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GENERALIZED TRANSLATION OPERATOR AND UNCERTAINTY PRINCIPLES ASSOCIATED WITH THE DEFORMED STOCKWELL TRANSFORM

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Dedicated to Professor Habib Maagli for his distinguished career in mathematics in general, and especially for his contribution to the Tunisian School of Potential.

Abstract. We study the generalized translation operator associated with the deformed Hankel transform on $\mathbb{R}$. Firstly, we prove the trigonometric form of the generalized translation operator. Next, we derive the positivity of this operator on a suitable space of even functions. Making use of the positivity of the generalized translation operator we introduce and study the deformed Stockwell transform. Knowing the fact that the study of uncertainty principles is both theoretically interesting and practically useful, we formulate several qualitative uncertainty principles for this new integral transform. Firstly, we mainly establish various versions of Heisenberg's uncertainty principles. Secondly, we derive some weighted uncertainty inequalities such as Pitt's and Beckner's uncertainty inequalities for the deformed Stockwell transform. We culminate our study by formulating several concentration-based uncertainty principles, including the Amrein–Berthier–Benedicks and local inequalities for the deformed Stockwell transform.

1. Introduction

Time-frequency analysis is undoubtedly a linchpin of modern communication systems, which deals with the study of localized spectral characteristics of transient and non-transient signals. The major breakthrough in the context of time-frequency analysis was witnessed in the form of the continuous wavelet transform, which offers efficient time-frequency representations of non-transient signals analysis in terms of time and frequency shifted basis functions, known as wavelets. The wavelets can be regarded as local decomposition filters which are adaptable to the spectral variations in the non-transient signals. By applying these local decomposition filters, the wavelet transform has proved to be of substantial importance in capturing the local characteristics of non-stationary signals and has paved its

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way to a number of fields including signal and image processing, sampling theory, geophysics, astrophysics, quantum mechanics and so on [10, 54]. Despite numerous success stories, the wavelet transform suffers from two apparent limitations: first, the detail measured by the wavelet transform is not directly analogue to the frequency, because the wavelet transform is essentially a time-scale transform with the inverse scale being interpreted as frequency; second, the phase-information is completely lost in the case of wavelet transform, because each wavelet component acts a local filter and the translation of the mother wavelet completely destroys the phase information. To circumvent these limitations, R. G. Stockwell [52] introduced the notion of Stockwell transform as a bridge between the short-time Fourier transform (STFT) and the wavelet transform. By adopting the progressive resolution of wavelets, the Stockwell transform is able to resolve a wider range of frequencies than the ordinary STFT and by using a Fourier-like basis and maintaining a phase of zero about the time $t = 0$, Fourier-based analysis could be performed locally. This unique feature of the Stockwell transform makes it a highly valuable tool for signal processing and is one of the hottest research areas of the contemporary era [52, 40, 41, 45, 46, 51]. We note that this transform has been successfully used to analyse signals in numerous applications, such as seismic recordings, ground vibrations, geophysics, medical imaging, hydrology, gravitational waves, power system analysis and many other areas. Many extensions of the Stockwell transform have been proposed in recent years. See, for example, [7, 8, 11, 13, 41].

Motivated by the previous works, we extend in this paper the Stockwell transform to the setup of the minimal unitary representation of the conformal group $\mathrm{O}(d+1,1)$, and we then establish its fundamental properties. More precisely, in [4] the authors gave a far reaching extension of the classical Fourier analysis by constructing a generalized Fourier transform $\mathcal{F}_{k,a}$ acting on a Hilbert space deforming $L^2(\mathbb{R}^d)$. The deformation parameters consists of a real parameter $a > 0$ coming from the interpolation of the minimal unitary representations of two different Lie groups and a real parameter k coming from Dunkl's theory of differential difference operators [14].

As it turned out, various known integral transforms are covered by $\mathcal{F}_{k,a}$:

$$\begin{tikzcd}
 & \boxed{\text{Dunkl transform [15]}} \arrow[r, "k\to0"] & \boxed{\text{Fourier transform [22]}} \\
\boxed{\mathcal{F}_{k,a}\ [4]} \arrow[ur, "a\to2"'] \arrow[dr, "a\to1"] & & \\
 & \boxed{k\text{-Hankel transform [4]}} \arrow[r, "k\to0"] & \boxed{\text{Hankel transform [25]}}
\end{tikzcd}$$

$(k, a\text{: general})$

Recently, there has been a growing interest to develop the analysis related to the (k, a)-generalized Fourier transform. Notably, maximal function and translation operator [3], uncertainty principles and Pitt inequalities [20, 23], Fourier multipliers [26], wavelets multipliers [31, 30], wavelet transform [32, 35], localization operators [36], Gabor transform [34, 33] and Hardy inequality [53] were explored by many researchers.

Motivated and inspired by these prolific developments in the deformed theory, we shall introduce the notion of the deformed Stockwell transform and also study some of the fundamental properties, such as Plancherel's formula, Calderón reproducing formula and inversion formula. Nevertheless, keeping in view that the uncertainty principles play a vital role in both quantum mechanics and harmonic analysis, we also present a comprehensive study of uncertainty principles associated with the deformed Stockwell transforms.

We recall that the notion of uncertainty principles is central in harmonic analysis and with the advent of time-frequency analysis, the study of uncertainty principles gained considerable attention and have been extended to a wide class of integral transforms ranging from the classical Fourier [21]. The pioneering Heisenberg's uncertainty principle asserts that a non-trivial function cannot be sharply localized in both time and frequency domains simultaneously. To date, several generalizations, modifications and variations of the uncertainty principles have appeared in the open literature, for instance, the Beckner-type uncertainty principles [2], Benedick's uncertainty principles [5], Donoho and Stark's uncertainty principles [12], Slepian and Pollak's uncertainty principles [50, 48, 49], Nazarov's uncertainty principles, local uncertainty principles and much more [17]. These uncertainty principles are broadly classified into qualitative and quantitative inequalities. We mention that the quantitative uncertainty principles have been studied by many authors for various Fourier transforms, we refer the reader to the survey [17], the book [21] and the references [1, 44, 18, 16, 19, 24, 27, 39, 28, 29, 55] for numerous versions of uncertainty principles for the Fourier transform in different settings.

In the present article, our second goal is to formulate some quantitative uncertainty principles associated with the deformed Stockwell transform. Nevertheless, we shall also present certain prerequisite developments regarding the notion of the deformed Stockwell transform. The proposed study is expected to have diverse applications signal processing, mathematical analysis, mathematical physics, geophysics, quantum mechanics and so on.

In this paper, we consider the case $a = \frac{2}{n}$, $n \in \mathbb{N}$, and $d = 1$. We shall call the generalized Fourier transform $\mathcal{F}_{k,\frac{2}{n}}$ the *deformed Hankel transform* and we will denote it (simply) by $\mathcal{F}_{k,n}$.

The purpose of the present paper is twofold. We first want to introduce and study the deformed Stockwell transform. For this we investigate the generalized translation operator on the deformed Hankel setting. In particular, we prove its positivity on suitable space of functions. Profiting of this positivity we study the harmonic analysis for the deformed Stockwell transform. Keeping in view the fact that the theory of uncertainty principles for the deformed Stockwell transforms is yet to be investigated exclusively, our second endeavour is to formulate some quantitative uncertainty principles associated with this transform. We mention that, Shah and his co-authors have studied some quantitative uncertainty principles for some generalized Stockwell transforms (see [46, 51]).

The salient contributions of this study are highlighted below:

- To obtain the trigonometric formula for the generalized translation operator.
- To derive the positivity of the generalized translation operator on a suitable space of even functions.
- To introduce and to study the generalized Stockwell transform in the setting of the deformed Hankel transform.
- To derive several versions of the Heisenberg uncertainty principle via different techniques including generalized entropy, the contraction semigroup method of the homogeneous integral transform and others.
- To study the concentration-based uncertainty principles, including the Benedick–Amrein–Berthier and the local-type uncertainty principles for the deformed Stockwell transform.
- To formulate Pitt's and Beckner's uncertainty principles for the deformed Hankel Stockwell transform.

The main content of the paper is organized as follows: Section 2 is divided into three sub-sections; the first sub-section deals with the preliminaries including the fundamental notions of the deformed Hankel transform, in the second sub-section we investigate the generalized translation operators, whereas the third subsection is completely devoted to the formulation and investigation of the deformed Stockwell transform. In Section 3, we formulate certain Heisenberg-type uncertainty principles. In Section 4, we obtain some concentration-based uncertainty principles, including the Benedick–Amrein–Berthier and the local-type uncertainty principles. Finally, in Section 5, we obtain the Beckner uncertainty principle and some other weighted uncertainty inequalities for the deformed Stockwell transform.

2. Deformed Hankel and Stockwell transforms

In this section, we first recall the harmonic analysis associated with the deformed Hankel transform and then introduce the deformed Stockwell transform. We continue our study by investigating some mathematical properties of the proposed transform including the orthogonality relation, energy preserving relation, and the inversion formula.

2.1. **Deformed Hankel transform.** Here, we shall take a survey of the deformed Hankel transform together with the fundamental properties; our main reference is [4]. To facilitate the narrative, we set some notation:

- $\mathbb{1}_U$ is the characteristic function of U, where $U \subset \mathbb{R}^d$, $d = 1, 2$.
- $\mathbb{R}^* := \mathbb{R}\backslash\{0\}$.
- $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
- $C_b(\mathbb{R})$ is the space of bounded continuous functions on $\mathbb{R}$.

- $C_{b,e}(\mathbb{R})$ is the space of even bounded continuous functions on $\mathbb{R}$.
- For $p \in [1, \infty]$, p' denotes the conjugate exponent of p.
- $M_{k,n} := \dfrac{n^{\frac{n(2k-1)}{2}}}{2^{\frac{n(2k-1)+2}{2}}\Gamma\left(\frac{n(2k-1)+2}{2}\right)}$.
- $d\gamma_{k,n}(x) := M_{k,n}|x|^{\frac{(2k-2)n+2}{n}}\,dx$, $k \geq \frac{n-1}{n}$.
- $L^p_{k,n}(\mathbb{R})$, $1 \leq p \leq \infty$, is the space of measurable functions on $\mathbb{R}$ such that

$$\|f\|_{L^p_{k,n}(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p \, d\gamma_{k,n}(x)\right)^{\frac{1}{p}} < \infty \qquad \text{if } 1 \leq p < \infty,$$
$$\|f\|_{L^\infty_{k,n}(\mathbb{R})} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty.$$

For $p = 2$, we provide this space with the scalar product

$$\langle f, g\rangle_{L^2_{k,n}(\mathbb{R})} := \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\gamma_{k,n}(x).$$

For $k \geq \frac{n-1}{n}$, and $f \in L^1_{k,n}(\mathbb{R})$, the deformed Hankel transform is defined by

$$\mathcal{F}_{k,n}(f)(\lambda) = \int_{\mathbb{R}} f(x)B_{k,n}(\lambda, x)\, d\gamma_{k,n}(x) \qquad \text{for all } \lambda \in \mathbb{R},$$

where $B_{k,n}(\lambda, x)$ is the deformed Hankel kernel given by

$$B_{k,n}(\lambda, x) = \jmath_{nk-\frac{n}{2}}\left(n|\lambda x|^{\frac{1}{n}}\right) + \left(\frac{-in}{2}\right)^n \frac{\Gamma(nk - \frac{n}{2} + 1)}{\Gamma(nk + \frac{n}{2} + 1)} \lambda x \jmath_{nk+\frac{n}{2}}\left(n|\lambda x|^{\frac{1}{n}}\right);$$

here

$$\jmath_\alpha(u) := \Gamma(\alpha + 1)\left(\frac{u}{2}\right)^{-\alpha} J_\alpha(u) = \Gamma(\alpha + 1)\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\alpha + m + 1)}\left(\frac{u}{2}\right)^{2m}$$

denotes the normalized Bessel function of index α.

Next, we give some properties of the deformed Hankel kernel.

Proposition 2.1.

(i) *For $z, t \in \mathbb{R}$, we have*

$$B_{k,n}(z, t) = B_{k,n}(t, z), \quad B_{k,n}(z, 0) = 1, \quad \overline{B_{k,n}(z, t)} = B_{k,n}((-1)^n z, t),$$

and

$$B_{k,n}(\lambda z, t) = B_{k,n}(z, \lambda t) \qquad \textit{for all } \lambda \in \mathbb{R}.$$

(ii) *There exists a finite positive constant C, depending only on n and k, such that for all $x, y \in \mathbb{R}$ we have*

$$|B_{k,n}(x, y)| \leq C.$$

Convention ([23]). We shall replace $B_{k,n}$ by the rescaled version $B_{k,n}/C$ but continue to use the same symbol $B_{k,n}$ and we obtain

$$\forall\, x, y \in \mathbb{R}, \quad |B_{k,n}(x, y)| \leq 1. \tag{2.1}$$

We note that the authors in [20] conjectured (2.1) when $k \geq \frac{n-1}{n}$.

Remark 2.2. (i) We note that the previous inequality implies that the deformed Hankel transform is bounded on the space $L^1_{k,n}(\mathbb{R})$, and we have

$$\|\mathcal{F}_{k,n}(f)\|_{L^\infty_{k,n}(\mathbb{R})} \leq \|f\|_{L^1_{k,n}(\mathbb{R})}$$

for all f in $L^1_{k,n}(\mathbb{R})$.

(ii) The deformed Hankel transform $\mathcal{F}_{k,n}$ provides a natural generalization of the Hankel transform. Indeed, if we set

$$\begin{aligned} B^{even}_{k,n}(x,y) &= \frac{1}{2}\big(B_{k,n}(x,y) + B_{k,n}(x,-y)\big) \\ &= \jmath_{nk-\frac{n}{2}}(n|xy|^{\frac{1}{n}}), \end{aligned}$$

then the deformed Hankel transform $\mathcal{F}_{k,n}$ of an even function f on the real line specializes to a Hankel-type transform on $\mathbb{R}_+$. In fact, when $f(x) = F(|x|)$ is an even function on $\mathbb{R}$ and belongs to $L^1_{k,n}(\mathbb{R})$, we have

$$\forall \xi \in \mathbb{R}, \quad \mathcal{F}_{k,n}(f)(\xi) = \frac{(\frac{n}{2})^{\left(\frac{2nk-n}{2}\right)}}{\Gamma\left(\frac{2nk+2-n}{2}\right)} \int_0^\infty F(r) \jmath_{\frac{2nk-n}{2}}\left(n(r|\xi|)^{\frac{1}{n}}\right) r^{\frac{2}{n}\left(\frac{2nk+2-n}{2}\right)-1}\, dr.$$

Example 2.3. The function α_t, $t > 0$, defined on $\mathbb{R}$ by

$$\alpha_t(x) = \frac{1}{(2t)^{\frac{2nk+2-n}{2}}} e^{-\frac{n|x|^{\frac{2}{n}}}{4t}},$$

satisfies

$$\forall\, \xi \in \mathbb{R}, \quad \mathcal{F}_{k,n}(\alpha_t)(\xi) = e^{-nt|\xi|^{\frac{2}{n}}}. \tag{2.2}$$

The authors in [4] have proved the following.

Proposition 2.4.

(i) ***Plancherel's theorem for*** $\mathcal{F}_{k,n}$. *The deformed Hankel transform* $f \mapsto \mathcal{F}_{k,n}(f)$ *is an isometric isomorphism on* $L^2_{k,n}(\mathbb{R})$ *and we have*

$$\int_{\mathbb{R}} |f(x)|^2\, d\gamma_{k,n}(x) = \int_{\mathbb{R}} |\mathcal{F}_{k,n}(f)(\lambda)|^2\, d\gamma_{k,n}(\lambda). \tag{2.3}$$

(ii) ***Parseval's formula for*** $\mathcal{F}_{k,n}$. *For all* f, g *in* $L^2_{k,n}(\mathbb{R})$ *we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\, d\gamma_{k,n}(x) = \int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\lambda)\overline{\mathcal{F}_{k,n}(g)(\lambda)}\, d\gamma_{k,n}(\lambda). \tag{2.4}$$

(iii) ***Inversion formula.*** *The deformed Hankel transform is an involutive unitary operator on* $L^1_{k,n}(\mathbb{R})$, *i.e., we have*

$$\mathcal{F}^{-1}_{k,n}(f)(x) = \mathcal{F}_{k,n}(f)((-1)^n x), \quad x \in \mathbb{R}. \tag{2.5}$$

Proposition 2.5. *Let* $f \in L^p_{k,n}(\mathbb{R})$, $p \in [1,2]$. *Then* $\mathcal{F}_{k,n}(f)$ *belongs to* $L^{p'}_{k,n}(\mathbb{R})$ *and we have*

$$\|\mathcal{F}_{k,n}(f)\|_{L^{p'}_{k,n}(\mathbb{R})} \leq \|f\|_{L^p_{k,n}(\mathbb{R})}.$$

2.2. **Generalized translation operator.**

Definition 2.6. Let $x \in \mathbb{R}$. We define the *generalized translation operator* $\tau_x^{k,n}$ on $L^2_{k,n}(\mathbb{R})$ by

$$\mathcal{F}_{k,n}(\tau_x^{k,n} f) = \overline{B_{k,n}(.,x)}\mathcal{F}_{k,n}(f). \tag{2.6}$$

It is useful to have a class of functions in which (2.6) holds pointwise. One such class is given by the generalized Wigner space $\mathcal{W}_{k,n}(\mathbb{R})$ given by

$$\mathcal{W}_{k,n}(\mathbb{R}) := \Big\{ f \in L^1_{k,n}(\mathbb{R}) : \mathcal{F}_{k,n}(f) \in L^1_{k,n}(\mathbb{R}) \Big\}.$$

We give below several properties of the generalized translation operator.

Proposition 2.7.

(i) *Let $f \in L^2_{k,n}(\mathbb{R})$. We have*

$$\|\tau_x^{k,n} f\|_{L^2_{k,n}(\mathbb{R})} \leq \|f\|_{L^2_{k,n}(\mathbb{R})} \quad \forall x \in \mathbb{R}. \tag{2.7}$$

(ii) *For all f in $\mathcal{W}_{k,n}(\mathbb{R})$ we have*

$$\tau_x^{k,n} f(y) = \int_{\mathbb{R}} B_{k,n}((-1)^n x, \xi) B_{k,n}((-1)^n y, \xi) \mathcal{F}_{k,n}(f)(\xi)\, d\gamma_{k,n}(\xi) \quad \forall x, y \in \mathbb{R}.$$

(iii) *For all f in $\mathcal{W}_{k,n}(\mathbb{R})$ and for all $x, y \in \mathbb{R}$, we have*

$$\tau_x^{k,n} f(y) = \tau_y^{k,n}(f)(x). \tag{2.8}$$

(iv) *For all f in $\mathcal{W}_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y) g(y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y) \tau_{(-1)^n x}^{k,n} g(y)\, d\gamma_{k,n}(y) \quad \forall\, x \in \mathbb{R}. \tag{2.9}$$

Proof. For part (i), it is enough to use (2.6), Plancherel's identity (2.3) and relation (2.1). For part (ii) we use (2.6), inversion formula (2.5) and Proposition 2.1 (i). For part (iii), it is enough to use the symmetry $B_{k,n}(x, y) = B_{k,n}(y, x)$. For part (iv), using Parseval's formula (2.4), we get for all $x \in \mathbb{R}$

$$\begin{aligned}\int_{\mathbb{R}} \tau_x^{k,n} f(y) g(y)\, d\gamma_{k,n}(y) &= \int_{\mathbb{R}} \overline{B_{k,n}(x,\xi)} \mathcal{F}_{k,n}(f)(\xi) \overline{\mathcal{F}_{k,n}(g)(\xi)}\, d\gamma_{k,n}(\xi) \\ &= \int_{\mathbb{R}} f(y) \tau_{(-1)^n x}^{k,n} g(y)\, d\gamma_{k,n}(y). \qquad \square\end{aligned}$$

Recently the authors in [6] gave an explicit formula for the generalized translation operators given by the following.

