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DRAZIN INVERTIBILITY OF LINEAR OPERATORS ON QUATERNIONIC BANACH SPACES

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Abstract. The paper studies the Drazin inverse for right linear operators on a quaternionic Banach space. Let A be a right linear operator on a two-sided quaternionic Banach space. It is shown that if A is Drazin invertible then the Drazin inverse of A is given by $f(A)$, where f is 0 in an axially symmetric neighborhood of 0 and $f(q) = q^{-1}$ in an axially symmetric neighborhood of the nonzero spherical spectrum of A. Some results analogous to the ones concerning the Drazin inverse of operators on complex Banach spaces are proved in the quaternionic context.

1. Introduction and preliminaries

We denote by $\mathbb{H}$ the algebra of quaternions, introduced by Hamilton in 1843. An element q of $\mathbb{H}$ is of the form

$$q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}, \quad a, b, c, d \in \mathbb{R}$$

where i, j and k are imaginary units. By definition, they satisfy

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1.$$

Given $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$, we have:

- the conjugate quaternion of q is $\bar{q} := a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$;
- the norm of q is $|q| := \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$;
- the real and the imaginary parts of q are respectively $\operatorname{Re}(q) := \frac{1}{2}(q + \bar{q}) = a$ and $\operatorname{Im}(q) := \frac{1}{2}(q - \bar{q}) = b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$.

The unit sphere of imaginary quaternions is given by

$$\mathbb{S} := \{q \in \mathbb{H} : q^2 = -1\}.$$

Let p and q be two quaternions; p and q are said to be conjugated if there is $s \in \mathbb{H} \setminus \{0\}$ such that $p = sqs^{-1}$. The set of all quaternions conjugated with q is equal to the 2-sphere

$$[q] = \{\operatorname{Re}(q) + |\operatorname{Im}(q)| j : j \in \mathbb{S}\} = \operatorname{Re}(q) + |\operatorname{Im}(q)|\mathbb{S}.$$

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For every $j \in \mathbb{S}$, we denote by $\mathbb{C}_j$ the real subalgebra of $\mathbb{H}$ generated by j; that is,

$$\mathbb{C}_j := \{u + vj \in \mathbb{H} : u, v \in \mathbb{R}\}.$$

We say that $U \subseteq \mathbb{H}$ is axially symmetric if $[q] \subset U$ for every $q \in U$.

For a thorough treatment of the algebra of quaternions $\mathbb{H}$, the reader is referred, for instance, to [3].

Definition 1.1 ([1, Definition 2.3.1])**.** Let $(X, +)$ be an abelian group. X is a *two-sided quaternionic vector space* if it is endowed with left and right quaternionic multiplications such that, for all $u, v \in X$ and all $p, q \in \mathbb{H}$,

$$\begin{aligned}&u(p+q) = up + uq, \quad (u+v)q = uq + vq, \quad (up)q = u(pq), \quad u1 = u,\\ &(p+q)u = pu + qu, \quad q(u+v) = qu + qv, \quad q(pu) = (qp)u, \quad 1u = u,\\ &(pu)q = p(uq), \quad ru = ur \text{ for all } r \in \mathbb{R}.\end{aligned}$$

Definition 1.2. Let X be a two-sided quaternionic vector space. A function $\|\cdot\| : X \to [0; +\infty)$ is called a *norm* on X if it satisfies

(i) $\|u\| = 0$ if and only if $u = 0$;
(ii) $\|uq\| = \|qu\| = \|u\||q|$ for all $u \in X$ and all $q \in \mathbb{H}$;
(iii) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in X$.

