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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR p-KIRCHHOFF-TYPE NEUMANN PROBLEMS

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Abstract. We establish, based on variational methods, existence theorems for a p-Kirchhoff-type Neumann problem under the Landesman–Lazer type condition and under the local coercive condition. In addition, multiple solutions for a p-Kirchhoff-type Neumann problem are established using a known three-critical-point theorem proposed by H. Brezis and L. Nirenberg.

1. Introduction and main results

We will study the following p-Kirchhoff-type Neumann problem:

$$\begin{cases} -\left[M(\int_\Omega |\nabla u|^p\,dx)\right]^{p-1}\Delta_p u = f(x,u) & \text{in } \Omega,\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, $p \geq 2$, N is a positive integer, n is the outer unit normal to $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian differential operator, and $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function satisfying the following condition:

(M0) There exists a constant $m_0 > 0$ such that $M(s) \geq m_0 > 0$ for all $s \geq 0$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$\sup_{|t|\leq s} |f(\cdot,t)| \in L^1(\Omega) \quad \text{for all } s > 0.$$

A distinguishing feature of the p-Kirchhoff-type Neumann equation (1.1) is that the equation contains a nonlocal expression of $(\int_\Omega |\nabla u|^p\,dx)$, and hence the equation is no longer a pointwise identity. In the case $p = 2$, the problem (1.1) reduces to the following nonlocal Kirchhoff elliptic problem:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u) & \text{in } \Omega,\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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This is related to the stationary analogue of the Kirchhoff problem

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u = f(x,u).$$

Such a model was first proposed by Kirchhoff [21] to describe transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations. The study of the Kirchhoff-type problems is one of the hot spots in nonlocal partial differential equations. In the case $M(s) = a+bs$, if we replace the Neumann problem with the Dirichlet problem, the problem (1.2) reduces to the following nonlocal Kirchhoff elliptic problem:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\, dx\right)\Delta u = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which received much attention only after Lions [22] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [3], [4], [10], [11], [12], [13], [16], [24], [27], [34].

To the best of our knowledge, a little information on the existence of solutions for the Neumann problem of Kirchhoff type can be found in the existing references, see [17], [36], [37]. If we set $M(s) = a + bs$ with $a > 0$, $b = 0$ in (1.1), it reduces to the quasilinear elliptic problem with Neumann boundary condition. The solvability of such problem has been studied by many authors. References [1], [2], [5], [6], [7], [9], [14], [15], [18], [23], [24], [25], [26], [29], [33] can be recommended to readers. In particular, when $p = 2$, in [18] and [30]–[31] existence and multiplicity of solutions were obtained under the Landesman–Lazer type condition and under a new Landesman–Lazer type condition, respectively. More precisely, when $p \neq 2$, Wu–Tan in [33] obtained existence and multiplicity of solutions under the Landesman–Lazer type condition. Recently, Jiang et al. in [20] made a generalization of [33].

Resonance Neumann problems for the p-Kirchhoff type have not been studied up to now. Motivated by the above facts and the references [12], [20], [19], and [32], in the present paper we will study the existence and multiplicity of solutions of the problem (1.1) under the Landesman–Lazer type condition and under the local coercive condition, respectively, by using the variational method. That is, we generalize the results of [20], [19], and [32] to the case of the p-Kirchhoff type.

Our main results are summarized as follows:

Theorem 1.1. *Suppose* (M0) *and the following conditions hold:*

(H1) *Whenever* $\{u_n\} \subset W^{1,p}(\Omega)$ *is such that* $\|u_n\| \to \infty$ *and* $\frac{|\bar u_n||\Omega|^{\frac{1}{p}}}{\|u_n\|} \to 1$ *as* $n \to \infty$, *we have*

$$\limsup_{n\to\infty} \int_\Omega f(x,u_n)\frac{\bar u_n}{|\bar u_n|}\, dx > 0.$$

(H2) *Uniformly for almost all* $x \in \Omega$ *we have* $\limsup_{|t|\to\infty} \frac{F(x,t)}{|t|^p} < \frac{\bar\lambda m_0^{p-1}}{p}$ *with* $F(x,t) = \int_0^t f(x,s)\, ds$.

Then the problem (1.1) *has at least one solution in* $W^{1,p}(\Omega)$ *for* $p \in (N, +\infty)$. *Here* $W^{1,p}(\Omega)$ *is endowed with the norm*

$$\|u\| = \left(\int_\Omega |u|^p \, dx + \int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$

and may be split in the following way:

$$W^{1,p}(\Omega) = \widetilde{W}^{1,p}(\Omega) \oplus \mathbb{R},$$

$$\widetilde{W}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \bar{u} = 0\}, \quad \bar{u} = |\Omega|^{-1} \int_\Omega u(x) \, dx, \quad \tilde{u} = u - \bar{u},$$

and $\bar{\lambda}$ *is a positive constant such that*

$$\int_\Omega |\nabla u|^p \, dx \geq \bar{\lambda} \int_\Omega |u|^p \, dx \quad \text{for all } 0 \neq u \in \widetilde{W}^{1,p}(\Omega). \tag{1.3}$$

Throughout the paper, $\bar{\lambda}$ is supposed to be the biggest constant satisfying the Poincaré–Wirtinger inequality (1.3). Details of the proof of the inequality (1.3) may be seen in [33, Proposition 1].

