}$ on the (left) symmetrization $\mathcal{H}$ of $\mathcal{B}$, with $L^{\mathcal{H}} = F(L)$, is defined on the pre-Hilbert space $P^{\mathcal{H}} := \mathcal{H}_0/\mathcal{N}^{\mathcal{H}}_0$ with inner product

$$\langle a + \mathcal{N}^{\mathcal{H}}_0, b + \mathcal{N}^{\mathcal{H}}_0\rangle_{P^{\mathcal{H}}} := \langle a, b\rangle_{\mathcal{H}_0} = L^{\mathcal{H}}(a^*b), \tag{3.4}$$

where $\mathcal{H}_0 := \ker \varepsilon_{\mathcal{H}}$. The representation $\rho^{\mathcal{H}}$ is defined by $\rho^{\mathcal{H}}(a)(b + \mathcal{N}^{\mathcal{H}}_0) := (a \cdot b) + \mathcal{N}^{\mathcal{H}}_0$ and the ρ-cocycle $\eta^{\mathcal{H}}$ is given by

$$\eta^{\mathcal{H}}(b \otimes a) = \big(b \otimes a - \varepsilon_{\mathcal{H}}(b \otimes a) \cdot \mathbb{1}_{\mathcal{H}}\big) + \mathcal{N}^{\mathcal{H}}_0. \tag{3.5}$$

First we want to show that the pre-Hilbert spaces P and $P^{\mathcal{H}}$ are isometrically isomorphic, i.e., there exists a linear bijective map $\widetilde{T} : P^{\mathcal{H}} \to P$ with

$$\langle a, b\rangle_{P^{\mathcal{H}}} = \langle \widetilde{T}(a), \widetilde{T}(b)\rangle_P$$

for all $a, b \in P^{\mathcal{H}}$. Let $b \otimes a$, $d \otimes c \in \mathcal{H}_0 = \ker(\varepsilon_{\mathcal{B}} \otimes \varepsilon_{\mathcal{A}})$, $b, d \in \mathcal{B}$ and $a, c \in \mathcal{A}$. We have

$$(L \otimes \varepsilon_{\mathcal{A}})((b \otimes a)^*(d \otimes c)) = \varepsilon_{\mathcal{A}}(a^*c)L(b^*d),$$

since

$$
\begin{aligned}
&L^{\mathcal{H}} \circ m_{\mathcal{H}} \circ (*_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{H}}) \\
&\quad = (L \otimes \varepsilon) \circ (m \otimes m) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \mathrm{id}) \\
&\qquad \circ \Big(\big((\alpha \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ (* \otimes *) \big) \otimes \mathrm{id} \otimes \mathrm{id} \Big) \\
&\quad = (L \otimes \varepsilon \otimes \varepsilon) \circ (m \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\alpha \otimes \alpha \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id}) \\
&\qquad \circ (\mathrm{id} \otimes ((\Delta \otimes \mathrm{id}) \circ \Delta) \otimes \mathrm{id} \otimes \mathrm{id}) \circ (* \otimes * \otimes \mathrm{id} \otimes \mathrm{id}) \\
&\quad = (L \otimes \varepsilon) \circ (m \otimes \mathrm{id}) \circ (\alpha \otimes \alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}) \\
&\qquad \circ (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (* \otimes * \otimes \mathrm{id} \otimes \mathrm{id}) \\
&\quad = (L \otimes \varepsilon) \circ (\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (* \otimes * \otimes \mathrm{id} \otimes \mathrm{id}) \\
&\quad = (\varepsilon \otimes \varepsilon) \circ (\mathrm{id} \otimes \mathrm{id} \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\tau \otimes \tau) \circ (* \otimes * \otimes \mathrm{id} \otimes \mathrm{id}) \\
&\quad = (\varepsilon \otimes L) \circ (m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\tau \otimes \tau) \circ (* \otimes * \otimes \mathrm{id} \otimes \mathrm{id}),
\end{aligned}
$$

and thus

$$
\begin{aligned}
&L^{\mathcal{H}}\Big((b \otimes a)^* (d \otimes c) \Big) \\
&\quad = L^{\mathcal{H}} \circ m_{\mathcal{H}} \circ (*_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{H}})(b \otimes a \otimes d \otimes c) \\
&\quad = (\varepsilon_{\mathcal{A}} \otimes L) \circ (m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\tau \otimes \tau) \circ (* \otimes * \otimes \mathrm{id} \otimes \mathrm{id})(b \otimes a \otimes d \otimes c) \\
&\quad = \varepsilon_{\mathcal{A}}(a^* c) L(b^* d).
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
\langle b \otimes a, d \otimes c \rangle_{\mathcal{H}_0} &= L^{\mathcal{H}}((b \otimes a)^*(d \otimes c)) = (L \otimes \varepsilon_{\mathcal{A}})((b \otimes a)^*(d \otimes c)) \\
&= \varepsilon_{\mathcal{A}}(a^* c) L(b^* d) = \varepsilon_{\mathcal{A}}(a^* c) \langle b, d \rangle_{\mathcal{B}_0}.
\end{aligned} \tag{3.6}
$$

That (ρ, η, L) is the triple of the generator L means that

$$
L(b^* d) = \langle \eta(b), \eta(d) \rangle_P + \varepsilon(b^*) L(d) + L(b^*) \varepsilon(d).
$$

Thus from equation (3.6) it follows that

$$
\begin{aligned}
\varepsilon_{\mathcal{A}}(a^* c) L(b^* d) = \varepsilon_{\mathcal{A}}(a^* c) &\langle \eta(b), \eta(d) \rangle_P \\
&+ \varepsilon_{\mathcal{A}}(a^*) \varepsilon_{\mathcal{A}}(a) \varepsilon_{\mathcal{B}}(b^*) L(d) + \varepsilon_{\mathcal{A}}(a^*) \varepsilon_{\mathcal{A}}(a) L(b^*) \varepsilon_{\mathcal{B}}(d).
\end{aligned}
$$

We have

$$
\begin{aligned}
\ker \varepsilon_{\mathcal{H}} &= (\ker \varepsilon_{\mathcal{B}} \otimes (\ker \varepsilon_{\mathcal{A}} \oplus \langle \mathbb{1}_{\mathcal{A}} \rangle)) \oplus ((\ker \varepsilon_{\mathcal{B}} \otimes \langle \mathbb{1}_{\mathcal{B}} \rangle)) \otimes \ker \varepsilon_{\mathcal{A}} \\
&= (\ker \varepsilon_{\mathcal{B}} \otimes \ker \varepsilon_{\mathcal{B}}) \oplus (\ker \varepsilon_{\mathcal{B}} \otimes \langle \mathbb{1}_{\mathcal{A}} \rangle) \oplus (\langle \mathbb{1}_{\mathcal{B}} \rangle \otimes \ker \varepsilon_{\mathcal{A}}),
\end{aligned}
$$

and the second and the third addend vanish. Hence we have

$$
\langle \eta(b), \eta(d) \rangle_P = L(b^* d) = \langle b, d \rangle_P, \tag{3.7}
$$

and with equations (3.6) and (3.7) it follows that

$$
\begin{aligned}
\langle b \otimes a, d \otimes c \rangle_{\mathcal{H}_0} &= \varepsilon_{\mathcal{A}}(a^* c) L(b^* c) \\
&= \varepsilon_{\mathcal{A}}(a^* c) \langle \eta(b), \eta(d) \rangle_P \\
&= \langle \varepsilon_{\mathcal{A}}(a) \eta(b), \varepsilon_{\mathcal{A}}(c) \eta(d) \rangle_P.
\end{aligned}
$$

Now we define the map

$$\begin{aligned} T : \mathcal{H}_0 &\to P \\ b \otimes a &\mapsto \varepsilon_{\mathcal{A}}(a)\eta(b). \end{aligned}$$

Since $T(b \otimes a) = 0$ for $b \otimes a \in \mathcal{N}_0^{\mathcal{H}}$, we can lift the map to $P^{\mathcal{H}}$ and we get the isometric isomorphism

$$\begin{aligned} \widetilde{T} : P^{\mathcal{H}} &\to P \\ \eta^{\mathcal{H}} &\mapsto \varepsilon_{\mathcal{A}}(a)\eta(b), \end{aligned}$$

where the injectivity follows from $\ker \widetilde{T} = \{0\}$ and the surjectivity of $\widetilde{T}$ from the surjectivity of η. Since we have

$$\eta(\mathbb{1}_{\mathcal{B}}) = (\mathbb{1}_{\mathcal{B}} - \varepsilon_{\mathcal{B}}(\mathbb{1}_{\mathcal{B}}) \cdot \mathbb{1}_{\mathcal{B}}) + \mathcal{N}_0^{\mathcal{H}} = 0 + \mathcal{N}_0^{\mathcal{H}}, \tag{3.8}$$

it follows that

$$\begin{aligned} \widetilde{T} \circ \eta^{\mathcal{H}}(b \otimes a) &\overset{(3.5)}{=} \widetilde{T}\Big((b \otimes a - \varepsilon_{\mathcal{H}}(b \otimes a) \cdot \mathbb{1}_{\mathcal{H}}) + \mathcal{N}_0^{\mathcal{H}}\Big) \\ &= \widetilde{T}\Big(b \otimes a + \mathcal{N}_0^{\mathcal{H}}\Big) - \widetilde{T}\Big(\varepsilon_{\mathcal{B}}(b)\varepsilon_{\mathcal{A}}(a) \cdot (\mathbb{1}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{A}}) + \mathcal{N}_0^{\mathcal{H}}\Big) \\ &= \varepsilon_{\mathcal{A}}(a)\eta(b) - \varepsilon_{\mathcal{B}}(b)\varepsilon_{\mathcal{A}}(a)\varepsilon_{\mathcal{A}}(\mathbb{1}_{\mathcal{A}})\eta(\mathbb{1}_{\mathcal{B}}) \\ &\overset{(3.8)}{=} \varepsilon_{\mathcal{A}}(a)\eta(b). \end{aligned}$$