If X is complete with respect to the metric induced by $\|\cdot\|$, we call X a *two-sided quaternionic Banach space.*

Definition 1.3. Let X be a two-sided quaternionic Banach space. A *right linear operator* on X is a map $T : X \to X$ such that

$$T(up + v) = (Tu)p + Tv \quad \text{for all } u, v \in X \text{ and all } p \in \mathbb{H}.$$

A right linear operator T on X is called *bounded* if

$$\|T\| := \sup\{\|Tu\| : u \in X, \|u\| = 1\} < \infty.$$

The set of all right linear bounded operators on X is denoted by $\mathcal{B}_R(X)$. The ring $\mathcal{B}_R(X)$ is viewed as a two-sided quaternionic vector space equipped with the metric $\mathcal{B}_R(X) \times \mathcal{B}_R(X) \ni (A, B) \mapsto \|A - B\|$.

In a two-sided quaternionic Banach space X, we can define a left and a right quaternionic multiplication on $\mathcal{B}_R(X)$ by

$$(Tq)u = T(qu) \quad \text{and} \quad (qT)u = q(Tu) \qquad \text{for all } q \in \mathbb{H}, u \in X \text{ and all } T \in \mathcal{B}_R(X).$$

Definition 1.4. Let $T \in \mathcal{B}_R(X)$. For $q \in \mathbb{H}$, we set

$$Q_q(T) := T^2 - 2\operatorname{Re}(q)T + |q|^2 I,$$

where I is the identity operator on X. We define the *S-resolvent set* $\rho_S(T)$ of T as

$$\rho_S(T) := \{q \in \mathbb{H} : Q_q(T) \text{ is invertible in } \mathcal{B}_R(X)\},$$

and we define the *S-spectrum* $\sigma_S(T)$ of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

Proposition 1.5 ([1, Proposition 3.1.8]). *Let $T \in \mathcal{B}_R(X)$. The sets $\sigma_S(T)$ and $\rho_S(T)$ are axially symmetric.*

Theorem 1.6 (Compactness of the S-spectrum, [1, Theorem 3.1.13]). *Let $T \in \mathcal{B}_R(X)$. The S-spectrum $\sigma_S(T)$ of T is a nonempty compact set contained in the closed ball $\{q \in \mathbb{H} : |q| \leq \|T\|\}$.*

The spectral theory over quaternionic Hilbert spaces has been developed in [3] and [6].

2. Generalized and Drazin inverses

Let X be a two-sided quaternionic Banach space. In this section, we study the generalized and Drazin invertibility of right linear operators on X.

Definition 2.1. An operator $B \in \mathcal{B}_R(X)$ is called a *generalized inverse* of $A \in \mathcal{B}_R(X)$ if $ABA = A$ and $BAB = B$.

The range and the kernel of an operator $T \in \mathcal{B}_R(X)$ are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively.

Theorem 2.2. *Suppose $A \in \mathcal{B}_R(X)$ with generalized inverse B such that $AB = BA$. Then*

$$\sigma_S(B) \setminus \{0\} = \{q^{-1} : q \in \sigma_S(A) \setminus \{0\}\}.$$

Proof. By [4, Theorem XI. 6.1], $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Then $A = T \oplus 0$ on $\mathcal{R}(A) \oplus \mathcal{N}(A)$ and $B = T^{-1} \oplus 0$. We have $Q_q(B) = Q_q(T^{-1}) \oplus Q_q(0)$ for all $q \in \mathbb{H}$. Then we have $\sigma_S(B) = \sigma_S(T^{-1}) \cup \sigma_S(0)$, since $Q_q(0) = |q|^2 I$ is always invertible (when $q \neq 0$), where I is the identity operator on $\mathcal{N}(A)$, and so

$$\sigma_S(B) \setminus \{0\} = \sigma_S(T^{-1}) \setminus \{0\}.$$

The function $f : \mathbb{H} \setminus \{0\} \ni q \mapsto q^{-1}$ is intrinsic slice hyperholomorphic (because $q^{-1} = \frac{\bar{q}}{|q|^2}$); then by [1, Theorem 4.2.1], $\sigma_S(T^{-1}) = \sigma_S(f(T)) = \{q^{-1} : q \in \sigma_S(T)\}$. Thus

$$\sigma_S(B) \setminus \{0\} = \{q^{-1} : q \in \sigma_S(A) \setminus \{0\}\}. \qquad \square$$

Now, we study the Drazin invertibility of right linear operators acting on a two-sided quaternionic Banach space.