Theorem 1.2. *Suppose* (M0) *and the following conditions hold:*

(H3) *Whenever* $\{u_n\} \subset W^{1,p}(\Omega)$ *is such that* $\|u_n\| \to \infty$ *and* $\frac{|\bar{u}_n||\Omega|^{\frac{1}{p}}}{\|u_n\|} \to 1$ *as* $n \to \infty$, *we have*

$$\lim_{n\to\infty} \int_\Omega F(x, u_n) \, dx = -\infty.$$

(H4) *Uniformly for almost all* $x \in \Omega$ *we have* $\limsup_{|t|\to\infty} \frac{F(x,t)}{|t|^p} < 0$.

Then the problem (1.1) *has at least one solution in* $W^{1,p}(\Omega)$.

Theorem 1.3. *Suppose* (M0), (H3), (H4), *and the following condition hold:*

(H5) *There exists a constant* $\delta > 0$ *such that*

$$0 \leq F(x,t) \leq \frac{\bar{\lambda} m_0^{p-1}}{p} |t|^p \quad \text{for all } |t| \leq \delta \text{ and almost all } x \in \Omega.$$

Then the problem (1.1) *has at least two nontrivial solutions in* $W^{1,p}(\Omega)$ *for* $p \in (N, +\infty)$.

Theorem 1.4. *Suppose* (M0) *and the following conditions hold:*

(H6) *There exist a constant* $C_1 > 0$ *and a real function* $\gamma \in L^1(\Omega)$ *such that*

$$|f(x,t)| \leq C_1 |t|^{p^*-1} + \gamma(x)$$

for all $t \in \mathbb{R}$ *and almost all* $x \in \Omega$, *where* $p^* = \frac{pN}{N-p}$ *for* $p < N$.

(H7) *There exists a subset* Ω_0 *of* Ω *with* $|\Omega_0| > 0$ *such that*

$$F(x,t) \to -\infty \text{ as } |t| \to \infty \text{ uniformly for almost all } x \in \Omega_0.$$

(H8) *There exists* $\kappa \in L^1(\Omega)$ *such that*

$$F(x,t) \leq \kappa(x)$$

for all $t \in \mathbb{R}$ *and almost all* $x \in \Omega$.

Then the problem (1.1) *has at least one solution in* $W^{1,p}(\Omega)$ *for* $p \in [2, N)$.

Remark 1.5. Both (H1) and (H3) are called the Landesman–Lazer type conditions. Theorems 2–4 in [20] correspond respectively to Theorems 1.1–1.3 in our paper, for the special case $M(s) = a + bs$ with $a > 0$, $b = 0$, and $m_0 = a$. Moreover, there are functions $F(x,t)$ satisfying our Theorem 1.1. For example,

$$F(x,t) = a|t|^p, \quad a < \frac{\bar{\lambda} m_0^{p-1}}{p}.$$

Of course, there are functions $F(x,t)$ satisfying our Theorem 1.3 but not satisfying those in other references. For example,

$$F(x,t) = \frac{\bar{\lambda} m_0^{p-1}}{p}|t|^p - b|t|^{p+1}, \quad b < \frac{\bar{\lambda} m_0^{p-1}}{p}.$$

Remark 1.6. Theorem 1.4 generalizes the results in [19] and [32]. Obviously, the corresponding results of [19] and [32] are the special case $M(s) = a + bs$ with $a > 0$, $b = 0$, $p = 2$, and $m_0 = a$ in Theorem 1.4. There are functions $F(x,t)$ satisfying our Theorem 1.4 but not satisfying those in other references. In fact, take

$$f(x,t) = -p(x - x_0)\frac{|t|^{p-2}t}{1 + |t|^p} + p^*|t|^{p^*-2}t \cos |t|^{p^*},$$

where $x_0 \in \bar{\Omega}$. An easy computation shows that the function

$$F(x,t) = -(x - x_0)\ln(1 + |t|^p) + \sin |t|^{p^*}$$

satisfies (H6)–(H8).

2. Proofs of the main results

We say that a function $u \in W^{1,p}(\Omega)$ is a weak solution of the problem (1.1) if

$$\left[M\left(\int_\Omega |\nabla u|^p \, dx\right)\right]^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx - \int_\Omega f(x,u)v \, dx = 0$$

for all $v \in W^{1,p}(\Omega)$. Thus, the corresponding energy functional of the problem (1.1) is defined by

$$\varphi(u) = \frac{1}{p}\hat{M}\left(\int_\Omega |\nabla u|^p \, dx\right) - \int_\Omega F(x,u) \, dx,$$

where $\hat{M}(t) = \int_0^t [M(s)]^{p-1} \, ds$. Obviously, $\hat{M}(t)$ is nondecreasing continuous for $t \geq 0$ by (M0). We all know that the weak solutions to the problem (1.1) are the critical points of the energy functional φ.