From equation (3.8) it also follows that

$$\widetilde{T} \circ \eta^{\mathcal{H}}(\mathbb{1}_{\mathcal{B}} \otimes a) = \varepsilon_{\mathcal{A}}(a)\eta(\mathbb{1}_{\mathcal{B}}) - \varepsilon_{\mathcal{B}}(\mathbb{1}_{\mathcal{B}})\varepsilon_{\mathcal{A}}(a)\varepsilon_{\mathcal{A}}(\mathbb{1}_{\mathcal{A}})\eta(\mathbb{1}_{\mathcal{B}}) = 0$$

for all $a \in \mathcal{A}$, and thus $\widetilde{T} \circ \eta^{\mathcal{H}}$ vanishes on $\mathbb{1}_{\mathcal{B}} \otimes \mathcal{A}$. For the proof of the last part let $b = b_0 + c_b \cdot \mathbb{1}_{\mathcal{B}} \in \ker \varepsilon_{\mathcal{B}} \oplus \mathbb{C} \cdot \mathbb{1}_{\mathcal{B}} = \mathcal{B}$ and $a = a_0 + c_a \cdot \mathbb{1}_{\mathcal{A}} \in \ker \varepsilon_{\mathcal{A}} \oplus \mathbb{C} \cdot \mathbb{1}_{\mathcal{A}} = \mathcal{A}$. We have

$$\begin{aligned} \langle b_0 \otimes a_0, b_0 \otimes a_0 \rangle_{\mathcal{H}_0} &\overset{(3.4)}{=} L^{\mathcal{H}}\Big((b_0 \otimes a_0)^*(b_0 \otimes a_0)\Big) \\ &= \varepsilon_{\mathcal{A}}\big((a_0)^*\big)\varepsilon_{\mathcal{A}}(a_0)L\big((b_0)^*b_0\big) = 0, \end{aligned}$$

and hence we get

$$\begin{aligned} \eta^{\mathcal{H}}(b \otimes a) &= \eta^{\mathcal{H}}\big(b_0 \otimes a_0\big) + \eta^{\mathcal{H}}\big(b_0 \otimes c_a \cdot \mathbb{1}_{\mathcal{A}}\big) + \eta^{\mathcal{H}}\big(c_b \cdot \mathbb{1}_{\mathcal{B}} \otimes a_0\big) \\ &\quad + \eta^{\mathcal{H}}\big(c_b \cdot \mathbb{1}_{\mathcal{B}} + c_a \cdot \mathbb{1}_{\mathcal{A}}\big) \\ &= \eta^{\mathcal{H}}\big(b_0 \otimes c_a \cdot \mathbb{1}_{\mathcal{A}}\big). \end{aligned}$$

Therefore it remains to show the assertion for $b_0 \otimes c_a \cdot \mathbb{1}_{\mathcal{A}} \in \ker \varepsilon_{\mathcal{B}} \oplus \mathbb{C} \cdot \mathbb{1}_{\mathcal{B}}$. Let $b \in \mathcal{B}$, $a \in \mathcal{A}$ and $b_0' \in \ker \varepsilon_{\mathcal{B}}$. We have $m_{\mathcal{H}}(b \otimes a \otimes d \otimes c) = b \cdot \alpha(a_{(1)} \otimes d) \otimes a_{(2)} \cdot c$. Because of that we get

$$\begin{aligned} &\widetilde{T}\Big(\rho^{\mathcal{H}}(b \otimes a)\eta^{\mathcal{H}}(b_0' \otimes \mathbb{1}_{\mathcal{A}})\Big) \\ &\quad \overset{(3.5)}{=} \widetilde{T}\Big(\rho^{\mathcal{H}}\big(b \otimes a\big)\big((b_0' \otimes \mathbb{1}_{\mathcal{A}} - \varepsilon_{\mathcal{H}}(b_0' \otimes \mathbb{1}_{\mathcal{A}}) \cdot \mathbb{1}_{\mathcal{H}} + \mathcal{N}_0^{\mathcal{H}})\Big) \end{aligned}$$

$$= \widetilde{T}\Big((b \otimes a) \cdot_{\mathcal{H}} (b'_0 \otimes \mathbb{1}_{\mathcal{A}})\Big) - \widetilde{T}\Big(\varepsilon_{\mathcal{H}}(b'_0 \otimes \mathbb{1}_{\mathcal{A}})(b \otimes a)\Big)$$
$$= \widetilde{T}\big(b \cdot \alpha(a_{(1)} \otimes b'_0) \otimes a_{(2)}\big) = \eta\big(b \cdot \alpha(a \otimes b'_0)\big).$$

Since $\rho^{\mathcal{H}}$ is an algebra homomorphism, we have

$$\widetilde{T}\Big(\rho^{\mathcal{H}}(\mathbb{1}_{\mathcal{B}} \otimes a)\eta^{\mathcal{H}}(b'_0 \otimes \mathbb{1}_{\mathcal{A}})\Big) = \eta\big(\alpha(a \otimes b'_0)\big) \text{ for } b \otimes a = \mathbb{1}_{\mathcal{B}} \otimes a,$$

and

$$\widetilde{T}\Big(\rho^{\mathcal{H}}(b \otimes \mathbb{1}_{\mathcal{A}})\eta^{\mathcal{H}}(b'_0 \otimes \mathbb{1}_{\mathcal{A}})\Big) = \eta\big(b \cdot \alpha(\mathbb{1}_{\mathcal{B}} \otimes b'_0\big) = \eta(bb'_0) \text{ for } b \otimes a = b \otimes \mathbb{1}_{\mathcal{A}}.$$

This completes the proof of the theorem for triples constructed in the way described above. But since the pre-Hilbert spaces belonging to two surjective triples which come from the same generator L are isometrically isomorphic, this gives rise to an isometric isomorphism $\check{T} : P^{\mathcal{H}} \to P$ with $\check{T} \circ \eta^{\mathcal{H}}(b \otimes a) = \varepsilon_{\mathcal{A}}(a)\eta(b)$ such that the assertions hold for arbitrary triples belonging to the same generator. □

4. A construction of braided ∗-spaces

In this section we will construct a large class of braided ∗-spaces and their symmetrizations.

Let $R \in \mathbb{C}^{n\times n} \otimes \mathbb{C}^{n\times n}$ be a universal R-matrix and thus R satisfies the quantum Yang–Baxter equation (1.2). We suppose that R is of real type I, i.e.,

$$\overline{R^{ij}_{kl}} = R^{lk}_{ji},$$

and that R is bi-invertible, i.e., that there exist matrices R^{-1} and $\widetilde{R} \in \mathbb{C}^{n\times n} \otimes \mathbb{C}^{n\times n}$ such that

$$\sum_{k,l} (R^{-1})^{ij}_{kl} R^{kl}_{pq} = \sum_{k,l} R^{ij}_{kl} (R^{-1})^{kl}_{pq} = \delta^i_p \delta^j_q,$$
$$\sum_{k,j} \widetilde{R}^{ij}_{kl} R^{kq}_{pj} = \sum_{k,j} R^{ij}_{kl} \widetilde{R}^{kq}_{pj} = \delta^i_p \delta^q_l.$$

Note that if R is of real type I and bi-invertible, then R^{-1} and $\widetilde{R}$ are also of real type I.

4.1. The FRT-∗-bialgebra.

We denote by $\mathcal{A}_*(R)$ the ∗-bialgebra generated by the elements $\{a^i_j\}_{i,j=1,2,\ldots,n}$ and their adjoints $\{b^k_l\}_{k,l=1,2,\ldots,n}$, where $b^k_l := (a^l_k)^*$, with the relations

$$\sum_{p,q} R^{ik}_{pq} a^p_j a^q_l = \sum_{p,q} a^k_q a^i_p R^{pq}_{jl},$$
$$\sum_{k,j} R^{ij}_{kl} b^k_q a^p_j = \sum_{k,j} a^k_l b^i_j R^{jp}_{qk},$$
$$\sum_k \Delta(a^i_j) = a^i_k \otimes a^k_j,$$
$$\varepsilon(a^i_j) = \delta^i_j.$$

The proof that these relations define indeed a $*$-bialgebra is similar to the construction of the FRT-bialgebra in [25]; see also [20].

