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ABEL ERGODIC THEOREMS FOR α-TIMES INTEGRATED SEMIGROUPS

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Dedicated to our professor Mohamed Akkar on the occasion of his 80th birthday.

Abstract. Let $\{T(t)\}_{t\geq 0}$ be an α-times integrated semigroup of bounded linear operators on the Banach space $\mathcal{X}$ and let A be their generator. In this paper, we study the uniform convergence of the Abel averages $\mathcal{A}(\lambda) = \lambda^{\alpha+1}\int_0^\infty e^{-\lambda t}T(t)\,dt$ as $\lambda \to 0^+$, with $\alpha \geq 0$. More precisely, we show that the following conditions are equivalent: (i) $T(t)$ is uniformly Abel ergodic; (ii) $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, with $\mathcal{R}(A)$ closed; (iii) $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$, and $\mathcal{R}(A^k)$ is closed for some integer k; (iv) A is a-Drazin invertible and $\mathcal{R}(A^k)$ is closed for some $k \geq 1$; where $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $R(\lambda, A)$ are the kernel, the range, and the resolvent function of A, respectively. Additionally, we show that if $T(t)$ satisfies $\lim_{t\to\infty}\|T(t)\|/t^{\alpha+1} = 0$, then $T(t)$ is uniformly Abel ergodic if and only if $\frac{1}{t^{\alpha+1}}\int_0^t T(s)\,ds$ converges uniformly as $t \to +\infty$. Finally, we examine simultaneously this theory with the uniform power convergence of the Abel averages $\mathcal{A}(\lambda)$ for some $\lambda > 0$.

1. Introduction

Throughout this paper $\mathcal{B}(\mathcal{X})$ denotes the Banach algebra of all bounded linear operators on a Banach space $\mathcal{X}$ into itself. Let A be a closed linear operator in $\mathcal{X}$ with domain $D(A) \subset \mathcal{X}$; we denote by $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\sigma(A)$, $\rho(A)$, and $R(.,A)$ the kernel, the range, the spectrum, the resolvent set, and the resolvent operator of A, respectively.

The family $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$ is called a *strongly continuous semigroup* (*C_0-semigroup* in short) if it has the following properties (see [18]):

(1) $T(0) = I$.
(2) $T(t)T(s) = T(t+s)$.
(3) The map $t \to T(t)x$ from $[0, +\infty[$ into $\mathcal{X}$ is continuous for all $x \in \mathcal{X}$.

Their infinitesimal generator A is defined by

$$Ax = \lim_{t\to 0^+}\frac{T(t)x - x}{t} \quad \text{for all } x \in D(A),$$

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where
$$D(A) = \left\{ x \in \mathcal{X} : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$
The Laplace transformation R_λ of a C_0-semigroup $T(t)$ on $\mathcal{B}(\mathcal{X})$ is defined as
$$R_\lambda x = \int_0^\infty e^{-\lambda t} T(t)x \, dt,$$
which is exactly the resolvent function of A. Moreover, the infinitesimal generator of a C_0-semigroup is a linear closed densely defined operator on a Banach space $\mathcal{X}$ (see, for instance, [8] and [18, p. 25]).

The α-times integrated semigroups, $\alpha \in \mathbb{R}^+$, and n-times integrated semigroups, $n \in \mathbb{N}$, of operators in a Banach space were introduced by Arendt [1] and studied by Arendt and Kellermann [2], Hieber [10], Thieme [24], and many others.

A relevant example is obtained if we assume that $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on $\mathcal{X}$; then $S(t) = \int_0^t T(r)\, dr$ defines an integrated semigroup $\{S(t)\}_{t\geq 0}$ having the following three properties:

(1) $S(0) = 0$.
(2) $S(s)S(t) = \int_0^s S(r+t) - S(r)\, dr \quad \text{for } t, s \geq 0.$
(3) The map $t \to S(t)$ from $[0, +\infty[$ into $\mathcal{X}$ is strongly continuous.

Let $\alpha \geq 0$ and let A be a linear operator on a Banach space $\mathcal{X}$. A is the generator of an α-times integrated semigroup [10, Definition 2.2] if, for some $\omega \in \mathbb{R}$, we have $]\omega, +\infty[\subseteq \rho(A)$ and there exists a strongly continuous mapping $T : [0, +\infty[\to \mathcal{B}(\mathcal{X})$ satisfying
$$\|T(t)\| \leq M e^{\omega t} \quad \text{for all } t \geq 0 \text{ and some } M > 0,$$
$$R(\lambda, A) = \lambda^\alpha \int_0^{+\infty} e^{-\lambda t} T(t)\, dt \quad \text{for all } \lambda > \max\{\omega, 0\}.$$
In this case, $\{T(t)\}_{t\geq 0}$ is called the *α-times integrated semigroup*, and the domain of its generator A is defined by
$$D(A) = \left\{ x \in \mathcal{X} : \int_0^t T(s)Ax\, ds = T(t)x - \frac{t^\alpha x}{\Gamma(\alpha+1)} \right\}.$$
From the uniqueness theorem of Laplace transforms, $\{T(t)\}_{t\geq 0}$ is uniquely determined. For convenience, we call a C_0-semigroup also a 0-*times integrated semigroup*, and the integrated semigroup is also a 1-times integrated semigroup.

Ergodic theorems [14] have a long tradition and are usually formulated via existence of the limits of the Cesàro averages:
$$C(t) := t^{-1} \int_0^t T(s)\, ds \quad \text{for } t \geq 0,$$
where $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup of bounded linear operators in a Banach space $\mathcal{X}$. The semigroup $\{T(t)\}_{t\geq 0}$ is said to be uniformly (resp., mean) Cesàro ergodic if the

Cesàro averages $C(t)$ converge in the norm (resp., the strong) operator topology. This notion is completely connected to studying the limit of the Abel averages of $T(t)$, defined by

$$\mathcal{A}(\lambda) = \lambda \int_0^\infty e^{-\lambda t} T(t)\, dt, \quad \text{where } \lambda > 0.$$

Recall that a semigroup $T(t)$ is called *uniformly Abel ergodic* if the limit of the Abel averages $\mathcal{A}(\lambda)$ when $\lambda \to 0^+$ exists in the norm operator topology.

We will denote the growth bound of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ by

$$\omega_0 = \inf\left\{w \in \mathbb{R} : \text{there exists } M \text{ such that } \|T(t)\| \leq Me^{wt},\, t \geq 0\right\}.$$

Usually one assumes $\omega_0 \leq 0$ or the even stronger condition $\|T(t)\|/t \longrightarrow 0$ as $t \to \infty$, to study the convergence of the Cesàro averages and the Abel averages of $\{T(t)\}_{t\geq 0}$. Generally, great attention has been focused on the study of the relationship between Cesàro ergodicity and Abel ergodicity for different classes of semigroups in $\mathcal{B}(\mathcal{X})$. The result of Hille and Phillips [11, Theorem 18.8.4] deals with the uniform Abel ergodicity of semigroups of class (A), a class slightly larger than C_0-semigroups, under the assumption $\omega_0 \leq 0$. More precisely, they have shown that $T(t)$ is uniformly Abel ergodic if and only if $\lambda^2 R(\lambda, A)x \longrightarrow 0$ as $\lambda \to 0^+$ for every $x \in \mathcal{X}$ and $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Furthermore, if $T(t)$ is uniformly Abel ergodic, then $\mathcal{R}(A^m) = \mathcal{R}(A)$ for all $m \in \mathbb{N}^*$. A relevant result was obtained by Shaw [19] for a locally integrated semigroup, under an assumption weaker than $\omega_0 \leq 0$, which means $T(t)$ is uniformly Cesàro ergodic if and only if it satisfies the following conditions:

(i) The Laplace transformation R_λ exists for every $\lambda > 0$.
(ii) $\|T(t)R_\lambda\|/t \longrightarrow 0$ as $t \to \infty$ for some $\lambda > 0$.
(iii) $T(t)$ is uniformly Abel ergodic.