Now, some lemmas are stated for the readers' convenience.

Lemma 2.1 (The least action principle [28]). *Let X be a reflexive Banach space. If a functional $\varphi \in C^1(X, \mathbb{R})$ is weakly lower semicontinuous and coercive, that is,*

$$\varphi(u) \to +\infty \quad \text{as } \|u\| \to \infty$$

for $u \in X$, then there exists $\tilde{x} \in E$ such that $\inf_{x\in X} \varphi(x) = \varphi(\tilde{x})$ and $\tilde{x}$ is also a critical point of φ, i.e., $\varphi'(\tilde{x}) = 0$.

Lemma 2.2 (Saddle point theorem [28]). *Let $X = X_1 \oplus X_2$, where X is a Banach space and $X_2 \neq \{0\}$ and $\dim X_2 < \infty$ is finite. Suppose $\varphi \in C(X, \mathbb{R})$ satisfies the (PS) condition and*

(i) *there is a constant r and a bounded neighborhood U of 0 in X_2 such that $\varphi|_{\partial U} \leq r$, and*
(ii) *there exists a constant $\alpha > r$ such that $\varphi|_{X_1} \geq \alpha$.*

Then φ possesses a critical value $c \geq \alpha$.

Lemma 2.3 ([8]). *Let X be a Banach space with a direct sum decomposition $X = X_1 \oplus X_2$ with $\dim X_2 < \infty$ and let φ be a C^1 function on X with $\varphi(0) = 0$, satisfying the (PS) condition. Assume that, for some $\delta_0 > 0$,*

$$\begin{aligned} \varphi(v) &\geq 0 \quad \text{for } v \in X_1 \text{ with } \|v\| \leq \delta_0, \\ \varphi(v) &\leq 0 \quad \text{for } v \in X_2 \text{ with } \|v\| \leq \delta_0. \end{aligned}$$

Assume also that φ is bounded from below and $\inf_X \varphi < 0$. Then φ has at least two nonzero critical points.

Lemma 2.4. *Suppose that F satisfies assumptions* (H6) *and* (H7). *Then there exist a real function $\alpha \in L^1(\Omega)$ and $G \in C(\mathbb{R}, \mathbb{R})$ which is subadditive, that is,*

$$G(s + t) \leq G(s) + G(t) \quad \text{for all } s, t \in \mathbb{R},$$

and coercive, that is,

$$G(t) \to +\infty \quad \text{as } |t| \to \infty,$$

and satisfies

$$G(t) \leq |t| + 4 \quad \text{for all } t \in \mathbb{R},$$

such that

$$F(x, t) \leq -G(t) + \alpha(x)$$

for all $t \in \mathbb{R}$ and a.e. $t \in \Omega_0$.

Proof. The proof of Lemma 2.4 is essentially the same one as the introductory part of the proof of Theorem 1 in [32]. □

Proof of Theorem 1.1. It suffices to show that all conditions of Lemma 2.2 with $X_1 = \widetilde{W}^{1,p}(\Omega)$, $X_2 = \mathbb{R}$ are fulfilled. Firstly, we show that there exist constants $M_1 > 0$ and $\mu > 0$ such that

$$\int_\Omega f(x, t)t \, dx \geq \mu|t| \tag{2.1}$$

for $t \in \mathbb{R}$ with $|t| \geq M_1$. If not, there exists a sequence $\{t_n\} \subset \mathbb{R}$ with $|t| \to \infty$ such that

$$\int_\Omega f(x, t_n) \frac{t_n}{|t_n|} \, dx < \frac{1}{n}$$

for any $n \geq 1$, which contradicts (H1).

It follows that

$$\begin{aligned}
\varphi(u) &= -\int_\Omega F(x, u) \, dx \\
&= -\int_\Omega [F(x, u) - F(x, 0)] \, dx - \int_\Omega F(x, 0) \, dx \\
&= -\int_\Omega \left[\int_0^1 f(x, us) u \, ds \right] dx - \int_\Omega F(x, 0) \, dx \\
&= -\int_\Omega \left[\int_0^{\frac{M_1}{|u|}} f(x, us) u \, ds + \int_{\frac{M_1}{|u|}}^1 f(x, us) u \, ds \right] dx - \int_\Omega F(x, 0) \, dx
\end{aligned}$$

for all $u \in X_2 = \mathbb{R}$ with $|u| \geq M_1$. Furthermore, from (2.1), we deduce that