Definition 2.3 ([2]). Let $A \in \mathcal{B}_R(X)$. An element $B \in \mathcal{B}_R(X)$ is a *Drazin inverse* of A, written $B = A^d$, if

$$AB = BA, \quad AB^2 = B, \quad A^{k+1}B = A^k \tag{2.1}$$

for some nonnegative integer k. The least nonnegative integer k for which these equations hold is the *Drazin index* $i(A)$ of A.

Definition 2.4. An element A of $\mathcal{B}_R(X)$ is called *quasinilpotent* if $\sigma_S(A) = \{0\}$. The set of all quasinilpotent elements in $\mathcal{B}_R(X)$ will be denoted by $QN(\mathcal{B}_R(X))$.

Proposition 2.5. *An element A of $\mathcal{B}_R(X)$ is quasinilpotent if and only if, for every T commuting with A, we have that $I-TA$ is invertible.*

Proof. Let $A \in \mathcal{B}_R(X)$. Assume that for every $T \in \mathcal{B}_R(X)$ commuting with A, we have that $I-TA$ is invertible. Let $T = \frac{-1}{|q|^2}A + \frac{2\operatorname{Re}(q)}{|q|^2}I$ with $q \in \mathbb{H}\setminus\{0\}$; clearly T commutes with A and $I - TA = \frac{1}{|q|^2}[A^2 - 2\operatorname{Re}(q)A + |q|^2 I]$ is invertible, hence $\sigma_S(A) = \{0\}$.

Conversely, if $\sigma_S(A) = \{0\}$, let $T \in \mathcal{B}_R(X)$ commuting with A; then by [1, Theorem 4.2.3], $r_S(TA) \le r_S(T)r_S(A) = 0$ and hence $\sigma_S(TA) = \{0\}$. Then by [1, Theorem 4.2.1], $\sigma_S(I-TA) = \{1\}$ and hence $I-TA$ is invertible. ☐

An operator $A \in \mathcal{B}_R(X)$ is said to be nilpotent if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The least nonnegative integer k for which $A^k = 0$ is called the nilpotency index of A and the set of all nilpotent elements in $\mathcal{B}_R(X)$ is denoted by $N(\mathcal{B}_R(X))$.

Koliha [5, Definition 2.3] generalized the notion of Drazin invertibility in a complex Banach algebra. According to this definition one can generalize the notion of Drazin invertibility in $\mathcal{B}_R(X)$.

Definition 2.6. Let $A \in \mathcal{B}_R(X)$. An element $B \in \mathcal{B}_R(X)$ is a *generalized Drazin inverse* of A, written $B = A^D$, if

$$AB = BA, \quad AB^2 = B, \quad A - A^2B \in QN(\mathcal{B}_R(X)). \tag{2.2}$$

Theorem 2.7 ([1, Theorem 4.1.5])**.** *Let $A \in \mathcal{B}_R(X)$ and assume that $\sigma_S(A) = \sigma_1 \cup \sigma_2$ with*

$$\operatorname{dist}(\sigma_1, \sigma_2) > 0.$$

We choose an open axially symmetric set O with $\sigma_1 \subset O$ and $\overline{O} \cap \sigma_2 = \emptyset$, and define a function χ_{σ_1} on $\mathbb{H}$ by $\chi_{\sigma_1}(s) = 1$ for $s \in O$ and $\chi_{\sigma_1}(s) = 0$ for $s \notin O$. Then $\chi_{\sigma_1} \in \mathcal{N}(\sigma_S(A))$, and for an arbitrary imaginary unit j in $\mathbb{S}$ and an arbitrary bounded slice Cauchy domain $U \subset \mathbb{H}$ such that $\sigma_1 \subset U \subset \overline{U} \subset O$, we have

$$P_{\sigma_1} := \chi_{\sigma_1}(A) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,A)\, ds_j$$

is a continuous projection that commutes with A. Hence $P_{\sigma_1}(X)$ is a right linear subspace of X that is invariant under A.