The condition (i) holds whenever $\omega_0 \leq 0$. Let us also mention that somewhat different necessary and sufficient conditions are obtained in [4, 20]. Clearly, if $T(t)$ is uniformly Cesàro ergodic, then it is uniformly Abel ergodic, but the reverse is not true; for more information see [14, Chapter 2]. It is useful to mention that the limits of Cesàro averages and of Abel averages of the C_0-semigroup $\{T(t)\}_{t\geq 0}$ are the same, namely, the projection P of $\mathcal{X}$ onto $\mathcal{N}(A)$ parallel to $\mathcal{R}(A)$, corresponding to the ergodic decomposition

$$\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A).$$

The classical uniform ergodic theorem for C_0-semigroups of bounded linear operators on a Banach space $\mathcal{X}$ goes back to Lin [16]. He treats the uniform ergodicity of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ under the assumption $\lim\limits_{t\to\infty} \|T(t)\|/t = 0$. It showed that $T(t)$ is uniformly ergodic if and only if its infinitesimal generator A has a closed range if and only if $T(t)$ is uniformly Abel ergodic. In this case, and under this latter assumption, we can easily check that $T(t)$ is uniformly Abel ergodic if and only if $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Furthermore, this theory also plays an important role in the study of power convergence of linear operators. Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is called *uniformly power convergent* if there exists an operator

$P \in \mathcal{B}(\mathcal{X})$ such that $\lim_{n\to\infty} \|T^n - P\| = 0$. Recently, Lin, Shoikhet, and Suciu [17] showed that, for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$ satisfying $\lim_{t\to\infty} \|T(t)\|/t = 0$, $T(t)$ is uniformly ergodic if and only if there exists some $\lambda > 0$ such that the Abel average $\mathcal{A}(\lambda)$ of $T(t)$ is uniformly power convergent on $\mathcal{B}(\mathcal{X})$. Kozitsky, Shoikhet, and Zemànek [13] obtained a necessary and sufficient condition for which the Abel average of $\{T(t)\}_{t\geq 0}$ can be uniformly power convergent. Further conditions have been obtained more recently by several authors [7, 17, 22].

In our paper [3], we studied the convergence of the Cesàro averages and the Abel averages of an integrated semigroup $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(\mathcal{X})$. More precisely, we have shown that, if $S(t)$ satisfies $S(t)/t^2 \longrightarrow 0$ as $t \to \infty$, then $S(t)$ is uniformly Cesàro ergodic if and only if $S(t)$ is uniformly Abel ergodic, if and only if $\mathcal{R}(A^k)$ is closed for some integer $k \geq 1$. In the same direction, we continue the development of ergodic theory in this class of α-times integrated semigroup $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(\mathcal{X})$. The purpose of this paper is to give necessary and sufficient conditions for the Abel averages of an α-times integrated semigroup to converge in the norm operator topology. If A is the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$, we show that the following conditions are equivalent:

(i) $T(t)$ is uniformly Abel ergodic.
(ii) $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, with $\mathcal{R}(A)$ closed.
(iii) $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$ and $\mathcal{R}(A)$ is closed.
(iv) $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$ and $\mathcal{R}(A^k)$ is closed for some integer k;
(v) A is a-Drazin invertible and $\mathcal{R}(A^k)$ is closed for some $k \geq 1$.
(vi) A is group invertible in the sense of Definition 2.1 with $A^d = A^{ad}$.

Also, we show that if $T(t)$ satisfies $\lim_{t\to\infty} \|T(t)\|/t^{\alpha+1} = 0$, then $T(t)$ is uniformly Abel ergodic if and only if $\dfrac{1}{t^{\alpha+1}} \displaystyle\int_0^t T(s)\,ds$ converges in $\mathcal{B}(\mathcal{X})$ as $t \to \infty$.

Additionally, we examine this theory with the uniform power convergence of the Abel average $\mathcal{A}(\lambda)$ for some $\lambda > 0$, until we prove that $T(t)$ is uniformly Abel ergodic if and only if the Abel average $\mathcal{A}(\lambda)$ for some $\lambda > 0$ is uniformly power convergent.

2. Preliminaries

We start by recalling an interesting concept in operator theory that we need in what follows. Let A be a closed linear operator with domain $D(A) \subset \mathcal{X}$; the smallest non-negative integer p such that $\mathcal{N}(A^p) = \mathcal{N}(A^{p+1})$ is called the *ascent* of A and denoted by $\operatorname{asc}(A)$. If such an integer does not exist, we set $\operatorname{asc}(A) = \infty$. Likewise, the smallest integer q such that $\mathcal{R}(A^q) = \mathcal{R}(A^{q+1})$ is called the *descent* of A and denoted by $\operatorname{des}(A)$. If such an integer does not exist, we set $\operatorname{des}(A) = \infty$. Let A be a bounded linear operator on a Banach space $\mathcal{X}$. If $\operatorname{asc}(A)$ and $\operatorname{des}(A)$ are both finite, then $\operatorname{asc}(A) = \operatorname{des}(A)$, which is not true if A is an operator (see [23, Theorem 6.2]). Generally, if A is a closed linear operator, we have the following equivalence:

$$\operatorname{asc}(A) = p < \infty \iff \mathcal{R}(A^p) \cap \mathcal{N}(A^j) = \{0\},\ j = 1, 2, \dots.$$

But the equivalence below is satisfied only if A belongs to $\mathcal{B}(\mathcal{X})$:

$$\operatorname{des}(A) = q < \infty \iff \mathcal{X} = \mathcal{R}(A^j) + \mathcal{N}(A^q),\ j = 1, 2, \dots.$$

The first implication is not satisfied when A is a closed linear operator.

Recall that, for A a closed linear operator with domain $D(A) \subset \mathcal{X}$, if there is an operator $S \in \mathcal{B}(\mathcal{X})$ with $\mathcal{R}(S) \subseteq D(A)$ such that $SAS = S$, $ASx = SAx$ for all $x \in D(A)$, and $A^k(I - AS) = 0$ for some $k \in \mathbb{N}$, then S is called a *Drazin inverse* of A. Note that a closed linear operator A has a Drazin inverse if and only if there exists $k \in \mathbb{N}$ such that $a(A) = d(A) = k$ and $\mathcal{X} = \mathcal{R}(A^k) \oplus \mathcal{N}(A^k)$; for more details we refer the reader to [5, 12].

The operator A has the conventional Drazin inverse if and only if 0 is at most a pole of the resolvent function of A; this occurs if and only if, for some $m \in \mathbb{N}$,

$$\mathcal{R}(A^{m+1}) = \mathcal{R}(A^m) \quad \text{and} \quad \mathcal{N}(A^{m+1}) = \mathcal{N}(A^m).$$

A special case of the Drazin inverse is the group inverse, defined as follows.

Definition 2.1 ([6, Definition 1.1])**.** Let A be a closed linear operator with domain $D(A) \subset \mathcal{X}$. We say that A is *group invertible* with the *group inverse* $A^d \in \mathcal{B}(\mathcal{X})$ if

(i) $\mathcal{R}(A^d) \cup \mathcal{R}(I - AA^d) \subset \mathcal{D}(A)$,
(ii) for all $x \in \mathcal{D}(A)$, $AA^dx = A^dAx$, $A^dAA^d = A^d$, and $AA^dAx = Ax$.

The definition was later extended by Butzer and Koliha [6] to introduce the a-Drazin inverse of a closed linear operator A defined as follows.

Definition 2.2 ([6, Definition 2.5])**.** Let A be a closed linear operator with domain $D(A) \subset \mathcal{X}$. Then A is called *a-Drazin invertible* if

(i) $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$ and the space $\overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ is closed in $\mathcal{X}$,
(ii) $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^2)}$.

The a-Drazin inverse of A, denoted by A^{ad}, is unique if it exists, and it is given by

$$A^{ad} = (I - P)(A + P)^{-1},$$

where P is the spectral projection of A at 0.

Generally, one of the main reasons for our interest in the a-Drazin inverse A^{ad} of the infinitesimal generator A of an operator semigroup is that, at least in the case of holomorphic semigroups, A^{ad} acts as the infinitesimal generator for an associated semigroup. For basic concepts of operator theory of closed linear operators, we refer the reader to [6, 9].

Now, we recall the notion of α-times integrated semigroup, which is a generalization of the C_0-semigroup. Let $\beta \geq -1$ and f be a continuous function. The convolution $j_\beta * f$ is defined, for all $t \geq 0$, by

$$j_\beta * f(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^\beta}{\Gamma(\beta+1)} f(s)\, ds & \text{if } \beta > -1, \\ \displaystyle\int_0^t f(t-s)\ d\delta_0(s) & \text{if } \beta = -1, \end{cases}$$

where Γ is the Euler integral given by $\Gamma(\beta+1) = \displaystyle\int_0^{+\infty} x^\beta e^{-x}\, dx$, $j_{-1} = \delta_0$ the Dirac measure, and, for all $\beta > -1$,

$$\begin{aligned} j_\beta :]0, +\infty[&\to \mathbb{R} \\ t &\mapsto \frac{t^\beta}{\Gamma(\beta+1)}. \end{aligned}$$

A strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$ is called an *α-times integrated semigroup* [10], where $\alpha > 0$, if $T(0) = 0$ and, for all $t, s \geq 0$,

$$T_n(t)T_n(s) = \int_t^{t+s} \frac{(s+t-r)^{n-1}}{\Gamma(n)} T_n(r)\, dr - \int_0^s \frac{(s+t-r)^{n-1}}{\Gamma(n)} T_n(r)\, dr, \quad (2.1)$$

where $n - 1 < \alpha \leq n$ and $T_n(t)(x) = (j_{n-\alpha-1} * T)(x)$ for all $x \in \mathcal{X}$.