$$\begin{aligned}
\left| \int_\Omega \int_0^{\frac{M_1}{|u|}} f(x, us) u \, ds \, dx \right| &\leq \int_\Omega \int_0^{\frac{M_1}{|u|}} |f(x, us)||u| \, ds \, dx \\
&\leq \int_\Omega \int_0^{\frac{M_1}{|u|}} g_1(x) |u| \, ds \, dx = M_1 \int_\Omega g_1(x) \, dx
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \int_{\frac{M_1}{|u|}}^1 f(x, us) u \, ds \, dx &= \int_{\frac{M_1}{|u|}}^1 \frac{1}{s} \left[\int_\Omega f(x, us) su \, dx \right] ds \\
&\geq \int_{\frac{M_1}{|u|}}^1 \frac{1}{s} (\mu |su|) \, ds = \mu |u| \left(1 - \frac{M_1 - 1}{|u|} \right) = \mu |u| + \mu M_1,
\end{aligned}$$

where $g_1(x) = \sup_{|su| < M_1} f(x, us) \in L^1(\Omega)$. Then we deduce

$$\varphi(u) \leq M_1 \int_\Omega g_1(x) \, dx - \mu |u| - \mu M_1 - \int_\Omega F(x, 0) \, dx.$$

Thus, φ is anti-coercive on $\mathbb{R}$, namely, $\varphi(u) \to -\infty$ as $|u| \to +\infty$ for $u \in \mathbb{R}$. That is, the condition (i) of Lemma 2.2 holds true.

Secondly, it follows from (H2) that there are constants $\varepsilon_0 \in \left(0, \frac{\bar{\lambda} m_0^{p-1}}{p}\right)$ and $M_2 > 0$ such that

$$F(x, t) < \left(\frac{\bar{\lambda} m_0^{p-1}}{p} - \varepsilon_0 \right) |t|^p$$

for all $t \in \mathbb{R}$ with $|t| > M_2$. Set $g_2(x) = \sup_{|t| \leq M_2} F(x, t)$. Then we obtain

$$F(x, t) < \left(\frac{\bar{\lambda} m_0^{p-1}}{p} - \varepsilon_0 \right) |t|^p + g_2(x) \tag{2.2}$$

for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$. In accordance with (M0), (2.2), and (1.3), we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\hat{M}\left(\int_\Omega |\nabla u|^p\,dx\right) - \int_\Omega F(x,u)\,dx \\
&> \frac{m_0^{p-1}}{p}\int_\Omega |\nabla u|^p\,dx - \left(\frac{\bar{\lambda} m_0^{p-1}}{p} - \varepsilon_0\right)\int_\Omega |u|^p\,dx - \int_\Omega g_2(x)\,dx \\
&\geq \frac{m_0^{p-1}}{p}\int_\Omega |\nabla u|^p\,dx - \left(\frac{\bar{\lambda} m_0^{p-1}}{p} - \varepsilon_0\right)\frac{1}{\bar{\lambda}}\int_\Omega |\nabla u|^p\,dx - \int_\Omega g_2(x)\,dx \\
&= \frac{\varepsilon_0}{\bar{\lambda}}\int_\Omega |\nabla u|^p\,dx - \int_\Omega g_2(x)\,dx = \frac{\varepsilon_0}{\bar{\lambda}}\|u\|^p - \int_\Omega g_2(x)\,dx
\end{aligned} \tag{2.3}$$

for all $u \in \widetilde{W}^{1,p}(\Omega)$. The inequality (2.3) means that $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$ for $u \in \widetilde{W}^{1,p}(\Omega)$. So the condition (ii) of Lemma 2.2 is fulfilled too.

Finally, we draw the conclusion that φ satisfies the (PS) condition. Assume that there is a sequence $\{u_n\}$ of $W^{1,p}(\Omega)$ such that

$$\{\varphi(u_n)\} \text{ is bounded} \quad \text{and} \quad \varphi'(u_n) \to 0 \text{ as } n \to \infty.$$

We claim that $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. Otherwise, we assume that any subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) satisfies

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty. \tag{2.4}$$

Let $v_n = \beta\frac{u_n}{\|u_n\|}$ with $\beta = \left(1/(1+\bar{\lambda})\right)^{\frac{1}{p}}$. Obviously, $\|v_n\| = \beta$ and $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$. Thus there exist a point $v \in W^{1,p}(\Omega)$ and a subsequence of $\{v_n\}$, say $\{v_n\}$, satisfying

$$v_n \rightharpoonup v \text{ weakly in } W^{1,p}(\Omega) \quad \text{and} \quad v_n \to v \text{ strongly in } L^p(\Omega).$$

