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ON CERTAIN REGULAR NICELY DISTANCE-BALANCED GRAPHS

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Abstract. A connected graph Γ is called *nicely distance-balanced*, whenever there exists a positive integer $\gamma = \gamma(\Gamma)$ such that, for any two adjacent vertices u, v of Γ, there are exactly γ vertices of Γ which are closer to u than to v, and exactly γ vertices of Γ which are closer to v than to u. Let d denote the diameter of Γ. It is known that $d \leq \gamma$, and that nicely distance-balanced graphs with $\gamma = d$ are precisely complete graphs and cycles of length $2d$ or $2d+1$. In this paper we classify regular nicely distance-balanced graphs with $\gamma = d+1$.

1. Introduction

Let Γ be a finite, undirected, connected graph with diameter d, and let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of Γ, respectively. For $u, v \in V(\Gamma)$, let $\Gamma(u)$ be the set of neighbors of u, and let $d(u,v) = d_\Gamma(u,v)$ denote the minimal path-length distance between u and v. For a pair of adjacent vertices u, v of Γ we let

$$W_{u,v} = \{x \in V(\Gamma) \mid d(x,u) < d(x,v)\}.$$

We say that Γ is *distance-balanced* (DB for short) whenever for an arbitrary pair of adjacent vertices u and v of Γ we have that

$$|W_{u,v}| = |W_{v,u}|.$$

The investigation of distance-balanced graphs was initiated in 1999 by Handa [10], although the name *distance-balanced* was coined nine years later by Jerebic, Klavžar, and Rall [13]. The family of distance-balanced graphs is very rich and its study is interesting from various purely graph-theoretic aspects where one focuses on particular properties such as symmetry [15], connectivity [10, 17] or complexity aspects of algorithms related to such graphs [6]. However, the balancedness property of these graphs makes them very appealing also in areas such as mathematical

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chemistry and communication networks. For instance, the investigation of such graphs is highly related to the well-studied Wiener index and Szeged index (see [2, 12, 13, 19]), and they present very desirable models in various real-life situations related to (communication) networks [2]. Recently, the relations between distance-balanced graphs and the traveling salesman problem were studied in [7]. It turns out that these graphs can be characterized by properties that at first glance do not seem to have much in common with the original definition from [13]. For example, in [3] it was shown that the distance-balanced graphs coincide with the *self-median* graphs, that is, graphs for which the sum of the distances from a given vertex to all other vertices is independent of the chosen vertex. Other such examples are *equal opportunity graphs* (see [2] for the definition). In [2] it is shown that even order distance-balanced graphs are also equal to opportunity graphs. Finally, let us also mention that various generalizations of the distance-balanced property were defined and studied in the literature (see, for example, [1, 8, 11, 14, 18]).

The notion of nicely distance-balanced graphs appears quite naturally in the context of DB graphs. We say that Γ is *nicely distance-balanced* (NDB for short) whenever there exists a positive integer $\gamma = \gamma(\Gamma)$ such that, for an arbitrary pair of adjacent vertices u and v of Γ,

$$|W_{u,v}| = |W_{v,u}| = \gamma$$

holds. Clearly, every NDB graph is also DB, but the opposite is not necessarily true. For example, if $n \geq 3$ is an odd positive integer, then the prism graph on $2n$ vertices is DB, but not NDB.

Assume now that Γ is NDB. Let us denote the diameter of Γ by d. In [16], where these graphs were first defined, it was proved that $d \leq \gamma$, and NDB graphs with $d = \gamma$ were classified. It turns out that Γ is NDB with $d = \gamma$ if and only if Γ is either isomorphic to a complete graph on $n \geq 2$ vertices, or to a cycle on $2d$ or $2d + 1$ vertices. In this paper we study NDB graphs for which $\gamma = d + 1$. The situation in this case is much more complex than in the case $\gamma = d$. Therefore, we will concentrate our study on the class of regular graphs (recall that Γ is said to be regular with valency k if $|\Gamma(u)| = k$ for every $u \in V(\Gamma)$). Our main result is the following theorem.

Theorem 1.1. *Let Γ be a regular NDB graph with valency k and diameter d. Then $\gamma = d + 1$ if and only if Γ is isomorphic to one of the following graphs:*

(1) *the Petersen graph (with $k = 3$ and $d = 2$);*
(2) *the complement of the Petersen graph (with $k = 6$ and $d = 2$);*
(3) *the complete multipartite graph $K_{t\times 3}$ with t parts of cardinality 3, $t \geq 2$ (with $k = 3(t - 1)$ and $d = 2$);*
(4) *the Möbius ladder graph on eight vertices (with $k = 3$ and $d = 2$);*
(5) *the Paley graph on 9 vertices (with $k = 4$ and $d = 2$);*
(6) *the 3-dimensional hypercube Q_3 (with $k = 3$ and $d = 3$);*
(7) *the line graph of the 3-dimensional hypercube Q_3 (with $k = 4$ and $d = 3$);*
(8) *the icosahedron (with $k = 5$ and $d = 3$).*

Our paper is organized as follows. After some preliminaries in Section 2 we prove certain structural results about NDB graphs with $\gamma = d+1$ in Section 3. In Section 4 we show that if Γ is a regular NDB graph with $\gamma = d+1$, then $d \le 5$ and the valency of Γ is either 3, 4 or 5. In Sections 5, 6 and 7 we consider each of these three cases separately.

2. Preliminaries

In this section we recall some preliminary results that we will find useful later in the paper. Let Γ be a simple, finite, connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. If $u, v \in V(\Gamma)$ are adjacent, then we simply write $u \sim v$ and we denote the corresponding edge by $uv = vu$. For $u \in V(\Gamma)$ and an integer i, we let $\Gamma_i(u)$ denote the set of vertices of $V(\Gamma)$ that are at distance i from u. We abbreviate $\Gamma(u) = \Gamma_1(u)$. We set $\epsilon(u) = \max\{d(u,z) \mid z \in V(\Gamma)\}$ and we call $\epsilon(u)$ the *eccentricity* of u. Let $d = \max\{\epsilon(u) \mid u \in V(\Gamma)\}$ denote the *diameter* of Γ. Pick adjacent vertices u, v of Γ. For any two non-negative integers i, j we let

$$D^i_j(u,v) = \Gamma_i(u) \cap \Gamma_j(v).$$

By the triangle inequality we observe that only the sets $D^{i-1}_i(u,v)$, $D^i_i(u,v)$, and $D^i_{i-1}(u,v)$ $(1 \le i \le d)$ can be nonempty. Moreover, the next result holds.

Lemma 2.1. *With the above notation, abbreviate $D^i_j = D^i_j(u,v)$. Then the following statements hold for $1 \le i \le d$:*

(i) *If $w \in D^i_{i-1}$ then $\Gamma(w) \subseteq D^{i-1}_{i-2} \cup D^{i-1}_{i-1} \cup D^{i-1}_i \cup D^i_{i-1} \cup D^i_i \cup D^{i+1}_i$.*
(ii) *If $w \in D^i_i$ then $\Gamma(w) \subseteq D^{i-1}_{i-1} \cup D^{i-1}_i \cup D^i_{i-1} \cup D^i_i \cup D^i_{i+1} \cup D^{i+1}_i \cup D^{i+1}_{i+1}$.*
(iii) *If $w \in D^{i-1}_i$ then $\Gamma(w) \subseteq D^{i-2}_{i-1} \cup D^{i-1}_{i-1} \cup D^{i-1}_i \cup D^i_{i-1} \cup D^i_i \cup D^i_{i+1}$.*
(iv) *If $D^i_{i+1} \neq \emptyset$ (resp., $D^{i+1}_i \neq \emptyset$) then $D^j_{j+1} \neq \emptyset$ (resp., $D^{j+1}_j \neq \emptyset$) for every $0 \le j \le i$.*

Proof. Straightforward (see also Figure 1). □

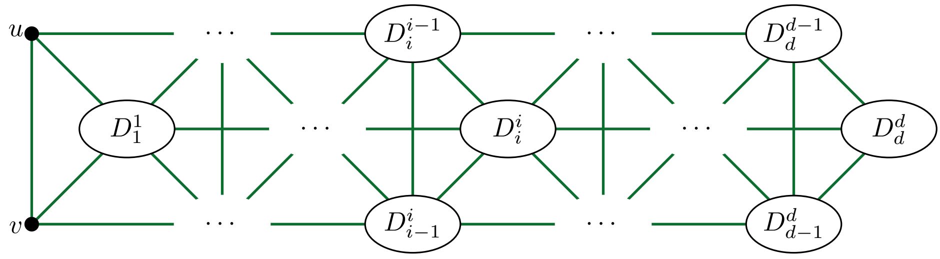

Figure 1. Graphical representation of the sets $D^i_j(u,v)$. The line between D^i_j and D^n_m indicates possible edges between vertices of D^i_j and D^n_m.

Let us recall the definition of the NDB graphs. For an edge uv of Γ we let

$$W_{u,v} = \{x \in V(\Gamma) \mid d(x,u) < d(x,v)\}.$$

We say that Γ is NDB whenever there exists a positive integer $\gamma = \gamma(\Gamma)$ such that, for any edge uv of Γ,

$$|W_{u,v}| = |W_{v,u}| = \gamma$$

holds. One can easily see that Γ is NDB if and only if, for every edge $uv \in E(\Gamma)$, we have

$$\sum_{i=1}^{d} |D^i_{i-1}(u,v)| = \sum_{i=1}^{d} |D^{i-1}_i(u,v)| = \gamma. \tag{2.1}$$

Pick adjacent vertices u, v of Γ. For the purposes of this paper we say that the edge uv is *balanced* if (2.1) holds for vertices u, v with $\gamma = d+1$.

A graph Γ is said to be *regular* if there exists a non-negative integer k such that $|\Gamma(u)| = k$ for every vertex $u \in V(\Gamma)$. In this case we also say that Γ is regular with *valency* k (or k-regular for short). The following simple observation about regular graphs will be very useful in the rest of the paper.

Lemma 2.2. *Let Γ be a connected regular graph. Then for every edge uv of Γ we have*

$$|D^1_2(u,v)| = |D^2_1(u,v)|.$$

Proof. Note that $\Gamma(u) = \{v\} \cup D^1_1(u,v) \cup D^1_2(u,v)$ and $\Gamma(v) = \{u\} \cup D^1_1(u,v) \cup D^2_1(u,v)$. As Γ is regular, the claim follows. □

Assume Γ is regular with valency k. If there exists a non-negative integer λ such that every pair u, v of adjacent vertices of Γ has exactly λ common neighbors (that is, if $|D^1_1(u,v)| = \lambda$), then we say that Γ is *edge-regular* (with parameter λ). Before we start with our study of regular NDB graphs with $\gamma = d+1$ we have a remark.

Remark 2.3. Let Γ be a regular NDB graph with diameter d and $\gamma = d+1$. Observe first that $d \geq 2$. Moreover, if $d = 2$ then it follows from [16, Theorem 5.2] that Γ is one of the following graphs:

(1) the Petersen graph,
(2) the complement of the Petersen graph,
(3) the complete multipartite graph $K_{t\times 3}$ with t parts of cardinality 3 ($t \geq 2$),
(4) the Möbius ladder graph on eight vertices,
(5) the Paley graph on 9 vertices.

In what follows we will therefore assume that $d \geq 3$.

Let Γ be an NDB graph with diameter $d \geq 3$ and with $\gamma = \gamma(\Gamma) = d+1$. Pick vertices x_0, x_d of Γ such that $d(x_0, x_d) = d$, and let $x_0, x_1, \ldots, x_d$ be a shortest path between x_0 and x_d. Consider the edge x_0x_1 and note that

$$\{x_1, x_2, \ldots, x_d\} \subseteq W_{x_1,x_0}.$$

It follows that there is a unique vertex $u \in W_{x_1,x_0} \setminus \{x_1, x_2, \dots, x_d\}$. Let $\ell = \ell(x_0, x_1)$ $(2 \le \ell \le d)$ be such that $u \in D_\ell^{\ell-1}(x_1, x_0)$, and so $D_\ell^{\ell-1}(x_1, x_0) = \{u, x_\ell\}$ and $D_i^{i-1}(x_1, x_0) = \{x_i\}$ for $2 \le i \le d$, $i \ne \ell$.

3. Some structural results

Let Γ be an NDB graph with diameter $d \ge 3$ and $\gamma = \gamma(\Gamma) = d + 1$. In this section we prove certain structural results about Γ. To do this, let us pick arbitrary vertices x_0, x_d of Γ with $d(x_0, x_d) = d$, and let us pick a shortest path $x_0, x_1, \dots, x_d$ between x_0 and x_d. Set $D_j^i = D_j^i(x_1, x_0)$ and $\ell = \ell(x_0, x_1)$. Recall that the unique vertex $u \in W_{x_1,x_0} \setminus \{x_1, x_2, \dots, x_d\}$ is contained in $D_\ell^{\ell-1}$. Observe that

$$\{x_0, x_1, \dots, x_{d-1}\} \subseteq W_{x_{d-1},x_d} \tag{3.1}$$

and

$$\{x_2, x_3, \dots, x_d\} \subseteq W_{x_2,x_1}. \tag{3.2}$$

Note that if $\ell \ge 3$, then also $u \in W_{x_2,x_1}$. In addition, we will use the following abbreviations:

$$A = \bigcup_{i=2}^{d} \big(\Gamma(x_i) \cap D_i^i\big),$$
$$B = \big(\Gamma(x_2) \cap D_1^2\big) \cup \big(\Gamma(x_d) \cap D_{d-1}^d\big).$$

Proposition 3.1. *With the notation above, the following statements hold:*

(i) *There are no edges between x_i and $D_{i-1}^i \cup D_{i-1}^{i-1}$ for $3 \le i \le d-1$.*
(ii) $|\Gamma(x_2) \cap (D_1^1 \cup D_1^2)| \le 1$.

Proof. (i) Assume that for some $3 \le i \le d-1$ we have that z is a neighbor of x_i contained in $D_{i-1}^i \cup D_{i-1}^{i-1}$. Let $x_0, y_1, \dots, y_{i-2}, z$ be a shortest path between x_0 and z. Observe that $\{y_1, \dots, y_{i-2}, z\} \cap \{x_0, x_1, \dots, x_{d-1}\} = \emptyset$ and that $\{y_1, \dots, y_{i-2}, z\} \subseteq W_{x_{d-1},x_d}$. These comments, together with (3.1), yield $|W_{x_{d-1},x_d}| \ge d + 2$, which contradicts the fact that $\gamma = d + 1$.

(ii) Let $z_1, z_2 \in \Gamma(x_2) \cap (D_1^1 \cup D_1^2)$, $z_1 \ne z_2$. Then $z_1, z_2 \in W_{x_{d-1},x_d}$. This, together with (3.1), contradicts the fact that $\gamma = d + 1$. □

Proposition 3.2. *With the notation above, the following statements hold:*

(i) $|A \cup B| \le 2$.
(ii) *If $\ell \ge 3$, then $|A \cup B \cup \big(\Gamma(u) \cap (D_\ell^\ell \cup D_{\ell-1}^\ell)\big)| = 1$.*

Proof. (i) Note that $A \cup B \subseteq W_{x_2,x_1}$ and that $(A \cup B) \cap \{x_2, \dots, x_d\} = \emptyset$. This, together with (3.2), forces $|A \cup B| \le 2$.

(ii) Note that in this case we have that $u \in W_{x_2,x_1}$. The proof that $|A \cup B \cup \big(\Gamma(u) \cap (D_\ell^\ell \cup D_{\ell-1}^\ell)\big)| \le 1$ is now similar to the proof of (i) above. On the other hand, if $|A \cup B \cup \big(\Gamma(u) \cap (D_\ell^\ell \cup D_{\ell-1}^\ell)\big)| = 0$, then $|W_{x_2,x_1}| = d$, contradicting the fact that $\gamma = d + 1$. □

4. Regular NDB graphs with $\gamma = d + 1$

Let Γ be a regular NDB graph with valency k, diameter $d \geq 3$, and $\gamma = \gamma(\Gamma) = d + 1$. In this section we use the results from Section 3 to find bounds on k and d. As in the previous section, let us pick arbitrary vertices x_0, x_d of Γ with $d(x_0, x_d) = d$, and let us pick a shortest path $x_0, x_1, \ldots, x_d$ between x_0 and x_d. Set $D^i_j = D^i_j(x_1, x_0)$ and $\ell = \ell(x_0, x_1)$.

Proposition 4.1. *Let Γ be a regular NDB graph with valency k, diameter $d = 3$, and $\gamma = 4$. Then for every $x \in V(\Gamma)$ we have eccentricity $\epsilon(x) = 3$.*

Proof. Since $d = 3$, there exists $y \in V(\Gamma)$ such that $\epsilon(y) = 3$. Pick $x \in \Gamma(y)$. By the triangle inequality we also observe that $\epsilon(x) \in \{2, 3\}$. Suppose that $\epsilon(x) = 2$. Then, the sets $D^3_2(x, y)$ and $D^3_3(x, y)$ are both empty. Recall that $\gamma = 4$, and so by Lemma 2.2 we have $|D^1_2(x, y)| = |D^2_1(x, y)| = 3$, which implies $D^2_3(x, y) = \emptyset$, contradicting that $\epsilon(y) = 3$. Therefore, $\epsilon(x) = 3$ for every $x \in \Gamma(y)$. Since Γ is connected, this finishes the proof as every neighbor of a vertex of eccentricity 3 has also eccentricity 3. □

Proposition 4.2. *There exists no regular NDB graph with valency $k = 6$, diameter $d = 3$, and $\gamma = 4$.*

Proof. Suppose to the contrary that there exists a regular NDB graph Γ with valency $k = 6$, diameter $d = 3$, and $\gamma = 4$. Then, by Proposition 4.1, every vertex $x \in V(\Gamma)$ has eccentricity $\epsilon(x) = 3$.

Let us pick an edge $xy \in E(\Gamma)$. By Lemma 2.2 we have that $|D^1_2(x, y)| = |D^2_1(x, y)|$, and so it follows from (2.1) that $|D^2_3(x, y)| = |D^3_2(x, y)|$ as well. We will prove that the sets $D^2_3(x, y)$ and $D^3_2(x, y)$ are nonempty.

Assume to the contrary that the sets $D^3_2(x, y)$ and $D^2_3(x, y)$ are empty. As $\gamma = d + 1 = 4$, we have that $|D^1_2(x, y)| = |D^2_1(x, y)| = 3$. Moreover, by Proposition 4.1 the set $D^3_3(x, y)$ is nonempty. Pick $z \in D^3_3(x, y)$ and note that there exists a vertex $w \in \Gamma(z) \cap D^2_2(x, y)$. Pick $x_1 \in D^1_2(x, y)$ and observe that $d(x_1, z) \in \{2, 3\}$. We first claim that $d(x_1, z) = 3$. Suppose to the contrary that $d(x_1, z) = 2$. Without loss of generality, we could assume that w and x_1 are adjacent. Notice that there exists a neighbor v of w in $D^1_1(x, y) \cup D^2_1(x, y)$ since $d(w, y) = 2$. Therefore, we have $\{x, y, x_1, v, w\} \subseteq W_{w,z}$, contradicting that $\gamma = 4$. This yields that $d(x_1, z) = 3$, and so there exists a shortest path x_1, v_1, w_1, z between x_1 and z of length 3. Note that by the above claim we have $w_1 \in D^2_2$, and so $\{x, y, x_1, v_1, w_1\} \subseteq W_{w_1,z}$. As $x_1 \notin \{x, y\}$, this yields a contradiction with $\gamma = 4$. This shows that the sets $D^2_3(x, y)$ and $D^3_2(x, y)$ are nonempty.

Assume for the moment that $|D^2_3(x, y)| = 2$. Since $\gamma = 4$, it follows from (2.1) that $|D^1_2(x, y)| = 1$. Let x_2 denote the unique vertex of Γ in $D^1_2(x, y)$ and let x_3 be a neighbor of x_2 which is in $D^2_3(x, y)$. Since the edge xx_2 is balanced and $D^2_3(x, y) \cup \{x_2\} \subseteq W_{x_2,x}$, we have that x_2 has at most one neighbor in $D^2_2(x, y) \cup D^2_1(x, y)$. However, as $k = 6$, this shows that x_2 has at least two neighbors in $D^1_1(x, y)$ and so the edge x_2x_3 is not balanced. Consequently, for every edge $xy \in E(\Gamma)$ we have that $|D^2_3(x, y)| = |D^3_2(x, y)| = 1$.

It follows from the above comments and (2.1) that $|D_2^1(x,y)| = |D_1^2(x,y)| = 2$ for every edge $xy \in E(\Gamma)$. This implies that $|D_1^1(x,y)| = 3$ for every edge $xy \in E(\Gamma)$ and so Γ is edge-regular with $\lambda = 3$.

Pick an edge $xy \in E(\Gamma)$. Let $D_2^1(x,y) = \{x_2, u\}$ and let x_3 be a neighbor of x_2 in $D_3^2(x,y)$. We observe that the three common neighbors of x_2 and x_3 are not all in $D_2^2(x,y)$, since the edge xx_2 is balanced. Then, u is a common neighbor of x_2 and x_3 and there exist two common neighbors of x_2 and x_3 in $D_2^2(x,y)$. Moreover, since the edge xx_2 is balanced, x_2 has no neighbors in $D_1^2(x,y)$. Furthermore, as $k = 6$ we have that x_2 has a neighbor, say z, in $D_1^1(x,y)$. It now follows that $\Gamma(x) \cap \Gamma(x_2) = \{u, z\}$, contradicting that $\lambda = 3$. □

Theorem 4.3. *Let Γ be a regular NDB graph with valency k, diameter $d \geq 3$, and $\gamma = d+1$. Then $k \in \{3,4,5\}$.*

Proof. First note that a cycle C_n $(n \geq 3)$ is NDB with $\gamma(C_n)$ equal to the diameter of C_n. Therefore, $k \geq 3$.

Assume first that $\ell = 2$ and recall that in this case the set $D_2^1 = \{x_2, u\}$. We observe that x_1 and x_3 are the only neighbors of x_2 in the set $D_1^0 \cup D_3^2$. Furthermore, by Proposition 3.1 (ii), x_2 has at most one neighbor in $D_1^1 \cup D_1^2$ and by Proposition 3.2 (i), x_2 has at most two neighbors in D_2^2. Moreover, since $\ell = 2$, we also notice that x_2 has at most one neighbor in D_2^1. If x_2 and u are not adjacent, then $k \leq 5$. Assume next that x_2 and u are adjacent. We consider the cases $d \geq 4$ and $d = 3$ separately. If $d \geq 4$, we also have that $u \in W_{x_{d-1},x_d}$, and so $W_{x_{d-1},x_d} = \{x_0, x_1, \ldots, x_{d-1}, u\}$ (recall that $\gamma = d+1$). If $w \in D_1^1 \cup D_1^2$ is adjacent to x_2, then we have that $w \in W_{x_{d-1},x_d}$, a contradiction. Therefore, x_2 has no neighbors in $D_1^1 \cup D_1^2$. As x_2 has at most 2 neighbors in D_2^2, it follows that $k \leq 5$. If x_2 and u are adjacent and $d = 3$, then $k \leq 6$. However, by Proposition 4.2, there exists no regular NDB graph with valency $k = 6$, diameter $d = 3$, and $\gamma = 4$. This shows that $k \leq 5$.

Assume next that $\ell \geq 3$. By Propositions 3.1 (ii) and 3.2 (ii), x_2 has at most one neighbor in $D_1^1 \cup D_1^2$, and at most one neighbor in D_2^2. Since x_2 has at most one neighbor in D_2^1 (namely u), it follows that $k \leq 5$. This concludes the proof. □

Theorem 4.4. *Let Γ be a regular NDB graph with valency k, diameter $d \geq 3$, and $\gamma = d+1$. Then the following statements hold:*

(i) *If $k = 3$, then $d \in \{3,4,5\}$.*
(ii) *If $k = 4$, then $d \in \{3,4\}$.*
(iii) *If $k = 5$, then $d = 3$.*

Proof. (i) Assume that $d \geq 6$ and consider first the case $\ell = 2$. Note that by Proposition 3.1 (i) x_4 and x_5 have a neighbor in D_4^4 and D_5^5 respectively. If x_3 has a neighbor in D_3^3 then this contradicts Proposition 3.2 (i). Therefore, x_3 and u are adjacent and so $u \in W_{x_{d-1},x_d}$. This and (3.1) implies that x_2 has no neighbors in $D_1^1 \cup D_1^2$. If x_2 and u are adjacent, then we have that $|W_{u,x_2}| = |W_{x_2,u}| = 1$, contradicting $\gamma = d+1$. Therefore, x_2 has a neighbor in D_2^2, contradicting Proposition 3.2 (i).

If $\ell = 3$, then by Proposition 3.1 (i) vertex x_5 has a neighbor in D_5^5. By Proposition 3.1 (i) and Proposition 3.2 (ii), x_3 and x_4 are both adjacent with u. But then $|W_{u,x_3}| = |W_{x_3,u}| = 1$, contradicting $\gamma = d+1$.

If $\ell = d-1$, then by Proposition 3.1 (i) vertex x_3 has a neighbor in D_3^3. Proposition 3.1 (i) and Proposition 3.2 (ii) now force that x_2 has a neighbor in D_1^1 and that x_{d-1} and u are adjacent. As $|W_{x_{d-1},x_d}| = d+1$ we have that also x_d and u are adjacent (otherwise $u \in W_{x_{d-1},x_d}$). But now $|W_{u,x_{d-1}}| = |W_{x_{d-1},u}| = 1$, contradicting $\gamma = d+1$.

If $\ell = d$, then x_3 and x_4 both have a neighbor in D_3^3 and D_4^4 respectively, contradicting Proposition 3.2 (ii).

Assume finally that $4 \le \ell \le d-2$. Similarly as above we see that x_ℓ and $x_{\ell+1}$ are not both adjacent to u, so either x_ℓ has a neighbor in D_ℓ^ℓ or $x_{\ell+1}$ has a neighbor in $D_{\ell+1}^{\ell+1}$ (but not both). Therefore we have that $u \in W_{x_{d-1},x_d}$, and so x_2 has no neighbors in $D_1^1 \cup D_1^2$. Consequently, x_2 has a neighbor in D_2^2, contradicting Proposition 3.2 (ii).

(ii) Assume $d \ge 5$. If $\ell = 2$, then by Proposition 3.1 (i) vertex x_3 has at least one neighbor in D_3^3, while vertex x_4 has two neighbors in D_4^4. However, this contradicts Proposition 3.2 (i).

If $\ell \ge 3$, then again by Proposition 3.1 (i) vertex x_3 (resp., vertex x_4) has at least one neighbor in D_3^3 (resp., D_4^4), contradicting Proposition 3.2 (ii).

(iii) Assume $d \ge 4$. It follows from the proof of Theorem 4.3 that in this case $\ell \in \{2,3\}$ holds. If $\ell = 2$, then by Proposition 3.1 (ii) and since $k = 5$, vertex x_2 has at least one neighbor in D_2^2, while vertex x_3 has at least two neighbors in D_3^3. However, this contradicts Proposition 3.2 (i).

If $\ell \ge 3$, then by Proposition 3.1 (i) vertex x_3 has at least two neighbors in D_3^3, again contradicting Proposition 3.2 (ii). This shows that $d = 3$. ☐

Proposition 4.5. *Let Γ be a regular NDB graph with valency k, diameter $d = 3$, and $\gamma = 4$. Then for every edge $xy \in E(\Gamma)$ we have that $|D_3^2(x,y)| = |D_2^3(x,y)| \ne 0$.*

Proof. Let us pick an edge $xy \in E(\Gamma)$. Recall that by Lemma 2.2 we have that $|D_2^1(x,y)| = |D_1^2(x,y)|$, and so it follows from (2.1) that $|D_3^2(x,y)| = |D_2^3(x,y)|$ as well. Therefore, it remains to prove that the sets $D_3^2(x,y)$ and $D_2^3(x,y)$ are nonempty.

Assume to the contrary that the sets $D_2^3(x,y)$ and $D_3^2(x,y)$ are empty. As $\gamma = d+1 = 4$ we have that $|D_2^1(x,y)| = |D_1^2(x,y)| = 3$. In view of Theorem 4.3 we therefore have $k \in \{4,5\}$. Moreover, by Proposition 4.1 the set $D_3^3(x,y)$ is nonempty. Pick $z \in D_3^3(x,y)$ and note that there exists a vertex $w \in \Gamma(z) \cap D_2^2(x,y)$.

Assume first that $k = 4$. