THE r-DYNAMIC EDGE COLORING OF A CLOSED HELM GRAPH

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ABSTRACT. As a natural generalization of the classical coloring problem in graph theory, the dynamic coloring problem deals with the existence of a proper coloring c of a graph so that $|c(N(v))| \geq \min\{r, d(v)\}$ for every vertex v. In this paper, we obtain the r-dynamic edge chromatic number of any given closed helm graph for any positive integer r. This coincides with the r-dynamic chromatic number of the line graph of a closed helm graph.

1. INTRODUCTION

In 2001, Montgomery [21] (see also [13]) introduced the concept of r-*dynamic proper k-coloring* of a graph $G = (V(G), E(G))$ as a proper k-coloring c such that

$$|c(N(v))| \geq \min\{r, d(v)\} \tag{1.1}$$

for every vertex $v \in V(G)$, where $N(v)$ and $d(v)$ denote, respectively, the neighborhood and the degree of the vertex v. In addition, he introduced the notion of r-*dynamic chromatic number* $\chi_r(G)$ of the graph G as the minimum positive integer k for which an r-dynamic proper k-coloring exists. Both concepts generalize in a natural way those classical ones concerning the proper coloring and the chromatic number of a graph, which arise indeed when $r = 1$. Since the original work of Montgomery, the case $r > 1$ has widely been dealt with in the literature for different types of graphs [8, 6, 7, 11, 12, 14, 15, 19, 22, 24], with particular emphasis in the case $r = 2$ [1, 2, 3, 4, 5]. Of particular interest for the topic of this paper, let us remark the studies of Mohanapriya et al. [20] and Furmańczyk et al. [9] concerning the r-dynamic coloring of different families of helm graphs. In particular, the last mentioned paper deals with the r-dynamic chromatic number of the line graph of a helm graph, which coincides as such with the so-called r-dynamic edge chromatic number of a helm graph. In this last regard, Meganingtyas [17] introduced the

concept of *r-dynamic proper k-edge-coloring* of a graph $G = (V(G), E(G))$ as a proper k-edge-coloring c such that

$$|c(N(uv))| \geq \min\{r, d(u) + d(v) - 2\}$$

for every pair of adjacent vertices $u, v \in V(G)$, where $N(uv)$ denotes the set of edges that are incident to that one containing both vertices u and v.

The *r-dynamic edge chromatic number* $\lambda_r(G)$ of the graph G is the minimum positive integer k for which an r-dynamic proper k-edge-coloring exists. Again, both concepts generalize the classical ones of proper edge-coloring and edge chromatic number, which arise when $r = 1$. Meganingtyas [17] dealt in particular with the r-dynamic edge chromatic number of paths, cycles, star graphs, wheel graphs, friendship graphs, prism graphs and ladder graphs. Some other kinds of graphs for which this value has been studied are lobster graphs, butterfly graphs, diamond graphs [16] and trees [18]. In addition, Nandini et al. [23] have recently dealt with the r-dynamic chromatic number of the line graph (and hence, the r-dynamic edge chromatic number) of any bistar graph. This paper delves into this topic by focusing on the r-dynamic chromatic number of the line graph of any closed helm graph.

The paper is organized as follows. In Section 2, we describe some preliminary concepts and results on graph theory that are used throughout the paper. Then, in Section 3, after establishing certain lower bounds for the r-dynamic edge chromatic number of a closed helm graph, we determine such a number for any positive integer r and any order of the closed helm graph under consideration.

2. Preliminaries

This section deals with some preliminary concepts and results on graph theory that are used throughout the paper. See [10] for more details about this topic.

A *graph* is any pair $G = (V(G), E(G))$ that is formed by a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*. A *subgraph* of the graph G is any other graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Each edge joins two vertices, which are then said to be *adjacent*. In addition, two edges sharing a vertex are said to be *incident*. From now on, we denote by vw the edge joining two vertices $v, w \in V(G)$. It constitutes a *loop* if $v = w$. Two edges joining the same pair of vertices are said to be *parallel*. A graph is *simple* if it contains no loops and no parallel edges. Further, the number of vertices and the number of edges of a graph are, respectively, its *order* and its *size*. If both of them are finite, then the graph is said to be *finite*. From now on, all the graphs are considered to be simple and finite. A graph is called *complete* if all its vertices are pairwise adjacent. The set of vertices of any complete subgraph of the graph G is called a *clique*. The complete graph of order n is denoted by K_n.

The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the subset of vertices in $V(G)$ that are adjacent to v. The cardinality $d_G(v)$ of this set is the *degree* of the vertex v. From now on, we write $N(v)$ and $d(v)$ when there is no risk of confusion. If $d(v) = 1$, then the vertex v is said to be *pendant*. A *pendant edge* is any edge

containing a pendant vertex. Furthermore, $\delta(G)$ and $\Delta(G)$ denote, respectively, the minimum and maximum vertex degree of the graph G.