Then, in light of (2.2), we derive

$$\begin{aligned}
\frac{\varphi(u_n)}{\|u_n\|^p} &= \frac{1}{p}\frac{\hat{M}\left(\int_\Omega |\nabla u_n|^p\,dx\right)}{\|u_n\|^p} - \frac{1}{\|u_n\|^p}\int_\Omega F(x,u_n)\,dx \\
&> \frac{m_0^{p-1}}{p\beta^p}\int_\Omega |\nabla v_n|^p\,dx - \left(\frac{\bar{\lambda} m_0^{p-1}}{p} - \varepsilon_0\right)\frac{1}{\beta^p}\int_\Omega |v_n|^p\,dx - \frac{\int_\Omega g_2(x)\,dx}{\|u_n\|^p} \\
&> \frac{m_0^{p-1}}{p\beta^p}\int_\Omega |\nabla v_n|^p\,dx - \frac{\bar{\lambda} m_0^{p-1}}{p\beta^p}\int_\Omega |v_n|^p\,dx - \frac{\int_\Omega g_2(x)\,dx}{\|u_n\|^p} \\
&= \frac{m_0^{p-1}}{p\beta^p} - \frac{m_0^{p-1}}{p\beta^p}\int_\Omega |v_n|^p\,dx - \frac{\bar{\lambda} m_0^{p-1}}{p\beta^p}\int_\Omega |v_n|^p\,dx - \frac{\int_\Omega g_2(x)\,dx}{\|u_n\|^p}.
\end{aligned} \tag{2.5}$$

Then, according to (2.4), (2.5), and the boundedness of $\{\varphi(u_n)\}$, we have

$$\int_\Omega |v|^p\,dx \geq \frac{1}{1+\bar{\lambda}} = \beta^p. \tag{2.6}$$

On the one hand, due to the weakly lower semicontinuity of the norm, we obtain

$$\|v\| \leq \liminf_{n\to+\infty} \|v_n\| = \beta.$$

Then we have

$$\int_\Omega |\nabla v|^p \, dx + \int_\Omega |v|^p \, dx = \|v\|^p \leq \beta^p. \tag{2.7}$$

In accordance with (2.6) and (2.7), we have

$$\int_\Omega |\nabla v|^p \, dx = 0 \quad \text{and} \quad \int_\Omega |v|^p \, dx = \frac{1}{1+\bar{\lambda}} = \beta^p. \tag{2.8}$$

This means that $|\nabla v(x)| = 0$, namely, $|v(x)| \equiv$ constant for all $x \in \Omega$. Therefore, it follows from (2.8) that $|v|^p = \frac{\beta^p}{|\Omega|}$. So we obtain

$$\begin{aligned} \frac{|\bar{u}_n|^p}{\|u_n\|^p} &= \left| \frac{1}{|\Omega|} \int_\Omega \frac{u_n}{\|u_n\|} \, dx \right|^p = \left| \frac{1}{|\Omega|\beta} \int_\Omega v_n(x) \, dx \right|^p \\ &\to \left| \frac{1}{|\Omega|\beta} \int_\Omega v(x) \, dx \right|^p = \left(\frac{1}{|\Omega|\beta} \int_\Omega |v(x)| \, dx \right)^p = \frac{1}{|\Omega|} \quad \text{as } n \to +\infty. \end{aligned}$$

This means that $\frac{|\bar{u}_n||\Omega|^{\frac{1}{p}}}{\|u_n\|} \to 1$ as $n \to +\infty$. In light of (2.4) and (H1), we deduce

$$\limsup_{n\to\infty} \int_\Omega f(x, u_n) \frac{\bar{u}_n}{|\bar{u}_n|} \, dx > 0. \tag{2.9}$$

On the other hand, since $\varphi'(u_n) \to 0$ as $n \to +\infty$, we obtain

$$\int_\Omega f(x, u_n) \frac{\bar{u}_n}{|\bar{u}_n|} \, dx = -\left\langle \varphi'(u_n), \frac{\bar{u}_n}{|\bar{u}_n|} \right\rangle \to 0 \quad \text{as } n \to +\infty,$$

which contradicts (2.9). This indicates that the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$.

Next, we claim that $\{u_n\}$ has a convergent subsequence in $W^{1,p}(\Omega)$. Indeed, by the reflexivity of $W^{1,p}(\Omega)$, there exist $u \in W^{1,p}(\Omega)$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega), \\ &u_n \to u \text{ strongly in } L^p(\Omega), \\ &u_n(x) \to u(x) \text{ a.e. in } \Omega \text{ as } n \to \infty, \end{aligned}$$

and hence

$$\|u_n - u\|_\infty \to 0, \quad \langle \varphi'(u_n), u_n - u \rangle \to 0 \text{ as } n \to \infty,$$

since $W^{1,p}(\Omega)$ can be compactly embedded in $C^{0,m}$ for all $m \in \left(0, 1 - \frac{N}{p}\right)$ for $p > N$. Consequently, there exists $\rho > 0$ such that

$$\|u_n\|_\infty \leq \rho \quad \text{for all } n \in \mathbb{N}.$$

Set the sequence

$$\begin{aligned}P_n &:= \langle \varphi'(u_n), u_n - u\rangle + \int_\Omega f(x,u_n)(u_n-u)\,dx \\ &= \left[M\left(\int_\Omega |\nabla u_n|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u_n-u)\,dx.\end{aligned}$$