By the identity (2.1) the following equality holds for all $t, s \geq 0$:

$$T(t)T(s) = T(s)T(t).$$

Conversely, let $\alpha \geq 0$ and let A be a linear operator on a Banach space $\mathcal{X}$. A is the generator of an α-times integrated semigroup [10, Definition 2.2] if, for some $\omega \in \mathbb{R}$, we have $]\omega, +\infty[\subseteq \rho(A)$ and there exists a strongly continuous mapping $T : [0, +\infty[\to \mathcal{B}(\mathcal{X})$ satisfying

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for all } t \geq 0 \text{ and some } M > 0,$$

$$R(\lambda, A) = \lambda^\alpha \int_0^{+\infty} e^{-\lambda t} T(t)\, dt \quad \text{for all } \lambda > \max\{\omega, 0\}.$$

In this case, $\{T(t)\}_{t\geq 0}$ is called the *α-times integrated semigroup*, and the domain of its generator A is defined by

$$D(A) = \left\{ x \in \mathcal{X} : \int_0^t T(s)Ax\, ds = T(t)x - \frac{t^\alpha x}{\Gamma(\alpha+1)} \right\}.$$

For convenience we call a C_0-semigroup also 0-times integrated semigroup and the integrated semigroup is also a 1-times integrated semigroup. Recall that, if $R(\lambda_0, A)$ exists for a number λ_0, then $R(\lambda, A)$ exists for all λ with $\operatorname{Re}\lambda > \operatorname{Re}\lambda_0$.

Let us denote

$$\mathcal{S}(A) := \inf\{u \in (\infty, \infty) : R(\lambda, A) \text{ exists for all } \lambda \text{ with } \operatorname{Re}\lambda > u\}.$$

Example 2.3. (1) An important example of generators of an α-times integrated semigroup, with $\alpha > 0$, is the adjoint A^* on $\mathcal{X}^*$, where A is the infinitesimal generator of a C_0-semigroup on a Banach space $\mathcal{X}$.

In particular [10, Examples 3.8], we consider $\mathcal{X} = L^1(\mathbb{R})$ and we define the linear operator by

$$Af = -f' \quad \text{for all } f \in D(A),$$

with $D(A) := \{f \in \mathcal{X} : f \text{ is continuous and } f' \in \mathcal{X}\}$. Since $\mathcal{X}^* = L^\infty(\mathbb{R})$, the adjoint A^* of A is defined by

$$A^* f = f' \quad \text{for all } f \in D(A^*),$$

where $D(A^*) = \{f \in \mathcal{X}^* : f \text{ continuous and } f' \in \mathcal{X}^*\}$. Therefore, A^* is a generator of an α-times integrated semigroup and $R(\lambda, A^*)$ exists for all λ with $\operatorname{Re}\lambda \geq 0$, which means that $\mathcal{S}(A^*) < 0$.

(2) We consider $\mathcal{X} = \ell^2$ and the family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on $\mathcal{X}$ defined by

$$T(t)(x_n)_{n\in\mathbb{N}^*} = \left(\int_0^t e^{a_n s}\, ds x_n\right)_{n\in\mathbb{N}^*}.$$

Then $\{T(t)\}_{t\geq 0}$ is an integrated semigroup on $\mathcal{X}$.

(3) Let $\mathcal{X} = C([0,\infty])$ and consider the derivation operator $Af = -f'$ for all $f \in D(A)$, with $D(A) = \{f \in C^1([0,1]) : f(0) = 0\}$. Since the domain $D(A)$ is not dense in $\mathcal{X}$, A cannot be an infinitesimal generator of a C_0-semigroup. Furthermore, the semigroup $T(t)$ generated by A is given by

$$\big(T(t)f\big)(x) = \begin{cases} -\displaystyle\int_x^{x-t} f(s)\, ds & \text{if } x > t, \\ \displaystyle\int_x^0 f(s)\, ds & \text{if } 0 \leq x \leq t. \end{cases}$$

Note that $T(t)$ is an integrated semigroup of type $\mathcal{S}(A) < 0$, which means that $R(\lambda, A)$ exists for all λ with $\operatorname{Re}\lambda \geq 0$.

Definition 2.4. Let $\{T(t)\}_{t\geq 0}$ be an integrated semigroup on $\mathcal{B}(\mathcal{X})$. We say that $\{T(t)\}_{t\geq 0}$ is *uniformly Abel ergodic* if the Abel average of $T(t)$ defined by

$$\mathcal{A}(\lambda) = \lambda^{\alpha+1} \int_0^\infty e^{-\lambda t} T(t)\, dt \quad \text{for } t \geq 0$$

converges in the norm operator topology as $\lambda \to 0^+$.

The next two propositions were investigated by Arendt [1] in the case of an n-times integrated semigroup on $\mathcal{B}(\mathcal{X})$, where $n \in \mathbb{N}$. These results have been generalized by Hieber [10] to the α-times integrated semigroup with $\alpha \in \mathbb{R}^+$.

Proposition 2.5 ([10, Proposition 2.4]). *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$, where $\alpha \geq 0$. Then for all $x \in D(A)$ and all $t \geq 0$:*

(1) *$T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$.*

(2) *$T(t)x = \dfrac{t^\alpha}{\Gamma(\alpha+1)} x + \displaystyle\int_0^t T(s)Ax\, ds$.*

(3) *For all* $x \in \mathcal{X}$, $\int_0^t T(s)x\,ds \in D(A)$, *and*
$$A\int_0^t T(s)x\,ds = T(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x.$$

Theorem 2.6 ([21, Theorem 2.7])**.** *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ in $\mathcal{B}(\mathcal{X})$, where $\alpha \geq 0$. If $\lim_{t\to\infty} \|T(t)\|/t = 0$ and $\mathcal{R}(A)$ is closed, then $\frac{1}{t^{\alpha+1}}\int_0^t T(s)\,ds$ converges uniformly for all $\alpha \geq 0$.*

3. Main results

The following lemmas are among the most widely used results of this paper. The first lemma was proved in our paper [21] and the second is obviously derived from the first.

Lemma 3.1 ([21, Lemma 2.3])**.** *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$ with $\alpha \geq 0$. Then we have the following assertions:*

(1) $\mathcal{R}(A) = \big(\lambda R(\lambda, A) - I\big)\mathcal{X}$.

(2) $\mathcal{N}(A) = \Big\{x \in \mathcal{X} : T(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x \text{ for all } t \geq 0\Big\}$
$= \{x \in \mathcal{X} : \lambda R(\lambda, A)x = x\}$.

Lemma 3.2. *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$ with $\alpha \geq 0$. Let X_0 be a closed subspace of $\mathcal{X}$ defined by $X_0 = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$ and P be the projection operator of X_0 onto $\mathcal{N}(A)$ parallel to $\overline{\mathcal{R}(A)}$. Then*
$$\lambda R(\lambda, A)x - Px = \lambda R(\lambda, A)(I-P)x \quad \textit{for all } x \in X_0.$$

Next, we need the following auxiliary results to prove our main theorem.

Lemma 3.3. *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$ with $\alpha \geq 0$. If $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$, then $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$, which yields* $\operatorname{asc}(A) \leq 1$.

Proof. We assume that $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$. Let $y \in \overline{\mathcal{R}(A)} \cap \mathcal{N}(A)$. It follows from the second assertion of Lemma 3.1 that
$$\lambda R(\lambda, A)y = y \quad \text{for all } \lambda \in \rho(A).$$
Since $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}\big(\lambda R(\lambda, A) - I\big)}$, there exist $x \in \mathcal{X}$ and $M > 0$ such that
$$y = \big(\lambda R(\lambda, A) - I\big)x \quad \text{and} \quad \|x\| \leq M\|y\|.$$
By the resolvent equation,
$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \quad \text{for all } \lambda \neq \mu \in \rho(A).$$
We get the following inequality for all $\lambda, \mu > 0$:
$$\begin{aligned}\|\lambda R(\lambda, A)y\| &\leq |\mu-\lambda|^{-1}\big[\|\lambda^2 R(\lambda, A)\| + |\lambda|\|\mu R(\mu, A)\|\big]\|x\| \\ &\leq M\,|\mu-\lambda|^{-1}\big[\|\lambda^2 R(\lambda, A)\| + |\lambda|\|\mu R(\mu, A)\|\big]\|y\|.\end{aligned}$$

Therefore, $\lambda R(\lambda, A)y \longrightarrow 0$ as $\lambda \to 0^+$. Since $\lambda R(\lambda, A)y = y$ for $\lambda > 0$, we have $y = 0$. Consequently, $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$, which yields $\operatorname{asc}(A) \leq 1$. □

The first main result of this paper is the following theorem.