The *cycle* C_n, with $n > 2$, is a graph for which $V(C_n) = \{v_0, \ldots, v_{n-1}\}$ and $E(C_n) = \{v_0 v_1, v_1 v_2, \ldots, v_{n-2} v_{n-1}, v_{n-1} v_0\}$. If every vertex v_i is joined to a new vertex v, then the resulting graph is the *wheel graph* W_{n+1}. If besides, each vertex v_i is joined to a pendant vertex p_i, then the resulting graph is the *helm graph* H_n. Finally, if the set of edges $\{p_0 p_1, \ldots, p_{n-2} p_{n-1}, p_{n-1} p_0\}$ is also added, then we get the *closed helm graph* CH_n. Figure 1 illustrates these four types of graphs.
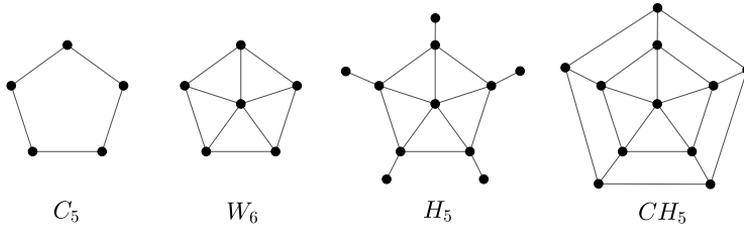

FIGURE 1. Examples of a cycle, a wheel graph, a helm graph and a closed helm graph.

The *line graph* $L(G)$ of a graph G is the graph having as vertices the edges of G, and such that two vertices in $L(G)$ are adjacent if and only if the associated two edges in G are incident. Figure 2 illustrates the line graph of the closed helm graph CH_5.
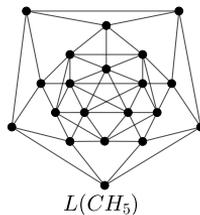

FIGURE 2. Line graph of the closed helm graph CH_5.

A *proper k-coloring* of a graph $G = (V(G), E(G))$ is any map $c : V(G) \to \{0, \ldots, k-1\}$ that assigns a set of k *colors* to the set of vertices $V(G)$ so that no two adjacent vertices share the same color. Similarly, a *proper k-edge-coloring* of the graph G is defined as any map $c : E(G) \to \{0, \ldots, k-1\}$ so that no two incident edges share the same color. The *chromatic number* $\chi(G)$ (respectively, the *edge chromatic number* $\lambda(G)$) of the graph G is the minimum positive integer k for which a proper k-coloring (respectively, a proper k-edge-coloring) of G exists. Particular examples are the *r-dynamic proper k-(edge-)coloring* and the *r-dynamic (edge) chromatic number* of a graph G, which have already been described in the

introductory section. In particular, $\chi_1(G) = \chi(G)$ and $\lambda_1(G) = \lambda(G)$. Moreover, as it has already been indicated in the introductory section, $\lambda_r(G) = \chi_r(L(G))$ for all positive integer r. The following result is also known.

Lemma 2.1 ([13]). *Let G be a graph and let r be a positive integer. Then,*

$$\min\{r, \Delta(G)\} + 1 \leq \chi_r(G) \leq \chi_{r+1}(G).$$

Moreover, $\chi_r(G) \leq \chi_{\Delta(G)}(G)$.

3. Dynamic edge coloring of a closed helm graph

In this section, we study the r-dynamic edge chromatic number of a closed helm graph CH_n, with $n > 2$, of set of vertices

$$V(CH_n) = \{v, v_0, \ldots, v_{n-1}, p_0, \ldots, p_{n-1}\},$$

where we follow the notation described in the preliminary section. To this end, we focus on the equivalent computation of the r-dynamic chromatic number of the line graph $L(CH_n)$, whose respective sets of vertices and edges are

$$V(L(CH_n)) = E(CH_n) = \{vv_i, \, v_i v_{i+1}, \, v_i p_i, \, p_i p_{i+1} \colon 0 \leq i < n\}$$

and

$$
\begin{aligned}
E(L(CH_n)) = \{&(vv_i)(vv_j) \colon 0 \leq i, j < n, \, i \neq j\} \\
\cup \{&(vv_i)(v_{i-1}v_i), \, (vv_i)(v_i v_{i+1}), \, (vv_i)(v_i p_i) \colon 0 \leq i < n\} \\
\cup \{&(v_i v_{i+1})(v_i p_i), \, (v_i v_{i+1})(v_{i+1} p_{i+1}) \colon 0 \leq i < n\} \\
\cup \{&(v_i p_i)(p_{i-1} p_i), \, (v_i p_i)(p_i p_{i+1}) \colon 0 \leq i < n\} \\
\cup \{&(v_i v_{i+1})(v_{i+1} v_{i+2}), \, (p_i p_{i+1})(p_{i+1} p_{i+2}) \colon 0 \leq i < n\},
\end{aligned}
$$

where all the indices are taken modulo n (this convention is considered throughout the whole paper). In particular, for each non-negative integer $i \leq n$, we have that

$$d(vv_i) = n + 2, \qquad d(v_i v_{i+1}) = 6, \qquad d(v_i p_i) = 5 \quad \text{and} \quad d(p_i p_{i+1}) = 4.$$

Thus, $\delta(L(CH_n)) = 4$ and

$$\Delta(L(CH_n)) = \begin{cases} 6 & \text{if } n = 3, \\ n+2 & \text{otherwise.} \end{cases}$$

The following results establish some lower bounds of the r-dynamic edge coloring of a closed helm graph.