Theorem 2.8. *Let $x \in \mathbb{R}$ and let $f \in C_b(\mathbb{R})$. For $k \geq \frac{n-1}{n}$, the generalized translation operator $\tau_x^{k,n}$ is given by*

$$\tau_x^{k,n} f(y) = \int_{\mathbb{R}} f(z)\, d\zeta_{x,y}^{k,n}(z); \tag{2.10}$$

here

$$d\zeta_{x,y}^{k,n}(z) = \begin{cases} \mathcal{K}_{k,n}(x,y,z)\, d\gamma_{k,n}(z) & \textit{if } xy \neq 0, \\ d\delta_x(z) & \textit{if } y = 0, \\ d\delta_y(z) & \textit{if } x = 0, \end{cases}$$

where $\mathcal{K}_{k,n}(x,y,.)$ *is supported on the set*

$$\left\{ z \in \mathbb{R} : \left| |x|^{\frac{1}{n}} - |y|^{\frac{1}{n}} \right| < |z|^{\frac{1}{n}} < |x|^{\frac{1}{n}} + |y|^{\frac{1}{n}} \right\}$$

and is given by

$$\mathcal{K}_{k,n}(x,y,z) = K_{\mathrm{B}}^{nk-\frac{n}{2}} \left(|x|^{\frac{1}{n}}, |y|^{\frac{1}{n}}, |z|^{\frac{1}{n}} \right) \nabla_{k,n}(x,y,z),$$

where

$$\begin{aligned} \nabla_{k,n}(x,y,z) := \frac{M_{k,n}}{2n} \Big\{ 1 &+ (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \left(\Delta\left(|x|^{\frac{2}{n}}, |y|^{\frac{2}{n}}, |z|^{\frac{2}{n}} \right) \right) \\ &+ \frac{n!\,\mathrm{sgn}(xz)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \left(\Delta\left(|z|^{\frac{2}{n}}, |x|^{\frac{2}{n}}, |y|^{\frac{2}{n}} \right) \right) \\ &+ \frac{n!\,\mathrm{sgn}(yz)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \left(\Delta\left(|z|^{\frac{2}{n}}, |y|^{\frac{2}{n}}, |x|^{\frac{2}{n}} \right) \right) \Big\}, \end{aligned} \tag{2.11}$$

$$\Delta(u,v,w) := \frac{1}{2\sqrt{uv}}(u+v-w) \quad \textit{for } u,v,w \in \mathbb{R}_+^*,$$

$C_n^{nk-\frac{n}{2}}$ *are the Gegenbauer polynomials, and* $K_{\mathrm{B}}^{nk-\frac{n}{2}}$ *is the positive kernel given by*

$$K_{\mathrm{B}}^{nk-\frac{n}{2}}(u,v,w) = \frac{2\Gamma(nk-\frac{n}{2}+1)}{\Gamma(nk-\frac{n-1}{2})\Gamma(\frac{1}{2})} \frac{\left\{ \left[(u+v)^2 - w^2 \right] \left[w^2 - (u-v)^2 \right] \right\}^{nk-\frac{n+1}{2}}}{(2uvw)^{2nk-n}}$$

for $|u-v| < w < u+v$ *and* $K_B^{nk-\frac{n}{2}}(u,v,w) = 0$ *elsewhere.*

The explicit formula implies the boundedness of $\tau_y^{k,n} f$. More precisely, we have the following result.

Proposition 2.9 ([6])**.** *For all* $f \in L_{k,n}^p(\mathbb{R})$, $1 \leq p \leq \infty$, *we have*

$$\|\tau_x^{k,n} f\|_{L_{k,n}^p(\mathbb{R})} \leq 4 \|f\|_{L_{k,n}^p(\mathbb{R})} \quad \forall\, x \in \mathbb{R}.$$

We will prove now the "trigonometric" form of the generalized translation operator.

Theorem 2.10. *For $f \in C_b(\mathbb{R})$, write $f = f_\mathrm{e} + f_\mathrm{o}$ as a sum of even and odd functions. Then*

$$\begin{aligned}\tau_x^{k,n} f(y) = \frac{M_{k,n}}{2n}\Bigg[\int_0^\pi & f_\mathrm{e}\big(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\big)\Big\{1+(-1)^n\frac{n!\operatorname{sgn}(xy)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\cos\phi\big)\Big\}\\ &+ f_\mathrm{o}\big(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\big)\Bigg\{\frac{n!\operatorname{sgn}(x)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\left(\frac{|x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right)\\ &+\frac{n!\operatorname{sgn}(y)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\left(\frac{|y|^{\frac{1}{n}}-|x|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right)\Bigg\}(\sin\phi)^{2nk-n}\,d\phi\Bigg],\end{aligned}$$

where

$$\langle\!\langle x,y\rangle\!\rangle_{\phi,n} := \big(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}-2|xy|^{\frac{1}{n}}\cos\phi\big)^{\frac{n}{2}}. \tag{2.12}$$

Proof. By (2.11), the even and odd parts of the function $\nabla_{k,n}(x,y,\cdot)$ are given respectively by

$$\begin{aligned}\nabla_{k,n,\mathrm{e}}(x,y,z) &:= \frac{M_{k,n}}{2n}\Big\{1+(-1)^n\frac{n!\operatorname{sgn}(xy)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\Delta(|x|^{\frac{2}{n}},|y|^{\frac{2}{n}},|z|^{\frac{2}{n}})\big)\Big\},\\ \nabla_{k,n,\mathrm{o}}(x,y,z) &:= \frac{M_{k,n}}{2n}\Big\{\frac{n!\operatorname{sgn}(xz)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\Delta(|z|^{\frac{2}{n}},|x|^{\frac{2}{n}},|y|^{\frac{2}{n}})\big)\\ &\qquad+\frac{n!\operatorname{sgn}(yz)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\Delta(|z|^{\frac{2}{n}},|y|^{\frac{2}{n}},|x|^{\frac{2}{n}})\big)\Big\}.\end{aligned}$$

Hence, equation (2.10) turns into

$$\begin{aligned}\tau_x^{k,n} f(y) &= 2\int_0^\infty f_\mathrm{e}(z)K_\mathrm{B}^{nk-\frac{n}{2}}(|x|^{\frac{1}{n}},|y|^{\frac{1}{n}},|z|^{\frac{1}{n}})\,\nabla_{k,n,\mathrm{e}}(x,y,z)\,d\gamma_{k,n}(z)\\ &\quad+2\int_0^\infty f_\mathrm{o}(z)K_\mathrm{B}^{nk-\frac{n}{2}}(|x|^{\frac{1}{n}},|y|^{\frac{1}{n}},|z|^{\frac{1}{n}})\,\nabla_{k,n,\mathrm{o}}(x,y,z)\,d\gamma_{k,n}(z).\end{aligned}$$

For

$$\Big||x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\Big| < |z|^{\frac{1}{n}} < |x|^{\frac{1}{n}}+|y|^{\frac{1}{n}},$$

we may substitute

$$\cos\phi := \frac{|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}-|z|^{\frac{2}{n}}}{2|xy|^{\frac{1}{n}}} = \Delta\big(|x|^{\frac{2}{n}},|y|^{\frac{2}{n}},|z|^{\frac{2}{n}}\big) \tag{2.13}$$

with $\phi\in[0,\pi]$. Moreover, using (2.13) and (2.12), we get

$$\Delta\big(|z|^{\frac{2}{n}},|x|^{\frac{2}{n}},|y|^{\frac{2}{n}}\big) = \frac{|z|^{\frac{2}{n}}+|x|^{\frac{2}{n}}-|y|^{\frac{2}{n}}}{2|x|^{\frac{1}{n}}|z|^{\frac{1}{n}}} = \frac{|x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}.$$

Thus, for $z>0$, we infer that

$$\frac{n!\operatorname{sgn}(xz)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\Delta(|z|^{\frac{2}{n}},|x|^{\frac{2}{n}},|y|^{\frac{2}{n}})\big) = \frac{n!\operatorname{sgn}(x)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\left(\frac{|x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right).$$

Similarly we prove that

$$\frac{n!\,\mathrm{sgn}(yz)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}\left(\Delta(|z|^{\frac{2}{n}},|y|^{\frac{2}{n}},|x|^{\frac{2}{n}})\right) = \frac{n!\,\mathrm{sgn}(y)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}\left(\frac{|y|^{\frac{1}{n}}-|x|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right).$$

Thus, the generalized translation operator takes the desired form. □

Below we will study the positivity of the generalized translation operator on even functions in $\mathcal{W}_{k,n}(\mathbb{R})$, which is far from being obvious. This result will be crucial for the rest of the paper. To do so, we will give an explicit expression of the generalized translation operator acting on such functions.

Corollary 2.11. *For all f in $C_{b,e}(\mathbb{R})$, we have*

$$\tau_x^{k,n} f(y) = \frac{M_{k,n}}{2n}\int_0^\pi f\left(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\right)\mathcal{N}_{k,n}(x,y,\phi)(\sin\phi)^{2nk-n}\,d\phi,$$

where

$$\mathcal{N}_{k,n}(x,y,\phi) := 1+(-1)^n\frac{n!\mathrm{sgn}(xy)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}(\cos\phi).$$

Using the previous corollary we infer the following

Lemma 2.12. *For every $\lambda>0$ and for every $x\in\mathbb{R}$, we have*

$$\tau_x^{k,n}(e^{-\lambda|.|^{\frac{2}{n}}})(y) = \frac{M_{k,n}}{2n}e^{-\lambda\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)}V_{k,n}(\lambda;x,y),$$

where

$$V_{k,n}(\lambda;x,y) := \int_0^\pi e^{2\lambda|xy|^{\frac{1}{n}}\cos\phi}\mathcal{N}_{k,n}(x,y,\phi)(\sin\phi)^{2nk-n}\,d\phi.$$

Remark 2.13. Using the previous lemma and the properties of the Gegenbauer polynomials, by simple calculations we infer that there exists a positive constant $C(k,n)$ such that

$$\left|\tau_x^{k,n}(e^{-\lambda|.|^{\frac{2}{n}}})(y)\right| \le C(k,n)e^{-\lambda\left(|x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\right)^2}.$$

Now, let us go back to the properties of the generalized translation operator.

Proposition 2.14. *Let f be an nonnegative even function of $\mathcal{W}_{k,n}(\mathbb{R})$. Then*

(i) *For any $x\in\mathbb{R}$, we have $\tau_x^{k,n} f\ge 0$.*
(ii) *For every $x\in\mathbb{R}$, we have $\tau_x^{k,n} f\in L^1_{k,n}(\mathbb{R})$ and*

$$\int_{\mathbb{R}}\tau_x^{k,n} f(y)\,d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\,d\gamma_{k,n}(y).$$

Proof. Using the explicit expression of the generalized translation operator given in Corollary 2.11, the properties of the Gegenbauer polynomials and by simple calculations we prove the first statement. To prove (ii), let us substitute $g(y)$ by $e^{-\lambda|y|^{\frac{2}{n}}}$ in the relation (2.9). Thus by Lemma 2.12, we get

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y) e^{-\lambda|y|^{\frac{2}{n}}}\, d\gamma_{k,n}(y) \\ = \frac{M_{k,n}}{2n} \int_{\mathbb{R}} f(y) e^{-\lambda\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)} V_{k,n}(\lambda; (-1)^n x, y)\, d\gamma_{k,n}(y). \tag{2.14}$$

Using the fact that $\tau_x^{k,n} f(y) e^{-\lambda|y|^{\frac{2}{n}}} \geq 0$ and the monotone convergence theorem we get

$$\lim_{\lambda\to 0} \int_{\mathbb{R}} \tau_x^{k,n} f(y) e^{-\lambda|y|^{\frac{2}{n}}}\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} \tau_x^{k,n} f(y)\, d\gamma_{k,n}(y). \tag{2.15}$$

Now we will estimate

$$\lim_{\lambda\to 0} \frac{M_{k,n}}{2n} \int_{\mathbb{R}} f(y) e^{-\lambda\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)} V_{k,n}(\lambda; (-1)^n x, y)\, d\gamma_{k,n}(y).$$

In view of the upper estimate for $\tau_x^{k,n}(e^{-\lambda|.|^{\frac{2}{n}}})(y)$ in Remark 2.13, the dominated convergence theorem gives

$$\lim_{\lambda\to 0} \frac{M_{k,n}}{2n} \int_{\mathbb{R}} f(y) e^{-\lambda\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)} V_{k,n}(\lambda; (-1)^n x, y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\, d\gamma_{k,n}(y). \tag{2.16}$$

Combining the relations (2.14), (2.15) and (2.16) we infer the desired result. □

We state now the second main result of this section.

Theorem 2.15. *Let $L^p_{k,n,e}(\mathbb{R})$ be the space of even functions in $L^p_{k,n}(\mathbb{R})$.*

(i) *Let $f \in L^1_{k,n,e}(\mathbb{R})$ be bounded and nonnegative. Then we have*

$$\forall\, x \in \mathbb{R}, \quad \tau_x^{k,n} f \geq 0, \quad \tau_x^{k,n} f \in L^1_{k,n}(\mathbb{R})$$

and

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\, d\gamma_{k,n}(y).$$

(ii) *The generalized translation operator initially defined on $L^1_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$ can be extended to all $L^p_{k,n,e}(\mathbb{R})$, $1 \leq p \leq \infty$, and for all f in $L^p_{k,n,e}(\mathbb{R})$ we have*

$$\forall\, x \in \mathbb{R}, \quad \|\tau_x^{k,n} f\|_{L^p_{k,n}(\mathbb{R})} \leq \|f\|_{L^p_{k,n}(\mathbb{R})}. \tag{2.17}$$

(iii) *For every $f \in L^1_{k,n}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\, d\gamma_{k,n}(y).$$

By means of the generalized translation operator, we define the generalized convolution product of two suitable functions f and g by

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau_x^{k,n} f((-1)^n y) g(y)\, d\gamma_{k,n}(y). \tag{2.18}$$

Remark 2.16. (i) It is clear that this convolution product is both commutative and associative.

(ii) This convolution structure carries a new commutative signed hypergroup in the sense of [42] or [43].

The generalized convolution product also satisfies the following properties.

Proposition 2.17 ([6])**.** *The following statements hold true:*

(i) *Let $f \in L^2_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R})$. Then the function $f *_{k,n} g$ defined almost everywhere on $\mathbb{R}$ by*

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau_x^{k,n} f((-1)^n y) g(y)\, d\gamma_{k,n}(y)$$

belongs to $L^2_{k,n}(\mathbb{R})$.

(ii) *Assume that $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. Then, for every f in $L^p_{k,n}(\mathbb{R})$ and $g \in L^q_{k,n}(\mathbb{R})$, the convolution product $f *_{k,n} g$ belongs to $L^r_{k,n}(\mathbb{R})$ and*

$$\|f *_{k,n} g\|_{L^r_{k,n}(\mathbb{R})} \leq 4 \|f\|_{L^p_{k,n}(\mathbb{R})} \|g\|_{L^q_{k,n}(\mathbb{R})}.$$

(iii) *For $f \in L^2_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R})$, we have*

$$\mathcal{F}_{k,n}(f *_{k,n} g) = \mathcal{F}_{k,n}(f)\mathcal{F}_{k,n}(g). \tag{2.19}$$

Proof of Theorem 2.15. Let f be a bounded and positive function in $L^1_{k,n,e}(\mathbb{R})$. In particular, $f \in L^2_{k,n}(\mathbb{R})$. Therefore, we may consider the function $f *_{k,n} \alpha_t$, $t > 0$. Using Proposition 2.14 we prove that the previous function belongs to $L^1_{k,n}(\mathbb{R})$. On the other hand, using (2.19), Cauchy–Schwarz's inequality and Plancherel's formula (2.3), we infer that $\mathcal{F}_{k,n}(f *_{k,n} \alpha_t)$ belongs to $L^1_{k,n}(\mathbb{R})$. Thus $f *_{k,n} \alpha_t$ belongs to $\mathcal{W}_{k,n}(\mathbb{R})$. As the function f is an even positive function we deduce also that $f *_{k,n} \alpha_t$ is an even positive function. The positivity of the generalized translation operator on the positive even function of $\mathcal{W}_{k,n}(\mathbb{R})$ implies that

$$\forall\, t > 0, \quad \tau_x^{k,n}\big(f *_{k,n} \alpha_t\big) \geq 0. \tag{2.20}$$

Using Plancherel's formula (2.3), the formula (2.2) and by a simple calculation we see that

$$\lim_{t \to 0} \|f - f *_{k,n} \alpha_t\|_{L^2_{k,n}(\mathbb{R})} = \|\mathcal{F}_{k,n}(f)\big(e^{-nt|\xi|^{\frac{2}{n}}} - 1\big)\|_{L^2_{k,n}(\mathbb{R})} = 0.$$

Using similar ideas as above and (2.7), we prove that

$$\lim_{t \to 0} \|\tau_x^{k,n}\big(f - f *_{k,n} \alpha_t\big)\|_{L^2_{k,n}(\mathbb{R})} = 0. \tag{2.21}$$

Thus up to sequences, (2.20) and (2.21) give that

$$\tau_x^{k,n} f(y) = \lim_{t \to 0} \tau_x^{k,n}\big(f *_{k,n} \alpha_t\big)(y) \geq 0$$

for almost every $y \in \mathbb{R}$. This finishes the proof of the first part of statement (i). For the second part, applying the monotone convergence theorem to the relation

(2.9) with $g(y)=e^{-\lambda|y|^{\frac{2}{n}}}$ and using the same argument used in Proposition 2.14 we prove that

$$\begin{aligned}\int_{\mathbb{R}}\tau_x^{k,n}f(y)\,d\gamma_{k,n}(y)&=\lim_{\lambda\to 0}\int_{\mathbb{R}}\tau_x^{k,n}f(y)e^{-\lambda|y|^{\frac{2}{n}}}\,d\gamma_{k,n}(y)\\&=\lim_{\lambda\to 0}\int_{\mathbb{R}}f(y)\tau_{(-1)^n x}^{k,n}\Big(e^{-\lambda|y|^{\frac{2}{n}}}\Big)\,d\gamma_{k,n}(y)\\&=\int_{\mathbb{R}}f(y)\,d\gamma_{k,n}(y).\end{aligned}$$

(ii) If $f\in L^1_{k,n,e}(\mathbb{R})\cap L^\infty_{k,n}(\mathbb{R})$, the previous result implies that

$$\|\tau_x^{k,n}f\|_{L^1_{k,n}(\mathbb{R})}\le\|\tau_x^{k,n}|f|\,\|_{L^1_{k,n}(\mathbb{R})}=\|f\|_{L^1_{k,n}(\mathbb{R})}.$$

On the other hand, if $f\in L^2_{k,n}(\mathbb{R})$, from (2.7) we have

$$\|\tau_x^{k,n}f\|_{L^2_{k,n}(\mathbb{R})}\le\|f\|_{L^2_{k,n}(\mathbb{R})}.$$

Thus by interpolation we deduce that, for any $p\in[1,2]$,

$$\|\tau_x^{k,n}f\|_{L^p_{k,n}(\mathbb{R})}\le\|f\|_{L^p_{k,n}(\mathbb{R})}.$$

Finally, by duality we infer the result.