Lemma 4.1. *Let V be a finite dimensional vector space and $R \in \mathrm{End}(V \otimes V)$ a bi-invertible solution of the quantum Yang–Baxter equation* (1.2)*. Then $\mathcal{A}_*(R)$ is a coquasi-triangular $*$-bialgebra, with the unique involutive r-form*

$$\boldsymbol{r} : \mathcal{A}_*(R) \otimes \mathcal{A}_*(R) \to \mathbb{C}$$

such that

$$\boldsymbol{r}(a^i_j \otimes a^k_l) = R^{ik}_{jl}, \quad \boldsymbol{r}(a^i_j \otimes b^k_l) = R^{ki}_{lj},$$
$$\boldsymbol{r}(a^i_j \otimes \mathbb{1}) = \boldsymbol{r}(\mathbb{1} \otimes a^i_j) = \delta^i_j.$$

The inverse (w.r.t. the convolution) $\overline{\boldsymbol{r}} : \mathcal{A}_*(R) \otimes \mathcal{A}_*(R) \to \mathbb{C}$ is defined by

$$\overline{\boldsymbol{r}}(a^i_j \otimes a^k_l) = (R^{-1})^{ik}_{jl}, \quad \overline{\boldsymbol{r}}(a^i_j \otimes b^k_l) = \widetilde{R}^{ki}_{lj},$$
$$\overline{\boldsymbol{r}}(a^i_j \otimes \mathbb{1}) = \overline{\boldsymbol{r}}(\mathbb{1} \otimes a^i_j) = \delta^i_j.$$

Proof. The proof is similar to that of [14, Ch. 10.1.2, Thm. 7]. See [20] for a more detailed treatment. □

4.2. **The left symmetrization $\mathcal{H} = \mathcal{V}(R) \otimes \mathcal{A}_*(R')$.** Consider the FRT-$*$-bialgebra $\mathcal{A}_*(R')$ for $R' \coloneqq \tau(R)$. From Lemma 4.1 it follows that $\mathcal{A}_*(R')$ is coquasi-triangular with the r-form $\boldsymbol{r'} : \mathcal{A}_*(R') \otimes \mathcal{A}_*(R') \to \mathbb{C}$, where

$$\boldsymbol{r'}(a^i_j \otimes a^k_l) = R^{ki}_{lj}, \qquad \boldsymbol{r'}(a^i_j \otimes b^k_l) = R^{ik}_{jl},$$
$$\boldsymbol{r'}(b^i_j \otimes a^k_l) = \widetilde{R}^{ki}_{lj}, \qquad \boldsymbol{r'}(b^i_j \otimes b^k_l) = (R^{-1})^{ik}_{jl},$$

and the inverse $\overline{\boldsymbol{r'}} : \mathcal{A}_*(R') \otimes \mathcal{A}_*(R') \to \mathbb{C}$ with

$$\overline{\boldsymbol{r'}}(a^i_j \otimes a^k_l) = (R^{-1})^{ki}_{lj}, \qquad \overline{\boldsymbol{r'}}(a^i_j \otimes b^k_l) = \widetilde{R}^{ik}_{jl},$$
$$\overline{\boldsymbol{r'}}(b^i_j \otimes a^k_l) = R^{ki}_{lj}, \qquad \overline{\boldsymbol{r'}}(b^i_j \otimes b^k_l) = R^{ik}_{jl}.$$

We extend $\boldsymbol{r'}$ to general elements such that equations (1.3) are satisfied. Let $\mathcal{V}(R)$ be the free algebra generated by the elements $x_1, \ldots, x_n$ and their adjoints $v^1 = (x_1)^*, \ldots, v^n = (x_n)^*$. The map $\gamma : \mathcal{V}(R) \to \mathcal{A}_*(R') \otimes \mathcal{V}(R)$ with

$$\gamma(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}, \quad \gamma(x_i) = \sum_j b^j_i \otimes x_j, \quad \gamma(v^i) = \sum_j a^i_j \otimes v^j$$

defines a left coaction on $\mathcal{V}(R)$. Thus the left action $\alpha = (\overline{\boldsymbol{r'}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma)$ introduced in Lemma 2.3 is given by

$$\alpha(a^i_j \otimes x_k) = \sum_l \widetilde{R}^{il}_{jk} x_l, \qquad \alpha(a^i_j \otimes v^k) = \sum_l (R^{-1})^{ki}_{lj} v^l,$$
$$\alpha(b^i_j \otimes x_k) = \sum_l R^{il}_{jk} x_l, \qquad \alpha(b^i_j \otimes v^k) = \sum_l R^{ki}_{lj} v^l,$$

and the braiding Ψ is given by

$$\Psi(x_i \otimes x_j) = \sum_{k,l} R^{kl}_{ij} x_l \otimes x_k,$$
$$\Psi(x_i \otimes v^j) = \sum_{k,l} R^{jk}_{li} v^l \otimes x_k,$$
$$\Psi(v^i \otimes x_j) = \sum_{k,l} \widetilde{R}^{il}_{kj} x_l \otimes v^k,$$
$$\Psi(v^i \otimes v^j) = \sum_{k,l} (R^{-1})^{ji}_{lk} v^l \otimes v^k$$

on the generators and extended to arbitrary elements by

$$\begin{aligned}\Psi(\mathbb{1} \otimes u) &= u \otimes \mathbb{1},\\ \Psi(u \otimes \mathbb{1}) &= \mathbb{1} \otimes u,\\ \Psi \circ (\mathrm{id} \otimes m) &= (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Psi) \circ (\Psi \otimes \mathrm{id}),\\ \Psi \circ (m \otimes \mathrm{id}) &= (\mathrm{id} \otimes m) \circ (\Psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Psi),\end{aligned}$$

such that the multiplication is Ψ-invariant. $\mathcal{V}(R)$ becomes a bialgebra with comultiplication Δ and counit ε given by

$$\Delta(x_i) = x_i \otimes \mathbb{1} + \mathbb{1} \otimes x_i, \qquad \Delta(v^i) = v^i \otimes \mathbb{1} + \mathbb{1} \otimes v^i,$$
$$\varepsilon(\mathbb{1}) = 1, \qquad \varepsilon(x_i) = \varepsilon(v^i) = 0$$

on the generators and extended such that it is a homomorphism from $\mathcal{V}(R)$ to $\mathcal{V}(R) \otimes \mathcal{V}(R)$, where the latter is equipped with the multiplication $(m \otimes m) \circ (\mathrm{id} \otimes \Psi \otimes \mathrm{id})$. Hence $\mathcal{V}(R)$ is a braided $*$-bialgebra in the braided category $\left({}^{\mathcal{A}_*(R')}\mathrm{C}_*, \Psi\right)$ and we can give the relations of the (left) symmetrization $\mathcal{H} = \mathcal{V}(R) \otimes \mathcal{A}(R')$ of $\mathcal{V}(R)$ (cf. Corollary 3.2). $\mathcal{H}$ is the $*$-bialgebra generated by the elements $x_i \otimes \mathbb{1}_{\mathcal{A}_*(R')}, v^i \otimes \mathbb{1}_{\mathcal{A}_*(R')} = (x_i)^* \otimes \mathbb{1}_{\mathcal{A}_*(R')}, \mathbb{1}_{\mathcal{V}(R)} \otimes a^i_j, \mathbb{1}_{\mathcal{V}_*(R)} \otimes b^j_i = \mathbb{1}_{\mathcal{V}_*(R)} \otimes (a^i_j)^*$ with the relations

$$\sum_{p,q} R^{ik}_{pq} a^p_j a^q_l = \sum_{p,q} a^k_q a^i_p R^{pq}_{jl},$$
$$\sum_{k,j} R^{ij}_{kl} b^k_q a^p_j = \sum_{k,j} a^k_l b^i_j R^{jp}_{qk},$$
$$a^j_k x_i = \sum_{p,l} \widetilde{R}^{jl}_{pi} x_l a^p_k,$$
$$b^j_k x_i = \sum_{p,l} R^{pl}_{ki} x_l b^j_p,$$
$$a^j_k v_i = \sum_{p,l} (R^{-1})^{ij}_{lp} v^l a^p_k,$$
$$b^j_k v^i = \sum_{p,l} R^{ip}_{lk} v^l b^j_p,$$

$$\begin{aligned}
\Delta(a^i_j) &= \sum_k a^i_k \otimes a^k_j,\\
\Delta(b^i_j) &= \sum_k b^k_j \otimes b^i_k,\\
\Delta(x_i) &= x_i \otimes \mathbb{1} + \sum_j b^j_i \otimes x_j,\\
\Delta(v^i) &= v^i \otimes \mathbb{1} + \sum_j a^i_j \otimes v^j,\\
\varepsilon(a^i_j) &= \varepsilon(b^i_j) = \delta^i_j,\\
\varepsilon(x_i) &= \varepsilon(v^i) = 0.
\end{aligned}$$