Theorem 3.4. *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$, where $\alpha \geq 0$. Then the following assertions are equivalent:*

(1) *$T(t)$ is uniformly Abel ergodic.*
(2) *$\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, with $\mathcal{R}(A)$ closed.*
(3) *$\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$ and $\mathcal{R}(A)$ is closed.*

Proof. (1) $\Longrightarrow$ (2) It is known from the mean ergodic theorem [25, p. 217] that if there exists an operator $P \in \mathcal{B}(\mathcal{X})$ such that $\|\lambda R(\lambda, A) - P\| \longrightarrow 0$ as $\lambda \to 0^+$, then P is the projection onto $\mathcal{N}\big(\lambda R(\lambda, A) - I\big)$ along $\big(\lambda R(\lambda, A) - I\big)\mathcal{X}$, and by Lemma 3.1, we get

$$\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A).$$

(2) $\Longrightarrow$ (3) Assume that $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, where $\mathcal{R}(A)$ is closed in $\mathcal{X}$. It is easy to show that $\|\lambda^2 R(\lambda, A)|_{\mathcal{N}(A)}\| \longrightarrow 0$ when $\lambda \to 0^+$. So, to complete the proof we show that $\|\lambda^2 R(\lambda, A)|_{\mathcal{R}(A)}\| \longrightarrow 0$ when $\lambda \to 0^+$. Set $Y = \mathcal{R}(A)$ and let A_1 be the generator of the restriction of $T(t)$ to Y, which is equal to the restriction of A to $Y \cap D(A)$. It is shown in Lemma 3.1 that $Y = \mathcal{R}\big(\lambda R(\lambda, A) - I\big)$ and by the decomposition, the operator $\big(\lambda R(\lambda, A) - I\big)$ is invertible on Y. Let $y \in Y \cap D(A)$ such that $A_1 y = 0$, hence

$$\begin{aligned} y &= R(\lambda, A)(\lambda - A)y \\ &= \lambda R(\lambda, A)y - R(\lambda, A)Ay \\ &= \lambda R(\lambda, A)y - R(\lambda, A)A_1 y \\ &= \lambda R(\lambda, A)y. \end{aligned}$$

Then $y \in \mathcal{N}\big(\lambda R(\lambda, A) - I\big)$, which implies that $y = 0$. Thus A_1 is one-to-one. Clearly, we have $R(\lambda, A)Y \subset Y$; hence we obtain that $\big(\lambda R(\lambda, A) - I\big)Y \subset \mathcal{R}(A_1)$. Then, we get the following:

$$Y \supset \mathcal{R}(A_1) \supset \big(\lambda R(\lambda, A) - I\big)Y = \big(\lambda R(\lambda, A) - I\big)\mathcal{X} = \mathcal{R}(A) = Y.$$

Hence $Y = \mathcal{R}(A_1)$, so A_1^{-1} is defined on all Y; since A_1 is closed, A_1^{-1} is also closed, and by the closed graph theorem A_1^{-1} is continuous.

Let $0 < \lambda < \delta < \dfrac{1}{\|A_1^{-1}\|}$ and $y \in Y$; we get

$$\begin{aligned} \|\lambda^2 R(\lambda, A)y\| &= \|\lambda^2 R(\lambda, A)A_1 A_1^{-1} y\| \\ &\leq \|\lambda^2\big(\lambda R(\lambda, A) - I\big)\| \, \|A_1^{-1}\| \, \|y\|. \end{aligned}$$

Hence

$$\|\lambda^2 R(\lambda, A)y\| \leq \lambda^2\big(\|\lambda R(\lambda, A)\| + 1\big) \, \|A_1^{-1}\| \, \|y\|.$$

Also, we have

$$\|\lambda R(\lambda, A)\| \le \delta\big(\|\lambda R(\lambda, A)\| + 1\big) \, \|A_1^{-1}\|.$$

Then, we get

$$\|\lambda R(\lambda, A)\| \le \frac{\delta\|A_1^{-1}\|}{1-\delta\|A_1^{-1}\|} = M.$$

Therefore,

$$\begin{aligned}\|\lambda^2 R(\lambda, A)y\| &\le \|\lambda^2\big(\lambda R(\lambda, A) - I\big)\| \, \|A_1^{-1}\| \, \|y\| \\ &\le \lambda^2\big(\|\lambda R(\lambda, A)\| + 1\big) \, \|A_1^{-1}\| \, \|y\| \\ &\le \lambda^2(M+1) \, \|A_1^{-1}\| \, \|y\|,\end{aligned}$$

which implies that $\|\lambda^2 R(\lambda, A)|_Y\| \longrightarrow 0$ as $\lambda \to 0^+$. Hence the assertion (3) holds.

(3) $\Longrightarrow$ (1) We suppose that $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$, and $\mathcal{R}(A)$ is closed. By Lemma 3.1, we have $\mathcal{R}(A) = \big(\lambda R(\lambda, A) - I\big)\mathcal{X}$, which means that, for all $\lambda > 0$, the operator $\lambda R(\lambda, A) - I$ has a closed range. Fix $\mu > 0$ such that, for each $y \in \big(\mu R(\mu, A) - I\big)\mathcal{X}$, there exists $M > 0$ and $x \in \mathcal{X}$ such that $y = \big(\mu R(\mu, A) - I\big)x$ and $\|x\| \le M\|y\|$. So we have

$$\lambda R(\lambda, A)\big(\mu R(\mu, A) - I\big) = \lambda\mu R(\lambda, A)R(\mu, A) - \lambda R(\lambda, A).$$

By the resolvent equation, we obtain

$$\begin{aligned}\lambda R(\lambda, A)\big(\mu R(\mu, A) - I\big) &= \lambda\mu R(\lambda, A)R(\mu, A) - \lambda(\mu-\lambda)R(\lambda, A)R(\mu, A) - \lambda R(\mu, A) \\ &= \lambda^2 R(\lambda, A)R(\mu, A) - \lambda R(\mu, A) \\ &= \lambda^2(\mu-\lambda)^{-1}\big[R(\lambda, A) - R(\mu, A)\big] - \lambda R(\mu, A) \\ &= (\mu-\lambda)^{-1}\big[\lambda^2 R(\lambda, A) - \lambda\mu R(\mu, A)\big].\end{aligned}$$

This gives

$$\begin{aligned}\|\lambda R(\lambda, A)y\| &= \|\lambda R(\lambda, A)\big(\mu R(\mu, A) - I\big)x\| \\ &= \big\|(\mu-\lambda)^{-1}\big[\lambda^2 R(\lambda, A) - \lambda\mu R(\mu, A)\big]x\big\| \\ &\le |\mu-\lambda|^{-1}\big[\|\lambda^2 R(\lambda, A)\| + |\lambda| \, \|\mu R(\mu, A)\|\big] \, M\|y\|.\end{aligned}$$

Hence $\|\lambda R(\lambda, A)|_{(\lambda R(\lambda,A)-I)\mathcal{X}}\| \longrightarrow 0$ as $\lambda \to 0^+$. Then for a small $\lambda > 0$, the operator $\lambda R(\lambda, A) - I$ is invertible on $\big(\lambda R(\lambda, A) - I\big)\mathcal{X}$; therefore,

$$\big(\lambda R(\lambda, A) - I\big)^2\mathcal{X} = \big(\lambda R(\lambda, A) - I\big)\mathcal{X},$$

which yields $\mathcal{X} = \big(\lambda R(\lambda, A) - I\big)\mathcal{X} + \mathcal{N}\big(\lambda R(\lambda, A) - I\big)$, and the summation is direct by Lemma 3.3. Since $\lambda R(\lambda, A)|_{\mathcal{N}(\lambda R(\lambda,A)-I)}$ converge to the identity I when $\lambda \to 0^+$, $\lambda R(\lambda, A)$ converges uniformly. Hence the assertion (1) holds. ☐

Now, we recall the following lemma.

Lemma 3.5 ([4, Lemma 3.10]). *Let $A \in \mathcal{C}(\mathcal{X})$ with domain $D(A) \subset \mathcal{X}$ such that $\operatorname{asc}(A) = d < \infty$. If either of the following hold,*

(i) $\mathcal{R}(A^n)$ *is closed for some* $n > d$*, or*

(ii) $\mathcal{R}(A^j)+\mathcal{N}(A^k)$ *is closed for some positive integers* j,k *with* $j+k=n\geq d$*,*

then $\mathcal{R}(A^n)$ *is closed for all* $n \geq d$*, and* $\mathcal{R}(A^j)+\mathcal{N}(A^k)$ *is closed for all integers* j,k *with* $j+k\geq d$*.*

From the previous lemma and Theorem 3.4, we infer the following corollary.