Lemma 3.1. *Let r and n be two positive integers, with $n > 2$. Then,*

$$\lambda_r(CH_n) = \chi_r(L(CH_n)) \geq \begin{cases} 4 & \text{if } n = 3, \\ n & \text{otherwise.} \end{cases}$$

Proof. Notice that the four vertices vv_0, $v_{n-1}v_0$, $v_0 v_1$ and $v_0 p_0$ determine a clique of order four within the line graph $L(CH_n)$. Thus, since every r-dynamic coloring constitutes a proper coloring, we have that $\chi_r(L(CH_n)) \geq \chi(K_4) = 4$ for all positive integer $n > 2$. In a similar way, since the subset of vertices $\{vv_i \colon 0 \leq i <$

$n\} \subset V(L(CH_n))$ determines a clique of order n within the line graph $L(CH_n)$, we have that $\chi_r(L(CH_n)) \geq \chi(K_n) = n$ for all $n \geq 4$. $\square$

Figures 3–5 illustrate how the lower bound in Lemma 3.1 is respectively reached for $r \leq n = 3$, $r < n \in \{4, 5\}$ and $r \leq n - 2 < n \in \{6, 7\}$. The following result shows that the case $(r, n) \in \{(5, 6), (6, 7)\}$ requires extra colors.
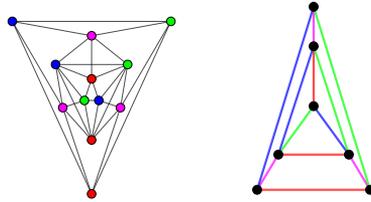

FIGURE 3. r-dynamic proper 4-coloring of the line graph $L(CH_3)$ (left) and r-dynamic proper 4-edge-coloring of the closed helm graph CH_3 for all $r \leq 3$ (right).
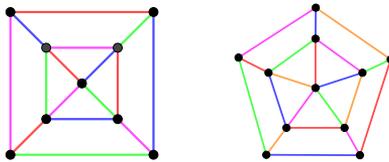

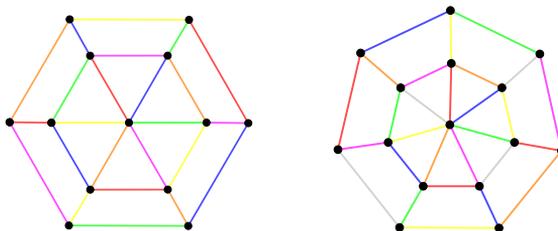FIGURE 4. r-dynamic proper n-edge-coloring of the graph CH_n for all $r < n \in \{4, 5\}$.

FIGURE 5. r-dynamic proper n-edge-coloring of the graph CH_n for all $r \leq n - 2 < n \in \{6, 7\}$.

Proposition 3.1. *Let $n \in \{6, 7\}$. Then,*

$$\lambda_{n-1}(CH_n) = \chi_{n-1}(L(CH_n)) \geq \begin{cases} 7 & \text{if } n = 6, \\ 9 & \text{if } n = 7. \end{cases}$$

Proof. From Lemma 2.1, we have that $n \leq \chi_{n-1}(L(CH_n))$. So, let us suppose the existence of an $(n-1)$-dynamic proper n-coloring c of $L(CH_n)$. From Condition (1.1), we have that $|c(N(v_i p_i))| = 5 = d(v_i p_i)$ for all non-negative integer $i < n$. Thus, the set $c(N(v_0 v_1))$ does not contain any of the two distinct colors $c(v_0 v_1)$ and $c(p_0 p_1)$. As such, $|c(N(v_0 v_1))| \leq n - 2$, which contradicts Condition (1.1). Hence, $\chi_{n-1}(L(CH_n)) \geq n + 1$ for all $n \in \{6, 7\}$.

Now, let c be a 6-dynamic proper 8-coloring of the line graph $L(CH_7)$. Condition (1.1) implies that $|c(N(v_i p_i))| = 5 = d(v_i p_i)$ and $|c(N(v_i v_{i+1}))| = 6 = d(v_i v_{i+1})$ for every non-negative integer $i < 7$. As a consequence, the set

$$\mathcal{S}_i := c(N(v_i v_{i+1})) \cup \{c(v_i v_{i+1}), c(p_i p_{i+1})\}$$

is formed by all the eight colors associated to the map c. Then, since Condition (1.1) also implies that $|c(N(p_i p_{i+1}))| = 4 = d(p_i p_{i+1})$ for all non-negative integer $i < 7$, exactly one of the following three conditions holds:

- $c(p_i p_{i+1}) = c(vv_{i-1}) = c(v_{i+2} v_{i+3}) = c(p_{i+4} p_{i+5})$;
- $c(p_i p_{i+1}) = c(v_{i-2} v_{i-1}) = c(vv_{i+2}) = c(p_{i+3} p_{i+4})$;
- $c(p_i p_{i+1}) = c(v_{i-2} v_{i-1}) = c(v_{i+2} v_{i+3})$.

In particular, if $i = 0$, then the following study of cases arises. All the stages in this study are sequentially described as immediate consequences of applying that $|c(N(v_i p_i))| = 5$, $|c(N(v_i v_{i+1}))| = 6$ and $|\mathcal{S}_i| = 8$ for every non-negative integer $i < 7$, together with the three described conditions and the fact that c is a proper coloring.