4.3. **The right symmetrization $\mathcal{H}_R = \mathcal{A}_*(R) \otimes \mathcal{V}(R)$.** For the second construction we choose the coquasi-triangular $*$-bialgebra $\mathcal{A}_*(R)$ with universal r-form $\boldsymbol{r} : \mathcal{A}_*(R) \otimes \mathcal{A}_*(R) \to \mathbb{C}$ from Lemma 4.1 given by

$$\begin{aligned}
\boldsymbol{r}(a^i_j \otimes a^k_l) &= R^{ik}_{jl}, & \boldsymbol{r}(a^i_j \otimes b^k_l) &= R^{ki}_{lj},\\
\boldsymbol{r}(b^i_j \otimes a^k_l) &= \widetilde{R}^{ik}_{jl}, & \boldsymbol{r}(b^i_j \otimes b^k_l) &= (R^{-1})^{ki}_{lj},
\end{aligned}$$

and inverse $\overline{\boldsymbol{r}} : \mathcal{A}_*(R) \otimes \mathcal{A}_*(R) \to \mathbb{C}$ given by

$$\begin{aligned}
\overline{\boldsymbol{r}}(a^i_j \otimes a^k_l) &= (R^{-1})^{ik}_{jl}, & \overline{\boldsymbol{r}}(a^i_j \otimes b^k_l) &= \widetilde{R}^{ki}_{lj},\\
\overline{\boldsymbol{r}}(b^i_j \otimes a^k_l) &= R^{ik}_{jl}, & \overline{\boldsymbol{r}}(b^i_j \otimes b^k_l) &= R^{ki}_{lj}.
\end{aligned}$$

The map $\overline{\gamma} : \mathcal{V}(R) \to \mathcal{V}(R) \otimes \mathcal{A}_*(R)$ with

$$\overline{\gamma}(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}, \quad \overline{\gamma}(x_i) = \sum_j x_j \otimes a^j_i, \quad \overline{\gamma}(v^i) = \sum_j v^j \otimes b^i_j$$

defines a right coaction on $\mathcal{V}(R)$. Thus $\overline{\alpha} = (\mathrm{id} \otimes \boldsymbol{r}) \circ (\overline{\gamma} \otimes \mathrm{id})$ gives us a right action $\overline{\alpha} : \mathcal{V}(R) \otimes \mathcal{A}_*(R) \to \mathcal{V}(R)$ as an analog of Lemma 2.3 which is given on the generators by

$$\begin{aligned}
\overline{\alpha}(x_i \otimes a^k_j) &= R^{lk}_{ij} x_l, & \overline{\alpha}(x_i \otimes b^k_j) &= R^{kl}_{ji} x_l,\\
\overline{\alpha}(v^i \otimes a^k_j) &= \widetilde{R}^{ik}_{lj} v^l, & \overline{\alpha}(v^i \otimes b^k_j) &= (R^{-1})^{ki}_{jl} v^l.
\end{aligned}$$

Furthermore, the braiding is defined by

$$\begin{aligned}
\overline{\Psi}(x_i \otimes x_j) &= \sum_l R^{lk}_{ij} x_k \otimes x_l,\\
\overline{\Psi}(x_i \otimes v^j) &= \sum_l R^{jl}_{ki} v^k \otimes x_l,\\
\overline{\Psi}(v^i \otimes x_j) &= \sum_l \widetilde{R}^{ik}_{lj} x_k \otimes v^l,\\
\overline{\Psi}(v^i \otimes v^j) &= \sum_k (R^{-1})^{ji}_{kl} v^k \otimes v^l.
\end{aligned}$$

Therefore we can define a braided category $((\mathrm{C}^{\mathcal{A}_*(R)})_*, \overline{\Psi})$ whose objects are right $\mathcal{A}_*(R)$-comodules in the same way as we constructed $({}^{\mathcal{A}}\mathrm{C}_*, \Psi)$ in Subsection 2.2, and $\mathcal{V}(R)$ is a braided $*$-bialgebra in the category $((\mathrm{C}^{\mathcal{A}_*(R)})_*, \overline{\Psi})$. Using a version of Corollary 3.2 for this category, we get the right symmetrization $\mathcal{H}_R = \mathcal{A}_*(R) \otimes \mathcal{V}(R)$. It is the $*$-bialgebra generated by the elements $\mathbb{1}_{\mathcal{A}_*(R)} \otimes x_i, \mathbb{1}_{\mathcal{A}_*(R)} \otimes v^i, a^i_j \otimes \mathbb{1}_{\mathcal{V}(R)}, b^i_j \otimes \mathbb{1}_{\mathcal{V}(R)}$ with the relations

$$\begin{aligned}
\sum_{p,q} R^{ik}_{pq} a^p_j a^q_l &= \sum_{p,q} a^k_q a^i_p R^{pq}_{jl},\\
\sum_{k,j} R^{ij}_{kl} b^k_q a^p_j &= \sum_{k,j} a^j_l b^i_k R^{kp}_{qj},\\
x_i a^j_k &= \sum_{p,l} R^{lp}_{ik} a^j_p x_l,\\
x_i b^j_k &= \sum_{p,l} R^{jl}_{pi} x_l b^j_p,\\
v^i a^j_k &= \sum_{p,l} \widetilde{R}^{ip}_{lk} v^l a^p_k,\\
v^i b^j_k &= \sum_{p,l} (R^{-1})^{ip}_{lk} v^l b^j_p,\\
\Delta(a^i_j) &= \sum_k a^i_k \otimes a^k_j,\\
\Delta(b^i_j) &= \sum_k b^k_j \otimes b^i_k,\\
\Delta(x_i) &= \sum_j x_j \otimes a^i_j + \mathbb{1} \otimes x_i,\\
\Delta(v^i) &= \sum_j v^j \otimes b^j_i + \mathbb{1} \otimes v^j,\\
\varepsilon(a^i_j) &= \varepsilon(b^i_j) = \delta^i_j,\\
\varepsilon(x_i) &= \varepsilon(v^i) = 0.
\end{aligned}$$

5. Realization of quantum Lévy processes on braided $*$-bialgebras

In this section we will show that there always exists a Lévy process on the braided $*$-spaces constructed in the previous section that can be considered as a standard Brownian motion on these spaces.

Definition 5.1 ([27, Section 5.1])**.** Let $\mathcal{B}$ be a braided bialgebra. A linear functional $\phi : \mathcal{B} \to \mathbb{C}$ is called *quadratic* (or *Gaussian*) if it satisfies

$$\phi(abc) = 0$$

for all $a, b, c \in \ker \varepsilon_{\mathcal{B}}$. A Lévy process whose generator is quadratic is called *Brownian motion.*

For the rest of this section R will denote a fixed bi-invertible R-matrix of real type I, and $\mathcal{V}(R)$ the associated free braided $*$-space from the previous section. For explicit calculations we will use the basis B consisting of the words in the generators $x_1,\dots,x_n,v^1,\dots,v^n$. Let $L:\mathcal{V}(R)\to\mathbb{C}$ be the functional defined by $L(x_iv^j)=\delta_i^j$ on basis elements of the form x_iv^j, and zero on all other basis elements.

Proposition 5.2. *The functional L is quadratic, Ψ-invariant, hermitian, and conditionally positive (i.e., positive on $\ker\varepsilon_{\mathcal{B}}$).*

Proof. For the Ψ-invariance we have to show that $(\mathrm{id}\otimes L)\circ\Psi=\Psi\circ(L\otimes\mathrm{id})$. We have

$$\begin{aligned}(\mathrm{id}\otimes L)\circ\Psi(x_iv^j\otimes x_k)&=\sum_{r,s,p,q}\widetilde{R}^{jq}_{pk}R^{rs}_{iq}x_s\otimes L(x_rv^p)=\sum_s\delta_i^j\delta_k^s x_s\\&=\Psi\circ(L\otimes\mathrm{id})(x_iv^j\otimes x_k).\end{aligned}$$

Similarly, we get

$$\begin{aligned}(\mathrm{id}\otimes L)\circ\Psi(x_iv^j\otimes v^k)&=\sum_{r,s,p,q}(R^{-1})^{kj}_{pq}R^{ps}_{ri}v^r\otimes L(x_sv^q)=\sum_r\delta_r^k\delta_i^j v^r\\&=\Psi\circ(L\otimes\mathrm{id})(x_iv^j\otimes v^k).\end{aligned}$$

A generator L is quadratic if and only if the associated representation is of the form $\rho=\varepsilon(\cdot)\mathrm{id}$. This is the case, since we have $\rho(x_i)=\rho(v^i)=0$ (see below). That L is hermitian and conditionally positive are straightforward calculations. □

We can now carry out the construction described in [27, Ch. 2] to obtain quantum stochastic differential equations for the symmetrization of the process $(j_{st})_{0\le s\le t}$ associated to the generator L. Let

$$\mathcal{N}:=\{b\in\mathcal{V}(R)\mid L\big((u-\varepsilon(u)\mathbb{1})^*(u-\varepsilon(u)\mathbb{1})\big)=0\}$$

and $P:=\mathcal{V}(R)/\mathcal{N}$. P is a Hilbert space with the inner product induced by $\langle u,v\rangle=L\big(u-\varepsilon(u)\mathbb{1})^*(v-\varepsilon(v)\mathbb{1})\big)$. Furthermore, $(\eta(v^i))_{i\in\{1,\dots,n\}}$, with

$$\begin{aligned}\eta:\mathcal{V}(R)&\to P\\ b&\mapsto b+\mathcal{N},\end{aligned}$$

forms an orthonormal basis in P, and we have

$$\eta(y)=\begin{cases}v^i+\mathcal{N} & \text{if } y=v^i,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore we have $P\cong\mathbb{C}^n$. Since η is a ρ-cocycle, we have

$$\begin{aligned}\rho(x_i)\eta(v^j)&=\eta(x_iv^j)-\eta(x_i)\varepsilon(v^j)=0,\\ \rho(v^i)\eta(v^j)&=\eta(v^iv^j)-\eta(v^i)\varepsilon(v^j)=0,\end{aligned}$$

and thus $\rho(x_i)=\rho(v^i)=0$. One verifies that L is α-invariant for the right action $\overline{\alpha}=(\mathrm{id}\otimes\boldsymbol{r})\circ(\overline{\gamma}\otimes\mathrm{id})$ from Lemma 2.3, i.e., $L(\alpha(u\otimes a))=\varepsilon_{\mathcal{A}}(a)L(u)$ for all $a\in\mathcal{A}(R)$, $u\in\mathcal{V}(R)$. Therefore we can use Theorem 3.7 to get the triple