Remark 2.8. *Let $q \in \mathbb{H}$. If $\sigma_1 = \{q\}$, we say that the projection P_{σ_1} is the Riesz projection of A corresponding to q.*

We denote by $\operatorname{acc} U$ (resp., $\operatorname{iso} U$) the set of all accumulation (resp., isolated) points of a set $U \subseteq \mathbb{H}$.

Theorem 2.9. *Let $A \in \mathcal{B}_R(X)$. Then $0 \notin \operatorname{acc}\sigma_S(A)$ if and only if there is a projection $P \in \mathcal{B}_R(X)$ commuting with A such that*

$$AP \in QN(\mathcal{B}_R(X)) \quad \text{and} \quad A + P \text{ is invertible in } \mathcal{B}_R(X). \tag{2.3}$$

Moreover, $0 \in \operatorname{iso}\sigma_S(A)$ if and only if $P \neq 0$, in which a case P is the Riesz projection of A corresponding to $q = 0$.

Proof. Clearly, $0 \notin \sigma_S(A)$ if and only if (2.3) holds with $P = 0$.

Assume that $0 \in \operatorname{iso} \sigma_S(A)$. Let P be the spectral projection of A corresponding to $q = 0$; then $P \neq 0$, commutes with A and $AP = id(A)\chi_{\{0\}}(A) = (id\chi_{\{0\}})(A)$, where $id : \mathbb{H} \to \mathbb{H}$, $q \mapsto q$. Hence $\sigma_S(AP) = id\chi_{\{0\}}(\sigma_S(A)) = \{0\}$, thus $AP \in QN(\mathcal{B}_R(X))$. Similarly, $A + P = id(A) + \chi_{\{0\}}(A) = (id + \chi_{\{0\}})(A)$; then $0 \notin \sigma_S(A + P) = (id + \chi_{\{0\}})\sigma_S(A)$, and therefore $A + P$ is invertible.

Conversely, assume that there is a nonzero projection P commuting with A such that (2.3) holds. For any $q \in \mathbb{H}$, we have

$$\begin{aligned} A^2 - 2\operatorname{Re}(q)A + |q|^2 I = P\big((AP)^2 - 2\operatorname{Re}(q)AP + |q|^2 I\big) \\ + (I - P)\big((A + P)^2 - 2\operatorname{Re}(q)(A + P) + |q|^2 I\big). \end{aligned}$$

There is an $r > 0$ such that if $|q| < r$ then $(A + P)^2 - 2\operatorname{Re}(q)(A + P) + |q|^2 I$ is invertible. Since $AP \in QN(\mathcal{B}_R(X))$, $(AP)^2 - 2\operatorname{Re}(q)AP + |q|^2 I$ is invertible for all $q \neq 0$. Hence, for all $0 < |q| < r$, it is easy to check that $A^2 - 2\operatorname{Re}(q)A + |q|^2 I$ is invertible and

$$\begin{aligned} \big(A^2 - 2\operatorname{Re}(q)A + |q|^2 I\big)^{-1} = P\big((AP)^2 - 2\operatorname{Re}(q)AP + |q|^2 I\big)^{-1} \\ + (I - P)\big((A + P)^2 - 2\operatorname{Re}(q)(A + P) + |q|^2 I\big)^{-1}. \end{aligned}$$

That is,

$$Q_q(A)^{-1} = PQ_q(AP)^{-1} + (I - P)Q_q(A + P)^{-1}. \tag{2.4}$$

Hence $0 \in \operatorname{iso} \sigma_S(A)$.