By the weak convergence, we have

$$\begin{aligned}&-\left[M\left(\int_\Omega |\nabla u_n|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla u_n\,dx \\ &\qquad\qquad + \left[M\left(\int_\Omega |\nabla u_n|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u|^p\,dx = o_n(1).\end{aligned}$$

Then we have

$$\begin{aligned}P_n + o_n(1) &= \left[M\left(\int_\Omega |\nabla u_n|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u_n-u)\,dx \\ &\quad - \left[M\left(\int_\Omega |\nabla u_n|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla u_n\,dx \\ &\quad + \left[M\left(\int_\Omega |\nabla u_n|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u|^p\,dx \\ &= \left[M\left(\int_\Omega |\nabla u_n|^p\,dx\right)\right]^{p-1}\int_\Omega \left\langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u\right\rangle.\end{aligned}$$

From (M0) and the following standard inequality in $\mathbb{R}^N$,

$$\left\langle |x|^{p-2}x - |y|^{p-2}y, x-y\right\rangle \geq C_p|x-y|^p \quad \text{for } p\geq 2,$$

we obtain

$$o_n(1) + P_n \geq m_0^{p-1}C_p\int_\Omega |\nabla u_n - \nabla u|^p\,dx.$$

Moreover, we know

$$\begin{aligned}P_n &= \langle \varphi'(u_n), u_n - u\rangle + \int_\Omega f(x,u_n)(u_n-u)\,dx \\ &\leq \langle \varphi'(u_n), u_n - u\rangle + \|u_n-u\|_\infty \int_\Omega \sup_{|t|\leq\rho} f(x,t)\,dx \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Thus the above inequalities mean that

$$\int_\Omega |\nabla u_n - \nabla u|^p\,dx \to 0 \quad \text{as } n\to\infty.$$

Consequently, we easily know that

$$\|u_n-u\|^p = \int_\Omega |\nabla(u_n-u)|^p\,dx + \int_\Omega |u_n-u|^p\,dx \to 0 \quad \text{as } n\to\infty,$$

which means that

$$u_n \to u \text{ in } W^{1,p}(\Omega) \quad \text{as } n \to \infty.$$

Hence φ satisfies the (PS) condition. Consequently, via Lemma 2.2, we conclude that the problem (1.1) has at least one solution. $\square$

Proof of Theorem 1.2. We use Lemma 2.1 for this proof.

Firstly, we can prove that $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$ for $u \in W^{1,p}(\Omega)$, using proof by contradiction. If not, there are a sequence $\{u_n\}$ in $W^{1,p}(\Omega)$ and a constant M_3 such that

$$\|u\| \to +\infty \text{ as } n \to +\infty \quad \text{and} \quad \varphi(u_n) \leq M_3. \tag{2.10}$$

Set $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$. Hence, there are a point $v_0 \in W^{1,p}(\Omega)$ and a subsequence of $\{v_n\}$, say $\{v_n\}$, such that

$$v_n \rightharpoonup v_0 \text{ weakly in } W^{1,p}(\Omega) \quad \text{and} \quad v_n \to v_0 \text{ strongly in } L^p(\Omega).$$

According to (H4), for any $\varepsilon > 0$ there is a constant $M_4 > 0$ such that

$$F(x,t) < \frac{\varepsilon}{p}|t|^p$$

for all $t \in \mathbb{R}$ with $|t| > M_4$. Put $g_3 = \sup_{|t| \leq M_4} F(x,t)$. Then we get

$$F(x,t) < \frac{\varepsilon}{p}|t|^p + g_3 \tag{2.11}$$

for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Then we have, by (2.11),

$$\begin{aligned}
\frac{M_3}{\|u_n\|^p} &\geq \frac{\varphi(u_n)}{\|u_n\|^p} \\
&= \frac{1}{p}\frac{\hat{M}\left(\int_\Omega |\nabla u_n|^p\,dx\right)}{\|u_n\|^p} - \frac{1}{\|u_n\|^p}\int_\Omega F(x,u_n)\,dx \\
&> \frac{m_0^{p-1}}{p}\int_\Omega |\nabla v_n|^p\,dx - \frac{\varepsilon}{p}\int_\Omega |v_n|^p\,dx - \frac{\int_\Omega g_3\,dx}{\|u_n\|^p} \\
&= \frac{m_0^{p-1}}{p} - \frac{m_0^{p-1}}{p}\int_\Omega |v_n|^p\,dx - \frac{\varepsilon}{p}\int_\Omega |v_n|^p\,dx - \frac{\int_\Omega g_3\,dx}{\|u_n\|^p}.
\end{aligned} \tag{2.12}$$

Thus by (2.10) and (2.12), we have

$$0 \geq \frac{m_0^{p-1}}{p} - \frac{m_0^{p-1}}{p}\int_\Omega |v_0|^p\,dx - \frac{\varepsilon}{p}\int_\Omega |v_0|^p\,dx.$$