$(\rho^{\mathcal{H}_R}, \eta^{\mathcal{H}_R}, L^{\mathcal{H}_R})$ on the right symmetrization $\mathcal{H}_R$. The pre-Hilbert space is given by $P^{\mathcal{H}_R} = (\mathcal{A}(R) \otimes \mathcal{V}(R))/\mathcal{N}^{\mathcal{H}_R}$ with

$$\mathcal{N}^{\mathcal{H}_R} := \big\{ a \otimes b \mid L^{\mathcal{H}_R}\big(a \otimes b - \delta_{\mathcal{H}_R}(a \otimes b)\mathbb{1}_{\mathcal{H}_R}\big)^* (a \otimes b - \delta_{\mathcal{H}_R}(a \otimes b)\mathbb{1}_{\mathcal{H}_R}) = 0 \big\}.$$

The ρ-cocycle $\eta^{\mathcal{H}_R} : \mathcal{H}_R \to P^{\mathcal{H}_R}$ is given by

$$\eta^{\mathcal{H}_R}(1 \otimes y) = \begin{cases} \eta(1 \otimes y) = v^i + \mathcal{N} & \text{for } y = v^i, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

and it follows that for the $*$-representation $\rho^{\mathcal{H}_R}$ we have

$$\rho^{\mathcal{H}_R}(\mathbb{1}_{\mathcal{A}} \otimes b)\eta^{\mathcal{H}_R}(\mathbb{1}_{\mathcal{A}} \otimes v^i) = \begin{cases} v^i + \mathcal{N} & \text{for } b = \mathbb{1}_{\mathcal{B}} \text{ (the empty word)}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\rho^{\mathcal{H}_R}(a \otimes \mathbb{1}_{\mathcal{B}})\eta^{\mathcal{H}_R}(\mathbb{1}_{\mathcal{A}} \otimes v^i) = \begin{cases} \widetilde{R}^{ki}_{pj}\eta^{\mathcal{H}_R}(\mathbb{1}_{\mathcal{A}} \otimes v^p) & \text{for } a = a^i_j, \\ (R^{-1})^{ik}_{jp}\eta^{\mathcal{H}_R}(\mathbb{1}_{\mathcal{A}} \otimes v^p) & \text{for } a = b^i_j. \end{cases}$$

The generator L is given by

$$L^{\mathcal{H}_R}(a \otimes b) = \begin{cases} \delta_{ij}\delta_{kl} & \text{for } a = a^i_j \text{ or } b^i_j \text{ and } b = x_k v^l, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Using the theory of Schürmann [27, Ch. 2] we get the following theorem.

Theorem 5.3. *Let j^H_{st} be the Lévy process in $\mathcal{H}_R$ with triple $(\eta^{\mathcal{H}_R}, \rho^{\mathcal{H}_R}, L^{\mathcal{H}_R})$ given in equations* (5.1)–(5.2). *Then a realization of the right symmetrization j^H_{st} on the Fock space $\Gamma(L^2(\mathbb{R}_+, \mathbb{C}^n))$ is given by the unique solution of the quantum stochastic differential equations*

$$\begin{aligned}
\mathrm{d}X_i &= \sum_j \mathrm{d}X_j \cdot \mathrm{d}\Lambda\left((\widetilde{R}^{kj}_{li} - \delta^j_i \delta^k_l)_{1 \le k,l \le n} \right) + \mathrm{d}A_i, \\
\mathrm{d}X^*_i &= \sum_j \mathrm{d}X^*_j \cdot \mathrm{d}\Lambda\left(\left((R^{-1})^{jk}_{ip} - \delta^j_i \delta^k_p \right)_{1 \le k,p \le n} \right) + \mathrm{d}A^*_i, \\
\mathrm{d}A^i_j &= \sum_k A^i_k \cdot \mathrm{d}\Lambda\left((\widetilde{R}^{lk}_{pj} - \delta^k_j \delta^l_p)_{1 \le l,p \le n} \right), \\
\mathrm{d}B^i_j &= \sum_k A^i_k \cdot \mathrm{d}\Lambda\left(\left((R^{-1})^{il}_{kp} - \delta^i_k \delta^l_p \right)_{1 \le l,p \le n} \right),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{d}X_i &:= \mathrm{d}j^{\mathcal{H}_R}_{st}(\mathbb{1} \otimes x_i), \\
\mathrm{d}X^*_i &:= \mathrm{d}j^{\mathcal{H}_R}_{st}(\mathbb{1} \otimes v_i), \\
\mathrm{d}A^i_j &:= \mathrm{d}j^{\mathcal{H}_R}_{st}(a^i_j \otimes \mathbb{1}), \\
\mathrm{d}B^i_j &:= \mathrm{d}j^{\mathcal{H}_R}_{st}(b^i_j \otimes \mathbb{1}).
\end{aligned}$$

6. Examples

6.1. **The one-dimensional R-matrix $R = (q)$.** Let us first consider the one-dimensional R-matrix $R_1 = (q)$. For $q \in \mathbb{R}$, $q \neq 0$, this is a bi-invertible R-matrix of real type I and defines therefore a braided $*$-space $\mathcal{V}(q)$. As an algebra, $\mathcal{V}(q)$ is the free algebra generated by x and $x^* = v$. We will use the words in x and v as a basis for $\mathcal{V}(q)$. The braiding is given by

$$\begin{aligned} \Psi(x \otimes x) &= qx \otimes x, & \Psi(x \otimes v) &= qv \otimes x, \\ \Psi(v \otimes x) &= q^{-1}x \otimes v, & \Psi(v \otimes v) &= q^{-1}v \otimes v. \end{aligned}$$

To be a generator of a Brownian motion a linear functional $L : \mathcal{V}(q) \to \mathbb{C}$ has to be Ψ-invariant, i.e., it has to satisfy

$$\begin{aligned} L(x)x &= \Psi \circ (L \otimes \mathrm{id})(x \otimes x) = (\mathrm{id} \otimes L) \circ \Psi(x \otimes x) = qL(x)x, \\ L(v)x &= \Psi \circ (L \otimes \mathrm{id})(v \otimes x) = (\mathrm{id} \otimes L) \circ \Psi(v \otimes x) = qL(v)x, \\ L(xx)x &= \Psi \circ (L \otimes \mathrm{id})(xx \otimes x) = (\mathrm{id} \otimes L) \circ \Psi(xx \otimes x) = q^2 L(xx)x, \\ L(xv)x &= \Psi \circ (L \otimes \mathrm{id})(xv \otimes x) = (\mathrm{id} \otimes L) \circ \Psi(xv \otimes x) = L(xv)x, \\ L(vx)x &= \Psi \circ (L \otimes \mathrm{id})(vx \otimes x) = (\mathrm{id} \otimes L) \circ \Psi(vx \otimes x) = qL(vx)x, \\ L(vv)x &= \Psi \circ (L \otimes \mathrm{id})(vv \otimes x) = (\mathrm{id} \otimes L) \circ \Psi(vv \otimes x) = q^{-2} L(vv)x, \end{aligned}$$

and a similar set of equations for v. Thus, for $q^2 \neq 1$, a Ψ-invariant quadratic functional can have non-zero values only on xv and vx. A quadratic functional on $\mathcal{V}(q)$ is conditionally positive if and only if the matrix

$$\hat{L} := \begin{pmatrix} L(xv) & L(xx) \\ L(vv) & L(vx) \end{pmatrix}$$

is positive semi-definite. It is also hermitian if we have $L(v) = \overline{L(x)}$. Thus we get the following classification for the quadratic generators on $\mathcal{V}(q)$.