Now, we show that P is the Riesz projection of A corresponding to $q = 0$. Since $S_L^{-1}(q, A) = -Q_q(A)^{-1}(A - \bar{q}I)$, because of (2.4) we have

$$S_L^{-1}(q, A) = PS_L^{-1}(q, AP) + (I - P)S_L^{-1}(q, A + P). \tag{2.5}$$

Let j and U be as in Theorem 2.7; then

$$\chi_{\{0\}}(A) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, A)\, ds_j.$$

If we take $U = \{q \in \mathbb{H} : |q| < \frac{r}{2}\}$, then by (2.5)

$$\begin{aligned} \chi_{\{0\}}(A) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, A)\, ds_j \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} PS_L^{-1}(s, AP) + (I - P)S_L^{-1}(s, A + P)\, ds_j \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} PS_L^{-1}(s, AP)\, ds_j + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} (I - P)S_L^{-1}(s, A + P)\, ds_j \\ &= P\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, AP)\, ds_j + (I - P)\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, A + P)\, ds_j. \end{aligned}$$

Since $S_L^{-1}(\cdot, A + P)$ is a right slice hyperholomorphic function on U (see [1, Lemma 3.1.11]),

$$\int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, A + P)\, ds_j = 0.$$

On the other hand,

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,AP)\,ds_j = I,$$

because $\sigma_S(AP)=\{0\}\subset U$. Hence $\chi_{\{0\}}(A)=P$. This completes the proof. □

Theorem 2.10. *Let $A\in\mathcal{B}_R(X)$. If $0\in\operatorname{iso}\sigma_S(A)$, then*

$$A^D = f(A),$$

where $f\in\mathcal{N}\big(\sigma_S(A)\big)$ is such that f is 0 in an axially symmetric neighborhood of 0 and $f(q)=q^{-1}$ in an axially symmetric neighborhood of $\sigma_S(A)\setminus\{0\}$, and

$$\sigma_S(A^D)\setminus\{0\} = \big\{q^{-1} : q\in\sigma_S(A)\setminus\{0\}\big\}.$$

Proof. Let O_1 be an axially symmetric open neighborhood of 0 and let O_2 be an axially symmetric open neighborhood of $\sigma_S(A)\setminus\{0\}$ with $\overline{O_1}\cap\overline{O_2}=\emptyset$. Define f by $f(q)=0$ if $q\in O_1$ and $f(q)=q^{-1}$ if $q\in O_2$; clearly $f\in\mathcal{N}(\sigma_S(A))$. By [1, Theorems 4.1.3 and 4.2.1], it is easy to see that (2.2) holds for A and $f(A)$.

By [1, Theorem 4.2.1], it follows that $\sigma_S(A^D)\setminus\{0\}=\sigma_S(f(A))\setminus\{0\}=\{f(q): q\in\sigma_S(A)\setminus\{0\}\}=\{q^{-1}: q\in\sigma_S(A)\setminus\{0\}\}$. □

Theorem 2.11. *Let $A\in\mathcal{B}_R(X)$. The following conditions are equivalent:*

(i) *A is generalized Drazin invertible;*
(ii) *$0\notin\operatorname{acc}\sigma_S(A)$;*
(iii) *$A=A_1\oplus A_2$, where A_1 is invertible on some closed subspace X_1 of X and A_2 is quasinilpotent on some complemented subspace X_1 of X.*

Proof. (i)⇔(ii) Already proved in [5, Lemma 2.4] and Theorem 2.9.

(i)⇒(iii) Set the projection $P:=I-AA^D$; then AP is quasinilpotent and $AP=PA$. Hence $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are invariant under A, that is, $A\mathcal{R}(P)\subset\mathcal{R}(P)$ and $A\mathcal{N}(P)\subset\mathcal{N}(P)$. Let $u\in\mathcal{N}(P)$; then $u=AA^Du$, thus the restriction of A to the kernel of P is injective and surjective, and so invertible. If we write $A=A_1\oplus A_2$ on $X=\mathcal{N}(P)\oplus\mathcal{R}(P)$, then $A_2\in\mathcal{B}_R(X_1)$ is quasinilpotent and $A_1\in\mathcal{B}_R(X_2)$ is invertible.