Let $\varepsilon \to 0$; we see the inequality

$$\int_\Omega |v_0|^p\,dx \geq 1$$

holds. Due to the weakly lower semicontinuity of the norm, we get

$$\|v_0\| \leq \liminf_{n \to +\infty} \|v_n\| = 1$$

and then

$$\int_\Omega |\nabla v_0|^p\,dx + \int_\Omega |v_0|^p\,dx = \|v_0\|^p \leq 1.$$

Using an argument analogous to that in the proof of Theorem 1.1, we can draw the following conclusion:

$$|v_0(x)| \equiv \text{constant for all } x \in \Omega \quad \text{and} \quad \frac{|\bar{u}_n||\Omega|^{\frac{1}{p}}}{\|u_n\|} \to 1 \text{ as } n \to +\infty.$$

Due to (2.10) and (H3), we have

$$\lim_{n\to\infty} \int_\Omega F(x, u_n)\, dx = -\infty.$$

From this it is easy to get

$$\lim_{n\to\infty} \varphi(u_n) \geq - \lim_{n\to\infty} \int_\Omega F(x, u_n)\, dx = +\infty,$$

which contradicts (2.10). Therefore $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$ for $u \in W^{1,p}(\Omega)$.

Lastly, we claim that the functional φ is weakly lower semicontinuous. Indeed, notice that the map $u \mapsto \int_\Omega |\nabla u|^p\, dx$ is weakly lower semicontinuous and $\hat{M}$ is nondecreasing and continuous, so $\varphi_1(u) = \hat{M}\left(\int_\Omega |\nabla u|^p\, dx\right)$ is weakly lower semicontinuous in $W^{1,p}(\Omega)$. Using the Sobolev embedding theorem and (2.11), we know the functional $\varphi_2(u) = \int_\Omega F(x, u)\, dx$ is weakly continuous. Obviously, we get that $\varphi = \varphi_1 - \varphi_2$ is weakly lower semicontinuous. It follows from Lemma 2.1 that φ has a minimum. Thus the problem (1.1) has at least one solution. □

Proof of Theorem 1.3. We know the space $W^{1,p}(\Omega)$ is directly divided into two spaces named $X_1 = \widetilde{W}^{1,p}(\Omega)$ and $X_2 = \mathbb{R}$ satisfying $W^{1,p}(\Omega) = X_1 \oplus X_2$. Then the proof of Theorem 1.3 relies on Lemma 2.3. Next, we will describe the detailed process.

Firstly, it follows from (H5) that $F(x, 0) = 0$ for a.e. $x \in \Omega$; then $u(x) = 0$ is a solution of the problem (1.1). φ is a C^1 function on $W^{1,p}(\Omega)$ with $\varphi(0) = 0$. Moreover, by the proof of Theorem 1.2, we know that φ is coercive and bounded from below. Therefore $\{u_n\}$ is bounded. Similar to the proof of Theorem 1.1, we get that $\{u_n\}$ has a convergent subsequence in $W^{1,p}(\Omega)$. This means that φ satisfies the (PS) condition. Then, it follows from (H5) that

$$\varphi(u) = -\int_\Omega F(x, u)\, dx \leq 0 \tag{2.13}$$

for all $u \in \mathbb{R}$ with $|u| \leq \delta$.

Furthermore, for all $u \in \widetilde{W}^{1,p}(\Omega)$ with $\|u\|_\infty \leq \delta$, by (H5) we obtain

$$\begin{aligned} \varphi(u) &= \frac{\hat{M}\left(\int_\Omega |\nabla u|^p\, dx\right)}{p} - \int_\Omega F(x, u)\, dx \\ &\geq \frac{m_0^{p-1}}{p} \int_\Omega |\nabla u|^p\, dx - \frac{\bar{\lambda} m_0^{p-1}}{p} \int_\Omega |u|^p\, dx \geq 0. \end{aligned}$$

Since $W^{1,p}(\Omega)$ can be compactly embedded in $C^{0,m}$ for all $m \in \left(0, 1 - \frac{N}{p}\right)$ for

$p > N$, there is a constant $C > 0$ such that

$$\|u\|_\infty \le C\|u\|, \quad u \in W^{1,p}(\Omega).$$

Put $\delta_0 = \min\left\{\frac{\delta}{C}, \delta|\Omega|^{\frac{1}{p}}\right\}$. Hence we easily obtain the following two inequalities:

(i) $\varphi(u) \le 0$ for every $u \in \mathbb{R}$ with $\|u\| \le \delta_0$,
(ii) $\varphi(u) \ge 0$ for $u \in \widetilde{W}^{1,p}(\Omega)$ with $\|u\| \le \delta_0$.

In the case of $\inf_{W^{1,p}(\Omega)} \varphi < 0$, it follows from Lemma 2.3 that the problem (1.1) has at least two distinct solutions in $W^{1,p}(\Omega)$.