- **Case 1**: $c(p_0 p_1) = c(vv_6) = c(v_2 v_3) = c(p_4 p_5)$.
  - **Subcase 1.1**: $c(p_1 p_2) = c(vv_0) = c(v_3 v_4) = c(p_5 p_6)$, which implies that $c(p_3 p_4) = c(v_1 v_2)$.
    - ∗ **Subcase 1.1.1**: $c(p_2 p_3) = c(vv_1) = c(v_4 v_5) = c(p_0 p_6)$, which implies that $c(v_5 v_6) = c(v_1 v_2)$ and $c(v_6 p_6) = c(v_1 p_1)$. But then, $c(v_1 p_1) \in \{c(vv_3), c(v_3 p_3)\} \cap \{c(vv_4), c(v_4 p_4)\}$. This implies that $|c(N(v_3 v_4))| < 6$, which is not possible.
    - ∗ **Subcase 1.1.2**: $c(p_2 p_3) = c(v_0 v_1) = c(v_4 v_5)$, which implies that $c(v_3 p_3) = c(vv_1)$ and $c(vv_3) = c(v_1 p_1) = c(v_5 v_6)$. Then, $c(vv_5) = c(v_1 v_2) = c(p_0 p_6)$ and $c(v_5 p_5) = c(vv_1)$. But then, $|\mathcal{S}_i| < 8$ for some $i \in \{3, 4, 5\}$, which is not possible.
  - **Subcase 1.2**: $c(p_1 p_2) = c(v_0 v_6) = c(v_3 v_4)$, which implies sequentially that $c(v_2 p_2) = c(vv_0)$, $c(vv_2) = c(v_0 p_0)$ and $c(p_3 p_4) = c(v_1 v_2)$.
    - ∗ **Subcase 1.2.1**: $c(p_2 p_3) = c(vv_1) = c(v_4 v_5) = c(p_0 p_6)$, which implies that $c(v_4 v_5) = c(v_1 v_2)$ and $c(v_6 p_6) = c(v_1 p_1)$. But then, $c(v_1 p_1) \in \{c(vv_3), c(v_3 p_3)\} \cap \{c(vv_4), c(v_4 p_4)\}$. This implies that $|\mathcal{S}_3| < 8$, which is not possible.

* **Subcase 1.2.2**: $c(p_2p_3) = c(v_0v_1)$, which implies sequentially that $c(v_3p_3) = c(vv_1)$, $c(vv_3) = c(v_1p_1) = c(v_5v_6)$, $c(vv_5) = c(v_1v_2) = c(p_0p_6)$, $c(v_6p_6) = c(vv_1)$ and $c(v_4v_5) = c(v_0p_0)$. But then, $|\mathcal{S}_i| < 8$ for some $i \in \{3, 4, 5\}$, which is not possible.
- **Case 2**: $c(p_0p_1) = c(v_5v_6) = c(vv_2) = c(p_3p_4)$. This case is symmetric to the precedent one. So, a similar reasoning gives rise to a contradiction.
- **Case 3**: $c(p_0p_1) = c(v_5v_6) = c(v_2v_3)$.
  - **Subcase 3.1**: $c(p_1p_2) = c(vv_0) = c(v_3v_4) = c(p_5p_6)$. Up to relabeling of indices, this particular condition coincides with the assumptions of the first case. So, no coloring is possible from here.
  - **Subcase 3.2**: $c(p_1p_2) = c(v_0v_6) = c(vv_3) = c(p_4p_5)$. Again, this particular condition is equivalent, up to relabeling of indices and symmetry, to the one described in the first case. So, no coloring is possible from here.
  - **Subcase 3.3**: $c(p_1p_2) = c(v_0v_6) = c(v_3v_4)$.
    * **Subcase 3.3.1**: $c(p_2p_3) = c(vv_1) = c(v_4v_5) = c(p_0p_6)$, which is equivalent to the first case and hence, no coloring is possible from here.
    * **Subcase 3.3.2**: $c(p_2p_3) = c(v_0v_1) = c(vv_4) = c(p_5p_6)$, which is again equivalent to the first case. No coloring is possible from here.
    * **Subcase 3.3.3**: $c(p_2p_3) = c(v_0v_1) = c(v_4v_5)$, which implies that $c(p_3p_4) = c(v_1v_2) = c(vv_5) = c(p_0p_6)$. Again, this last condition is equivalent to the one described in the first case. So, no coloring is possible from here.

As a consequence, $\chi_6(L(CH_7)) \geq 9$. $\qquad\qquad\square$

We establish now a lower bound for the case $4 \leq r = n$ (notice in this regard that the case $r = n = 3$ is already dealt with in Figure 3).

Proposition 3.2. *Let $n \geq 4$ be a positive integer. Then,*

$$\lambda_n(CH_n) = \chi_n(L(CH_n)) \geq \begin{cases} 8 & \text{if } n = 5, \\ n+2 & \text{otherwise.} \end{cases}$$

Proof. The following study of cases arises.

- **Case $n = 4$.**

  Let us suppose the existence of a 4-dynamic proper 5-coloring c of the line graph $L(CH_4)$. From Condition (1.1), we have that $|c(N(p_ip_{i+1}))| = 4 = d(p_ip_{i+1})$ for all non-negative integer $i < n$. Thus, since $N(p_0p_1) \cap N(p_2p_3) = \{p_1p_2, p_3p_0\}$ and the map c is a proper coloring, we have that

  $$c(p_2p_3) \in \{c(p_0p_1), c(v_0p_0), c(v_1p_1)\}.$$

If $c(p_2p_3) = c(v_0p_0)$, then $|c(N(p_3p_0))| \leq 3$, which contradicts Condition (1.1). Similarly, $c(p_2p_3) \neq c(v_1p_1)$ and hence, $c(p_2p_3) = c(p_0p_1)$. But then, $|c(N(p_3p_0)| \leq 3$, which contradicts again Condition (1.1). As a consequence, no such map c exists and hence, $\chi_4(L(CH_4)) \geq 6$.