(iii) Choose even functions $f_j\in\mathcal{W}_{k,n}(\mathbb{R})$ such that $f_j\to f$ and $\tau_x^{k,n}f_j\to\tau_x^{k,n}f$ in $L^1_{k,n}(\mathbb{R})$. Since

$$\int_{\mathbb{R}}\tau_x^{k,n}f_j(y)g(y)\,d\gamma_{k,n}(y)=\int_{\mathbb{R}}f_j(y)\tau_{(-1)^n x}^{k,n}g(y)\,d\gamma_{k,n}(y)$$

for every $g\in\mathcal{W}_{k,n}(\mathbb{R})$ we get, taking limit as j tends to infinity,

$$\int_{\mathbb{R}}\tau_x^{k,n}f(y)g(y)\,d\gamma_{k,n}(y)=\int_{\mathbb{R}}f(y)\tau_{(-1)^n x}^{k,n}g(y)\,d\gamma_{k,n}(y).$$

Now take $g(y)=e^{-\lambda|y|^{\frac{2}{n}}}$ and using the same argument used in Proposition 2.14 we prove the result. □

Using Theorem 2.15 we improve the estimate given in Proposition 2.17 (ii). More precisely, we have:

Corollary 2.18. *Assume that $1\le p,q,r\le\infty$ satisfy $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$. Then, for every $f\in L^p_{k,n,e}(\mathbb{R})$ and $g\in L^q_{k,n}(\mathbb{R})$, the convolution product $f*_{k,n}g$ belongs to $L^r_{k,n}(\mathbb{R})$ and*

$$\|f*_{k,n}g\|_{L^r_{k,n}(\mathbb{R})}\le\|f\|_{L^p_{k,n}(\mathbb{R})}\|g\|_{L^q_{k,n}(\mathbb{R})}.$$

We close this section by recalling the following results which will play a significant role.

Proposition 2.19 ([32]).

(i) *Let f and g be in $L^2_{k,n}(\mathbb{R})$. Then $f*_{k,n}g \in L^2_{k,n}(\mathbb{R})$ if and only if $\mathcal{F}_{k,n}(f)\mathcal{F}_{k,n}(g)$ belongs to $L^2_{k,n}(\mathbb{R})$, and in this case we have*
$$\mathcal{F}_{k,n}(f *_{k,n} g) = \mathcal{F}_{k,n}(f)\mathcal{F}_{k,n}(g).$$

(ii) *Let f and g be in $L^2_{k,n}(\mathbb{R})$. Then, we have*
$$\int_{\mathbb{R}} |f *_{k,n} g(x)|^2 \, d\gamma_{k,n}(x) = \int_{\mathbb{R}} |\mathcal{F}_{k,n}(f)(\xi)|^2 |\mathcal{F}_{k,n}(g)(\xi)|^2 \, d\gamma_{k,n}(\xi)$$
whenever both sides are finite.

2.3. **Deformed Stockwell transforms.**

Definition 2.20. For any function h in $L^2_{k,n,e}(\mathbb{R})$ and any $\nu \in \mathbb{R}$, we define the *modulation* of h by ν as
$$\mathcal{M}_\nu h := \mathcal{F}_{k,n}\Big(\sqrt{\tau^{k,n}_\nu\big(|\mathcal{F}_{k,n}(h)|^2\big)}\Big), \tag{2.22}$$
where $\tau^{k,n}_\nu$, $\nu \in \mathbb{R}$, are the generalized translation operators.

Let $a \in \mathbb{R}$. The dilation operator Δ_a of a measurable function h is defined by
$$\forall\, x \in \mathbb{R}, \quad \Delta_a h(x) := |a|^{\frac{(2k-1)n+2}{2n}} h(ax). \tag{2.23}$$
By simple calculations we prove that these operators satisfy the following properties.

Proposition 2.21.

(i) *For all a, b in $\mathbb{R}^*$, we have*
$$\Delta_a \Delta_b = \Delta_{ab}$$
and
$$\Delta_a M_b = M_{ab} \Delta_a.$$

(ii) *Let $a \in \mathbb{R}^*$. For all h in $L^2_{k,n}(\mathbb{R})$, the function $\Delta_a h$ belongs to $L^2_{k,n}(\mathbb{R})$ and we have*
$$\|\Delta_a h\|_{L^2_{k,n}(\mathbb{R})} = \|h\|_{L^2_{k,n}(\mathbb{R})}$$
and
$$\mathcal{F}_{k,n}(\Delta_a h)(y) = |a|^{-\frac{(2k-1)n+2}{2n}} \mathcal{F}_{k,n}(h)(\frac{y}{a}), \quad y \in \mathbb{R}. \tag{2.24}$$

(iii) *Let $a \in \mathbb{R}^*$. For all h, g in $L^2_{k,n}(\mathbb{R})$, we have*
$$\langle \Delta_a h, g\rangle_{L^2_{k,n}(\mathbb{R})} = \langle h, \Delta_{\frac{1}{a}} g\rangle_{L^2_{k,n}(\mathbb{R})}.$$

(iv) *Let $a \in \mathbb{R}^*$ and $x \in \mathbb{R}$. We have*
$$\Delta_a \tau^{k,n}_x = \tau^{k,n}_{\frac{x}{a}} \Delta_a. \tag{2.25}$$

(v) *Let $a \in \mathbb{R}^*$ and $h \in L^2_{k,n}(\mathbb{R})$. We have*
$$|\Delta_a h|^2 = |a|^{\frac{(2k-1)n+2}{2n}} \Delta_a |h|^2. \tag{2.26}$$

Definition 2.22. A *deformed Stockwell wavelet* on $\mathbb{R}$ is an even measurable function h on $\mathbb{R}$ satisfying, for almost all $\xi \in \mathbb{R}^*$, the condition

$$0 < C_h := \int_{\mathbb{R}} |\mathcal{F}_{k,n}(\mathcal{M}_\nu \Delta_\nu h)(\xi)|^2 \, d\gamma_{k,n}(\nu) < \infty. \tag{2.27}$$

Proposition 2.23. *Let h be a deformed Stockwell wavelet on $\mathbb{R}$. We have*

$$C_h := \int_{\mathbb{R}} |\mathcal{F}_{k,n}(\mathcal{M}_\nu \Delta_\nu h)(\xi)|^2 \, d\gamma_{k,n}(\nu) = M_{k,n} \int_{\mathbb{R}} \tau_1^{k,n}\big(|\mathcal{F}_{k,n}(h)|^2\big) \left(\frac{(-1)^n \xi}{\nu}\right) \frac{d\nu}{|\nu|}.$$

Proof. Let $\nu \in \mathbb{R}^*$. Using the relations (2.22), (2.23), (2.24), (2.25) and (2.26) we deduce that

$$\begin{aligned} |\mathcal{F}_{k,n}(\mathcal{M}_\nu \Delta_\nu h)(\xi)|^2 &= \tau_\nu^{k,n}(|\mathcal{F}_{k,n}(\Delta_\nu h)|^2)((-1)^n \xi) \\ &= \frac{1}{|\nu|^{\frac{(2k-1)n+2}{n}}} \tau_1^{k,n}\big(|\mathcal{F}_{k,n}(h)|^2\big) \left(\frac{(-1)^n \xi}{\nu}\right). \end{aligned} \tag{2.28}$$

Then (2.27) is written as

$$\begin{aligned} C_h &:= \int_{\mathbb{R}} |\mathcal{F}_{k,n}(\mathcal{M}_\nu \Delta_\nu h)(\xi)|^2 \, d\gamma_{k,n}(\nu) \\ &= \int_{\mathbb{R}} \tau_1^{k,n}\big(|\mathcal{F}_{k,n}(h)|^2\big) \left(\frac{(-1)^n \xi}{\nu}\right) \frac{d\gamma_{k,n}(\nu)}{|\nu|^{\frac{(2k-1)n+2}{n}}} \\ &= M_{k,n} \int_{\mathbb{R}} \tau_1^{k,n}\big(|\mathcal{F}_{k,n}(h)|^2\big) \left(\frac{(-1)^n \xi}{\nu}\right) \frac{d\nu}{|\nu|}. \end{aligned}$$

Thus we obtain the desired result. □

Let $\nu \in \mathbb{R}^*$ and let h be a deformed Stockwell wavelet in $L^2_{k,n}(\mathbb{R})$. We consider the family $h_{x,\nu}$, $x \in \mathbb{R}$, of functions on $\mathbb{R}$ in $L^2_{k,n}(\mathbb{R})$ defined by

$$h_{x,\nu}(y) := \tau_x^{k,n} \mathcal{M}_\nu(\Delta_\nu h)((-1)^n y), \quad y \in \mathbb{R},$$

where $\tau_x^{k,n}$, $x \in \mathbb{R}$, are the generalized translation operators given by (2.6).

We note that we have

$$\forall\, (x,\nu) \in \mathbb{R}^2, \quad \|h_{x,\nu}\|_{L^2_{k,n}(\mathbb{R})} \leq \|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{2.29}$$

For $1 \leq p \leq \infty$, let $L^p_{\mu_{k,n}}(\mathbb{R}^2)$, $p \in [1,\infty]$, be the space of measurable functions f on $\mathbb{R}^2$ such that

$$\begin{aligned} \|f\|_{L^p_{\mu_{k,n}}(\mathbb{R}^2)} &:= \left(\int_{\mathbb{R}^2} |f(x,\nu)|^p \, d\mu_k(x,\nu)\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \\ \|f\|_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)} &:= \operatorname*{ess\,sup}_{(x,\nu)\in\mathbb{R}^2} |f(x,\nu)| < \infty, \end{aligned}$$

where the measure $\mu_{k,n}$ is defined by

$$\forall\, (x,\nu) \in \mathbb{R}^2, \quad d\mu_{k,n}(x,\nu) = d\gamma_{k,n}(x)\, d\gamma_{k,n}(\nu).$$

Definition 2.24. Let h be a deformed Stockwell wavelet on $\mathbb{R}$ in $L^2_{k,n}(\mathbb{R})$. The *deformed Stockwell continuous transform* $\mathcal{S}_h^{k,n}$ on $\mathbb{R}$ is defined for regular functions f on $\mathbb{R}$ by

$$\forall\,(x,\nu)\in\mathbb{R}^2,\quad \mathcal{S}_h^{k,n}(f)(x,\nu)=\int_{\mathbb{R}} f(y)\overline{h_{x,\nu}(y)}\,d\gamma_{k,n}(y). \tag{2.30}$$

This transform can also be written in the form

$$\mathcal{S}_h^{k,n}(f)(x,\nu)=f*_{k,n}\overline{\mathcal{M}_\nu\Delta_\nu h}(x), \tag{2.31}$$

where $*_{k,n}$ is the generalized convolution product given by (2.18).

Remark 2.25. (i) Let h be a deformed Stockwell wavelet in $L^2_{k,n}(\mathbb{R})$. Using relation (2.30), Cauchy–Schwarz's inequality and relation (2.29) we get, for all f in $L^2_{k,n}(\mathbb{R})$,

$$\|\mathcal{S}_h^{k,n}(f)\|_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)}\le\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{2.32}$$

(ii) Using Proposition 2.21 and by a standard computation it is easy to see that, for every $f\in L^2_{k,n}(\mathbb{R})$ and h in $L^2_{k,n,e}(\mathbb{R})$, for all $\lambda>0$ and for all $(x,\nu)\in\mathbb{R}^2$, we have

$$\mathcal{S}_h^{k,n}(f_\lambda)(x,\nu)=\mathcal{S}_h^{k,n}(f)\left(\frac{x}{\lambda},\lambda\nu\right), \tag{2.33}$$

where

$$\forall\,t>0,\ \forall\,x\in\mathbb{R},\quad g_t(x):=\frac{1}{t^{\frac{(2k-1)n+2}{2n}}}g\left(\frac{x}{t}\right).$$

Henceforth, the function h will denote a deformed Stockwell wavelet on $\mathbb{R}$ in $L^2_{k,n}(\mathbb{R})$. By simple calculations we prove the following:

Lemma 2.26. *For any $f\in L^2_{k,n}(\mathbb{R})$, we have*

$$\mathcal{F}_{k,n}\Big(\mathcal{S}_h^{k,n}(f)(.,\nu)\Big)(\xi)=\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu\Delta_\nu h})(\xi)\mathcal{F}_{k,n}(f)(\xi). \tag{2.34}$$

Theorem 2.27 (Parseval's formula for $\mathcal{S}_h^{k,n}$)**.** *Let f,g be in $L^2_{k,n}(\mathbb{R})$. Then, we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\,d\gamma_{k,n}(x)=\frac{1}{C_h}\int_{\mathbb{R}^2}\mathcal{S}_h^{k,n}(f)(x,\nu)\overline{\mathcal{S}_h^{k,n}(g)(x,\nu)}\,d\mu_{k,n}(x,\nu).$$

Proof. Using Fubini's Theorem, relation (2.31) and Parseval's formula (2.4), we get

$$\begin{aligned}&\int_{\mathbb{R}^2}\mathcal{S}_h^{k,n}(f)(x,\nu)\overline{\mathcal{S}_h^{k,n}(g)(x,\nu)}\,d\mu_{k,n}(x,\nu)\\
&\quad=\int_{\mathbb{R}^2}\Big(f*_{k,n}\overline{\mathcal{M}_\nu\Delta_\nu h}(x)\Big)\Big(\overline{g*_{k,n}\overline{\mathcal{M}_\nu\Delta_\nu h}(x)}\Big)\,d\mu_{k,n}(x,\nu)\\
&\quad=\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{F}_{k,n}(f)(\xi)\overline{\mathcal{F}_{k,n}(g)(\xi)}|\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu\Delta_\nu h})(\xi)|^2\,d\gamma_{k,n}(\xi)\,d\gamma_{k,n}(\nu)\\
&\quad=\int_{\mathbb{R}}\mathcal{F}_{k,n}(f)(\xi)\overline{\mathcal{F}_{k,n}(g)(\xi)}\Big(\int_{\mathbb{R}}|\mathcal{F}_{k,n}(\mathcal{M}_\nu\Delta_\nu h)((-1)^n\xi)|^2\,d\gamma_{k,n}(\nu)\Big)\,d\gamma_{k,n}(\xi).\end{aligned}$$

As h is a deformed Stockwell wavelet, (2.27) yields

$$\int_{\mathbb{R}} |\mathcal{F}_{k,n}(\mathcal{M}_\nu \Delta_\nu h)((-1)^n\xi)|^2 \, d\gamma_{k,n}(\nu) = C_h.$$

Thus we obtain

$$\int_{\mathbb{R}^2} \mathcal{S}_h^{k,n}(f)(x,\nu)\overline{\mathcal{S}_h^{k,n}(g)(x,\nu)} \, d\mu_{k,n}(x,\nu) = C_h \int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\xi)\overline{\mathcal{F}_{k,n}(g)(\xi)} \, d\gamma_{k,n}(\xi).$$

Finally, using Parseval's formula (2.4) we obtain the result. □

Corollary 2.28 (Plancherel's formula for $\mathcal{S}_h^{k,n}$)**.** *For all f in $L^2_{k,n}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} |f(x)|^2 \, d\gamma_{k,n}(x) = \frac{1}{C_h} \int_{\mathbb{R}^2} |\mathcal{S}_h^{k,n}(f)(x,\nu)|^2 \, d\mu_{k,n}(x,\nu). \tag{2.35}$$

By Riesz–Thorin's interpolation theorem we derive the following.

Proposition 2.29. *Let $f \in L^2_{k,n}(\mathbb{R})$ and $p \in [2,\infty]$. We have*

$$\|\mathcal{S}_h^{k,n}(f)\|_{L^p_{\mu_{k,n}}(\mathbb{R}^2)} \leq (C_h)^{\frac{1}{p}} \left(\|h\|_{L^2_{k,n}(\mathbb{R})}\right)^{\frac{p-2}{p}} \|f\|_{L^2_{k,n}(\mathbb{R})}.$$

Theorem 2.30 (Calderón's reproducing formula)**.** *Let h be a deformed Stockwell wavelet in $L^2_{k,n}(\mathbb{R})$ such that $\mathcal{F}_{k,n}(h)$ belongs to $L^\infty_{k,n}(\mathbb{R})$. Then, for any f in $L^2_{k,n}(\mathbb{R})$ and $0 < \epsilon < \delta < \infty$, the function*

$$f^{\epsilon,\delta}(x) = \frac{1}{C_h} \int_{C(\epsilon,\delta)} \int_{\mathbb{R}} \mathcal{S}_h^{k,n}(f)(y,\nu) h_{y,\nu}(x) \, d\gamma_{k,n}(y) \, d\gamma_{k,n}(\nu), \quad x \in \mathbb{R}, \tag{2.36}$$

belongs to $L^2_{k,n}(\mathbb{R})$ and satisfies

$$\lim_{\epsilon \to 0, \, \delta \to \infty} \|f^{\epsilon,\delta} - f\|_{L^2_{k,n}(\mathbb{R})} = 0, \tag{2.37}$$

where

$$C(\varepsilon,\delta) := \{x \in \mathbb{R} : \varepsilon \leq |x| \leq \delta\}.$$

To prove this theorem we need the following lemmas.

Lemma 2.31. *Retain the assumption of Theorem 2.30. Then, the function $K_{\epsilon,\delta}$ defined by*

$$K_{\epsilon,\delta}(\lambda) = \frac{1}{C_h} \int_{C(\epsilon,\delta)} |\mathcal{F}_{k,n}(M_\nu \Delta_\nu h)(\lambda)|^2 \, d\gamma_{k,n}(\nu), \quad \lambda \in \mathbb{R}, \tag{2.38}$$

satisfies, for almost all $\lambda \in \mathbb{R}$,

$$0 < K_{\epsilon,\delta}(\lambda) \leq 1$$

and

$$\lim_{\epsilon \to 0, \, \delta \to \infty} K_{\epsilon,\delta}(\lambda) = 1.$$

Proof. The result follows immediately from the hypothesis on the deformed Stockwell wavelet function h and relation (2.27). □

Lemma 2.32. *Let h be the deformed Stockwell wavelet satisfying the assumption of Theorem 2.30. Then the function $f^{\epsilon,\delta}$ defined by the relation* (2.36) *belongs to* $L^2_{k,n}(\mathbb{R})$ *and satisfies*

$$\mathcal{F}_{k,n}(f^{\epsilon,\delta})(\lambda) = \mathcal{F}_{k,n}(f)(\lambda)K_{\epsilon,\delta}(\lambda), \quad \lambda \in \mathbb{R}, \tag{2.39}$$

where $K_{\epsilon,\delta}$ is the function given by the relation (2.38)*.*

Proof. We prove first that the function $f^{\epsilon,\delta}$ belongs to $L^2_{k,n}(\mathbb{R})$. From (2.18), (2.31) and relation (2.8) we can write $f^{\epsilon,\delta}$ as

$$f^{\epsilon,\delta}(x) = \frac{1}{C_h}\int_{C(\epsilon,\delta)} (f *_{k,n} \overline{M_\nu \Delta_\nu h}) *_{k,n} M_\nu \Delta_\nu h(x)\, d\gamma_{k,n}(\nu). \tag{2.40}$$

By using Hölder's inequality for the measure $d\gamma_{k,n}(\nu)$, we get

$$|f^{\epsilon,\delta}(x)|^2 \leq \frac{1}{\left(C_h\right)^2}\Big(\int_{C(\epsilon,\delta)} d\gamma_{k,n}(\nu)\Big) \times \int_{C(\epsilon,\delta)} |(f *_{k,n} \overline{M_\nu \Delta_\nu h}) *_{k,n} M_\nu \Delta_\nu h(x)|^2\, d\gamma_{k,n}(\nu).$$

So, by applying Fubini–Tonelli's theorem, we obtain

$$\int_{\mathbb{R}} |f^{\epsilon,\delta}(x)|^2\, d\gamma_{k,n}(x) \leq \frac{1}{\left(C_h\right)^2}\Big(\int_{C(\epsilon,\delta)} d\gamma_{k,n}(\nu)\Big) \times \int_{C(\epsilon,\delta)} \Big(\int_{\mathbb{R}} |(f *_{k,n} \overline{M_\nu \Delta_\nu h}) *_{k,n} M_\nu \Delta_\nu h(x)|^2\, d\gamma_{k,n}(x)\Big)\, d\gamma_{k,n}(\nu).$$

From Plancherel's formula (2.3) and Proposition 2.19 (ii), we deduce that

$$\int_{\mathbb{R}} |f^{\epsilon,\delta}(x)|^2\, d\gamma_{k,n}(x) \leq \frac{1}{\left(C_h\right)^2}\Big(\int_{C(\epsilon,\delta)} d\gamma_{k,n}(\nu)\Big) \times \int_{\mathbb{R}} |\mathcal{F}_{k,n}(f)(\lambda)|^2 \Big(\int_{C(\epsilon,\delta)} |\mathcal{F}_{k,n}(M_\nu \Delta_\nu h)(\lambda)|^4\, d\gamma_{k,n}(\nu)\Big)\, d\gamma_{k,n}(\lambda).$$

On the other hand, using (2.27), (2.28), (2.17) and the fact that $\mathcal{F}_{k,n}(h)$ belongs to $L^\infty_{k,n}(\mathbb{R})$, we deduce that

$$\int_{C(\epsilon,\delta)} |\mathcal{F}_{k,n}(M_\nu \Delta_\nu h)(\lambda)|^4\, d\gamma_{k,n}(\nu) \leq \frac{C_h}{\varepsilon^{\frac{(2k-1)n+2}{n}}} \|\mathcal{F}_{k,n}(h)\|^2_{L^\infty_{k,n}(\mathbb{R})}.$$

Thus,

$$\int_{\mathbb{R}} |f^{\epsilon,\delta}(x)|^2\, d\gamma_{k,n}(x) \leq \frac{1}{\varepsilon^{\frac{(2k-1)n+2}{n}} C_h}\Big(\int_{C(\epsilon,\delta)} d\gamma_{k,n}(\nu)\Big) \|\mathcal{F}_{k,n}(h)\|^2_{L^\infty_{k,n}(\mathbb{R})} \|\mathcal{F}_{k,n}(f)\|^2_{L^2_{k,n}(\mathbb{R})},$$

and Plancherel's formula (2.3) implies

$$\int_{\mathbb{R}} |f^{\epsilon,\delta}(x)|^2 \, d\gamma_{k,n}(x) \leq \frac{1}{\varepsilon^{\frac{(2k-1)n+2}{n}} C_h} \Big(\int_{C(\epsilon,\delta)} d\gamma_{k,n}(\nu) \Big) \|\mathcal{F}_{k,n}(h)\|^2_{L^\infty_{k,n}(\mathbb{R})} \|f\|^2_{L^2_{k,n}(\mathbb{R})} < \infty.$$

Then, $f^{\epsilon,\delta}$ belongs to $L^2_{k,n}(\mathbb{R})$.

We prove now the relation (2.39). Let $\psi \in \mathcal{S}(\mathbb{R})$. We have that the function $(\mathcal{F}_{k,n})^{-1}(\psi)$ belongs to $\mathcal{S}(\mathbb{R})$. From the relation (2.40), we have

$$\int_{\mathbb{R}} f^{\epsilon,\delta}(x)(\mathcal{F}_{k,n})^{-1}(\psi)(x) \, d\gamma_{k,n}(x) = \int_{\mathbb{R}} \Big(\frac{1}{C_h} \int_{C(\epsilon,\delta)} (f *_{k,n} \overline{M_\nu \Delta_\nu h}) *_{k,n} M_\nu \Delta_\nu h(x) \, d\gamma_{k,n}(\nu) \Big) \times (\mathcal{F}_{k,n})^{-1}(\psi)(x) \, d\gamma_{k,n}(x). \tag{2.41}$$

We proceed as above. We prove that the second member of the relation (2.41) can also be written in the form

$$\frac{1}{C_h} \int_{C(\epsilon,\delta)} \Big(\int_{\mathbb{R}} (f *_{k,n} \overline{M_\nu \Delta_\nu h}) *_{k,n} M_\nu \Delta_\nu h(x) (\mathcal{F}_{k,n})^{-1}(\psi)(x) \, d\gamma_{k,n}(x) \Big) \, d\gamma_{k,n}(\nu). \tag{2.42}$$

But, by using Parseval's formula (2.4), the relation (2.42) is equal to

$$\frac{1}{C_h} \int_{C(\epsilon,\delta)} \Big(\int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\lambda) |\mathcal{F}_{k,n}(M_\nu \Delta_\nu h)(\lambda)|^2 \psi((-1)^n \lambda) \, d\gamma_{k,n}(\lambda) \Big) \, d\gamma_{k,n}(\nu).$$

By applying Fubini–Tonelli's theorem and next Fubini's theorem to this integral, it takes the form

$$\int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\lambda) \Big(\frac{1}{C_h} \int_{C(\epsilon,\delta)} |\mathcal{F}_{k,n}(M_\nu \Delta_\nu h)(\lambda)|^2 \, d\gamma_{k,n}(\nu) \Big) \psi((-1)^n \lambda) \, d\gamma_{k,n}(\lambda) = \int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\lambda) K_{\epsilon,\delta}(\lambda) \psi((-1)^n \lambda) \, d\gamma_{k,n}(\lambda). \tag{2.43}$$

On the other hand, by applying Parseval's formula (2.4) to the first member of the relation (2.41), we get

$$\int_{\mathbb{R}} \mathcal{F}_{k,n}(f^{\epsilon,\delta})(\lambda) \psi((-1)^n \lambda) \, d\gamma_{k,n}(\lambda). \tag{2.44}$$

From (2.43) and (2.44), we obtain, for all ψ in $\mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \Big(\mathcal{F}_{k,n}(f^{\epsilon,\delta})(\lambda) - \mathcal{F}_{k,n}(f)(\lambda) K_{\epsilon,\delta}(\lambda) \Big) \psi((-1)^n \lambda) \, d\gamma_{k,n}(\lambda) = 0.$$

Thus

$$\mathcal{F}_{k,n}(f^{\epsilon,\delta})(\lambda) = \mathcal{F}_{k,n}(f)(\lambda) K_{\epsilon,\delta}(\lambda), \quad \lambda \in \mathbb{R}. \qquad \square$$

Proof of Theorem 2.30. From Lemma 2.32, the function $f^{\epsilon,\delta}$ belongs to $L^2_{k,n}(\mathbb{R})$. By using the Plancherel formula (2.3) and Lemma 2.32, we obtain

$$\begin{aligned}\|f^{\epsilon,\delta} - f\|^2_{L^2_{k,n}(\mathbb{R})} &= \int_{\mathbb{R}} |\mathcal{F}_{k,n}(f^{\epsilon,\delta} - f)(\lambda)|^2 \, d\gamma_{k,n}(\lambda) \\ &= \int_{\mathbb{R}} |\mathcal{F}_{k,n}(f)(\lambda)(K_{\epsilon,\delta}(\lambda) - 1)|^2 \, d\gamma_{k,n}(\lambda) \\ &= \int_{\mathbb{R}} |\mathcal{F}_{k,n}(f)(\lambda)|^2 |1 - K_{\epsilon,\delta}(\lambda)|^2 \, d\gamma_{k,n}(\lambda).\end{aligned}$$

But from Lemma 2.31, for almost all $\lambda \in \mathbb{R}$, we have

$$\lim_{\epsilon\to 0,\, \delta\to\infty} |\mathcal{F}_{k,n}(f)(\lambda)|^2 \, |1 - K_{\epsilon,\delta}(\lambda)|^2 = 0,$$

and

$$|\mathcal{F}_{k,n}(f)(\lambda)|^2 \, |1 - K_{\epsilon,\delta}(\lambda)|^2 \leq C|\mathcal{F}_{k,n}(f)(\lambda)|^2,$$

with $|\mathcal{F}_{k,n}(f)(\lambda)|^2$ in $L^1_{k,n}(\mathbb{R})$. So, the relation (2.37) follows from the dominated convergence theorem.

Theorem 2.33 (Inversion formula for $\mathcal{S}^{k,n}_h$)**.** *For all f in $L^2_{k,n}(\mathbb{R})$, we have*

$$f(y) = \frac{1}{C_h} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{S}^{k,n}_h(f)(x,\nu) h_{y,\nu}(x) \, d\gamma_{k,n}(x) \, d\gamma_{k,n}(\nu) \quad a.e., \tag{2.45}$$

where, for each $y \in \mathbb{R}$, both the inner integral with respect to $d\gamma_{k,n}(x)$ and the outer integral with respect to $d\gamma_{k,n}(\nu)$ are absolutely convergent, but possibly not the integral with respect to $d\gamma_{k,n}(x)\, d\gamma_{k,n}(\nu)$.

Proof. Using similar ideas as in the proof of Theorem 6.III.3 in [54, p. 99], we obtain the relation (2.45). □

3. Heisenberg-type uncertainty principles for the deformed Stockwell transform

The uncertainty principle is one of the cornerstones of harmonic analysis; it stems from Heisenberg's uncertainty principle in quantum mechanics asserting that the position and momentum of particles can't be determined explicitly but only in a probabilistic sense [21]. In signal analysis, the uncertainty principle is also known as the duration-bandwidth theorem, due to the fact that the principle states that the widths of a signal in the time domain (duration) and in the frequency domain (bandwidth) are constrained and cannot be made arbitrarily narrow. In this section, we shall establish certain Heisenberg-type uncertainty inequalities in the context of the deformed Stockwell transform $\mathcal{S}^{k,n}_h$ by choosing the window function h as a non trivial even function in the space $L^2_{k,n}(\mathbb{R})$.

3.1. **Generalized Heisenberg's uncertainty principle.** In order to facilitate the formulation of the generalized Heisenberg uncertainty principle for the deformed Stockwell transform, we ought to recall the fundamental uncertainty inequality associated with the deformed Hankel transform $\mathcal{F}_{k,n}$.

Proposition 3.1 ([4, 23])**.** *For $s,t>0$, there exists a positive constant $\mathcal{C}_{k,n}(s,t)$ such that, for every $f\in L^2_{k,n}(\mathbb{R})$, the following inequality holds:*

$$\Big\| |\xi|^s\mathcal{F}_{k,n}(f)(\xi)\Big\|^{\frac{t}{s+t}}_{L^2_{k,n}(\mathbb{R})}\Big\| |x|^t f(x)\Big\|^{\frac{s}{s+t}}_{L^2_{k,n}(\mathbb{R})}\geq \mathcal{C}_{k,n}(s,t)\|f\|_{L^2_{k,n}(\mathbb{R})}. \tag{3.1}$$

For $s,t\geq \frac{1}{n}$, $\mathcal{C}_{k,n}(s,t)=\Big(\frac{(2k-1)n+2}{2n}\Big)^{\frac{nst}{s+t}}$.

Theorem 3.2 (Heisenberg's uncertainty principle for $\mathcal{S}^{k,n}_h$)**.** *Let $s,t>0$. For every f that belongs to $L^2_{k,n}(\mathbb{R})$, we have*

$$\left(\int_{\mathbb{R}^2}|y|^{2t}|\mathcal{S}^{k,n}_h(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)\right)^{\frac{s}{s+t}}\left(\int_{\mathbb{R}}|\xi|^{2s}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi)\right)^{\frac{t}{s+t}}\\ \geq \big(\mathcal{C}_{k,n}(s,t)\big)^2\big(C_h\big)^{\frac{s}{s+t}}\|f\|^2_{L^2_{k,n}(\mathbb{R})}. \tag{3.2}$$

Here $\mathcal{C}_{k,n}(s,t)$ is the constant given in Proposition 3.1.

Proof. Let us consider the non-trivial case where both integrals on the left hand side of (3.2) are finite. Fixing ν arbitrary, Heisenberg's inequality (3.1) gives

$$\left(\int_{\mathbb{R}}|\xi|^{2s}|\mathcal{F}_{k,n}(\mathcal{S}^{k,n}_h(f)(.,\nu))(\xi)|^2\,d\gamma_{k,n}(\xi)\right)^{\frac{t}{s+t}}\\ \times\left(\int_{\mathbb{R}}|y|^{2t}|\mathcal{S}^{k,n}_h(f)(y,\nu)|^2\,d\gamma_{k,n}(y)\right)^{\frac{s}{s+t}}\\ \geq \big(\mathcal{C}_{k,n}(s,t)\big)^2\int_{\mathbb{R}}|\mathcal{S}^{k,n}_h(f)(y,\nu)|^2\,d\gamma_{k,n}(y).$$

Integrating over ν with respect to the measure $d\gamma_{k,n}(\nu)$, and using the Cauchy–Schwarz inequality, we get

$$\left(\int_{\mathbb{R}^2}|\xi|^{2s}|\mathcal{F}_{k,n}(\mathcal{S}^{k,n}_h(f)(.,\nu))(\xi)|^2\,d\mu_{k,n}(\xi,\nu)\right)^{\frac{t}{s+t}}\\ \times\left(\int_{\mathbb{R}^2}|y|^{2t}|\mathcal{S}^{k,n}_h(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)\right)^{\frac{s}{s+t}}\\ \geq \big(\mathcal{C}_{k,n}(s,t)\big)^2\int_{\mathbb{R}^2}|\mathcal{S}^{k,n}_h(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu).$$

Further, using (2.31), Proposition 2.19 (ii) and the hypothesis on h, we infer that

$$\int_{\mathbb{R}^2}|\xi|^{2s}|\mathcal{F}_{k,n}(\mathcal{S}^{k,n}_h(f)(.,\nu))(\xi)|^2\,d\mu_{k,n}(\xi,\nu)=C_h\int_{\mathbb{R}}|\xi|^{2s}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi).$$

Thus, we deduce that

$$\begin{aligned}
&\big(C_h\big)^{\frac{t}{s+t}} \left(\int_{\mathbb{R}} |\xi|^{2s} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\mu_{k,n}(\xi)\right)^{\frac{t}{t+s}} \\
&\qquad\qquad \times \left(\int_{\mathbb{R}^2} |y|^{2t} |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2 \, d\mu_{k,n}(y,\nu)\right)^{\frac{s}{s+t}} \\
&\qquad \geq \big(\mathcal{C}_{k,n}(s,t)\big)^2 \int_{\mathbb{R}^2} |\mathcal{S}_h^{k,n} f(y,\nu)|^2 \, d\mu_{k,n}(y,\nu) = \big(\mathcal{C}_{k,n}(s,t)\big)^2 C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})}.
\end{aligned}$$

This proves the result. □

Theorem 3.3. *For $s,t>0$ and for all f in $L^2_{k,n}(\mathbb{R})$, the following inequality holds:*

$$\begin{aligned}
&\Big\| \, |\nu|^s \mathcal{S}_h^{k,n}(f)(y,\nu) \Big\|^{\frac{t}{s+t}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \Big\| \, |x|^t f(x) \Big\|^{\frac{s}{s+t}}_{L^2_{k,n}(\mathbb{R})} \\
&\qquad\qquad \geq \mathcal{C}_{k,n}(s,t) \big(\mathcal{M}_{k,n}(h)(2s)\big)^{\frac{t}{2(s+t)}} \|f\|_{L^2_{k,n}(\mathbb{R})},
\end{aligned}$$

where

$$\mathcal{M}_{k,n}(h)(2s) = \int_0^\infty \Big(\tau_1^{k,n} |\mathcal{F}_{k,n}(h)|^2(r) + \tau_1^{k,n} |\mathcal{F}_{k,n}(h)|^2(-r)\Big) \frac{dr}{r^{2s+1}}$$

and $\mathcal{C}_{k,n}(s,t)$ is the constant given in Proposition 3.1.

Proof. In the following we assume that

$$\int_{\mathbb{R}^2} |\nu|^{2s} |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2 \, d\mu_{k,n}(y,\nu) < \infty \quad \text{and} \quad \int_{\mathbb{R}} |x|^{2t} |f(x)|^2 \, d\gamma_{k,n}(x) < \infty.$$

Otherwise, the inequality is trivially satisfied. Using Fubini's theorem, Plancherel's formula (2.3) and (2.34), we get

$$\int_{\mathbb{R}^2} |\nu|^{2s} |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2 \, d\mu_{k,n}(y,\nu) = \int_{\mathbb{R}} \Lambda_{k,n}(\xi) |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi),$$

with

$$\Lambda_{k,n}(\xi) = \int_{\mathbb{R}} |\nu|^{2s} \, |\mathcal{F}_{k,n}(M_\nu \Delta_\nu h)(\xi)|^2 \, d\gamma_{k,n}(\nu).$$

By simple calculations, we see that $\Lambda_{k,n}(\xi)$ is just a function of $|\xi|^{2s}$. Indeed, using (2.28) we get

$$\begin{aligned}
\Lambda_{k,n}(\xi) &= \int_{\mathbb{R}} \tau_1^{k,n} |\mathcal{F}_{k,n}(h)|^2 \left(\frac{\xi}{\nu}\right) \frac{d\nu}{|\nu|^{1-2s}} \\
&= \Big(\int_0^\infty \big(\tau_1^{k,n} |\mathcal{F}_{k,n}(h)|^2(r) + \tau_1^{k,n} |\mathcal{F}_{k,n}(h)|^2(-r)\big) \frac{dr}{r^{2s+1}} \Big) |\xi|^{2s} \\
&= \mathcal{M}_{k,n}(h)(2s) \, |\xi|^{2s}.
\end{aligned}$$

Thus

$$\begin{aligned}\left(\int_{\mathbb{R}^2}|\nu|^{2s}|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)\right)^{\frac{t}{s+t}}\left(\int_{\mathbb{R}}|x|^{2t}|f(x)|^2\,d\gamma_{k,n}(x)\right)^{\frac{s}{s+t}}\\ =\Big(\mathcal{M}_{k,n}(h)(2s)\Big)^{\frac{t}{s+t}}\left(\int_{\mathbb{R}}|\xi|^{2s}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi)\right)^{\frac{t}{s+t}}\\ \times\left(\int_{\mathbb{R}}|x|^{2t}|f(x)|^2\,d\gamma_{k,n}(x)\right)^{\frac{s}{s+t}}.\end{aligned}$$

Now, the result is obtained from Proposition 3.1. □

Corollary 3.4. *For $s,t>0$ and for all f in $L^2_{k,n}(\mathbb{R})$, the following inequality holds:*

$$\begin{aligned}\Big\||y|^t\mathcal{S}_h^{k,n}(f)(y,\nu)\Big\|^{\frac{2s}{s+t}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\Big\||\nu|^s\mathcal{S}_h^{k,n}(f)(y,\nu)\Big\|^{\frac{2t}{s+t}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\\ \geq\big(\mathcal{C}_{k,n}(s,t)\big)^2\big(C_h\big)^{\frac{s}{s+t}}\big(\mathcal{M}_{k,n}(h)(2s)\big)^{\frac{t}{s+t}}\|f\|^2_{L^2_{k,n}(\mathbb{R})}.\end{aligned}$$

Here $\mathcal{C}_{k,n}(s,t)$ is the constant given in Proposition 3.1.