In the case of $\inf_{W^{1,p}(\Omega)} \varphi \ge 0$, it follows from (2.13) that

$$\varphi(u) = \inf_{W^{1,p}(\Omega)} \varphi = 0 \quad \text{for all } u \in \mathbb{R} \text{ with } \|u\| \le \delta_0,$$

which implies that all $u \in \mathbb{R}$ with $|u| \le \delta_0$ are solutions of the problem (1.1). That is, the problem (1.1) has infinitely many solutions in $W^{1,p}(\Omega)$. Consequently, Theorem 1.3 is proved. □

Proof of Theorem 1.4. According to Lemma 2.1, we need to prove that the functional φ is coercive, namely, $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$ for $u \in W^{1,p}(\Omega)$.

It follows from Lemma 2.4, (H8), (1.3), and Hölder's inequality that

$$\begin{aligned}
\int_\Omega F(x,u)\,dx &= \int_{\Omega_0} F(x,u)\,dx + \int_{\Omega\setminus\Omega_0} F(x,u)\,dx \\
&\le -\int_{\Omega_0} G(u)\,dx + \int_{\Omega_0} \alpha(x)\,dx + \int_{\Omega\setminus\Omega_0} \kappa(x)\,dx \\
&\le -\int_{\Omega_0} G(\bar{u})\,dx + \int_{\Omega_0} G(-\tilde{u})\,dx + \int_{\Omega_0} \alpha(x)\,dx + \int_{\Omega\setminus\Omega_0} \kappa(x)\,dx \\
&\le -|\Omega_0|G(\bar{u}) + \int_{\Omega_0} G(-\tilde{u})\,dx + \int_\Omega |\alpha(x)|\,dx + \int_\Omega |\kappa(x)|\,dx \\
&\le -|\Omega_0|G(\bar{u}) + \int_{\Omega_0} (|\tilde{u}| + 4)\,dx + M_5 \\
&\le -|\Omega_0|G(\bar{u}) + |\Omega_0|^{1/q}\left(\int_{\Omega_0} |\tilde{u}|^p\,dx\right)^{1/p} + 4|\Omega_0| + M_5 \\
&\le |\Omega_0|(4 - G(\bar{u})) + |\Omega_0|^{1/q}(1/\bar{\lambda})^{1/p}\left(\int_\Omega |\nabla\tilde{u}|^p\,dx\right)^{1/p} + M_5 \\
&= |\Omega_0|(4 - G(\bar{u})) + M_6\left(\int_\Omega |\nabla\tilde{u}|^p\,dx\right)^{1/p} + M_5
\end{aligned} \tag{2.14}$$

for all $u \in W^{1,p}(\Omega)$, where $M_5 = \int_\Omega |\alpha(x)|\,dx + \int_\Omega |\kappa(x)|\,dx$, $M_6 = |\Omega_0|^{1/q}(1/\bar{\lambda})^{1/p}$. By (M0) and (2.14), we have

$$\begin{aligned}
\varphi(u) &= \frac{\hat{M}\left(\int_\Omega |\nabla u|^p\,dx\right)}{p} - \int_\Omega F(x,u)\,dx \\
&\geq \frac{m_0^{p-1}}{p}\int_\Omega |\nabla u|^p\,dx + |\Omega_0|(G(\bar{u})-4) - M_6\left(\int_\Omega |\nabla \tilde{u}|^p\,dx\right)^{1/p} - M_5 \\
&= \frac{m_0^{p-1}}{p}\int_\Omega |\nabla \tilde{u}|^p\,dx + (G(\bar{u})-4)|\Omega_0| - M_6\|\tilde{u}\| - M_5 \\
&= \frac{m_0^{p-1}}{p}\|\tilde{u}\|^p + (G(\bar{u})-4)|\Omega_0| - M_6\|\tilde{u}\| - M_5
\end{aligned}$$

for all $u \in W^{1,p}(\Omega)$, which implies that φ is coercive by Lemma 2.4 and the fact that

$$\|\tilde{u}\|^p + \|\bar{u}\|^p = \|u\|^p.$$

Lastly, we claim the functional φ is weakly lower semicontinuous. Indeed, we know $\varphi_1(u) = \hat{M}\left(\int_\Omega |\nabla u|^p\,dx\right)$ is weakly lower semicontinuous. Moreover, if $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ as $n \to \infty$, without loss of generality we may assume that

$$u_n \to u \text{ strongly in } L^p(\Omega) \quad \text{and} \quad u_n(x) \to u(x) \text{ a.e. in } \Omega$$

as $n \to \infty$. Since $F(x,u_n(x)) \to F(x,u(x))$ as $n \to \infty$ for a.e. $x \in \Omega$, we have

$$\limsup_{n\to\infty} \int_\Omega F(x,u_n)\,dx \leq \int_\Omega F(x,u)\,dx$$

by (H8) and the Lebesgue–Fatou lemma [35]. This means that the functional $\varphi_2(u) = \int_\Omega F(x,u)\,dx$ is weakly upper semicontinuous. Hence $\varphi = \varphi_1 - \varphi_2$ is weakly lower semicontinuous. By the least action principle (see Lemma 2.1), φ has a minimum. Hence the problem (1.1) has at least one solution. ☐

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