Corollary 3.6. *Let* A *be the generator of an* α*-times integrated semigroup* $\{T(t)\}_{t\geq 0}$ *on* $\mathcal{B}(\mathcal{X})$*, where* $\alpha \geq 0$*. Then the following assertions are equivalent:*

(1) $T(t)$ *is uniformly Abel ergodic.*
(2) $\|\lambda^2 R(\lambda,A)\| \longrightarrow 0$ *as* $\lambda \to 0^+$*, and* $\mathcal{R}(A^k)$ *is closed for some integer* k*.*
(3) $\|\lambda^2 R(\lambda,A)\| \longrightarrow 0$ *as* $\lambda \to 0^+$*, and* $\mathcal{R}(A^k)+\mathcal{N}(A^j)$ *is closed for some integers* k *and* j*.*

The second main result of this paper can be stated as follows.

Theorem 3.7. *Let* A *be the generator of an* α*-times integrated semigroup* $\{T(t)\}_{t\geq 0}$ *on* $\mathcal{B}(\mathcal{X})$*, where* $\alpha \geq 0$*. Then* $T(t)$ *is uniform Abel ergodic if and only if the a-Drazin inverse* A^{ad} *of* A *exists and is bounded, with*

$$A^{ad}x = \lim_{\lambda\to 0^+} \lambda^{-2}Px - R(\lambda^2, A) \quad \text{for all } x \in \mathcal{X}.$$

Proof. By means of Theorem 3.4, $T(t)$ is uniformly Abel ergodic; then there exists an operator $P \in \mathcal{B}(\mathcal{X})$ such that $\|\mathcal{A}(\lambda) - P\| \longrightarrow 0$ as $\lambda \to 0^+$, where P is the projection onto $\mathcal{N}(A)$ along $\mathcal{R}(A)$ corresponding to the ergodic decomposition

$$\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A).$$

From Definition 2.2, we easily check that A is a-Drazin invertible and

$$A^{ad} = (I-P)(A+P)^{-1}.$$

Let us show that A^{ad} is bounded. Indeed, let $x \in \mathcal{D}(A^{ad})$; then $x = Ag + Px$ for some $g \in \mathcal{R}(A)\cap\mathcal{D}(A)$, and $A^{ad}x = g$. Moreover, if $x \in \mathcal{N}(A)$, then $x = Px$, so we get $Ag = 0$, which means that $g \in \mathcal{N}(A)$. Since $g \in \mathcal{R}(A)\cap\mathcal{D}(A)$ and $\mathcal{R}(A)\cap\mathcal{N}(A) = \{0\}$, we have $g = 0$. Consequently, $\mathcal{N}(A) \subset \mathcal{N}(A^{ad})$. On the other hand, if $x \in \mathcal{R}(A)$, then $Px = 0$, which gives $x = Ag$. Since $\mathcal{R}(A)$ is closed, there exists $M > 0$ such that $\|g\| \leq M\|x\|$. Therefore,

$$\begin{aligned}
\|A^{ad}x\| = \|g\| &\leq \|g + \big(R(\lambda,A)x - \lambda^{-1}Px\big)\| \\
&\leq \|g + R(\lambda,A)Ag\| \\
&\leq \|g + \big(\lambda R(\lambda,A)g - g\big)\| \\
&\leq \|\lambda R(\lambda,A)g\| \\
&\leq M\|\lambda R(\lambda,A)\|\|x\|.
\end{aligned}$$

Then, from the decomposition of $\mathcal{X}$, it follows that A^{ad} is bounded.

Next, we show that $A^{ad}x = \lim_{\lambda\to 0^+} \lambda^{-1}Px - R(\lambda,A)x$. Indeed, let $x \in \mathcal{D}(A^{ad})$, where $\mathcal{D}(A^{ad})$ is the domain of A^{ad}; then we have

$$x = Ag + Px \text{ for some } g \in \mathcal{R}(A)\cap\mathcal{D}(A) \text{ and } A^{ad}x = g.$$

From Lemma 3.2, we get

$$\begin{aligned} R(\lambda,A)x-\lambda^{-1}Px+A^{ad}x &= R(\lambda,A)(I-P)x+Ax \\ &= R(\lambda,A)Ag+g. \end{aligned}$$

Using the identity $AR(\lambda,A)x=\big(\lambda R(\lambda,A)-I\big)$, we get

$$\begin{aligned} R(\lambda,A)x-\lambda^{-1}Px+A^{ad}x &= \big(\lambda R(\lambda,A)g-Ig\big)+g \\ &= \lambda R(\lambda,A)g. \end{aligned}$$

Since $\mathcal{R}(A)=\mathcal{N}(P)$ and $g\in\mathcal{R}(A)\cap\mathcal{D}(A)$, $\lambda R(\lambda,A)g\longrightarrow 0$ as $\lambda\to 0^+$. Therefore, $A^{ad}x=\lim\limits_{\lambda\to 0^+}\lambda^{-1}Px-R(\lambda,A)$.

Conversely, suppose that the a-Drazin inverse A^{ad} of A exists and is bounded, which means $\mathcal{D}(A^{ad})=\mathcal{X}$; then by Definition 2.2, we get

$$\mathcal{X}=\overline{\mathcal{R}(A)}+\mathcal{N}(A) \quad \text{and} \quad \mathcal{R}(A)\subseteq\overline{\mathcal{R}(A^2)}. \tag{3.1}$$

Then, for any $x\in\mathcal{X}$, we have $x=Ay+Px$ with $A^{ad}x=y$ and P is the spectral projection of A on 0, corresponding to the above decomposition. Then, we can write $x=x_1+x_2$ such that $x_1\in\overline{\mathcal{R}(A)}$ and $x_2\in\mathcal{N}(A)$. Since $\mathcal{N}(A)=\mathcal{N}(\lambda R(\lambda,A)-I)$ which coincides with the set of fixed points of $T(t)$, $\lambda R(\lambda,A)$ converges to I on $\mathcal{N}(A)$. To complete the proof, let us show that $\lambda R(\lambda,A)$ converges to 0 on $\overline{\mathcal{R}(A)}$. Indeed, let $x\in\mathcal{X}$; then there exists $x_1\in\overline{\mathcal{R}(A)}$ and $x_2\in\mathcal{N}(A)$ such that $x=x_1+x_2$. So, we get $x=Ay+Px=Ay+Px_2$ with $A^{ad}x=y$, hence we obtain $x_1=Ay$ and $y=A^{ad}x_1$.

Now, let $0<\lambda<\delta<\dfrac{1}{\|A^{ad}\|}$; then

$$\begin{aligned} \|\lambda R(\lambda,A)x_1\| &= \|\lambda R(\lambda,A)Ay\| \\ &\le \|\lambda\big(\lambda R(\lambda,A)y-Iy\big)\| \\ &\le \|\lambda\big(\lambda R(\lambda,A)-I\big)\|\|A^{ad}\|\|x_1\| \\ &\le \lambda\big(\|\lambda R(\lambda,A)\|+1\big)\|A^{ad}\|\|x_1\|. \end{aligned}$$

So, we get $\|\lambda R(\lambda,A)\| \ \le\delta\|\big(\lambda R(\lambda,A)\|+1\big)\|A^{ad}\|$.

It follows that

$$\|\lambda R(\lambda,A)\|\le\delta\big(\|\lambda R(\lambda,A)\|+1\big)\,\|A^{ad}\|.$$

Then, we obtain $\|\lambda R(\lambda,A)\|\le\dfrac{\delta\|A^{ad}\|}{1-\delta\|A^{ad}\|}$. Therefore,

$$\begin{aligned} \|\lambda R(\lambda,A)x_1\| &\le \|\lambda\big(\lambda R(\lambda,A)-I\big)\|\,\|A^{ad}\|\,\|x_1\| \\ &\le \lambda\left(1+\frac{\delta\|A^{ad}\|}{1-\delta\|A^{ad}\|}\right)\|A^{ad}\|\,\|x_1\|, \end{aligned}$$

which implies that $\lambda R(\lambda,A)\longrightarrow 0$ as $\lambda\to 0^+$ on $\overline{\mathcal{R}(A)}$.

Finally, by the decomposition (3.1), we get that $T(t)$ is uniformly Abel ergodic. □

The following corollary is an immediate consequence of Theorem 3.4, Theorem 3.7, and Lemma 3.5.