Proof. From above we have

$$\begin{aligned}\left(\int_{\mathbb{R}^2}|\nu|^{2s}|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)\right)^{\frac{t}{s+t}}\\ \times\left(\int_{\mathbb{R}^2}|y|^{2t}|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)\right)^{\frac{s}{s+t}}\\ =\big(\mathcal{M}_{k,n}(h)(2s)\big)^{\frac{t}{s+t}}\left(\int_{\mathbb{R}}|\xi|^{2s}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi)\right)^{\frac{t}{s+t}}\\ \times\left(\int_{\mathbb{R}^2}|y|^{2t}|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)\right)^{\frac{s}{s+t}}.\end{aligned}$$

The desired relation follows from (3.2). □

As a consequence of the previous corollary, we have the following local-type uncertainty principle.

Corollary 3.5. *Let $s,t>0$ and let $U\subset\mathbb{R}^2$ be such that $0<\mu_{k,n}(U):=\int_U d\mu_{k,n}(y,\nu)<\infty$. For all f in $L^2_{k,n}(\mathbb{R})$, the following inequality holds:*

$$\begin{aligned}\int_U|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)\\ \leq\mathcal{C}(k,n,s,t)\Big\||y|^t\mathcal{S}_h^{k,n}(f)(y,\nu)\Big\|^{\frac{2s}{s+t}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\Big\||\nu|^s\mathcal{S}_h^{k,n}(f)(y,\nu)\Big\|^{\frac{2t}{s+t}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)},\end{aligned}$$

where

$$\mathcal{C}(k,n,s,t) := \frac{\mu_{k,n}(U)\|h\|^2_{L^2_{k,n}(\mathbb{R})}}{\big(\mathcal{C}_{k,n}(s,t)\big)^2 \big(C_h\big)^{\frac{s}{s+t}} \Big(\mathcal{M}_{k,n}\big(\tau_1^{k,n}|\mathcal{F}_{k,n}(h)|^2(2s)\big)\Big)^{\frac{t}{s+t}}}$$

and $\mathcal{C}_{k,n}(s,t)$ *is the constant given in Proposition 3.1.*

Proof. From the relation (2.32), we have

$$\int_{\mathbb{R}^2} \mathbb{1}_U(y,\nu)|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\, d\mu_{k,n}(y,\nu) \leq \mu_{k,n}(U)\|h\|^2_{L^2_{k,n}(\mathbb{R})}\|f\|^2_{L^2_{k,n}(\mathbb{R})}.$$

On the other hand, from Corollary 3.4 we have

$$\|f\|^2_{L^2_{k,n}(\mathbb{R})} \leq \frac{\Big\| |y|^t\mathcal{S}_h^{k,n}(f)(y,\nu)\Big\|^{\frac{2s}{s+t}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \Big\| |\nu|^s\mathcal{S}_h^{k,n}(f)(y,\nu)\Big\|^{\frac{2t}{s+t}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}}{\big(\mathcal{C}_{k,n}(s,t)\big)^2 \big(C_h\big)^{\frac{s}{s+t}} \Big(\mathcal{M}_{k,n}\big(\tau_1^{k,n}|\mathcal{F}_{k,n}(h)|^2(2s)\big)\Big)^{\frac{t}{s+t}}}.$$

Thus the result is immediate. □

Proposition 3.6 (Nash's uncertainty principle for $\mathcal{S}_h^{k,n}$)**.** *For every* $s>0$, *there exists a positive constant* $\mathcal{C}(k,n,s,h)$ *such that, for all* $f\in L^2_{k,n}(\mathbb{R})$, *we have*

$$\|f\|_{L^2_{k,n}(\mathbb{R})} \leq \mathcal{C}(k,n,s,h)\Big\| \|(y,\nu)\|^s\mathcal{S}_h^{k,n}(f)\Big\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}. \tag{3.3}$$

Proof. It is clear that the relation (3.3) holds if $f=0$. Assume that $0\neq f\in L^2_{k,n}(\mathbb{R})$ and let $R>0$. From Plancherel's formula (2.35) we have

$$\begin{aligned} C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})} &= \|\mathcal{S}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &= \|\mathbb{1}_{B(0,R)}\mathcal{S}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} + \|(1-\mathbb{1}_{B(0,R)})\mathcal{S}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}, \end{aligned}$$

where $B(0,R) := \big\{(y,\nu)\in\mathbb{R}^2 : \|(y,\nu)\| \leq R\big\}$ and $\mathbb{1}_{B(0,R)}$ is its characteristic function. By (2.32), we have

$$\begin{aligned} \|\mathbb{1}_{B(0,R)}\mathcal{S}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} &\leq \|h\|^2_{L^2_{k,n}(\mathbb{R})}\|f\|^2_{L^2_{k,n}(\mathbb{R})}\int_{\mathbb{R}^2}\mathbb{1}_{B(0,R)}\, d\mu_{k,n}(y,\nu) \\ &\leq CR^{\frac{2(2k-1)n+2}{n}}\|h\|^2_{L^2_{k,n}(\mathbb{R})}\|f\|^2_{L^2_{k,n}(\mathbb{R})}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Big\|(1-\mathbb{1}_{B(0,R)})\mathcal{S}_h^{k,n}(f)\Big\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} &\leq R^{-2s}\Big\|(1-\mathbb{1}_{B(0,R)})\,\|(y,\nu)\|^s\mathcal{S}_h^{k,n}(f)\Big\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\leq R^{-2s}\Big\| \|(y,\nu)\|^s\mathcal{S}_h^{k,n}(f)\Big\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}. \end{aligned}$$

It follows then that

$$\begin{aligned}C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})} \leq CR^{\frac{2(2k-1)n+2}{n}}\|h\|^2_{L^2_{k,n}(\mathbb{R})}\|f\|^2_{L^2_{k,n}(\mathbb{R})}\\ + R^{-2s}\Big\| \, \|(y,\nu)\|^s \mathcal{S}^{k,n}_h(f)\Big\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.\end{aligned}$$

Minimizing over $R>0$ the right hand side of the above inequality we obtain

$$\begin{aligned}C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})} \leq C(k,n,s)\|h\|^{\frac{2ns}{(2k-1+s)n+2}}_{L^2_{k,n}(\mathbb{R})}\|f\|^{\frac{2ns}{(2k-1+s)n+2}}_{L^2_{k,n}(\mathbb{R})}\\ \times \Big\| \, \|(y,\nu)\|^s \mathcal{S}^{k,n}_h(f)\Big\|^{\frac{2(2k-1)n+4}{(2k-1+s)n+2}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\end{aligned} \tag{3.4}$$

for some positive constant $C(k,n,s)$. The desired result follows immediately from (3.4). □

3.2. **Heisenberg's uncertainty principle via (k,n)-entropy.** Let ρ be a probability density function on $\mathbb{R}^2$, i.e. a nonnegative measurable function on $\mathbb{R}^2$ satisfying

$$\int_{\mathbb{R}^2} \rho(y,\nu)\, d\mu_{k,n}(y,\nu) = 1.$$

Following Shannon [47], the (k,n)-entropy of a probability density function ρ on $\mathbb{R}^2$ is defined by

$$E_{k,n}(\rho) := -\int_{\mathbb{R}^2} \ln(\rho(y,\nu))\rho(y,\nu)\, d\mu_{k,n}(y,\nu).$$

Henceforth, we extend the definition of the (k,n)-entropy of a nonnegative measurable function ρ on $\mathbb{R}^2$ whenever the previous integral on the right hand side is well defined.

The aim of this part is to study the localization of the (k,n)-entropy of the deformed Stockwell transform. Indeed, we have the following result.

Proposition 3.7. *For all $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$E_{k,n}\big(|\mathcal{S}^{k,n}_h(f)|^2\big) \geq -2C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}\ln\big(\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}\big).$$

Proof. Assume that $\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})} = 1$. By (2.32),

$$|\mathcal{S}^{k,n}_h(f)(y,\nu)| \leq \|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})} = 1.$$

In particular, $E_{k,n}\big(|\mathcal{S}^{k,n}_h(f)|^2\big) \geq 0$. Next, let us drop the above assumption, and let

$$\phi := \frac{f}{\|f\|_{L^2_{k,n}(\mathbb{R})}} \quad \text{and} \quad \psi := \frac{h}{\|h\|_{L^2_{k,n}(\mathbb{R})}}.$$

Then, $\phi,\psi \in L^2_{k,n}(\mathbb{R})$ and $\|\phi\|_{L^2_{k,n}(\mathbb{R})}\|\psi\|_{L^2_{k,n}(\mathbb{R})} = 1$.

Therefore, $E_{k,n}\big(|\mathcal{S}^{k,n}_\psi(\phi)|^2\big) \geq 0$. Moreover,

$$\mathcal{S}^{k,n}_\psi(\phi) = \frac{1}{\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}}\mathcal{S}^{k,n}_h(f),$$

which implies

$$E_{k,n}\big(|\mathcal{S}_{\psi}^{k,n}(\phi)|^2\big) = \frac{1}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}\|h\|^2_{L^2_{k,n}(\mathbb{R})}} E_{k,n}\big(|\mathcal{S}_h^{k,n}(f)|^2\big) + \frac{2C_h}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \ln\big(\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}\big).$$

Using the fact that $E_{k,n}\big(|\mathcal{S}_{\psi}^{k,n}(\phi)|^2\big) \geq 0$, we deduce that

$$E_{k,n}\big(|\mathcal{S}_h^{k,n}(f)|^2\big) \geq -2C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})} \ln\big(\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}\big). \qquad \square$$

Using the (k,n)-entropy of the deformed Stockwell transform, we can obtain another version of the Heisenberg uncertainty principle for $\mathcal{S}_h^{k,n}$.

Theorem 3.8. *Let $p,q>0$. Then for every $f \in L^2_{k,n}(\mathbb{R})$ we have*

$$\left(\int_{\mathbb{R}^2} |y|^p |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\, d\mu_{k,n}(y,\nu)\right)^{\frac{q}{p+q}} \left(\int_{\mathbb{R}^2} |\nu|^q |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\, d\mu_{k,n}(y,\nu)\right)^{\frac{p}{p+q}} \geq \mathfrak{M}_{p,q}(k,n) C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})},$$

where

$$\mathfrak{M}_{p,q}(k,n) = \frac{(2k-1)n+2}{nep^{\frac{q}{p+q}}q^{\frac{p}{p+q}}} \times \exp\left(\frac{npq}{((2k-1)n+2)(p+q)} \ln\left(\frac{pq}{4\Gamma\big(\frac{(2k-1)n+2}{np}\big)\Gamma\big(\frac{(2k-1)n+2}{nq}\big)}\right)\right).$$

Proof. For every triple of positive real numbers t,p,q, let $\eta^{k,n}_{t,p,q}$ be the function defined on $\mathbb{R}^2$ by

$$\eta^{k,n}_{t,p,q}(y,\nu) := \frac{pq}{4\Gamma\big(\frac{(2k-1)n+2}{np}\big)\Gamma\big(\frac{(2k-1)n+2}{nq}\big)} \frac{\exp\left(-\frac{|y|^p+|\nu|^q}{t}\right)}{t^{\frac{((2k-1)n+2)(p+q)}{npq}}}.$$

By a simple computation, we see that

$$\int_{\mathbb{R}^2} \eta^{k,n}_{t,p,q}(y,\nu)\, d\mu_{k,n}(y,\nu) = 1.$$

In particular, the measure $d\sigma^{k,n}_{t,p,q}(y,\nu) := \eta^{k,n}_{t,p,q}(y,\nu)\, d\mu_{k,n}(y,\nu)$ is a probability measure on $\mathbb{R}^2$. Since the function $\varphi(t) = t\ln(t)$ is convex over $(0,\infty)$, by using Jensen's inequality for convex functions we get

$$\int_{\mathbb{R}^2} \frac{|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2}{\eta^{k,n}_{t,p,q}(y,\nu)} \ln\left(\frac{|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2}{\eta^{k,n}_{t,p,q}(y,\nu)}\right) d\sigma^{k,n}_{t,p,q}(y,\nu) \geq 0,$$

which implies in terms of (k,n)-entropy that

$$E_{k,n}\big(|\mathcal{S}_h^{k,n}(f)|^2\big)+\ln\left(\frac{pq}{4\Gamma\big(\frac{(2k-1)n+2}{np}\big)\Gamma\big(\frac{(2k-1)n+2}{nq}\big)}\right)C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}$$
$$\le \ln\left(t^{\frac{((2k-1)n+2)(p+q)}{npq}}\right)C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}$$
$$+\frac{1}{t}\int_{\mathbb{R}^2}(|y|^p+|\nu|^q)|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu).$$

Assume that $\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}=1$. Then, by Proposition 3.7 we get

$$\int_{\mathbb{R}^2}(|y|^p+|\nu|^q)|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)$$
$$\ge t\left(\ln\left(\frac{pq}{4\Gamma\big(\frac{(2k-1)n+2}{np}\big)\Gamma\big(\frac{(2k-1)n+2}{nq}\big)}\right)-\ln\left(t^{\frac{((2k-1)n+2)(p+q)}{npq}}\right)\right)C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}.$$

However, the expression

$$t\left(\ln\left(\frac{pq}{4\Gamma\big(\frac{(2k-1)n+2}{np}\big)\Gamma\big(\frac{(2k-1)n+2}{nq}\big)}\right)-\ln\left(t^{\frac{((2k-1)n+2)(p+q)}{npq}}\right)\right)$$

attains its upper bound at

$$t_0=\exp\left(\frac{npq}{((2k-1)n+2)(p+q)}\ln\left(\frac{pq}{4\Gamma\big(\frac{(2k-1)n+2}{np}\big)\Gamma\big(\frac{(2k-1)n+2}{nq}\big)}\right)-1\right),$$

and consequently

$$\int_{\mathbb{R}^2}(|y|^p+|\nu|^q)|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)\ge C_{p,q}(k,n)C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})},$$

where

$$C_{p,q}(k,n)=\frac{((2k-1)n+2)(p+q)}{nepq}$$
$$\times\exp\left(\frac{npq}{((2k-1)n+2)(p+q)}\ln\left(\frac{pq}{4\Gamma\big(\frac{(2k-1)n+2}{np}\big)\Gamma\big(\frac{(2k-1)n+2}{nq}\big)}\right)\right).$$

Therefore, for every $f\in L^2_{k,n}(\mathbb{R})$ and $h\in L^2_{k,e}(\mathbb{R})$ such that $\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}=1$, we have

$$\int_{\mathbb{R}^2}|y|^p|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)+\int_{\mathbb{R}^2}|\nu|^q|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu)$$
$$\ge C_{p,q}(k,n)C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}.$$

Now, for every $\lambda>0$, the dilate f_λ belongs to $L^2_{k,n}(\mathbb{R})$. Then, by substituting f by f_λ and using the fact that $\|f_\lambda\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}=\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}=1$, the above inequality gives

$$\int_{\mathbb{R}^2} |y|^p |\mathcal{S}_h^{k,n}(f_\lambda)(y,\nu)|^2\, d\mu_{k,n}(y,\nu) + \int_{\mathbb{R}^2} |\nu|^q |\mathcal{S}_h^{k,n}(f_\lambda)(y,\nu)|^2\, d\mu_{k,n}(y,\nu) \geq C_{p,q}(k,n) C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})}.$$

Using (2.33), we deduce that

$$\lambda^p \int_{\mathbb{R}^2} |y|^p |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\, d\mu_{k,n}(y,\nu) + \lambda^{-q} \int_{\mathbb{R}^2} |\nu|^q |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\, d\mu_{k,n}(y,\nu) \geq C_{p,q}(k,n) C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})}.$$

In particular, the inequality holds at the point

$$\lambda = \left(\frac{q \int_{\mathbb{R}^2} |\nu|^q |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\, d\mu_{k,n}(y,\nu)}{p \int_{\mathbb{R}^2} |y|^p |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\, d\mu_{k,n}(y,\nu)} \right)^{\frac{1}{p+q}},$$

which implies that

$$\left(\int_{\mathbb{R}^2} |y|^p |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\, d\mu_{k,n}(y,\nu) \right)^{\frac{q}{p+q}} \left(\int_{\mathbb{R}^2} |\nu|^q |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\, d\mu_{k,n}(y,\nu) \right)^{\frac{p}{p+q}} \geq \mathfrak{M}_{p,q}(k,n) C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})},$$

where

$$\begin{aligned} \mathfrak{M}_{p,q}(k,n)(k,n) &= C_{p,q}(k,n) \frac{p^{\frac{p}{p+q}} q^{\frac{q}{p+q}}}{p+q} \\ &= \frac{(2k-1)n+2}{np^{\frac{q}{p+q}} q^{\frac{p}{p+q}}} \exp\Bigg(\frac{npq}{((2k-1)n+2)(p+q)} \\ &\qquad \times \ln\left(\frac{pq}{4\Gamma\big(\frac{(2k-1)n+2}{np}\big)\Gamma\big(\frac{(2k-1)n+2}{nq}\big)} \right) - 1 \Bigg). \end{aligned}$$

Now, the general formula follows from above by substituting f by $f/\{\|f\|_{L^2_{k,n}(\mathbb{R})}\}$ and h by $h/\|h\|_{L^2_{k,n}(\mathbb{R})}$. □

Remark 3.9. When $p = q = 2$, we get

$$\Big\| |y|\, \mathcal{S}_h^{k,n}(f) \Big\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \Big\| |\nu|\, \mathcal{S}_h^{k,n}(f) \Big\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \frac{(2k-1)n+2}{2ne} \left(\frac{1}{\Gamma\big(\frac{(2k-1)n+2}{2n}\big)} \right)^{\frac{2n}{(2k-1)n+2}} C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})}.$$

3.3. **L^p-Heisenberg's uncertainty principle.** In this subsection, we shall derive a general form of L^p-Heisenberg's uncertainty inequality for the deformed Stockwell transform. Our strategy of the proof is motivated by [9], wherein the authors studied the L^2-Heisenberg's uncertainty inequality on Lie groups. To facilitate the narrative, we set the notation

$$\Gamma_t(y,\nu) := e^{-t\|(y,\nu)\|^2}, \qquad (y,\nu)\in\mathbb{R}^2,\ t>0.$$

By simple calculations it is easy to check that, for every $1\le q<\infty$, there exists a positive constant C such that

$$\|\Gamma_t\|_{L^q_{\mu_{k,n}}(\mathbb{R}^2)} = Ct^{-\frac{(2k-1)n+2}{nq}}. \tag{3.5}$$

Lemma 3.10. *Let $1<p\le 2$ and $0<\alpha<\frac{(2k-1)n+2}{2np'}$, where p' denotes the conjugate exponent of p. Then, there exists a positive constant $C(k,n)$ such that, for all $f\in L^2_{k,n}(\mathbb{R})$ and $t>0$,*

$$\begin{aligned}\left\|\Gamma_t\,\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} \le C(k,n)&\frac{\|h\|_{L^2_{k,n}(\mathbb{R})}^{\frac{4n\alpha}{(2k-1)n+2}-\frac{2}{p'}+1}}{\left(C_h\right)^{\frac{2n\alpha}{(2k-1)n+2}-\frac{1}{p'}}}\\ &\times t^{-2\alpha}\Big[\|\,|y|^\alpha f\|_{L^2_{k,n}(\mathbb{R})}+\|\,|y|^\alpha f\|_{L^{2p}_{k,n}(\mathbb{R})}\Big].\end{aligned} \tag{3.6}$$

Proof. Inequality (3.6) holds whenever $\|\,|y|^\alpha f\|_{L^2_{k,n}(\mathbb{R})}+\|\,|y|^\alpha f\|_{L^{2p}_{k,n}(\mathbb{R})}=\infty$. Let us assume that

$$\|\,|y|^\alpha f\|_{L^2_{k,n}(\mathbb{R})}+\|\,|y|^\alpha f\|_{L^{2p}_{k,n}(\mathbb{R})}<\infty.$$

For $s>0$, let $f_s=\mathbb{1}_{(-s,s)}f$ and $f^s=f-f_s$. Since

$$|f^s(y)|\le s^{-\alpha}|\,|y|^\alpha f(y)|,$$

we deduce from Proposition 2.29 that

$$\begin{aligned}\left\|\Gamma_t\,\mathcal{S}_h^{k,n}(\mathbb{1}_{\mathbb{R}\setminus(-s,s)}f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} &\le \|\Gamma_t\|_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)}\left\|\mathcal{S}_h^{k,n}(\mathbb{1}_{\mathbb{R}\setminus(-s,s)}f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}\\ &\le (C_h)^{\frac{1}{p'}}\left(\|h\|_{L^2_{k,n}(\mathbb{R})}\right)^{\frac{p'-2}{p'}}\left\|\mathbb{1}_{\mathbb{R}\setminus(-s,s)}f\right\|_{L^2_{k,n}(\mathbb{R})}\\ &\le (C_h)^{\frac{1}{p'}}\left(\|h\|_{L^2_{k,n}(\mathbb{R})}\right)^{\frac{p'-2}{p'}}s^{-\alpha}\,\|\,|y|^\alpha f\|_{L^2_{k,n}(\mathbb{R})}.\end{aligned}$$

On the other hand, by (2.32) and Hölder's inequality,

$$\begin{aligned}\left\|\Gamma_t\,\mathcal{S}_h^{k,n}(\mathbb{1}_{(-s,s)}f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} &\le \|\Gamma_t\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}\left\|\mathcal{S}_h^{k,n}(\mathbb{1}_{(-s,s)}f)\right\|_{L^\infty_{k,n}(\mathbb{R})}\\ &\le \|h\|_{L^2_{k,n}(\mathbb{R})}\|\Gamma_t\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}\left\|\mathbb{1}_{(-s,s)}f\right\|_{L^2_{k,n}(\mathbb{R})}\\ &\le \|h\|_{L^2_{k,n}(\mathbb{R})}\|\Gamma_t\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}\left\|\,|y|^{-\alpha}\mathbb{1}_{(-s,s)}\right\|_{L^{2p'}_{k,n}(\mathbb{R})}\\ &\qquad\times\|\,|y|^\alpha f\|_{L^{2p}_{k,n}(\mathbb{R})}.\end{aligned}$$

A simple calculation shows that there exists a positive constant C such that

$$\big\| \, |y|^{-\alpha}\mathbb{1}_{(-s,s)} \big\|_{L^{2p'}_{k,n}(\mathbb{R})} = C s^{-\alpha + \frac{(2k-1)n+2}{2np'}}.$$

Therefore,

$$\begin{aligned}
\left\| \Gamma_t\, \mathcal{S}_h^{k,n}(f) \right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} &\leq \left\| \Gamma_t\, \mathcal{S}_h^{k,n}(f_s) \right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} + \left\| \Gamma_t\, \mathcal{S}_h^{k,n}(f^s) \right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} \\
&\leq C s^{-\alpha} \|h\|_{L^2_{k,n}(\mathbb{R})} \Bigg[\left(\frac{C_h}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \right)^{\frac{1}{p'}} \| \, |y|^{\alpha} f \|_{L^2_{k,n}(\mathbb{R})} \\
&\qquad + s^{\frac{(2k-1)n+2}{2np'}} \|\Gamma_t\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} \| \, |y|^{\alpha} f \|_{L^{2p}_{k,n}(\mathbb{R})} \Bigg].
\end{aligned}$$

Using (3.5), we get

$$\begin{aligned}
\left\| \Gamma_t\, \mathcal{S}_h^{k,n}(f) \right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} &\leq C s^{-\alpha} \|h\|_{L^2_{k,n}(\mathbb{R})} \Bigg[\left(\frac{C_h}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \right)^{\frac{1}{p'}} \| \, |y|^{\alpha} f \|_{L^2_{k,n}(\mathbb{R})} \\
&\qquad + s^{\frac{(2k-1)n+2}{2np'}} t^{-\frac{(2k-1)n+2}{np'}} \| \, |y|^{\alpha} f \|_{L^{2p}_{k,n}(\mathbb{R})} \Bigg].
\end{aligned}$$

Thus, choosing $s = \left(\frac{C_h}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \right)^{\frac{2n}{(2k-1)n+2}} t^2$ we obtain the desired inequality. ☐

Theorem 3.11. *Let $1 < p \leq 2$, $0 < \alpha < \frac{(2k-1)n+2}{2np'}$ and $\beta > 0$. Then, there exists a positive constant $\mathcal{C}(k,n)$ such that, for all $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\begin{aligned}
\left\| \mathcal{S}_h^{k,n}(f) \right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} &\leq \mathcal{C}(k,n) \left(\frac{\|h\|_{L^2_{k,n}(\mathbb{R})}^{\frac{4n\alpha}{(2k-1)n+2} - \frac{2}{p'} + 1}}{\left(C_h\right)^{\frac{2n\alpha}{(2k-1)n+2} - \frac{1}{p'}}} \right)^{\frac{\beta}{\alpha+\beta}} \\
&\quad \times \left[\| \, |y|^{\alpha} f \|_{L^2_{k,n}(\mathbb{R})} + \| \, |y|^{\alpha} f \|_{L^{2p}_{k,n}(\mathbb{R})} \right]^{\frac{\beta}{\alpha+\beta}} \\
&\quad \times \left\| \|(y,\nu)\|^{4\beta} \mathcal{S}_h^{k,n}(f) \right\|^{\frac{\alpha}{\alpha+\beta}}_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}.
\end{aligned} \tag{3.7}$$

Proof. Inequality (3.7) holds whenever $\mathcal{S}_h^{k,n}(f) = 0$. Assume that $\mathcal{S}_h^{k,n}(f) \neq 0$. Let $1 < p \leq 2$ and $0 < \alpha < \frac{(2k-1)n+2}{2np'}$.

Let us assume that $\beta \leq \frac{1}{2}$. From the previous lemma, for all $t > 0$, we have

$$\begin{aligned}
\left\|\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} &\leq \left\|\Gamma_t\, \mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} + \left\|(1-\Gamma_t)\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} \\
&\leq C(k,n)\frac{\|h\|_{L^2_{k,n}(\mathbb{R})}^{\frac{4n\alpha}{(2k-1)n+2}-\frac{2}{p'}+1}}{\left(C_h\right)^{\frac{2n\alpha}{(2k-1)n+2}-\frac{1}{p'}}} \\
&\quad \times t^{-2\alpha}\Big[\|\,|y|^{\alpha} f\|_{L^2_{k,n}(\mathbb{R})} + \|\,|y|^{\alpha} f\|_{L^{2p}_{k,n}(\mathbb{R})}\Big] \\
&\quad + \left\|(1-\Gamma_t)\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\|(1-\Gamma_t)\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} \\
&\qquad = t^{2\beta}\left\|(t\|(y,\nu)\|^2)^{-2\beta}(1-\Gamma_t)\|(y,\nu)\|^{4\beta}\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}.
\end{aligned}$$

Since $(1-e^{-u})u^{-2\beta}$ is bounded for $u \geq 0$ if $\beta \leq \frac{1}{2}$, we obtain

$$\begin{aligned}
&\left\|\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} \\
&\quad \leq C(k,n)\frac{\|h\|_{L^2_{k,n}(\mathbb{R})}^{\frac{4n\alpha}{(2k-1)n+2}-\frac{2}{p'}+1}}{\left(C_h\right)^{\frac{2n\alpha}{(2k-1)n+2}-\frac{1}{p'}}} t^{-2\alpha}\Big[\|\,|y|^{\alpha} f\|_{L^2_{k,n}(\mathbb{R})} + \|\,|y|^{\alpha} f\|_{L^{2p}_{k,n}(\mathbb{R})}\Big] \\
&\qquad + Ct^{2\beta}\left\|\|(y,\nu)\|^{4\beta}\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)},
\end{aligned}$$

from which, optimizing in t, we obtain (3.7) for $0 < \alpha < \frac{(2k-1)n+2}{2np'}$ and $\beta \leq \frac{1}{2}$.

Next, we assume that $\beta > \frac{1}{2}$. For $u \geq 0$ and $\beta' \leq \frac{1}{2} < \beta$, we have $u^{4\beta'} \leq 1 + u^{4\beta}$, which for $u = \|(y,\nu)\|/\varepsilon$ becomes

$$\left(\frac{\|(y,\nu)\|}{\varepsilon}\right)^{4\beta'} < 1 + \left(\frac{\|(y,\nu)\|}{\varepsilon}\right)^{4\beta} \qquad \text{for all } \varepsilon > 0.$$

It follows that

$$\begin{aligned}
\left\|\|(y,\nu)\|^{4\beta'}\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} &\leq \varepsilon^{4\beta'}\left\|\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} \\
&\quad + \varepsilon^{4(\beta'-\beta)}\left\|\|(y,\nu)\|^{4\beta}\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}.
\end{aligned}$$

Upon optimizing over ε, we choose

$$\varepsilon = \left(\frac{(\beta-\beta')\left\|\|(y,\nu)\|^{4\beta}\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}}{\beta'\left\|\mathcal{S}_h^{k,n}(f)\right\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}}\right)^{\frac{1}{4\beta}}$$

and we obtain

$$\Big\| \|(y,\nu)\|^{4\beta'} \mathcal{S}_h^{k,n}(f)\Big\|_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} \leq \Big\|\mathcal{S}_h^{k,n}(f)\Big\|^{\frac{\beta-\beta'}{\beta}}_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)} \Big\| \|(y,\nu)\|^{4\beta} \mathcal{S}_h^{k,n}(f)\Big\|^{\frac{\beta'}{\beta}}_{L^{p'}_{\mu_{k,n}}(\mathbb{R}^2)}.$$

Together with (3.7) for β', we get the result for $\beta > \frac{1}{2}$. $\square$

Corollary 3.12. *Let $0 < \alpha < \frac{(2k-1)n+2}{4n}$ and $\beta > 0$. For all $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\begin{aligned}\|f\|_{L^2_{k,n}(\mathbb{R})} \leq \mathcal{C}(k,n) &\left(\frac{\|h\|^{\frac{4n\beta}{(2k-1)n+2}}_{L^2_{k,n}(\mathbb{R})}}{\big(C_h\big)^{\frac{1}{2}+\frac{2n\beta}{(2k-1)n+2}}} \right)^{\frac{\alpha}{\alpha+\beta}} \\ &\times \Big[\| \, |y|^\alpha f\|_{L^2_{k,n}(\mathbb{R})} + \| |y|^\alpha f\|_{L^4_{k,n}(\mathbb{R})} \Big]^{\frac{\beta}{\alpha+\beta}} \Big\| \|(y,\nu)\|^{4\beta} \mathcal{S}_h^{k,n}(f)\Big\|^{\frac{\alpha}{\alpha+\beta}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.\end{aligned}$$

Here $\mathcal{C}(k,n)$ is the constant given in Theorem 3.11.

Proof. The statement follows from Theorem 3.11 with $p = 2$ and Plancherel's formula (2.35). $\square$

4. Concentration-based inequalities for the deformed Stockwell transforms

In this section, we derive some concentration-based uncertainty inequalities for the deformed Stockwell transform as an analogue of the Benedick–Amrein–Berthier and local uncertainty principles in time-frequency analysis.

4.1. Benedick–Amrein–Berthier's uncertainty principle.

T. R. Johansen [23] proved the Benedicks–Amrein–Berthier uncertainty principle for the generalized Fourier transform $\mathcal{F}_{k,n}$, which states that if E_1 and E_2 are two subsets of $\mathbb{R}$ with finite measure, then there exists a positive constant $C_{k,n}(E_1,E_2)$ such that, for any $f \in L^2_{k,n}(\mathbb{R})$,

$$\begin{aligned}&\int_{\mathbb{R}} |f(t)|^2 \, d\gamma_{k,n}(y) \\ &\quad \leq C_{k,n}(E_1,E_2)\Big\{ \int_{\mathbb{R}\setminus E_1} |f(t)|^2 \, d\gamma_{k,n}(y) + \int_{\mathbb{R}\setminus E_2} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi)\Big\}. \quad (4.1)\end{aligned}$$

In this section, our primary interest is to establish the Benedick–Amrein–Berthier uncertainty principle for the deformed Stockwell transforms by employing the inequality (4.1). In this direction, we have the following main theorem.

Theorem 4.1. *For any arbitrary function $f \in L^2_{k,n}(\mathbb{R})$, we have the following uncertainty inequality:*

$$\int_{\mathbb{R}}\int_{\mathbb{R}\setminus E_1}\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu)+C_h\int_{\mathbb{R}\setminus E_2}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi)$$
$$\geq \frac{C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}}{C_{k,n}(E_1,E_2)}, \quad (4.2)$$

where $C_{k,n}(E_1,E_2)$ is the constant given in relation (4.1)*.*

Proof. Since, for all $\nu \in \mathbb{R}$, $\mathcal{S}_h^{k,n}(f)(.,\nu) \in L^2_{k,n}(\mathbb{R})$ whenever $f \in L^2_{k,n}(\mathbb{R})$, we can replace the function f appearing in (4.1) with $\mathcal{S}_h^{k,n}(f)(.,\nu)$ to get

$$\int_{\mathbb{R}}\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\gamma_{k,n}(y)\leq C_{k,n}(E_1,E_2)\bigg\{\int_{\mathbb{R}\setminus E_1}\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\gamma_{k,n}(y)$$
$$+\int_{\mathbb{R}\setminus E_2}\left|\mathcal{F}_{k,n}\big[\mathcal{S}_h^{k,n}(f)(.,\nu)\big](\xi)\right|^2 d\gamma_{k,n}(\xi)\bigg\}.$$

By integrating this inequality with respect to the measure $d\gamma_{k,n}(\nu)$, we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu)$$
$$\leq C_{k,n}(E_1,E_2)\bigg\{\int_{\mathbb{R}}\int_{\mathbb{R}\setminus E_1}\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu)$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}\setminus E_2}\left|\mathcal{F}_{k,n}\big[\mathcal{S}_h^{k,n}(f)(.,\nu)\big](\xi)\right|^2 d\mu_{k,n}(\xi,\nu)\bigg\}.$$

Using Lemma 2.26 together with Plancherel's formula (2.35), the above inequality becomes

$$\int_{\mathbb{R}}\int_{\mathbb{R}\setminus E_1}\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu)$$
$$+\int_{\mathbb{R}\setminus E_2}\int_{\mathbb{R}}|\mathcal{F}_{k,n}(f)(\xi)|^2\,|\mathcal{F}_{k,n}(M_\nu\Delta_\nu h)(\xi)|^2\,d\mu_{k,n}(\xi,\nu)\geq \frac{C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}}{C_{k,n}(E_1,E_2)}.$$

Thus using the fact that h is deformed Stockwell on $\mathbb{R}$, we derive the inequality

$$\int_{\mathbb{R}}\int_{\mathbb{R}\setminus E_1}\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu)+C_h\int_{\mathbb{R}\setminus E_2}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi)$$
$$\geq \frac{C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}}{C_{k,n}(E_1,E_2)},$$

which is the desired Benedick–Amrein–Berthier's uncertainty principle for the deformed Stockwell transforms. □

Theorem 4.1 allows us to obtain another version of Heisenberg-type uncertainty inequality for the deformed Stockwell transforms.

Corollary 4.2. *Let $p,q>0$. Then there exists a positive constant $\mathcal{C}_{k,n}(p,q)$ such that, for any arbitrary function $f\in L^2_{k,n}(\mathbb{R})$, we have the following uncertainty inequality:*

$$\Big(\int_{\mathbb{R}^2}|y|^{2p}\big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2\,d\mu_{k,n}(y,\nu)\Big)^{\frac{q}{2}}\Big(\int_{\mathbb{R}}|\xi|^{2q}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi)\Big)^{\frac{p}{2}}$$
$$\geq \mathcal{C}_{k,n}(p,q)\big(C_h\big)^{\frac{q}{2}}\|f\|^{p+q}_{L^2_{k,n}(\mathbb{R})}.$$

Proof. Let $p,q>0$ and let $f\in L^2_{k,n}(\mathbb{R})$. Take $E_1=E_2=(-1,1)$. Then by (4.2)

$$\int_{\mathbb{R}\setminus(-1,1)}\int_{\mathbb{R}}\big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2\,d\mu_{k,n}(y,\nu)+C_h\int_{\mathbb{R}\setminus(-1,1)}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi)$$
$$\geq\frac{C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}}{C(k,n)}.$$

Here $C(k,n):=C_{k,n}(E_1,E_2)$. It follows that

$$\int_{\mathbb{R}^2}|y|^{2p}\big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2\,d\mu_{k,n}(y,\nu)+C_h\int_{\mathbb{R}}|\xi|^{2q}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi)$$
$$\geq\frac{C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}}{C(k,n)}.$$

Now replacing f by f_λ, we get by (2.33)

$$\int_{\mathbb{R}^2}|y|^{2p}\bigg|\mathcal{S}_h^{k,n}(f)\left(\frac{y}{\lambda},\lambda\nu\right)\bigg|^2\,d\mu_{k,n}(y,\nu)$$
$$+\lambda^{\frac{(2k-1)n+2}{n}}C_h\int_{\mathbb{R}}|\xi|^{2q}|\mathcal{F}_{k,n}(f)(\lambda\xi)|^2\,d\gamma_{k,n}(\xi)\geq\frac{C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}}{C(k,n)}.$$

Thus

$$\lambda^{2p}\int_{\mathbb{R}^2}|y|^{2p}\big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2\,d\mu_{k,n}(y,\nu)$$
$$+\lambda^{-2q}C_h\int_{\mathbb{R}}|\xi|^{2q}|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi)\geq\frac{C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}}{C(k,n)}.$$

The desired result follows by minimizing the right hand side over $\lambda>0$. ☐

4.2. **Local-type uncertainty principles.** We begin this subsection by recalling the local uncertainty principle for the generalized transforms.

Proposition 4.3 [19]). *Let $E\subset\mathbb{R}$ be such that $0<\gamma_{k,n}(E):=\displaystyle\int_E d\gamma_{k,n}(x)<\infty$. For $0<p<\frac{(2k-1)n+2}{2n}$, there exists a positive constant $\mathfrak{C}(k,n,p)$ such that, for any $f\in L^2_{k,n}(\mathbb{R})$,*

$$\int_E\big|\mathcal{F}_{k,n}(f)(\xi)\big|^2\,d\gamma_{k,n}(\xi)\leq\mathfrak{C}(k,n,p)\big(\gamma_{k,n}(E)\big)^{\frac{2np}{(2k-1)n+2}}\|\,|x|^pf\|^2_{L^2_{k,n}(\mathbb{R})}.\qquad(4.3)$$

The first main objective of this subsection is to establish the local uncertainty principles for the deformed Stockwell transforms by employing the previous inequality.

Theorem 4.4. *Let $E \subset \mathbb{R}$ be as above with finite measure and $0 < s < \frac{(2k-1)n+2}{2n}$. Then, for any $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}^2} |y|^{2s} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu) \geq \frac{C_h}{\mathfrak{C}(k,n,s)\big(\gamma_{k,n}(E)\big)^{\frac{2ns}{(2k-1)n+2}}} \int_E |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi),$$

where $\mathfrak{C}(k,n,s)$ is the constant given in the relation (4.3).

Proof. As, for all $\nu \in \mathbb{R}$, we have $\mathcal{S}_h^{k,n}(f)(.,\nu) \in L^2_{k,n}(\mathbb{R})$ whenever $f \in L^2_{k,n}(\mathbb{R})$, we can replace the function f appearing in (4.3) with $\mathcal{S}_h^{k,n}(f)(.,\nu)$ to get

$$\int_E \big|\mathcal{F}_{k,n}\big[\mathcal{S}_h^{k,n}(f)(.,\nu)\big](\xi)\big|^2 \, d\gamma_{k,n}(\xi) \leq \mathfrak{C}(k,n,s)\big(\gamma_{k,n}(E)\big)^{\frac{2ns}{(2k-1)n+2}} \Big\| |y|^s \mathcal{S}_h^{k,n}(f)(.,\nu)\Big\|^2_{L^2_{k,n}(\mathbb{R})}. \quad (4.4)$$

For an explicit expression of (4.4), we shall integrate this inequality with respect to the measure $d\gamma_{k,n}(\nu)$ to get

$$\int_{\mathbb{R}} \int_E \big|\mathcal{F}_{k,n}\big[\mathcal{S}_h^{k,n}(f)(.,\nu)\big](\xi)\big|^2 \, d\mu_{k,n}(\xi,\nu) \leq \mathfrak{C}(k,n,s)\big(\gamma_{k,n}(E)\big)^{\frac{2ns}{(2k-1)n+2}} \times \int_{\mathbb{R}^2} |y|^{2s} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu),$$

which together with Lemma 2.26 and Fubini's theorem gives

$$\int_E |\mathcal{F}_{k,n}(f)(\xi)|^2 \bigg(\int_{\mathbb{R}} |\mathcal{F}_{k,n}(M_\nu \Delta_\nu h)(\xi)|^2 \, d\gamma_{k,n}(\nu)\bigg) d\gamma_{k,n}(\xi) \leq \mathfrak{C}(k,n,s)\big(\gamma_{k,n}(E)\big)^{\frac{2ns}{(2k-1)n+2}} \int_{\mathbb{R}^2} |y|^{2s} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu). \quad (4.5)$$

Using the hypothesis on h, inequality (4.5) reduces to

$$C_h \int_E \big|\mathcal{F}_{k,n}(f)(\xi)\big|^2 \, d\gamma_{k,n}(\xi) \leq \mathfrak{C}(k,n,s)\big(\gamma_{k,n}(E)\big)^{\frac{2ns}{(2k-1)n+2}} \int_{\mathbb{R}^2} |y|^{2s} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu).$$

Or equivalently for $0 < s < \frac{(2k-1)n+2}{2n}$,

$$\int_{\mathbb{R}^2} |y|^{2s} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu) \geq \frac{C_h}{\mathfrak{C}(k,n,s)\big(\gamma_{k,n}(E)\big)^{\frac{2ns}{(2k-1)n+2}}} \int_E |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi).$$

This completes the proof of Theorem 4.4. □

Let E be a subset of $\mathbb{R}$. We define the *Paley–Wiener space* $PW_{k,n}(E)$ as follows:

$$PW_{k,n}(E) := \Big\{ f \in L^2_{k,n}(\mathbb{R}) : \operatorname{supp} \mathcal{F}_{k,n}(f) \subset E \Big\}.$$

From Plancherel's formula (2.3), the definition of the Paley–Wiener space $PW_{k,n}(E)$ and the previous theorem we obtain the following:

Corollary 4.5. *Let $E \subset \mathbb{R}$ be such that $0 < \gamma_{k,n}(E) < \infty$. Let $0 < s < \frac{(2k-1)n+2}{2n}$. For any $f \in PW_{k,n}(E)$, we have*

$$\|f\|^2_{L^2_{k,n}(\mathbb{R})} \leq \frac{\mathfrak{C}(k,n,s)(\gamma_{k,n}(E))^{\frac{2ns}{(2k-1)n+2}}}{C_h} \int_{\mathbb{R}^2} |y|^{2s} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu),$$

where $\mathfrak{C}(k,n,s)$ is the constant given in Proposition 4.3.

By interchanging the roles of f and $\mathcal{F}_{k,n}(f)$ in Proposition 4.3, we get the following:

Corollary 4.6. *Let $F \subset \mathbb{R}$ be such that $0 < \gamma_{k,n}(F) < \infty$. For $0 < t < \frac{(2k-1)n+2}{2n}$ and for any $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\int_F |f(y)|^2 \, d\gamma_{k,n}(y) \leq \mathfrak{C}(k,n,t)\big(\gamma_{k,n}(F)\big)^{\frac{2nt}{(2k-1)n+2}} \| \, |\xi|^t \mathcal{F}_{k,n}(f)\|^2_{L^2_{k,n}(\mathbb{R})},$$

where $\mathfrak{C}(k,n,t)$ is the constant given in Proposition 4.3.

Using Corollary 4.6 and similar ideas to those given in the proof of Theorem 4.4, we prove the following.

Corollary 4.7. *Let $F \subset \mathbb{R}$ be such that $0 < \gamma_{k,n}(F) < \infty$. Let $0 < t < \frac{(2k-1)n+2}{2n}$. For any $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} \int_F \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu) \leq C_h \mathfrak{C}(k,n,t)(\gamma_{k,n}(F))^{\frac{2nt}{(2k-1)n+2}} \int_{\mathbb{R}} |\xi|^{2t} \big|\mathcal{F}_{k,n}(f)(\xi)\big|^2 \, d\gamma_{k,n}(\xi),$$

where $\mathfrak{C}(k,n,t)$ is the constant given in Proposition 4.3.

Let $F \subset \mathbb{R}$. We define the *generalized Paley–Wiener space* $GPW_{k,n}(F)$ as follows:

$$GPW_{k,n}(F) := \Big\{ f \in L^2_{k,n}(\mathbb{R}) : \forall \, \nu \in \mathbb{R}, \operatorname{supp} \mathcal{S}_h^{k,n}(f)(.,\nu) \subset F \Big\}.$$

Applying Plancherel's formula (2.35), the definition of the generalized Paley–Wiener space $GPW_{k,n}(F)$ and the previous corollary we obtain the following:

Corollary 4.8. *Let E and F be two subsets of $\mathbb{R}$ such that $0 < \gamma_{k,n}(E)$, $\gamma_{k,n}(F) < \infty$. Let $0 < s, t < \frac{(2k-1)n+2}{2n}$.*

(i) *For any $f \in GPW_{k,n}(F)$, we have*

$$\|f\|^2_{L^2_{k,n}(\mathbb{R})} \leq \mathfrak{C}(k,n,t)(\gamma_{k,n}(F))^{\frac{2nt}{(2k-1)n+2}} \int_{\mathbb{R}} |\xi|^{2t} \big|\mathcal{F}_{k,n}(f)(\xi)\big|^2 \, d\gamma_{k,n}(\xi).$$

(ii) *For any $f \in PW_{k,n}(E) \bigcap GPW_{k,n}(F)$, we have*

$$\begin{aligned}\|f\|^{s+t}_{L^2_{k,n}(\mathbb{R})} \leq &\big(\mathfrak{C}(k,n,t)\big)^{\frac{s}{2}} \big(\mathfrak{C}(k,n,s)\big)^{\frac{t}{2}} (\gamma_{k,n}(E)\gamma_{k,n}(F))^{\frac{nts}{(2k-1)n+2}} \\ &\times \left(\int_{\mathbb{R}} |\xi|^{2t} \big|\mathcal{F}_{k,n}(f)(\xi)\big|^2 \, d\gamma_{k,n}(\xi) \right)^{\frac{s}{2}} \\ &\times \left(\int_{\mathbb{R}^2} |y|^{2s} \big|\mathcal{S}^{k,n}_h(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu) \right)^{\frac{t}{2}}.\end{aligned}$$

We finish this subsection by establishing another version of Heisenberg-type uncertainty inequality for the deformed Stockwell transforms.

Theorem 4.9. *Let $0 < p < \frac{(2k-1)n+2}{2n}$ and $q > 0$. Then for any $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\|f\|^2_{L^2_{k,n}(\mathbb{R})} \leq \mathcal{C}(k,n,p,q) \left\| \, |y|^p \, \mathcal{S}^{k,n}_h(f) \right\|^{\frac{2q}{p+q}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \left\| \, |\xi|^q \mathcal{F}_{k,n}(f) \right\|^{\frac{2p}{p+q}}_{L^2_{k,n}(\mathbb{R})},$$

where

$$\mathcal{C}(k,n,p,q) = \left(\frac{\mathfrak{C}(k,n,p)}{\left(\frac{(2k-1)n+2}{2n}\right)^{\frac{2np}{(2k-1)n+2}} C_h} \right)^{\frac{q}{p+q}} \left[\left(\frac{p}{q}\right)^{\frac{q}{p+q}} + \left(\frac{q}{p}\right)^{\frac{p}{p+q}} \right],$$

with $\mathfrak{C}(k,n,p)$ the constant given in Proposition 4.3.

Proof. Let $0 < p < \frac{(2k-1)n+2}{2n}$, $q > 0$ and $r > 0$. Then

$$\begin{aligned}\|f\|^2_{L^2_{k,n}(\mathbb{R})} &= \|\mathcal{F}_{k,n}(f)\|^2_{L^2_{k,n}(\mathbb{R})} \\ &= \int_{-r}^{r} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi) + \int_{\mathbb{R}\setminus(-r,r)} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi).\end{aligned} \tag{4.6}$$

From Theorem 4.4 and by a simple calculation, we have

$$\begin{aligned}\int_{-r}^{r} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi) \leq &\frac{\mathfrak{C}(k,n,p)}{\left(\frac{(2k-1)n+2}{2n}\right)^{\frac{2np}{(2k-1)n+2}} C_h} \\ &\times r^{2p} \int_{\mathbb{R}^2} |y|^{2p} \big|\mathcal{S}^{k,n}_h(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu).\end{aligned} \tag{4.7}$$

Moreover, it is easy to see that

$$\int_{\mathbb{R}\setminus(-r,r)} |\mathcal{F}_{k,n}(f)(\xi)|^2\, d\gamma_{k,n}(\xi) \le r^{-2q} \int_{\mathbb{R}} |\xi|^{2q} |\mathcal{F}_{k,n}(f)(\xi)|^2\, d\gamma_{k,n}(\xi). \tag{4.8}$$

Combining the relations (4.6), (4.7) and (4.8), we get

$$\begin{aligned}\|f\|^2_{L^2_{k,n}(\mathbb{R})} \le{}& \frac{\mathfrak{C}(k,n,p)}{\left(\frac{(2k-1)n+2}{2n}\right)^{\frac{2np}{(2k-1)n+2}} C_h} r^{2p} \left\| |y|^p \mathcal{S}^{k,n}_h(f)\right\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &+ r^{-2q} \left\| |\xi|^q \mathcal{F}_{k,n}(f)\right\|^2_{L^2_{k,n}(\mathbb{R})}.\end{aligned}$$

We choose

$$r = \left[\frac{q\left(\frac{(2k-1)n+2}{2n}\right)^{\frac{2np}{(2k-1)n+2}} C_h}{p\mathfrak{C}(k,n,p)}\right]^{\frac{1}{2p+2q}} \left(\frac{\left\| |\xi|^q \mathcal{F}_{k,n}(f)\right\|_{L^2_{k,n}(\mathbb{R})}}{\left\| |y|^p \mathcal{S}^{k,n}_h(f)\right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}}\right)^{\frac{1}{p+q}},$$

and obtain the desired inequality. □

Building on the ideas of Faris [16] and Price [37, 38, 39] for the Fourier transform, we show another local uncertainty principle for the deformed Stockwell transform. More precisely, we will prove the following result.

Theorem 4.10 (Faris–Price's uncertainty principle for $\mathcal{S}^{k,n}_h$)**.** *Let η, p be two real numbers such that $0 < \eta < \frac{(2k-1)n+2}{n}$ and $p \ge 1$. Then, there is a positive constant $C(k,n,\eta,p)$ such that, for every function $f \in L^2_{k,n}(\mathbb{R})$, and for every measurable subset $T \subset \mathbb{R}^2$ such that*

$$0 < \mu_{k,n}(T) := \int_T d\mu_{k,n}(y,\nu) < \infty,$$

we have

$$\begin{aligned}&\left(\int_T |\mathcal{S}^{k,n}_h(f)(y,\nu)|^p\, d\mu_{k,n}(y,\nu)\right)^{\frac{1}{p}} \\ &\quad \le C(k,n,\eta,p)\big(\mu_{k,n}(T)\big)^{\frac{1}{p(p+1)}} \left\| \|(y,\nu)\|^\eta \mathcal{S}^{k,n}_h(f)\right\|^{\frac{4nk}{\left(\frac{(2k+\eta-1)n+2}{n}\right)(p+1)}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\qquad \times \left(\|f\|_{L^2_{k,n}(\mathbb{R})} \|h\|_{L^2_{k,n}(\mathbb{R})}\right)^{\frac{((2k+\eta-1)n+2)(p+1)-4nk}{((2k+\eta-1)n+2)(p+1)}}.\end{aligned}$$

Proof. We can assume that $\|f\|_{L^2_{k,n}(\mathbb{R})} \|h\|_{L^2_{k,n}(\mathbb{R})} = 1$. Then, for every positive real number $s > 1$, we have

$$\|\mathcal{S}^{k,n}_h(f)\|_{L^p_{\mu_{k,n}}(T)} \le \|\mathbb{1}_{B(0,s)} \mathcal{S}^{k,n}_h(f)\|_{L^p_{\mu_{k,n}}(T)} + \|\mathbb{1}_{B^c(0,s)} \mathcal{S}^{k,n}_h(f)\|_{L^p_{\mu_{k,n}}(T)}.$$

However, by Hölder's inequality and (2.32), for every $\eta \in \big(0, \frac{2(2k-1)n+2}{n}\big)$ we get

$$\begin{aligned}\left\|\mathbb{1}_{B(0,s)}\mathcal{S}_h^{k,n}(f)\right\|_{L^p_{\mu_{k,n}}(T)} &= \left(\int_{\mathbb{R}^2} |\mathcal{S}_h^{k,n}(f)(y,\nu)|^p \mathbb{1}_{B(0,s)}(y,\nu)\mathbb{1}_T(y,\nu)\, d\mu_{k,n}(y,\nu)\right)^{\frac{1}{p}} \\ &\leq \big(\mu_{k,n}(T)\big)^{\frac{1}{p(p+1)}} \left\|\mathcal{S}_h^{k,n}(f)\mathbb{1}_{B(0,s)}\right\|^{\frac{1}{p+1}}_{L^1_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\leq \big(\mu_{k,n}(T)\big)^{\frac{1}{p(p+1)}} \left\|\|(y,\nu)\|^\eta \mathcal{S}_h^{k,n}(f)\right\|^{\frac{1}{p+1}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\qquad \times \left\|\|(y,\nu)\|^{-\eta}\mathbb{1}_{B(0,s)}\right\|^{\frac{1}{p+1}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.\end{aligned}$$

On the other hand, by a simple calculation we can see that

$$\left\|\|(y,\nu)\|^{-\eta}\mathbb{1}_{B(0,s)}\right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leq \left(\frac{\Gamma\big(\frac{(2k-1)n+2}{2n}\big)}{\sqrt{\big(\frac{(2k-1)n+2}{n}-\eta\big)\Gamma\big(\frac{(2k-1)n+2}{n}\big)}}\right) s^{\frac{(2k-1)n+2}{n}-\eta}.$$

Thus we get

$$\begin{aligned}\left\|\mathbb{1}_{B(0,s)}\mathcal{S}_h^{k,n}(f)\right\|_{L^p_{\mu_{k,n}}(T)} &\leq \big(\mu_{k,n}(T)\big)^{\frac{1}{p(p+1)}} \left(\frac{\Gamma\big(\frac{(2k-1)n+2}{2n}\big)}{\sqrt{\big(\frac{(2k-1)n+2}{n}-\eta\big)\Gamma\big(\frac{(2k-1)n+2}{n}\big)}}\right)^{\frac{1}{p+1}} \\ &\qquad \times s^{\frac{\frac{(2k-1)n+2}{n}-\eta}{p+1}} \left\|\|(y,\nu)\|^\eta \mathcal{S}_h^{k,n}(f)\right\|^{\frac{1}{p+1}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.\end{aligned}$$

On the other hand, using again Hölder's inequality and (2.32), we deduce that

$$\begin{aligned}&\left\|\mathbb{1}_{B^c(0,s)}\mathcal{S}_h^{k,n}(f)\right\|_{L^p_{\mu_{k,n}}(T)} \\ &\quad\leq \big(\mu_{k,n}(T)\big)^{\frac{1}{p(p+1)}} \left(\int_{\mathbb{R}^2} |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2 \mathbb{1}_{B^c(0,s)}(y,\nu)\, d\mu_{k,n}(y,\nu)\right)^{\frac{1}{p+1}} \\ &\quad\leq \big(\mu_{k,n}(T)\big)^{\frac{1}{p(p+1)}} s^{-\frac{2\eta}{p+1}} \left\|\|(y,\nu)\|^\eta \mathcal{S}_h^{k,n}(f)\right\|^{\frac{2}{p+1}}_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.\end{aligned}$$

Hence, for every $\eta \in \left(0, \frac{2(2k-1)n+2}{n}\right)$, we have

$$\left(\int_T |\mathcal{S}_h^{k,n}(f)(y,\nu)|^p \, d\mu_{k,n}\right)^{\frac{1}{p}} \leq \left(\mu_{k,n}(T)\right)^{\frac{1}{p(p+1)}} \left\| \|(y,\nu)\|^\eta \mathcal{S}_h^{k,n}(f) \right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}^{\frac{1}{p+1}}$$
$$\times \left(\left(\frac{\Gamma\left(\frac{(2k-1)n+2}{2n}\right)}{\sqrt{\left(\frac{(2k-1)n+2}{n} - \eta\right)\Gamma\left(\frac{(2k-1)n+2}{n}\right)}} \right)^{\frac{1}{p+1}} s^{\frac{\frac{(2k-1)n+2}{n} - \eta}{p+1}} + \left\| \|(y,\nu)\|^\eta \, \mathcal{S}_h^{k,n}(f) \right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}^{\frac{1}{p+1}} s^{-\frac{2\eta}{p+1}} \right).$$

In particular, the inequality holds for

$$s_0 = \left(\frac{\Gamma\left(\frac{(2k-1)n+2}{2n}\right)}{\sqrt{\left(\frac{(2k-1)n+2}{n} - \eta\right)\Gamma\left(\frac{(2k-1)n+2}{n}\right)}} \right)^{\frac{-n}{(2k+\eta-1)n+2}} \times \left(\frac{2n\eta}{(2k-\eta-1)n+2}\right)^{\frac{n(p+1)}{(2k+\eta-1)n+2}} \left\| \|(y,\nu)\|^\eta \mathcal{S}_h^{k,n}(f) \right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}^{\frac{n}{(2k+\eta-1)n+2}},$$

and therefore

$$\left(\int_T |\mathcal{S}_h^{k,n}(f)(y,\nu)|^p \, d\mu_{k,n}\right)^{\frac{1}{p}} \leq \left(\frac{\Gamma(\frac{(2k-1)n+2}{2n})}{\sqrt{(\frac{(2k-1)n+2}{n} - \eta)\Gamma(\frac{(2k-1)n+2}{n})}} \right)^{\frac{2n\eta}{((2k+\eta-1)n+2)(p+1)}}$$
$$\times \left(\frac{2n\eta}{(2k-\eta-1)n+2}\right)^{\frac{-2n\eta}{(2k+\eta-1)n+2}} \left(\frac{(2k+\eta-1)n+2}{(2k-\eta-1)n+2}\right)$$
$$\times \left(\mu_{k,n}(T)\right)^{\frac{1}{p(p+1)}} \left\| \|(y,\nu)\|^\eta \mathcal{S}_h^{k,n}(f) \right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}^{\frac{4nk}{((2k+\eta-1)n+2)(p+1)}}.$$

□

5. Weighted inequalities for the deformed Stockwell transform

The Pitt inequality in the generalized setting expresses a fundamental relationship between a sufficiently smooth function and the corresponding generalized Fourier transform. This subject was studied by Gorbachev et al. in [20], where the authors gave the sharp Pitt's inequality and logarithmic uncertainty principle for the generalized Fourier transform $\mathcal{F}_{k,n}$ on $\mathbb{R}$. More precisely, they proved that, for every $f \in \mathcal{S}(\mathbb{R}) \subseteq L^2_{k,n}(\mathbb{R})$,

$$\int_{\mathbb{R}} |\xi|^{-2\lambda} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi) \\ \leq C_{k,n}(\lambda) \int_{\mathbb{R}} |x|^{2\lambda} |f(x)|^2 \, d\gamma_{k,n}(x), \quad 0 \leq \lambda < \tfrac{(2k-1)n+2}{2n}, \tag{5.1}$$

where

$$C_{k,n}(\lambda) := \left(\frac{n}{2}\right)^{2n\lambda} \left[\frac{\Gamma\left(\frac{(2k-1-2\lambda)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1+2\lambda)n+2}{4}\right)} \right]^2 . \tag{5.2}$$

The first main objective of this section is to formulate an analogue of Pitt's inequality (5.1) for the deformed Stockwell transform.

Theorem 5.1. *For $0 \leq \lambda < \frac{(2k-1)n+2}{2n}$ and for any arbitrary $f \in S(\mathbb{R}) \subseteq L^2_{k,n}(\mathbb{R})$, the Pitt inequality for the deformed Stockwell transform is given by*

$$C_h \int_{\mathbb{R}} |\xi|^{-2\lambda} \left|\mathcal{F}_{k,n}(f)(\xi)\right|^2 d\gamma_{k,n}(\xi) \leq C_{k,n}(\lambda) \int_{\mathbb{R}^2} |y|^{2\lambda} \left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu), \tag{5.3}$$

where $C_{k,n}(\lambda)$ is given by (5.2).

Proof. As a consequence of the inequality (5.1), we can write

$$\forall\, \nu \in \mathbb{R}, \quad \int_{\mathbb{R}} |\xi|^{-2\lambda} \left|\mathcal{F}_{k,n}[\mathcal{S}_h^{k,n}(f)(.,\nu)](\xi)\right|^2 d\gamma_{k,n}(\xi) \\ \leq C_{k,n}(\lambda) \int_{\mathbb{R}} |y|^{2\lambda} \left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\gamma_{k,n}(y),$$

which upon integration with respect to the Haar measure $d\gamma_{k,n}(\nu)$ yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\xi|^{-2\lambda} \left|\mathcal{F}_{k,n}[\mathcal{S}_h^{k,n}(f)(.,\nu)](\xi)\right|^2 d\mu_{k,n}(\xi,\nu) \\ \leq C_{k,n}(\lambda) \int_{\mathbb{R}^2} |y|^{2\lambda} \left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu). \tag{5.4}$$

Invoking Lemma 2.26, we can express the inequality (5.4) in the following manner:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\xi|^{-2\lambda} |\mathcal{F}_{k,n}(f)(\xi)|^2 |\mathcal{F}_{k,n}(M_\nu \Delta_\nu h)(\xi)|^2 \, d\mu_{k,n}(\xi,\nu) \\ \leq C_{k,n}(\lambda) \int_{\mathbb{R}^2} |y|^{2\lambda} \left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu).$$

Equivalently, we have

$$\int_{\mathbb{R}} |\xi|^{-2\lambda} \big|\mathcal{F}_{k,n}(f)(\xi)\big|^2 \Big\{ \int_{\mathbb{R}} |\mathcal{F}_{k,n}(M_\nu \Delta_\nu h)(\xi)|^2 \, d\gamma_{k,n}(\nu) \Big\} \, d\gamma_{k,n}(\xi) \\ \leq C_{k,n}(\lambda) \int_{\mathbb{R}^2} |y|^{2\lambda} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu).$$

Using the hypothesis on h, we obtain

$$C_h \int_{\mathbb{R}} |\xi|^{-2\lambda} \big|\mathcal{F}_{k,n}(f)(\xi)\big|^2 \, d\gamma_{k,n}(\xi) \leq C_{k,n}(\lambda) \int_{\mathbb{R}^2} |y|^{2\lambda} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu),$$

which establishes the Pitt inequality for the deformed Stockwell transform. □

Remark 5.2. For $\lambda = 0$, equality holds in (5.3), which is in consonance with Plancherel's formula (2.35).

Theorem 5.3. *For any function $f \in S(\mathbb{R})$, the following inequality holds:*

$$\int_{\mathbb{R}^2} \log |y| \, |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2 \, d\mu_{k,n}(y,\nu) + C_h \int_{\mathbb{R}} \log |\xi| \, |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi) \\ \geq n \left[\frac{\Gamma'\left(\frac{(2k-1)n+2}{2}\right)}{\Gamma\left(\frac{(2k-1)n+2}{2}\right)} - \ln\left(\frac{n}{2}\right) \right] C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})}. \tag{5.5}$$

Proof. For every $0 \leq \lambda < \frac{(2k-1)n+2}{2n}$, we define

$$P(\lambda) = C_h \int_{\mathbb{R}} |\xi|^{-2\lambda} \big|\mathcal{F}_{k,n}(f)(\xi)\big|^2 \, d\gamma_{k,n}(\xi) \\ - C_{k,n}(\lambda) \int_{\mathbb{R}^2} |y|^{2\lambda} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu). \tag{5.6}$$

On differentiating (5.6) with respect to λ, we obtain

$$P'(\lambda) = -2C_h \int_{\mathbb{R}} |\xi|^{-2\lambda} \log|\xi| \big|\mathcal{F}_{k,n}(f)(\xi)\big|^2 \, d\gamma_{k,n}(\xi) \\ - 2C_{k,n}(\lambda) \int_{\mathbb{R}^2} |y|^{2\lambda} \log|y| \, |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2 \, d\mu_{k,n}(y,\nu) \\ - C'_{k,n}(\lambda) \int_{\mathbb{R}^2} |y|^{2\lambda} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2 \, d\mu_{k,n}(y,\nu),$$

where

$$C'_{k,n}(\lambda) = -nC_{k,n}(\lambda) \left(\frac{\Gamma'\left(\frac{(2k-1-2\lambda)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1-2\lambda)n+2}{4}\right)} + \frac{\Gamma'\left(\frac{(2k-1+2\lambda)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1+2\lambda)n+2}{4}\right)} - 2\ln\left(\frac{n}{2}\right) \right). \tag{5.7}$$

For $\lambda = 0$, equation (5.7) yields

$$C'_{k,n}(0) = -2n \left(\frac{\Gamma'\left(\frac{(2k-1)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1)n+2}{4}\right)} - \ln\left(\frac{n}{2}\right) \right). \tag{5.8}$$

By virtue of the deformed Stockwell Pitt inequality (5.3), it follows that

$$P(\lambda) \leq 0 \quad \text{for all } \lambda \in \left[0, \tfrac{(2k-1)n+2}{2n}\right)$$

and

$$\begin{aligned} P(0) &= C_h \int_{\mathbb{R}} \left|\mathcal{F}_{k,n}(f)(\xi)\right|^2 d\gamma_{k,n}(\xi) - C_{k,n}(0) \int_{\mathbb{R}^2} \left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu) \\ &= C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})} - C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})} = 0. \end{aligned}$$

Therefore,

$$P'(0^+) := \lim_{\lambda \to 0^+} \frac{P(\lambda)}{\lambda} = \lim_{\lambda \to 0^+} P'(\lambda) \leq 0,$$

which is equivalent to

$$\begin{aligned} &-2C_h \int_{\mathbb{R}} \log|\xi| \left|\mathcal{F}_{k,n}(f)(\xi)\right|^2 d\gamma_{k,n}(\xi) \\ &\qquad - 2C_{k,n}(0) \int_{\mathbb{R}^2} \log|y| \, |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2 \, d\mu_{k,n}(y,\nu) \\ &\qquad\qquad - C'_{k,n}(0) \int_{\mathbb{R}^2} \left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu) \leq 0. \end{aligned}$$

Applying Plancherel's formula (2.35) and the obtained estimate (5.8) of $C'_{k,n}(0)$, we get

$$\begin{aligned} &-2C_h \int_{\mathbb{R}} \log|\xi| \left|\mathcal{F}_{k,n}(f)(\xi)\right|^2 d\gamma_{k,n}(\xi) - 2 \int_{\mathbb{R}^2} \log|y| \left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu) \\ &\qquad + 2n \left(\frac{\Gamma'\left(\frac{(2k-1)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1)n+2}{4}\right)} - \ln\left(\frac{n}{2}\right) \right) C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})} \leq 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} &\int_{\mathbb{R}^2} \log|y| \left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu) + C_h \int_{\mathbb{R}} \log|\xi| \, |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi) \\ &\qquad \geq n \left[\frac{\Gamma'\left(\frac{(2k-1)n+2}{2}\right)}{\Gamma\left(\frac{(2k-1)n+2}{2}\right)} - \ln\left(\frac{n}{2}\right) \right] C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})}. \end{aligned}$$

This completes the proof of Theorem 5.3. $\square$

The generalized Beckner's inequality [20] is given by

$$\int_{\mathbb{R}} \log|y|\,|f(t)|^2\,d\gamma_{k,n}(y) + \int_{\mathbb{R}} \log|\xi|\,|\mathcal{F}_{k,n}(f)(\xi)|^2\,d\gamma_{k,n}(\xi) \\ \geq n\left[\frac{\Gamma'\left(\frac{(2k-1)n+2}{2}\right)}{\Gamma\left(\frac{(2k-1)n+2}{2}\right)} - \ln\left(\frac{n}{2}\right)\right]\int_{\mathbb{R}} |f(t)|^2\,d\gamma_{k,n}(y) \quad (5.9)$$

for all $f \in \mathcal{S}(\mathbb{R})$. This inequality is related to Heisenberg's uncertainty principle and for that reason it is often referred as the logarithmic uncertainty principle. Considerable attention has been paid to this inequality for its various generalizations, improvements, analogues, and their applications [23].

We now present an alternate proof of Theorem 5.3. The strategy of the proof differs from the one given in the previous section and is obtained directly from the generalized Beckner's inequality (5.9).

Proof of Theorem 5.3. Let $\nu \in \mathbb{R}$. We replace f in (5.9) with $\mathcal{S}_h^{k,n}(f)(.,\nu)$, so that

$$\int_{\mathbb{R}} \log|y|\,|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\gamma_{k,n}(y) + \int_{\mathbb{R}} \log|\xi|\big|\mathcal{F}_{k,n}[\mathcal{S}_h^{k,n}(f)(.,\nu)](\xi)\big|^2\,d\gamma_{k,n}(\xi) \\ \geq n\left(\frac{\Gamma'\left(\frac{(2k-1)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1)n+2}{4}\right)} - \ln\left(\frac{n}{2}\right)\right)\int_{\mathbb{R}} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2\,d\gamma_{k,n}(y). \quad (5.10)$$

Integrating (5.10) with respect to the measure $d\gamma_{k,n}(\nu)$, we obtain

$$\int_{\mathbb{R}^2} \log|y|\,|\mathcal{S}_h^{k,n}(f)(y,\nu)|^2\,d\mu_{k,n}(y,\nu) \\ + \int_{\mathbb{R}}\int_{\mathbb{R}} \log|\xi|\big|\mathcal{F}_{k,n}[\mathcal{S}_h^{k,n}(f)(.,\nu)](\xi)\big|^2\,d\mu_{k,n}(\xi,\nu) \\ \geq n\left(\frac{\Gamma'\left(\frac{(2k-1)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1)n+2}{4}\right)} - \ln\left(\frac{n}{2}\right)\right)\int_{\mathbb{R}^2} \big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2\,d\mu_{k,n}(y,\nu).$$

Using Plancherel's formula (2.35), we get

$$\int_{\mathbb{R}^2} \log|y|\,\big|\mathcal{S}_h^{k,n}(f)(y,\nu)\big|^2\,d\mu_{k,n}(y,\nu) \\ + \int_{\mathbb{R}^2} \log|\xi|\,\big|\mathcal{F}_{k,n}\big[\mathcal{S}_h^{k,n}(f)(.,\nu)\big](\xi)\big|^2\,d\mu_{k,n}(y,\nu) \\ \geq n\left(\frac{\Gamma'\left(\frac{(2k-1)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1)n+2}{4}\right)} - \ln\left(\frac{n}{2}\right)\right) C_h\|f\|^2_{L^2_{k,n}(\mathbb{R})}. \quad (5.11)$$

We shall now simplify the second integral of (5.11). By using Lemma 2.26 we infer that

$$\int_{\mathbb{R}^2} \log|\xi| \left|\mathcal{F}_{k,n}\left[\mathcal{S}_h^{k,n}(f)(.,\nu)\right](\xi)\right|^2 d\mu_{k,n}(\xi,\nu) = C_h \int_{\mathbb{R}} \log|\xi| \, |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi). \tag{5.12}$$

Plugging the estimate (5.12) in (5.11) gives the desired inequality for the deformed Stockwell transforms as

$$\begin{aligned}\int_{\mathbb{R}^2} \log|y| \, |\mathcal{S}_h^{k,n}(f)(y,\nu)|^2 \, d\mu_{k,n}(y,\nu) + C_h \int_{\mathbb{R}} \log|\xi| \, |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi) \\ \geq n \left(\frac{\Gamma'\left(\frac{(2k-1)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1)n+2}{4}\right)} - \ln\left(\frac{n}{2}\right) \right) C_h \|f\|^2_{L^2_{k,n}(\mathbb{R})}.\end{aligned}$$

The previous inequality is the desired Beckner's uncertainty principle for the deformed Stockwell transform. □

Corollary 5.4. *Let h be a deformed Stockwell wavelet on $\mathbb{R}$ in $L^2_{k,n}(\mathbb{R})$ such that $C_h = 1$. For any function $f \in S(\mathbb{R})$, the following inequality holds:*

$$\begin{aligned}\left\{ \int_{\mathbb{R}^2} |y|^2 \left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2 d\mu_{k,n}(y,\nu) \right\}^{1/2} \left\{ \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\gamma_{k,n}(\xi) \right\}^{1/2} \\ \geq \exp\left(n \left[\frac{\Gamma'\left(\frac{(2k-1)n+2}{2}\right)}{\Gamma\left(\frac{(2k-1)n+2}{2}\right)} - \ln\left(\frac{n}{2}\right) \right] \right) \|f\|^2_{L^2_{k,n}(\mathbb{R})}.\end{aligned}$$

Proof. Using Jensen's inequality in (5.5) and the fact that $C_h = 1$, we obtain

$$\begin{aligned}&\log\left\{ \int_{\mathbb{R}^2} |y|^2 \frac{\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}} d\mu_{k,n}(y,\nu) \int_{\mathbb{R}} |\xi|^2 \frac{|\mathcal{F}_{k,n}(f)(\xi)|^2}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}} d\gamma_{k,n}(\xi) \right\}^{1/2} \\ &= \log\left\{ \int_{\mathbb{R}^2} |y|^2 \frac{\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}} d\mu_{k,n}(y,\nu) \right\}^{1/2} \\ &\quad + \log\left\{ \int_{\mathbb{R}} |\xi|^2 \frac{|\mathcal{F}_{k,n}(f)(\xi)|^2}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}} d\gamma_{k,n}(\xi) \right\}^{1/2} \\ &\geq \int_{\mathbb{R}^2} \log|y| \frac{\left|\mathcal{S}_h^{k,n}(f)(y,\nu)\right|^2}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}} d\mu_{k,n}(y,\nu) + \int_{\mathbb{R}} \log|\xi| \frac{|\mathcal{F}_{k,n}(f)(\xi)|^2}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}} d\gamma_{k,n}(\xi) \\ &\geq n \left[\frac{\Gamma'\left(\frac{(2k-1)n+2}{2}\right)}{\Gamma\left(\frac{(2k-1)n+2}{2}\right)} - \ln\left(\frac{n}{2}\right) \right],\end{aligned}$$

which upon simplification yields the result. □

Remark 5.5. (i) Using the approximation identity

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - 2\int_0^{\infty} \frac{t}{(t^2+z^2)(e^{2\pi t}-1)}\, dt \tag{5.13}$$

we infer that

$$\exp\left[n\left(\frac{\Gamma'\left(\frac{(2k-1)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1)n+2}{4}\right)} - \ln\left(\frac{n}{2}\right)\right)\right] \approx \left(\frac{(2k-1)n+2}{2n}\right)^n$$

for $(2k-1)n+2 \gg 2n$, which is the constant of the Heisenberg uncertainty principle for the deformed Stockwell transform given in Theorem 3.2.

(ii) Proceeding as above in the logarithmic uncertainty inequality (5.9) we deduce the following Heisenberg uncertainty inequality:

$$\begin{aligned}\left(\int_{\mathbb{R}} |y|^2\, |f(t)|^2\, d\gamma_{k,n}(t)\right)^{\frac{1}{2}} &\left(\int_{\mathbb{R}} |\xi|^2\, |\mathcal{F}_{k,n}(f)(\xi)|^2\, d\gamma_{k,n}(\xi)\right)^{\frac{1}{2}} \\ &\geq \exp\left[n\left(\frac{\Gamma'\left(\frac{(2k-1)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1)n+2}{4}\right)} - \ln\left(\frac{n}{2}\right)\right)\right] \int_{\mathbb{R}} |f(t)|^2\, d\gamma_{k,n}(t). \end{aligned} \tag{5.14}$$

(iii) Using the approximation relation (5.13) we deduce that the constant in the right-hand side of (5.14),

$$\exp\left[n\left(\frac{\Gamma'\left(\frac{(2k-1)n+2}{4}\right)}{\Gamma\left(\frac{(2k-1)n+2}{4}\right)} - \ln\left(\frac{n}{2}\right)\right)\right] \approx \left(\frac{(2k-1)n+2}{2n}\right)^n$$

for $(2k-1)n+2 \gg 2n$, which is the constant of the Heisenberg uncertainty principle for the generalized Fourier transform given in Proposition 3.1.

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