Theorem 6.1. *A quadratic functional $L : \mathcal{V}(q) \to \mathbb{C}$ is a generator of a Lévy process on $\mathcal{V}(q)$ if and only if*

1. *for $q = 1$: $\hat{L}$ is positive semi-definite and $L(v) = \overline{L(x)}$;*
2. *for $q = -1$: $\hat{L}$ is positive semi-definite and $L(x) = L(v) = 0$;*
3. *for $q^2 \neq 1$: $L(xv), L(vx) \geq 0$ and L vanishes on all other basis elements.*

The symmetrization of this braided $*$-space gives for $\mathcal{A}(q)$ the free commutative algebra with group-like generator a and its adjoint b. $\mathcal{H}$ is generated by a, x and their adjoints $b = a^*$ and $v = x^*$. The algebraic relations are

$$ab = ba, \qquad xa = qax, \qquad bx = qxb,$$

and the coalgebraic relations are

$$\Delta(a) = a \otimes a, \qquad \Delta(x) = x \otimes a + \mathbb{1} \otimes x.$$

Let L now be the generator with $L(xv) = 1$ and $L(u) = 0$ on all other basis elements. The construction of the triple gives the pre-Hilbert space $D^{\mathcal{H}_R} = \mathbb{C}$, the

ρ-cocycle

$$\eta^{\mathcal{H}_R}(m) = \begin{cases} 0 & \text{for } m \in \{x, a, b\}, \\ 1 & \text{for } m = v, \end{cases}$$

and the representation

$$\rho^{\mathcal{H}_R}(m) = \begin{cases} q^{-1} & \text{for } m \in \{a, b\}, \\ 0 & \text{for } m \in \{x, v\}. \end{cases}$$

Thus we get the stochastic differential equations

$$\begin{aligned} \mathrm{d}X &= X\,\mathrm{d}\Lambda(q^{-1}-1) + \mathrm{d}A(1), \\ \mathrm{d}V &= V\,\mathrm{d}\Lambda(q^{-1}-1) + \mathrm{d}A^*(1), \\ \mathrm{d}A &= A\,\mathrm{d}\Lambda(q^{-1}-1), \\ \mathrm{d}B &= B\,\mathrm{d}\Lambda(q^{-1}-1), \end{aligned}$$

for the processes $X(t) = j_t(x)$, $V(t) = j_t(v)$, $A(t) = j_t(a)$, $B(t) = j_t(b)$. The solution of this system of quantum stochastic differential equations is the quantum Azéma martingale (see [23, 26]).

6.2. **The sl_2-R-matrix.** Let R_2 be the R-matrix of the standard two-dimensional quantum plane, i.e.,

$$R_2 = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2-1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}$$

(cf. [19, Example 10.2.2]). Then tR_2 with $q, t \in \mathbb{R}$, $q, t \neq 0$ is bi-invertible and of real type I, and we can therefore define a braided $*$-space $\mathcal{V}(tR_2)$ for it. As an algebra, this is the free algebra generated by x_1, x_2 and their adjoints $x_1^* = v^1$, $x_2^* = v^2$. We will use the words in these four elements as a basis of $\mathcal{V}(tR_2)$. It turns out that the Ψ-invariance restricts very much the possible generators.

Proposition 6.2. *Let L be a quadratic functional on $\mathcal{V}(tR_2)$, $q, t \neq 0$. Then L is characterized by*

$$\begin{aligned} A &= (A_{ij}) = (L(x_i x_j)), & B &= (B_i{}^j) = (L(x_i v^j)), \\ C &= (C^i{}_j) = (L(v^i x_j)), & D &= (D^{ij}) = (L(v^i v^j)), \\ a &= (a_i) = (L(x_i)), & b &= (b^i) = (L(v^i)). \end{aligned}$$

The functional L is Ψ-invariant if and only if

(1) *for $q = 1$ and*
 (a) *$t = 1$: all functionals are Ψ-invariant;*
 (b) *$t = -1$: $a = b = 0$ and A, B, C and D are arbitrary;*
 (c) *$t^2 \neq 1$: $A = D = 0$ and $a = b = 0$ and B and C are arbitrary;*
(2) *for $q = -1$ and*
 (a) *$t^2 = 1$: A, B, C and D are diagonal and a, b vanish;*

(b) $t^2 \neq 1$: *B, C are diagonal and A, D, a, b vanish;*

(3) *for $q^2 \neq 1$ and*

(a) $t^2q^3 = 1$: $qA_{12} + A_{21} = 0$, ${B_1}^1 = {B_2}^2$, ${C^1}_1 = q^2{C^2}_2$, $D^{12} + qD^{21} = 0$ *and all other coefficients vanish;*

(b) $t^2q^3 \neq 1$: ${B_1}^1 = {B_2}^2$, ${C^1}_1 = q^2{C^2}_2$ *and all other coefficients vanish.*

Proof. Let L be an arbitrary quadratic functional. L is Ψ-invariant if and only if the following equations are satisfied:

$$
\begin{aligned}
A_{ij}\delta^l_k &= \sum_{n_1,n_2,n_3} t^2 A_{n_3n_1} {R^{n_3}}_{i}{}^{l}{}_{n_2} {R^{n_1}}_{j}{}^{n_2}{}_{k},\\
A_{ij}\delta^k_l &= \sum_{n_1,n_2,n_3} t^2 A_{n_3n_1} {R^{n_2}}_{l}{}^{n_3}{}_{i} {R^{k}}_{n_2}{}^{n_1}{}_{j},\\
{B_i}^j\delta^l_k &= \sum_{n_1,n_2,n_3} {B_{n_3}}^{n_1} {R^{n_3}}_{i}{}^{l}{}_{n_2} {\widetilde{R}^{j}}{}_{n_1}{}^{n_2}{}_{k},\\
{B_i}^j\delta^k_l &= \sum_{n_1,n_2,n_3} {B_{n_3}}^{n_1} {R^{n_2}}_{l}{}^{n_3}{}_{i} {(R^{-1})^{k}}_{n_2}{}^{j}{}_{n_1},\\
{C^i}_j\delta^l_k &= \sum_{n_1,n_2,n_3} {C^{n_3}}_{n_1} {\widetilde{R}^{i}}{}_{n_3}{}^{l}{}_{n_2} {R^{n_1}}_{j}{}^{n_2}{}_{k},\\
{C^i}_j\delta^k_l &= \sum_{n_1,n_2,n_3} {C^{n_3}}_{n_1} {(R^{-1})^{n_2}}_{l}{}^{i}{}_{n_3} {R^{k}}_{n_2}{}^{n_1}{}_{j},\\
D^{ij}\delta^l_k &= \sum_{n_1,n_2,n_3} t^{-2} D^{n_3n_1} {\widetilde{R}^{i}}{}_{n_3}{}^{l}{}_{n_2} {\widetilde{R}^{j}}{}_{n_1}{}^{n_2}{}_{k},\\
D^{ij}\delta^k_l &= \sum_{n_1,n_2,n_3} t^{-2} D^{n_3n_1} {(R^{-1})^{n_2}}_{l}{}^{i}{}_{n_3} {(R^{-1})^{k}}_{n_2}{}^{j}{}_{n_1},
\end{aligned}
$$

$$
\begin{aligned}
a_i\delta^k_j &= \sum_{n_1} t {R^{n_1}}_{i}{}^{k}{}_{j} a_{n_1}, & a_i\delta^j_k &= \sum_{n_1} t {R^{j}}_{k}{}^{n_1}{}_{i} a_{n_1},\\
b^i\delta^k_j &= \sum_{n_1} t^{-1} {\widetilde{R}^{i}}{}_{n_1}{}^{k}{}_{j} b^{n_1} & b^i\delta^j_k &= \sum_{n_1} t^{-1} {(R^{-1})^{j}}_{k}{}^{i}{}_{n_1} b^{n_1}
\end{aligned}
$$

for all $i, j, k, l = 1, \ldots, n$. These equations follow directly from the invariance condition. For the first equation, e.g., we apply $\Psi \circ (L \otimes \mathrm{id}) = (\mathrm{id} \otimes L) \circ \Psi$ to $x_ix_j \otimes x_k$. Solving this system of linear equations (using, e.g., a computer program for symbolic computation like Maple) one arrives at the results listed in the proposition. □

The functional L is conditionally positive semi-definite if and only if the matrix

$$
\hat{L} := \left(L \begin{pmatrix} x_iv^j & x_ix_j \\ v^iv^j & v^ix_j \end{pmatrix} \right) = \begin{pmatrix} B & A \\ D & C \end{pmatrix}
$$

is positive semi-definite. For L to be hermitian, we need to impose furthermore $a_i = L(x_i) = \overline{L(v^i)} = \overline{b^i}$, for $i = 1, \ldots, n$. This leads to the following classification.

Theorem 6.3. *Suppose $q^2 \neq 1$.*

(a) *If $t^2q^3 = 1$, then all Ψ-invariant generators on $\mathcal{V}(tR_2)$ are of the form*

$$\hat{L} = \begin{pmatrix} b & 0 & 0 & -qa \\ 0 & b & a & 0 \\ 0 & \bar{a} & q^2c & 0 \\ -q\bar{a} & 0 & 0 & c \end{pmatrix},$$

and $L(x_i) = L(v^i) = 0$ for $i = 1, \ldots, n$, where $b, c \geq 0$, $bc \geq q^2|a|^2$, and $bc \geq q^{-2}|a|^2$.

(b) *If $t^2q^3 \neq 1$, then all Ψ-invariant generators on $\mathcal{V}(tR_2)$ are of the form*

$$\hat{L} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & q^2c & 0 \\ 0 & 0 & 0 & c \end{pmatrix},$$

and $L(x_i) = L(v^i) = 0$ for $i = 1, \ldots, n$, where $b, c \geq 0$.