(iii)⇒(i) It is easy to check that $A^D=A_1^{-1}\oplus 0$. □

Corollary 2.12. *Let $A\in\mathcal{B}_R(X)$. The following conditions are equivalent:*

(i) *A is Drazin invertible;*
(iii) *$A=A_1\oplus A_2$, where A_1 is invertible on some closed subspace X_1 of X, A_2 is nilpotent on some complemented subspace X_1 of X and the nilpotency index of A_2 is the Drazin index of A.*

Proof. Assume that A is Drazin invertible; then by Theorem 2.11 (iii), $A=A_1\oplus A_2$ and $A^d=A_1^{-1}\oplus 0$. Hence, by (2.1), $A^{k+1}A^d=A^k$, then $A_1^k\oplus 0=A_1^k\oplus A_2^k$, thus $A_2^k=0$, so that the nilpotency index of A_2 is less than the Drazin index of A.

Conversely, let $B=A_1^{-1}\oplus 0$, where A_1 is invertible and A_2 is nilpotent; then (2.1) holds for A, B and the nilpotency index of A_2. Hence A is Drazin invertible and the Drazin index of A is less than the nilpotency index of A_2. □

Definition 2.13. A *two-sided quaternionic Banach algebra* is a two-sided quaternionic Banach space $\mathcal{A}$ that is endowed with a product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that:

(i) The product is associative and distributive over the sum in $\mathcal{A}$;
(ii) $(qx)y = q(xy)$ and $x(yq) = (xy)q$ for all $x, y \in \mathcal{A}$ and all $q \in \mathbb{H}$;
(iii) $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{A}$.

If in addition there exists $e \in \mathcal{A}$ such that $ex = xe = x$ for all $x \in \mathcal{A}$, then $\mathcal{A}$ is called a *two-sided quaternionic Banach algebra with unit.*

One can prove that $\mathcal{B}_R(X)$ is a two-sided quaternionic Banach algebra with unit.

Definition 2.14. Let $\mathcal{A}$ be a two-sided quaternionic Banach algebra and $a \in \mathcal{A}$. An element $b \in \mathcal{A}$ is a *Drazin inverse* of a, written $b = a^d$, if

$$ab = ba, \quad ab^2 = b, \quad a^{k+1}b = a^k,$$

for some nonnegative integer k. The least nonnegative integer k for which these equations hold is the *Drazin index* $i(a)$ of a.

Let $\mathcal{A}$ be a two-sided quaternionic Banach algebra and $a \in \mathcal{A}$. For any $a \in \mathcal{A}$ we define the left multiplication of a by $L_a(b) = ab$, for all $b \in \mathcal{A}$. Then $L_a \in \mathcal{B}_R(\mathcal{A})$, and we have $\|L_a\| = \|a\|$.

Theorem 2.15. *Let $\mathcal{A}$ be a two-sided quaternionic Banach algebra and let $a \in \mathcal{A}$ with unit. Then a is Drazin invertible if and only if L_a is Drazin invertible. In such a case, $L_a^d = L_{a^d}$ and $i(L_a) = i(a)$.*

Proof. Let $a \in \mathcal{A}$ such that a is Drazin invertible. For every $b \in \mathcal{A}$, we have $L_a L_b = L_{ab}$, hence it is easy to check that $L_{a^d} = L_a^d$ and then $i(L_a) \leq i(a)$.

Conversely, assume that L_a is Drazin invertible and let $b = L_a^d(e)$. Since $L_a^{k+1} L_a^d = L_a^k$, $a^{k+1}b = a^k$. Hence $L_a^{k+1} L_b = L_a^k = L_a^d L_a^{k+1}$, and then by [2, Theorem 4] and its proof, $L_a^d = L_a^k L_b^{k+1} = L_{a^k b^{k+1}}$. Let $c = a^k b^{k+1}$; then $L_a L_c = L_c L_a$, $L_a L_c^2 = L_c$, $L_a^{k+1} L_c = L_a^k$, hence $ac = ca$, $ac^2 = c$, $a^{k+1}c = a^k$. Thus a is Drazin invertible and therefore $i(a) \leq i(L_a)$. ☐

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