- **Case** $n = 5$.

  Let us suppose the existence of a 5-dynamic proper 7-coloring c of the line graph $L(CH_5)$. Condition (1.1) implies that $|c(N(v_i p_i))| = 5 = d(v_i p_i)$ and $|c(N(p_i p_{i+1}))| = 4 = d(p_i p_{i+1})$ for all $i < 5$. As a consequence, $|c(N(v_i p_i)) \cup \{c(v_i p_i)\}| = 6$. All these conditions together imply that either $c(p_i p_{i+1}) = c(vv_{i-1}) = c(v_{i+2} v_{i+3})$ for all $i < 5$, or $c(p_i p_{i+1}) = c(v_{i-2} v_{i-1}) = c(vv_{i+2})$ for all $i < 5$. Otherwise, it would be either $|c(N(v_{i-1} v_i))| < 5$, or $|c(N(v_{i+1} v_{i+2}))| < 5$ for, at least, one non-negative integer $i < 5$, which is not possible.

  The resulting coloring requires that $c(v_i p_i) = c(v_{i+1} p_{i+1})$ for some $i < 5$. This would imply that $|c(N(p_i p_{i+1}))| < 4$, which is not possible. As a consequence, $\chi_5(L(CH_5)) \geq 8$.

- **Case** $n \geq 6$.

  Let us suppose the existence of an n-dynamic proper $(n + 1)$-coloring c of the line graph $L(CH_n)$. Without loss of generality, we may assume that $c(vv_i) = i$ for all $i < n$. Then, Condition (1.1) implies that

  $$n \in \{c(v_{i-1} v_i), c(v_i, v_{i+1}), c(v_i p_i)\} \quad \text{for all } i < n, \tag{3.1}$$

  because $d(vv_i) = n + 2 > n$. Nevertheless, $c(v_i v_{i+1}) \neq n$, because Condition (1.1) implies that $|c(N(v_{i+1} v_{i+2}))| = 6 = d(v_{i+1} v_{i+2})$, and hence, it should be

  $$n \notin \{c(v_{i+1} v_{i+2}), c(v_{i+2} v_{i+3}), c(v_{i+2} p_{i+2})\},$$

  which contradicts (3.1). Similarly, we get that $c(v_{i-1} v_i) \neq n$. As a consequence, it must be $c(v_i p_i) = n$ for all non-negative integer $i < n$. But then, $|c(N(v_i v_{i+1}))| \leq 5 < 6 = d(v_i v_{i+1}) \leq n$, which contradicts Condition (1.1). Hence, the existence of the map c is not possible and the result holds. $\square$

We focus now on the case $r = n + 1$.

Proposition 3.3. *Let $n \geq 3$ be a positive integer. Then,*

$$\lambda_{n+1}(CH_n) = \chi_{n+1}(L(CH_n)) \geq \begin{cases} 10 & \text{if } n = 5, \\ n + 3 & \text{if } n \equiv 0 \,(\text{mod}\, 3) \text{ or } n = 4, \\ n + 4 & \text{otherwise.} \end{cases}$$

Proof. The following study of cases arises.

- **Case** $n = 3$.

  Let c be a 4-dynamic proper 5-coloring c of the line graph $L(CH_3)$. From Condition (1.1), we have that $|c(N(p_i p_{i+1}))| = 4 = d(p_i p_{i+1})$ for all positive integer $i < 3$. Then, since $|N(p_0 p_1) \cap N(p_1 p_2)| = 2$ and both vertices $p_0 p_1$ and $p_1 p_2$ are adjacent, it must be $c(v_0 p_0) = c(v_2 p_2)$. But then, $|c(N(p_2 p_0))| \leq 3$, which contradicts Condition (1.1). Hence, $\chi_4(L(CH_3)) \geq 6$.

- **Case** $n = 4$.

  Let us suppose the existence of a 5-dynamic proper 6-coloring c of the line graph $L(CH_4)$. From Condition (1.1), the set $\{c(p_i p_{i+1}) \colon 0 \leq i < 4\}$ is

formed by four distinct colors. Moreover, $c(v_0 p_0) = c(v_2 p_2)$ and $c(v_1 p_1) = c(v_3 p_3)$ are the remaining two colors. Then, since the map c is a proper coloring and $|c(N(v_i p_{i+1}))| = 5 = d(v_i p_{i+1})$ because of Condition (1.1), it must be $c(vv_i) = c(v_{i+1} p_{i+1})$ for all positive integer $i < 4$. But then, there is a clique of order four within the line graph CH_4 whose vertices are colored with only two colors. This contradicts the fact that the map c is a proper coloring and hence, $\chi_5(L(CH_4)) \geq 7$.

- **Case $n = 5$.**

  It is simply verified that the existence of three vertices sharing the same color within $L(CH_5)$ contradicts Condition (1.1). Then, the result follows straightforwardly from the fact that this line graph contains 20 vertices.

- **Case $n \geq 6$.**

  Let c be an $(n+1)$-dynamic proper $(n+2)$-coloring of the line graph $L(CH_n)$. Without loss of generality, we may assume that $c(vv_i) = i$ for all positive integer $i < n$. Then, Condition (1.1) implies that $\{n, n+1\} \subseteq c(N(vv_i)) \cap c(N(vv_{i+1}))$ for all $i < n$. Thus, $|c(N(v_i v_{i+1}))| \leq 5$, which contradicts Condition (1.1), and hence, $\chi_{n+1}(L(CH_n)) \geq n + 3$.