Corollary 3.8. *Let $\{T(t)\}_{t\geq 0}$ be an α-times integrated semigroup on $\mathcal{B}(\mathcal{X})$ generated by A, with $\alpha \geq 0$. The following conditions are equivalent:*

1. *$T(t)$ is uniformly Abel ergodic.*
2. *The point 0 is a simple pole of the resolvent $R(\lambda, A)$ of A.*
3. *A is a-Drazin invertible and $\mathcal{R}(A)$ is closed.*
4. *A is a-Drazin invertible and $\mathcal{R}(A^k)$ is closed for some $k \geq 1$.*
5. *A is a-Drazin invertible and $\mathcal{R}(A) + \mathcal{N}(A)$ is closed.*
6. *A is a-Drazin invertible and the descent $d(A)$ of A is finite.*
7. *A is group invertible in the sense of Definition 2.1 with $A^d = A^{ad}$.*

Next, we give the following result which proves that the study of the convergence of Abel averages $\mathcal{A}(\lambda)$ of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ can be limited to studying the convergence $\|\mathcal{A}(\lambda)|_{\mathcal{R}(A)}\| \longrightarrow 0$ as $\lambda \to 0^+$, where $\mathcal{R}(A)$ is closed.

Proposition 3.9. *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$, with $\alpha \geq 0$. Then, $T(t)$ is uniformly Abel ergodic if and only if $\mathcal{R}(A)$ is closed and $\|\mathcal{A}(\lambda)|_{\mathcal{R}(A)}\| \longrightarrow 0$ as $\lambda \to 0^+$.*

Proof. The necessary part of this proposition is obvious.

Conversely, let $\mathcal{R}(A)$ be closed and $\|\lambda R(\lambda, A)|_{\mathcal{R}(A)}\| \longrightarrow 0$ as $\lambda \to 0^+$, where $R(\lambda, A)|_{\mathcal{R}(A)}$ is the restriction of $R(\lambda, A)$ to $\mathcal{R}(A)$. Since $\mathcal{R}(A) = \big(\lambda R(\lambda, A) - I\big)\mathcal{X}$, $\big\|\lambda R(\lambda, A)|_{(\lambda R(\lambda,A)-I)\mathcal{X}}\big\| \longrightarrow 0$ as $\lambda \to 0^+$. Then, for a small λ, the operator $\big(\lambda R(\lambda, A) - I\big)|_{(\lambda R(\lambda,A)-I)\mathcal{X}}$ is invertible. Therefore,

$$\mathcal{R}\big(\lambda R(\lambda, A) - I\big) = \mathcal{R}\big((\lambda R(\lambda, A) - I)|_{\mathcal{R}(A)}\big) = \mathcal{R}\big[\big(\lambda R(\lambda, A) - I\big)^2\big].$$

Hence

$$\mathcal{X} = \mathcal{R}\big(\lambda R(\lambda, A) - I\big) + \mathcal{N}\big(\lambda R(\lambda, A) - I\big). \tag{3.2}$$

Now, let $y \in \mathcal{R}\big(\lambda R(\lambda, A) - I\big) \cap \mathcal{N}\big(\lambda R(\lambda, A) - I\big)$, so $\lambda R(\lambda, A)y = y$ for all $\lambda > 0$, and by assumption $\lambda R(\lambda, A)y \longrightarrow 0$ as $\lambda \to 0^+$; hence $y = 0$, which means that

$$\mathcal{R}\big(\lambda R(\lambda, A) - I\big) \cap \mathcal{N}\big(\lambda R(\lambda, A) - I\big) = \{0\}.$$

Then, the summation in (3.2) is direct. Finally, Theorem 3.4 implies that $T(t)$ is uniformly Abel ergodic. □

Now, we present our third main result as follows. Theorems of this nature are referred to in the literature as ergodic theorems.

Theorem 3.10. *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$, with $\alpha \geq 0$. Assume that $\lim_{t\to\infty} \|T(t)\|/t^{\alpha+1} = 0$. Then the following assertions are equivalent:*

1. *$T(t)$ is uniformly Abel ergodic.*
2. *$\mathcal{R}(A^k)$ is closed for some integer $k \geq 1$.*
3. *There exists $P \in \mathcal{B}(\mathcal{X})$ such that $\lim_{t\to\infty} \left\| \frac{1}{t^{\alpha+1}} \int_0^t T(s)\,ds - P \right\| = 0$.*

We need the following auxiliary result to prove this theorem.

Lemma 3.11. *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$, with $\alpha \geq 0$. If $T(t)$ satisfies $\lim_{t\to\infty} \|T(t)\|/t^{\alpha+1} = 0$, then*

$$\lim_{\lambda\to 0^+} \|\lambda^2 R(\lambda, A)\| = 0.$$

Proof. Let $\{T(t)\}_{t\geq 0}$ be an α-times integrated semigroup on $\mathcal{B}(\mathcal{X})$, where $\alpha \geq 0$ such that $\lim_{t\to\infty} \|T(t)\|/t^{\alpha+1} = 0$. Then, there exist $\varepsilon > 0$ and $a > 0$ such that

$$\|T(t)\| \leq \varepsilon t^{\alpha+1} \quad \text{for all } t > a.$$

Using the resolvent equation, we obtain, for all $x \in \mathcal{X}$,

$$\begin{aligned}
\|\lambda^2 R(\lambda, A)x\| &= \big\|\lambda^2\big[R(\mu, A) + (\mu-\lambda)R(\lambda, A)R(\mu, A)\big]x\big\| \\
&\leq \|\lambda^2 R(\mu, A)\|\|x\| + |\mu-\lambda|\lambda^2\big\|R(\lambda, A)R(\mu, A)x\big\| \\
&\leq \|\lambda^2 R(\mu, A)\|\|x\| + |\mu-\lambda|\lambda^{\alpha+2}\int_0^\infty e^{-\lambda t}\big\|T(t)R(\mu, A)x\big\|\,dt \\
&\leq \|\lambda^2 R(\mu, A)\|\|x\| + |\mu-\lambda|\Big[\lambda^{\alpha+2}\int_0^a e^{-\lambda t}\big\|T(t)R(\mu, A)x\big\|\,dt \\
&\quad + \varepsilon\lambda^{\alpha+2}\int_a^\infty e^{-\lambda t}t^{\alpha+1}\|R(\mu, A)x\|\,dt\Big].
\end{aligned}$$

It is known that, for any operator $P \in \mathcal{B}(\mathcal{X})$ and all $\lambda \in \mathbb{C}$,

$$\lambda^{\alpha+2}\int_0^\infty e^{-\lambda t}t^{\alpha+1}P\,dt = (\alpha+1)!P \quad \text{for all } \alpha, t \geq 0.$$

Therefore,

$$\begin{aligned}
\|\lambda^2 R(\lambda, A)x\| \leq \|\lambda^2 R(\mu, A)\|\|x\| + |\mu-\lambda|\Big[\lambda^{\alpha+2}a\Big(\sup_{t\leq a}\|T(t)\|\|R(\mu, A)\|\Big) \\
+ \varepsilon(\alpha+1)!\|R(\mu, A)\|\Big]\|x\|.
\end{aligned}$$

It is easily seen from the above estimate that $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ when $\lambda \to 0^+$. □

Proof of Theorem 3.10. (1) $\Longleftrightarrow$ (2) It follows from Lemma 3.11 and Corollary 3.6.

(1) $\Longrightarrow$ (3) Assume that $T(t)$ is uniformly Abel ergodic. Then by Theorem 3.4, we obtain the decomposition $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, with $\mathcal{R}(A)$ closed, and from Lemma 3.1, we have

(i) $\mathcal{R}(A) = \big(\lambda R(\lambda, A) - I\big)\mathcal{X}$.

(ii) $\mathcal{N}(A) = \Big\{x \in \mathcal{X} : T(t)x = \dfrac{t^\alpha}{\Gamma(\alpha+1)}x \text{ for all } t \geq 0\Big\}$
$= \{x \in \mathcal{X} : \lambda R(\lambda, A)x = x\}$.