Proof. (a) Proposition 6.2 (3) (a) implies that a Ψ-invariant functional

$$L : \mathcal{V}(tR_2) \to \mathbb{C}$$

is of the form

$$\hat{L} = \begin{pmatrix} b & 0 & 0 & -qa \\ 0 & b & a & 0 \\ 0 & d & q^2c & 0 \\ -qd & 0 & 0 & c \end{pmatrix},$$

and $L(x_i) = L(v^i) = 0$, where $a = L(x_2x_1)$, $b = L(x_1v^1)$, $c = L(v^2x_2)$ and $d = L(v^1x_2)$. Such an invariant functional L is a generator if and only if this matrix is positive semi-definite. This is the case if and only if the matrices $\begin{pmatrix} b & -qa \\ -qd & c \end{pmatrix}$ and $\begin{pmatrix} b & a \\ d & q^2c \end{pmatrix}$ are positive semi-definite, which leads immediately to the conditions given in the theorem.

(b) Proposition 6.2 (3) (b) shows us that if $t^2q^3 \neq 1$, then L is Ψ-invariant if and only if we also have $a = d = 0$. □

6.3. **The sl_3-R-matrix.** Let now

$$R_3 = \begin{pmatrix} q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & q^2-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & q^2-1 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & q^2-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 \end{pmatrix}$$

be the sl_3-R-matrix. We get a similar classification for the generators on $\mathcal{V}(tR_3)$ as in the previous subsection, but there are no additional generators for the special case $t^2q^3 = 1$ (as in Proposition 6.2).

Theorem 6.4. *Suppose $q^2 \neq 1$. Then all Ψ-invariant generators on $\mathcal{V}(tR_3)$ are of the form*

$$\hat{L} = \begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & q^4c & 0 & 0 \\ 0 & 0 & 0 & 0 & q^2c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}$$

and $L(x_i) = L(v^i) = 0$ for $i = 1, \ldots, n$, where $b, c \geq 0$.

Proof. We first determine all invariant quadratic functionals and then we check positivity, as in Proposition 6.2 and Theorem 6.3. □

6.4. **The braided quantum** SU(2) **groups** $\mathrm{SU}_q(2)$**,** $q \in \mathbb{C}\backslash\{0\}$**.** Finally, let us treat the braided Hopf-$*$-algebra underlying the braided compact quantum groups introduced and studied by Kasprzak, Meyer, Roy, and Woronowicz [12]. The braiding is not actually defined in [12], instead the authors define a monoidal category of C*-algebras. Here we show how this example fits into the framework of braided $*$-bialgebras that we used in this paper.

Let $\mathbb{CZ} \cong \mathrm{Pol}(\mathbb{T})$ be the group algebra of $\mathbb{Z}$. We will write $[z]$ for the canonical basis element of $\mathbb{CZ}$ associated to an integer $z \in \mathbb{Z}$. Recall that $\mathbb{CZ}$ is a Hopf-$*$-algebra with the following operations: for $z, z_1, z_2 \in \mathbb{Z}$, $c \in \mathbb{C}$, we set

$$\begin{aligned} m_{\mathbb{CZ}}([z_1] \otimes [z_2]) &= [z_1 + z_2], \\ \mathbb{1}(c) &= c[0], \\ \Delta([z]) &= [z] \otimes [z], \\ \varepsilon([z]) &= 1, \\ S([z]) &= [-z] = [z]^*, \end{aligned}$$

and extend $m_{\mathbb{CZ}}, \mathbb{1}, \Delta, \delta$ as algebra homomorphisms, S as an algebra anti-homomorphism, and $*$ as an anti-linear algebra anti-homomorphism.

Let $\mathcal{B}$ be a $\mathbb{CZ}$-graded $*$-bialgebra. In particular, we have $\deg(vw) = \deg(v) + \deg(w)$ and $\deg(v^*) = -\deg(v)$. Let $\zeta = \frac{q}{\bar{q}}$ for $0 < |q| < 1$.

We can use the grading to define an action of $\mathbb{CZ}$.

Lemma 6.5.

$$\begin{aligned}\tilde{\vartheta} : \mathbb{C}\mathbb{Z} \otimes \mathcal{B} &\to \mathcal{B},\\ [z] \otimes v &\mapsto \zeta^{-z \deg(v)} \cdot v\end{aligned}$$

defines a left action and

$$\begin{aligned}\tilde{\rho} : \mathcal{B} &\to \mathbb{C}\mathbb{Z} \otimes \mathcal{B},\\ v &\mapsto [\deg(v)] \otimes v\end{aligned}$$

defines a left coaction.

Lemma 6.6. *$\tilde{\vartheta}$ and $\tilde{\rho}$ satisfy the Yetter–Drinfeld condition, i.e.,*

$$\begin{aligned}&(m_{\mathbb{C}\mathbb{Z}} \otimes \tilde{\vartheta}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\tilde{\Delta} \otimes \tilde{\rho})\\ &\qquad = (m_{\mathbb{C}\mathbb{Z}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\tilde{\rho} \otimes \mathrm{id}) \circ (\tilde{\vartheta} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\tilde{\Delta} \otimes \mathrm{id}).\end{aligned}$$

Proof. For the left hand side we get

$$\begin{aligned}&(m_{\mathbb{C}\mathbb{Z}} \otimes \tilde{\vartheta}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\tilde{\Delta} \otimes \tilde{\rho})([z] \otimes v)\\ &\qquad = (m_{\mathbb{C}\mathbb{Z}} \otimes \tilde{\vartheta}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})([z] \otimes [z] \otimes \deg(v) \otimes v)\\ &\qquad = (m_{\mathbb{C}\mathbb{Z}} \otimes \tilde{\vartheta})([z] \otimes [\deg(v)] \otimes [z] \otimes v)\\ &\qquad = \zeta^{-z\cdot\deg(v)}([z + \deg(v)]) \otimes v,\end{aligned}$$

and the right hand side can be reduced to the same expression,

$$\begin{aligned}&(m_{\mathbb{C}\mathbb{Z}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\tilde{\rho} \otimes \mathrm{id}) \circ (\tilde{\vartheta} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\tilde{\Delta} \otimes \mathrm{id})([z] \otimes v)\\ &\quad = (m_{\mathbb{C}\mathbb{Z}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\tilde{\rho} \otimes \mathrm{id}) \circ (\tilde{\vartheta} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau)([z] \otimes [z] \otimes v)\\ &\quad = (m_{\mathbb{C}\mathbb{Z}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\tilde{\rho} \otimes \mathrm{id}) \circ (\tilde{\vartheta} \otimes \mathrm{id})([z] \otimes v \otimes [z])\\ &\quad = (m_{\mathbb{C}\mathbb{Z}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\tilde{\rho} \otimes \mathrm{id})(\zeta^{-z\cdot\deg v} \cdot v \otimes [z])\\ &\quad = (m_{\mathbb{C}\mathbb{Z}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau)(\zeta^{-z\cdot\deg v}[\deg(v)] \otimes v \otimes [z])\\ &\quad = (m_{\mathbb{C}\mathbb{Z}} \otimes \mathrm{id})(\zeta^{-z\cdot\deg v} \cdot [\deg(v)] \otimes [z] \otimes v)\\ &\quad = \zeta^{-z\cdot\deg v} \cdot (\deg(v) + z) \otimes v.\end{aligned}$$

□

Lemma 6.7. *The following equations hold:*

$$* \circ \tilde{\vartheta} = \tilde{\vartheta} \circ (* \otimes *) \circ (S \otimes \mathrm{id}), \qquad \tilde{\rho} \circ * = (* \otimes *) \circ \tilde{\rho}.$$

Proof. For the first equation:

$$\begin{aligned}* \circ \tilde{\vartheta}([z] \otimes v) &= (\zeta^{-z\deg(v)} \cdot v)^* = \overline{\zeta^{-z\deg(v)} \cdot v} = \overline{\zeta}^{-z\deg(v)} \cdot v^*\\ &= \zeta^{z\cdot\deg(v)} \cdot v^* = \zeta^{-z\cdot\deg(v^*)} \cdot v^* = \tilde{\vartheta}([z] \otimes v^*)\\ &= \tilde{\vartheta} \circ (* \otimes *)([-z] \otimes v) = \tilde{\vartheta} \circ (* \otimes *)(S \otimes \mathrm{id})([z] \otimes v).\end{aligned}$$

For the second equation:

$$\begin{aligned}\tilde{\rho} \circ *(v) &= \tilde{\rho}(v^*) = [\deg(v^*)] \otimes v^* = [-\deg(v)] \otimes v^*\\ &= [\deg(v)]^* \otimes v^* = (* \otimes *)([\deg(v)] \otimes v) = (* \otimes *) \circ \tilde{\rho}(v).\end{aligned}$$

□

Corollary 6.8. *$(\mathcal{B}, \tilde{\vartheta}, \tilde{\rho})$ is an involutive Yetter–Drinfeld module and thus an object in the category ${}^{\mathbb{CZ}}_{\mathbb{CZ}}\mathcal{YD}_*$.*

The category ${}^{\mathbb{CZ}}_{\mathbb{CZ}}\mathcal{YD}_*$ is braided with the braiding Ψ given by

$$\Psi(v \otimes w) = (\tilde{\vartheta} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\tilde{\rho} \otimes \mathrm{id})(v \otimes w) = \zeta^{-\deg(v)\deg(w)} \cdot w \otimes v.$$

Thus the multiplication on the tensor product is given by

$$\begin{aligned} m_{\mathcal{B}\otimes\mathcal{B}}(v_1 \otimes v_2 \otimes w_1 \otimes w_2) &= (m \otimes m) \circ (\mathrm{id} \otimes \Psi \otimes \mathrm{id})(v_1 \otimes v_2 \otimes w_1 \otimes w_2) \\ &= \zeta^{-\deg(v_2)\deg(w_1)} v_1 w_1 \otimes v_2 w_2, \end{aligned}$$

and the involution is given by

$$\begin{aligned} *_{\mathcal{B}\otimes\mathcal{B}}(v \otimes w) &= \Psi \circ (* \otimes *) \circ \tau(v \otimes w) = \zeta^{-\deg(w^*)\deg(v^*)} v^* \otimes w^* \\ &= \zeta^{-\deg(w)\deg(v)} v^* \otimes w^*. \end{aligned}$$

Let $\tilde{\mathcal{A}} \coloneqq \mathbb{C}\langle \alpha, \alpha^*, \gamma, \gamma^* \rangle$ be equipped with a $\mathbb{CZ}$-graduation determined by $\deg(\alpha) = \deg(\alpha^*) = 0$, $\deg(\gamma) = 1$, $\deg(\gamma^*) = -1$.

Lemma 6.9. *$\tilde{\Delta} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}}$ given by*

$$\tilde{\Delta}(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \tilde{\Delta}(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

is coassociative, and $\tilde{\varepsilon} : \tilde{\mathcal{A}} \to \mathbb{C}$ with $\tilde{\varepsilon}(\alpha) = 1$ and $\tilde{\varepsilon}(\gamma) = 0$ satisfies the counit axiom.

Proof. First we show that $(\mathrm{id} \otimes \tilde{\Delta}) \circ \tilde{\Delta}(\alpha) = (\tilde{\Delta} \otimes \mathrm{id}) \circ \tilde{\Delta}(\alpha)$. We have

$$\begin{aligned} (\mathrm{id} \otimes \tilde{\Delta}) \circ \tilde{\Delta}(\alpha) &= (\mathrm{id} \otimes \tilde{\Delta})(\alpha \otimes \alpha - q\gamma^* \otimes \gamma) \\ &= \alpha \otimes \tilde{\Delta}(\alpha) - q\gamma^* \otimes \tilde{\Delta}(\gamma) \\ &= \alpha \otimes (\alpha \otimes \alpha - q\gamma^* \otimes \gamma) - q\gamma^* \otimes (\gamma \otimes \alpha + \alpha^* \otimes \gamma) \\ &= \alpha \otimes \alpha \otimes \alpha - q\alpha \otimes \gamma^* \otimes \gamma - q\gamma^* \otimes \gamma \otimes \alpha + q\gamma^* \otimes \alpha^* \otimes \gamma \\ &= \alpha \otimes \alpha \otimes \alpha - q\gamma^* \otimes \gamma \otimes \alpha - q(\gamma^* \otimes \alpha^* + \alpha \otimes \gamma^*) \otimes \gamma \\ &= (\alpha \otimes \alpha - q\gamma^* \otimes \gamma) \otimes \alpha - q(\gamma \otimes \alpha + \alpha^* \otimes \gamma)^* \otimes \gamma \\ &= \tilde{\Delta}(\alpha) \otimes \alpha - q\tilde{\Delta}(\gamma)^* \otimes \gamma \\ &= (\tilde{\Delta} \otimes \mathrm{id})(\alpha \otimes \alpha - q\gamma^* \otimes \gamma) = (\tilde{\Delta} \otimes \mathrm{id}) \circ \tilde{\Delta}(\alpha). \end{aligned}$$

Similarly, we can check that $(\mathrm{id} \otimes \tilde{\Delta}) \circ \tilde{\Delta}(\gamma) = (\tilde{\Delta} \otimes \mathrm{id}) \circ \tilde{\Delta}(\gamma)$.

The counit property for α is verified as follows:

$$\begin{aligned} (\mathrm{id} \otimes \tilde{\varepsilon}) \circ \tilde{\Delta}(\alpha) &= (\mathrm{id} \otimes \tilde{\varepsilon})(\alpha \otimes \alpha - q\gamma^* \otimes \gamma) \\ &= \alpha \otimes \tilde{\delta}(\alpha) - q\gamma^* \otimes \tilde{\varepsilon}(\gamma) \\ &= \alpha \\ &= \tilde{\varepsilon}(\alpha) \otimes \alpha - q\tilde{\varepsilon}(\gamma^*) \otimes \gamma \\ &= (\tilde{\varepsilon} \otimes \mathrm{id})(\alpha \otimes \alpha - q\gamma^* \otimes \gamma) = (\tilde{\varepsilon} \otimes \mathrm{id}) \circ \tilde{\Delta}(\alpha). \end{aligned}$$

The counit property for γ can be verified in the same way. ☐

It is also straightforward to check that $\tilde{\Delta}$ preserves the grading.

Let I be the two-sided $*$-ideal in $\tilde{\mathcal{A}}$ generated by the relations

$$\begin{aligned} I_1 &:= \alpha^*\alpha + \gamma^*\gamma - \mathbb{1}, \\ I_2 &:= \alpha\alpha^* + |q|^2\gamma^*\gamma - \mathbb{1}, \\ I_3 &:= \gamma\gamma^* - \gamma^*\gamma, \\ I_4 &:= \alpha\gamma - \bar{q}\gamma\alpha, \\ I_5 &:= \alpha\gamma^* - q\gamma^*\alpha, \end{aligned}$$

and $\mathcal{A} := \tilde{\mathcal{A}}/I$ the quotient algebra.

Lemma 6.10. *I is a coideal in $\tilde{\mathcal{A}}$.*

Proof. One checks that

$$\begin{aligned} \tilde{\Delta}(I) &\subseteq \tilde{\mathcal{A}} \otimes I + I \otimes \tilde{\mathcal{A}}, \\ \tilde{\varepsilon}(I) &= \{0\}. \end{aligned}$$ □

Since I is a coideal in $\tilde{\mathcal{A}}$ and $\tilde{\Delta}$ is coassociative, there exists a unique bialgebra-structure Δ, δ on the quotient algebra $\mathcal{A} = \tilde{\mathcal{A}}/\tilde{\mathcal{I}}$. We have $\deg I = 0$, so $\mathcal{A}$ inherits a grading, and we can lift the action and the coaction on $\mathcal{A}$ to an action $\vartheta : \mathbb{C}\mathbb{Z} \otimes \mathcal{A} \to \mathcal{A}$ and a coaction $\rho : \mathcal{A} \to \mathbb{C}\mathbb{Z} \otimes \mathcal{A}$ on $\mathcal{A}$. It follows that $(\mathcal{A}, \vartheta, \rho)$ is again an object in ${}^{\mathbb{C}\mathbb{Z}}_{\mathbb{C}\mathbb{Z}}\mathcal{YD}_*$. With these structures, $\mathcal{A} = \mathrm{Pol}(\mathrm{SU}_q(2))$ is the braided Hopf-$*$-algebra of the braided compact quantum groups $\mathrm{SU}_q(2)$, $0 < |q| < 1$, defined and studied in [12].

Schürmann and Skeide described all generators on $\mathrm{Pol}(\mathrm{SU}_q(2))$ for $q \in \mathbb{R}\backslash\{0\}$, i.e., in the unbraided case [28]. We extend the classification of the quadratic generators in [28, Corollary 3.3] to the braided SU(2) quantum groups.

Proposition 6.11. *Let H be a Hilbert space. For any vector $v \in H$ and real number λ there exists a unique triple (ε, η, L) such that*

$$\begin{aligned} \eta(\gamma) &= \eta(\gamma^*) = 0, \\ \eta(\alpha) &= -\eta(\alpha^*) = v, \\ L(\gamma) &= L(\gamma^*) = 0, \\ L(\alpha) &= i\lambda - \frac{\|v\|^2}{2}, \\ L(\alpha^*) &= -i\lambda - \frac{\|v\|^2}{2}. \end{aligned}$$

Two such triples determined by pairs (v, λ) and (v', λ') have the same generator L if and only if $\|v\| = \|v'\|$ and $\lambda = \lambda'$. Furthermore, all quadratic generators on $\mathrm{Pol}(\mathrm{SU}_q(2))$ *arise in this way.*

Proof. Similar to [28]. □

It turns out that all quadratic generators on $\mathrm{Pol}(\mathrm{SU}_q(2))$ are invariant and therefore define Brownian motions. In fact, as in [28], ε, η, and L vanish on γ and γ^*, which implies that the triple factorizes through the quotient map

$$\mathcal{A} \to \mathcal{A}/\mathcal{J}_\gamma \cong \mathbb{C}\mathbb{Z} \cong \mathrm{Pol}(\mathbb{T}),$$

where $\mathcal{J}_\gamma$ denotes the $*$-ideal generated by γ. This means that the Brownian motions on $\mathrm{SU}_q(2)$ are induced from Brownian motions of the undeformed subgroup $\mathbb{T} \subseteq \mathrm{SU}_q(2)$.

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