  Now, let c' be an $(n+1)$-dynamic proper $(n+3)$-coloring of $L(CH_n)$. Again, we can suppose that $c'(vv_i) = i$ for all $i < n$. From Condition (1.1), it must be $|c'(N(vv_i))| \geq n + 1$ and $|c'(N(v_i v_{i+1}))| = 6$ for all $i < n$. This implies that $c'(v_i v_{i+1}) \notin \{n, n+1, n+2\}$ for all $i < n$. As a consequence, $\{n, n+1, n+2\} = \{c'(v_{i-1} v_i), c'(v_i v_{i+1}), c'(v_{i+1} v_{i+2})\}$ and $c'(v_i v_{i+1}) = c'(v_{i+3} v_{i+4})$ for all $i < n$.   □

Finally, the following result deals with the case $r = n + 2$.

Proposition 3.4. *Let $n \geq 3$ be a positive integer. Then,*

$$\lambda_{n+2}(CH_n) = \chi_{n+2}(L(CH_n)) \geq \begin{cases} n + 5 & \text{if } n \in \{3, 5k, 6k\}, \text{ with } k > 0, \\ n + 6 & \text{otherwise.} \end{cases}$$

Proof. Let us study separately each case.

- **Case $n \in \{3, 5k, 6k\}$, with $k > 0$.**

  Let c be an $(n+2)$-dynamic proper $(n+4)$-coloring of $L(CH_n)$. From Condition (1.1), we have that $|c(N(vv_i))| = n + 2 = d(vv_i)$ for all $i < n$. Without loss of generality, we may assume that $c(vv_i) = i$ for all $i < n$, and hence, $\{c(v_{i-1} v_i), c(v_i p_i), c(v_i v_{i+1})\} \subset \{n, n+1, n+2, n+3\}$ for all $i < n$. Then, the following assertions are simply verified.
  - If $n = 3$, then $|c(N(v_i v_{i+1}))| < 5$ for some $i < 3$.
  - If $n \in \{5k, 6k\}$ for some $k > 0$, then $|c(N(v_i v_{i+1}))| < 6 = d(v_i v_{i+1})$ for all $i < n$.

  Both assertions contradict Condition (1.1) and hence, the result holds.

- **Case $3 < n \notin \{5k, 6k\}$ for all $k > 0$.**

  Let c be an $(n+2)$-dynamic proper $(n+5)$-coloring of $L(CH_n)$. Again, we may assume that $c(vv_i) = i$ for all $i < n$. Then, Condition (1.1) implies that, for each positive integer $i < n$, we have $\{c(v_{i-1} v_i), c(v_i p_i), c(v_i v_{i+1}),$

$c(v_{i+1}p_{i+1})$, $c(v_{i+1}v_{i+2})\} = \{n, \ldots, n+4\}$ and $\{c(v_{i-1}v_i), c(v_ip_i)\})\} = \{c(v_{i+2}p_{i+2}), c(v_{i+2}v_{i+3})\})\}$. This is not possible when $n \notin \{5k, 6k\}$ for all $k > 0$. Hence, $\chi_{n+2}(L(CH_n)) \geq n+6$. $\qquad\square$

The following theorem is the main result of the paper. It establishes the r-dynamic chromatic number of the line graph $L(CH_n)$.

Theorem 3.2. *Let r and n be two positive integers, with $n > 2$. Then,*

$$\lambda_r(CH_n) = \chi_r(L(CH_n)) = \begin{cases} n & \text{if } \max\{3, r\} < n \text{ and } (r,n) \neq \begin{cases}(5,6),\\(6,7),\end{cases} \\ n+1 & \text{if } \begin{cases} r \leq n = 3, \\ (r,n) = (5,6), \end{cases} \\ n+2 & \text{if } \begin{cases} (r,n) = (6,7), \\ 4 \leq r = n \neq 5, \end{cases} \\ n+3 & \text{if } \begin{cases} r = 5 \in \{n, n+1\}, \\ r-1 = n \equiv 0 \,(\text{mod } 3), \end{cases} \\ n+4 & \text{if } 5 \neq r-1 = n \not\equiv 0 \,(\text{mod } 3), \\ n+5 & \text{if } \begin{cases} r-1 = n = 5, \\ r-2 = n = 3, \\ r-2 \geq n \in \{5k, 6k\}, \text{ with } k > 0, \end{cases} \\ n+6 & \text{otherwise.} \end{cases}$$

Proof. The case $r \leq n = 3$ follows from Lemma 3.1 and Figure 3. Let us study separately the remaining cases by defining an appropriate r-dynamic proper coloring c of the corresponding line graph $L(CH_n)$ satisfying Condition (1.1). Without loss of generality, we may define this map c so that $c(vv_i) = i$ for all non-negative integer $i < n$.

- **Case** $\max\{3, r\} < n$.

  Except for $(r,n) \in \{(5,6), (6,7)\}$, the case $4 \leq n \leq 7$ follows simply from Lemma 3.1 and Figures 4 and 5. The case $(r,n) = (5,6)$ holds from Lemma 3.1 and Figure 5 once we replace the colors of any three vertices v_ip_i, $v_{i+2}p_{i+2}$ and $v_{i+4}p_{i+4}$ in the left graph of that figure by a seventh color. In addition, the case $(r,n) = (6,7)$ follows from Proposition 3.1 and Figure 6 (center). So, let us focus on the case $n > 7$. From Lemma 3.1, we have that $n \leq \chi_r(L(CH_n))$. In order to prove that this lower bound is reached, we define the map c so that, for each non-negative integer $i < n$,

$$c(w) = \begin{cases} (i-3) \bmod n & \text{if } w = v_iv_{i+1}, \\ (i+2) \bmod n & \text{if } w = v_ip_i, \\ (i-1) \bmod n & \text{if } w = p_ip_{i+1}. \end{cases}$$

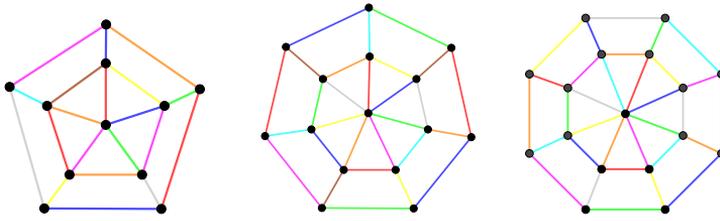 Condition (1.1) holds and hence, $\chi_r(L(CH_n)) = n$. Figure 6 (right) illustrates the case $n = 8$.

FIGURE 6. 5-dynamic proper 8-edge-coloring of the closed helm graph CH_5 (left); r-dynamic proper 9-edge-coloring of the closed helm graph CH_7 for $r \in \{6, 7\}$ (center); and r-dynamic proper 8-edge-coloring of the closed helm graph CH_8 for all $r \leq 7$ (right).

- **Case** $4 \leq r = n$.

  The case $n = 5$ holds from Proposition 3.2 and Figure 6 (left). So, let us focus on the case $n \neq 5$. From Proposition 3.2, it is $n + 2 \leq \chi_n(L(CH_n))$. Let the map c be defined so that, for each non-negative integer $i < n$,

  $$c(v_i v_{i+1}) = \begin{cases} n & \text{if } n \text{ is odd and } i = n - 1, \\ n + 1 & \text{if } n \text{ is odd and } i = 0, \\ (i - 3) \bmod n & \text{otherwise,} \end{cases}$$

  $$c(v_i p_i) = \begin{cases} i + 2 & \text{if } n \text{ is odd and } i \in \{0, 1, n - 1\}, \\ n & \text{if } \begin{cases} \text{both } n \text{ and } i \text{ are even,} \\ n \text{ is odd and } i \notin \{0, n - 1\} \text{ is even,} \end{cases} \\ n + 1 & \text{otherwise,} \end{cases}$$

  and

  $$c(p_i p_{i+1}) = (i - 1) \bmod n.$$

Condition (1.1) holds and hence, $\chi_r(L(CH_n)) = n + 2$. Figure 7 illustrates the case $r = n \in \{6, 9\}$.

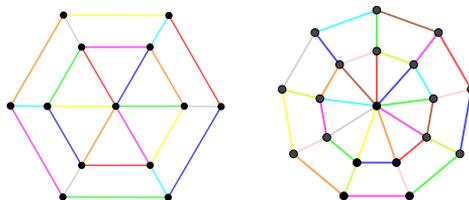

FIGURE 7. n-dynamic proper $(n + 2)$-edge-coloring of the closed helm graph CH_n for $n \in \{6, 9\}$.

- **Case** $r = n + 1$.

  The case $n \leq 5$ follows from Proposition 3.3 and Figure 8. Now, for $n > 5$, we define the map c so that, for each non-negative integer $i < n$,

$$c(v_i v_{i+1}) = \begin{cases} n + (i \bmod 3) & \text{if } n \equiv 0 \,(\mathrm{mod}\,3), \\ n + (i \bmod 4) & \text{otherwise,} \end{cases}$$

$$c(v_i p_i) = (i + 2) \bmod n \qquad \text{and} \qquad c(p_i p_{i+1}) = (i - 1) \bmod n.$$

  Condition (1.1) holds and hence, Proposition 3.3 implies $\chi_{n+1}(L(CH_n)) = n + 3$ if $n \equiv 0 \,(\mathrm{mod}\,3)$, and $\chi_{n+1}(L(CH_n)) = n + 4$ otherwise. Figure 9 illustrates the case $n \in \{6, 7\}$.
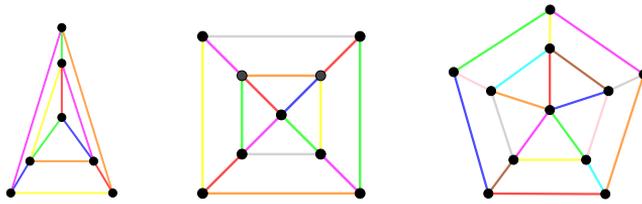

FIGURE 8. $(n + 1)$-dynamic proper $(n + 3)$-edge-coloring of the closed helm graph CH_n for $n \in \{3, 4, 6\}$, and 6-dynamic proper 10-edge-coloring of the closed helm graph CH_5.
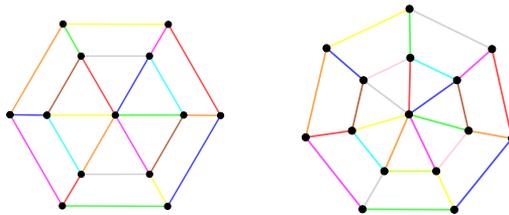

FIGURE 9. 7-dynamic proper 9-edge-coloring of the closed helm graph CH_6 (left) and 8-dynamic proper 11-edge-coloring of the closed helm graph CH_7 (right).

- **Case** $r = n + 2$.

  The case $n = 3$ follows from Proposition 3.4 and Figure 10 (left). Further, from Proposition 3.4, we have that $n + 5 \leq \chi_{n+2}(L(CH_n))$ for all $n \in \{5k, 6k\}$ for some $k > 0$. Let us prove that this lower bound is reached in both cases.
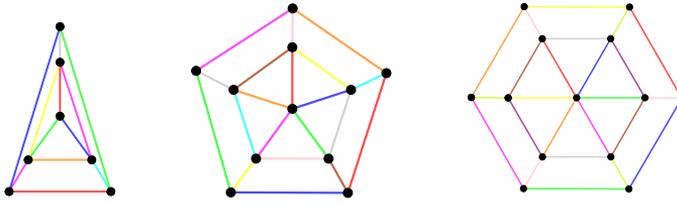
FIGURE 10. $(n+2)$-dynamic proper $(n+5)$-edge-coloring of the graph CH_n for all $n \in \{3, 5, 6\}$.

– **Subcase** $n = 5k$ for some $k > 0$.
Let the map c be defined so that, for each non-negative integer $i < n$,

$$
c(w) = \begin{cases} n + (i \bmod 5) & \text{if } w = v_i v_{i+1}, \\ n + ((i+2) \bmod 5) & \text{if } w = v_i p_i, \\ (i-1) \bmod n & \text{if } w = p_i p_{i+1}. \end{cases}
$$

Condition (1.1) holds and hence, $\chi_{n+2}(L(CH_n)) = n + 5$. Figure 10 (center) illustrates the case $n = 5$.

– **Subcase** $n = 6k$ for some $k > 0$.
Let the map c be defined so that, for each non-negative integer $i < n$,

$$
c(w) = \begin{cases} n + (i \bmod 3) & \text{if } w = v_i v_{i+1}, \\ n + 3 + (i \bmod 2) & \text{if } w = v_i p_i, \\ (i-1) \bmod n & \text{if } w = p_i p_{i+1}. \end{cases}
$$

Condition (1.1) holds and hence, $\chi_{n+2}(L(CH_n)) = n + 5$. Figure 10 (right) illustrates the case $n = 6$.

Finally, if $n \notin \{3, 5k, 6k\}$ for all $k > 0$, then Proposition 3.4 implies that $n + 6 \leq \chi_{n+2}(L(CH_n))$. In order to prove that this lower bound is reached, let the map c be defined so that, for each non-negative integer $i < n$,

$$
c(w) = \begin{cases} n + (i \bmod 3) & \text{if } w = v_i v_{i+1} \text{ and } n \equiv 0 \, (\mathrm{mod}\, 3), \\ n + (i \bmod 4) & \text{if } w = v_i v_{i+1} \text{ and } n \not\equiv 0 \, (\mathrm{mod}\, 3), \\ n + 3 + (i \bmod 3) & \text{if } w = v_i p_i \text{ and } n \equiv 0 \, (\mathrm{mod}\, 3), \\ n + 4 + (i \bmod 2) & \text{if } w = v_i p_i \text{ and } \begin{cases} n \text{ is even,} \\ n \not\equiv 0 \, (\mathrm{mod}\, 3) \text{ is odd} \\ \qquad\qquad \text{and } i > 0, \end{cases} \\ n + 3 & \text{if } w = v_0 p_0 \text{ and } n \not\equiv 0 \, (\mathrm{mod}\, 3) \text{ is odd}, \\ (i-1) \bmod n & \text{if } w = p_i p_{i+1}. \end{cases}
$$

Condition (1.1) holds and hence, $\chi_{n+2}(L(CH_n)) = n + 6$. Figure 11 (left) illustrates the case $n \in \{4, 7, 9\}$.
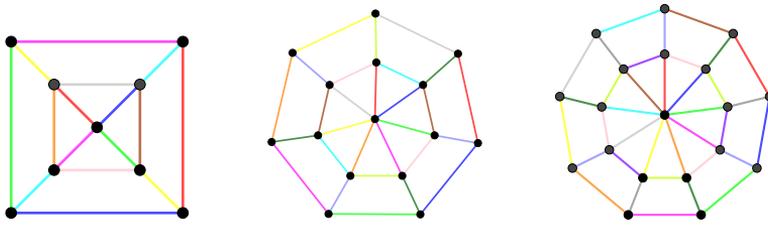
FIGURE 11. $(n+2)$-dynamic proper $(n+6)$-edge-coloring of the graph CH_n for all $n \in \{4, 7, 9\}$.

- **Case** $r > n + 2$.

  It follows simply from Lemma 2.1 and the case $r = n - 2$. □

4. CONCLUSION AND FURTHER WORK

In this paper, we have determined the r-dynamic chromatic number of the line graph of any closed helm graph CH_n for all positive integers r and $n > 2$, and hence, the r-dynamic edge chromatic number of any given closed helm graph. In this regard, Theorem 3.2 is the main result of the paper; there it has in particular been obtained that $n \le \lambda_r(CH_n) = \chi_r(L(CH_n)) \le n + 6$.

Similarly to recent works dealing with the r-dynamic coloring of certain products of graphs [7, 6], of particular interest for the continuation of this paper is the study of the r-dynamic edge coloring of different products of graphs concerning closed helm graphs.

ACKNOWLEDGMENTS

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