By hypothesis and through a simple calculation, we get

$$\lim_{t\to\infty}\Big\|\frac{1}{t^{\alpha+1}}\int_0^t T(s)x\,ds - \frac{Ix}{(\alpha+1)\Gamma(\alpha+1)}\Big\| = 0 \quad \text{for all } x \in \mathcal{N}(A).$$

So, to complete the proof, we show that $\left\|\frac{1}{t^{\alpha+1}}\int_0^t T(s)y\,ds\right\| \longrightarrow 0$ when $t \to \infty$ for all $y \in \mathcal{R}(A)$. Let A_1 be the generator of the restriction of $T(t)$ to $\mathcal{R}(A)$, which is equal to the restriction of A to $\mathcal{R}(A) \cap D(A)$. It was shown in the proof of Theorem 3.4 that A_1^{-1} is defined on all $\mathcal{R}(A)$ and continuous, then for all $y \in \mathcal{R}(A)$, there exists $x \in D(A)$ such that $y = A_1 x$ and $\|x\| \leq \|A_1^{-1}\|\|y\|$. The second assertion of Proposition 2.5 implies that, for all $x \in D(A)$, we have

$$\int_0^t T(s)Ax\,ds = T(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x.$$

It follows that we get

$$\begin{aligned}\left\|\frac{1}{t^{\alpha+1}}\int_0^t T(s)y\,ds\right\| &= \left\|\frac{1}{t^{\alpha+1}}\Big[T(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x\Big]\right\| \\ &\leq \|A_1^{-1}\|\left[\left\|\frac{T(t)}{t^{\alpha+1}}\right\| + \left\|\frac{1}{t\Gamma(\alpha+1)}\right\|\right]\|y\|.\end{aligned}$$

Then $\lim_{t\to\infty}\left\|\frac{1}{t^{\alpha+1}}\int_0^t T(s)y\,ds\right\| = 0$ for all $y \in \mathcal{R}(A)$. Hence the assertion (3) holds.

(3) $\Longrightarrow$ (1) Let $\mathcal{I}(t) = \int_0^t T(s)\,ds$ for all $t \geq 0$, and $\mathcal{C}(t) = \frac{1}{t^{\alpha+1}}\int_0^t T(s)\,ds$. We assume that there exists an operator $P \in \mathcal{B}(\mathcal{X})$ such that $\lim_{t\to\infty}\|\mathcal{C}(t) - P\| = 0$. So, there exist $\varepsilon > 0$ and $a > 0$ such that $\|\mathcal{C}(t) - P\| \leq \varepsilon$ for all $t > a$.

Now, we use integration by parts to get the following identity:

$$R(\lambda, A) = \lambda^{\alpha+1}\int_0^\infty e^{-\lambda t}\mathcal{I}(t)\,dt \quad \text{for all } \lambda > 0 \text{ and } t \geq 0.$$

Then for every $x \in \mathcal{X}$, we have

$$\begin{aligned}\left\|\lambda R(\lambda, A)x - (\alpha+1)!Px\right\| &= \left\|\lambda R(\lambda, A) - \lambda^{\alpha+2}\int_0^\infty e^{-\lambda t}t^{\alpha+1}P\,dt\right\| \\ &= \left\|\lambda^{\alpha+2}\int_0^\infty e^{\lambda t}\mathcal{I}(t)\,dt - \lambda^{\alpha+2}\int_0^\infty e^{-\lambda t}t^{\alpha+1}E\,dt\right\| \\ &= \lambda^{\alpha+2}\left\|\int_0^\infty e^{-\lambda t}\big(\mathcal{I}(t) - t^{\alpha+1}P\big)\,dt\right\| \\ &\leq \Big[\big|\lambda^{\alpha+2}\big|\int_0^a e^{-\lambda t}\big(\big\|\mathcal{I}(t)\big\| + t^{\alpha+1}\big\|P\big\|\big)\,dt \\ &\quad + \big|\lambda^{\alpha+2}\big|\int_a^\infty e^{-\lambda t}t^{\alpha+1}\big\|\mathcal{C}(t) - P\big\|\,dt\Big]\|x\| \\ &\leq \Big[\big|\lambda^{\alpha+2}\big|a\Big(\sup_{t\leq a}\big\|\mathcal{I}(t)\big\| + a^{\alpha+1}\big\|P\big\|\Big) + (\alpha+1)!\varepsilon\Big]\|x\|.\end{aligned}$$

Then the above estimate implies that $\big\|\lambda R(\lambda, A) - (\alpha+1)!P\big\| \longrightarrow 0$ when $\lambda \to 0^+$, which means that $T(t)$ is uniformly Abel ergodic, and the proof is finished. □

Remark 3.12. Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$, where $\alpha \geq 0$.

(1) If we assume that $\alpha = 0$ in Theorem 3.10, we get the uniform ergodic theorem proved by Lin in [16].
(2) If $T(t)$ is of type $\alpha \geq 1$ satisfying $\lim_{t\to\infty} \|T(t)\|/t = 0$, then it follows from Lemma 3.1 that A is one-to-one. In this case, if the Abel average is convergent, it will converge to zero.
(3) If $T(t)$ is of type $\alpha > 0$ satisfying $\lim_{t\to\infty} \|T(t)\|/t = 0$ and their generator A has a closed range, the strong limit of Cesàro averages $C(t) := \frac{1}{t}\int_0^t T(s)\,ds$ may be divergent, as the following example shows.

Example. Hieber showed in [10] that if an operator A generates a C_0-semigroup on a Banach space $\mathcal{X}$, then its adjoint A^* generates an α-times integrated semigroup on $\mathcal{X}^*$ for all $\alpha > 0$. In particular, let $\mathcal{X}$ be the set of all Lebesgue measurable functions and let $f : \mathcal{X} \to [0,\infty]$ such that

$$\|f\| := \left(\int_0^\infty e^{ps^2}|f(s)|^p\,ds\right)^{\frac{1}{p}} + \left(\int_0^\infty |f(s)|^q\,ds\right)^{\frac{1}{q}} < \infty \quad \text{for } 1 \leq p < q < \infty.$$

Then $(\mathcal{X}, \|.\|)$ is a reflexive Banach space whenever $p > 1$.

Now, let $\{T(t)\}_{t\geq 0}$ be the C_0-semigroup defined by

$$\big(T(t)f\big)(s) := f(t+s) \quad \text{for all } f \in \mathcal{X} \text{ and } s,t \geq 0.$$

Hence $T(t)$ is of type $\omega_0 = 0$ and $\|T(t)\| = 1$ for all $t \geq 0$, where ω_0 is the growth bound of $T(t)$. Thus $\lim_{t\to\infty} \|T(t)\|/t = 0$. Further, their infinitesimal generator is defined by $A = d/dt$ and has empty spectrum. Then

$$\big\|\lambda R(\lambda, A)\big\| = \big\|\lambda(\lambda - A)^{-1}\big\| \to 0 \quad \text{as } \lambda \to 0^+.$$

Hence $T(t)$ is uniformly Abel ergodic to 0. Since $T(t)$ are positive operators for all $t \geq 0$, we have, for every function $f \in \mathcal{X}$,

$$\begin{aligned} \frac{1}{t}\int_0^t T(s)f\,ds &\leq \frac{1}{t}\int_0^t e^{1-s/t}T(s)f\,ds \\ &\leq e\mu\int_0^\infty e^{-\mu s}T(s)f\,ds \quad \text{with } \mu = \frac{1}{t}. \end{aligned}$$

Then,

$$\left\|\frac{1}{t}\int_0^t T(s)\,ds\right\| \leq \big\|e\mu R(\mu, A)\big\|.$$

Consequently, it follows from the above estimate that $T(t)$ is uniformly Cesàro ergodic to 0 when $t \to \infty$, and the ergodic decomposition is given by $\mathcal{X} = \mathcal{R}(A)$.

By Hieber's remark, the adjoint A^* generates an α-times integrated semigroup $\{T^*(t)\}_{t\geq 0}$ on $(\mathcal{X}^*, \|.\|)$, where $\alpha \geq 1$. Thus $\mathcal{R}(A^*)$ is closed and $\lim_{t\to\infty} \|T^*(t)\|/t = 0$. Hence Theorem 3.10 implies that $T^*(t)$ is uniformly Abel ergodic but is not mean Cesàro ergodic. Indeed, assume that $T^*(t)$ is mean Cesàro ergodic; then there exists an operator P such that $\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t T^*(s)g\,ds - Pg \right\| = 0$ for all $g \in \mathcal{X}^*$. Hence $P^2 = P$ and $\mathcal{X}^* = \mathcal{R}(\mathcal{X}^*) \oplus \mathcal{N}(A^*)$, with $P(\mathcal{X}^*) = \mathcal{N}(A^*)$ and $\mathcal{N}(P) = \mathcal{R}(A^*)$ by the mean ergodic decomposition. Since $\lim_{t\to\infty} \|T^*(t)\|/t = 0$, A^* is one-to-one. Therefore, $P(\mathcal{X}^*) = \{0\}$, $\mathcal{X}^* = \mathcal{R}(A^*)$, and

$$\lim_{t\to+\infty} \left\| \frac{1}{t} \int_0^t T^*(s)g\,ds \right\| = 0 \quad \text{for all } g \in \mathcal{X}^*.$$

Let $g \in \mathcal{X}^* \backslash \{0\}$; applying Proposition 2.5, we get

$$A^* \frac{1}{t} \int_0^t T^*(s)g\,ds = \frac{T^*(t)g}{t} - \frac{t^{\alpha-1}g}{\Gamma(\alpha+1)}.$$

Since A^* is invertible, we get the following inequality:

$$\left\| \frac{t^{\alpha-1}g}{\Gamma(\alpha+1)} \right\| \leq \frac{1}{\|(A^*)^{-1}\|} \left\| \frac{1}{t} \int_0^t T^*(s)g\,ds \right\| + \frac{\|T^*(t)g\|}{t}.$$

It follows that

$$\lim_{t\to+\infty} \frac{t^{\alpha-1}g}{\Gamma(\alpha+1)} = 0.$$

Since $\alpha \geq 1$, we get $g = 0$, absurd. Hence $T^*(t)$ is not mean Cesàro ergodic.

Proposition 3.13. *Let A be the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$ with $\alpha \geq 0$. If $\sup_{t\geq 0} \left\| \frac{1}{t^{\alpha+1}} \int_0^t T(s)\,ds \right\| \leq M$ for some $M > 0$, then $\mathcal{S}(A) \leq 0$, which means that $R(\lambda, A)$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.*

Proof. Assume that there exists $M > 0$ such that

$$\sup_{t\geq 0} \left\| \frac{1}{t^{\alpha+1}} \int_0^t T(s)\,ds \right\| \leq M \quad \text{for all } \alpha \geq 0.$$

Set $\mathcal{I}(t) = \int_0^t T(s)\,ds$ for all $t \geq 0$, and let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$. Then, for all $0 < u < v$ and $x \in \mathcal{X}$, we have

$$\begin{aligned}
\left\| \lambda^\alpha \int_u^v e^{-\lambda t} T(t)x\,dt \right\| &= \left\| \lambda^\alpha \left[e^{-\lambda t}\mathcal{I}(t)x \right]_u^v + \lambda^{\alpha+1} \int_u^v e^{-\lambda t}\mathcal{I}(t)x\,dt \right\| \\
&\leq M \left\| \lambda^\alpha \left[e^{-\lambda t} t^{\alpha+1} \right]_u^v + \lambda^{\alpha+1} \int_u^v e^{-\lambda t} t^{\alpha+1}\,dt \right\| \|x\| \\
&\leq M \Big[|\lambda^\alpha| \left(e^{-\lambda v} v^{\alpha+1} + e^{-\lambda u} u^{\alpha+1} \right) \\
&\qquad + |\lambda^{\alpha+1}| \int_u^v e^{-\operatorname{Re}\lambda\, t} t^{\alpha+1}\,dt \Big] \|x\|.
\end{aligned}$$

Hence, for all $\alpha \geq 0$, we get $\left\| \lambda^\alpha \int_u^v e^{-\lambda t} T(t)\,dt \right\| \longrightarrow 0$ when $u \to \infty$. Therefore, $R(\lambda, A)$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, which means that $\mathcal{S}(A) \leq 0$. □

The power convergence of the Abel average $\mathcal{A}(\lambda)$ has been studied by several authors for the class of C_0-semigroups acting on $\mathcal{B}(\mathcal{X})$ (see, for instance, [13] and [17]). In the same direction, we obtain the following results.

Theorem 3.14. *Let $\{T(t)\}_{t\geq 0}$ be an α-times integrated semigroup on $\mathcal{B}(\mathcal{X})$, with $\alpha \geq 0$. If $T(t)$ is uniformly Abel ergodic, then, for small enough $\lambda > 0$, the sequence $\{\mathcal{A}(\lambda)^n\}_{n\in\mathbb{N}}$ converges in $\mathcal{B}(\mathcal{X})$.*

Proof. We assume that $T(t)$ is uniformly Abel ergodic; then there exists $P \in \mathcal{B}(\mathcal{X})$ such that $\lim\limits_{\lambda\to 0^+} \|\mathcal{A}(\lambda) - P\| = 0$, with $P = P^2 = T(t)P = PT(t)$ for all $t \geq 0$, which is equivalent to $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$, and $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$ by Theorem 3.4.

Moreover, we have from Lemma 3.1 $\mathcal{R}(P) = \mathcal{N}(A) = \mathcal{N}\big(\lambda R(\lambda, A) - I\big)$; hence, for $\lambda > 0$ and each $n \in \mathbb{N}$, we get

$$\lambda R(\lambda, A)\, P = P \quad \text{and} \quad \big(\lambda R(\lambda, A)\big)^n\, P = P.$$

Clearly, $I - P$ is the projection of $\mathcal{X}$ onto $\mathcal{R}(A)$ along $\mathcal{N}(A)$. Then, we have

$$\mathcal{R}(I - P) = \mathcal{R}(A) = \mathcal{R}\big(\lambda R(\lambda, A) - I\big).$$

So, for $x \in \mathcal{X}$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \big\|\big[\big(\lambda R(\lambda, A)\big)^n - P\big]x\big\| &= \big\|\big[\big(\lambda R(\lambda, A)\big)^n - \big(\lambda R(\lambda, A)\big)^n P\big]x\big\| \\ &= \big\|\big(\lambda R(\lambda, A)\big)^n (I - P)x\big\| \\ &\leq \big\|\big(\lambda R(\lambda, A)\big)^n|_{\mathcal{R}(A)}\big\|\, \|I - P\|\, \|x\|. \end{aligned}$$

As mentioned in Proposition 3.9, $T(t)$ is uniformly Abel ergodic if and only if $\mathcal{R}(A)$ is closed and $\|\lambda R(\lambda, A)|_{\mathcal{R}(A)}\| \longrightarrow 0$ as $\lambda \to 0^+$. Then, for a small enough $\lambda > 0$, the operator $\lambda R(\lambda, A)$ is a strict contraction on $\mathcal{R}(A)$, which means that $\|\lambda R(\lambda, A)|_{\mathcal{R}(A)}\| < 1$. Consequently,

$$\big\|\big(\lambda R(\lambda, A)\big)^n|_{\mathcal{R}(A)}\big\| \longrightarrow 0 \quad \text{as } n \to \infty.$$

Then, it is easy to see from the above estimate that, for such fixed λ, where $0 < \lambda < \delta$, the sequence $\{\mathcal{A}(\lambda)^n\}_{n\in\mathbb{N}}$ converges in $\mathcal{B}(\mathcal{X})$. □

Corollary 3.15. *Let $\{T(t)\}_{t\geq 0}$ be an α-times integrated semigroup on $\mathcal{B}(\mathcal{X})$, with $\alpha \geq 0$. $T(t)$ is uniformly Abel ergodic if and only if the sequence $\{\mathcal{A}(\lambda)^n\}_{n\in\mathbb{N}}$ for some $\lambda > 0$ converges in $\mathcal{B}(\mathcal{X})$.*

Proof. The first implication follows from Theorem 3.14.

Conversely, we assume that there exists $\lambda > 0$ such that the Abel average $\mathcal{A}(\lambda)$ is uniformly power convergent; then the discrete Cesàro mean $\mathcal{M}_n\big(\lambda R(\lambda, A)\big)$ defined by

$$\mathcal{M}_n\big(\lambda R(\lambda, A)\big) = \frac{1}{n}\sum_{k=0}^{n-1} \big(\lambda R(\lambda, A)\big)^k$$

converges uniformly in $\mathcal{B}(\mathcal{X})$, and by the uniform ergodic theorem [15], we have

$$\mathcal{X} = (\lambda R(\lambda, A) - I)\mathcal{X} \oplus \mathcal{N}(\lambda R(\lambda, A) - I).$$

It follows that $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$ by Lemma 3.1. Therefore, Theorem 3.4 implies that $T(t)$ is uniformly Abel ergodic. □

Corollary 3.16. *Let $\{T(t)\}_{t\geq 0}$ be an α-times integrated semigroup on $\mathcal{B}(\mathcal{X})$, with $\alpha \geq 0$. The following statements are equivalent:*

(1) *$T(t)$ is uniformly Abel ergodic.*
(2) *The sequence $\{\mathcal{A}(\lambda)^n\}_{n\in\mathbb{N}}$, for some $\lambda > 0$, converges in $\mathcal{B}(\mathcal{X})$.*
(3) *The sequence $\{\mathcal{A}(\lambda)^n\}_{n\in\mathbb{N}}$, for all $\lambda > 0$, converges in $\mathcal{B}(\mathcal{X})$.*
(4) *The discrete Cesàro mean $\mathcal{M}_n\big(\lambda R(\lambda, A)\big)$, for some $\lambda > 0$, converges in $\mathcal{B}(\mathcal{X})$.*
(5) *The discrete Cesàro mean $\mathcal{M}_n\big(\lambda R(\lambda, A)\big)$, for all $\lambda > 0$, converges in $\mathcal{B}(\mathcal{X})$.*

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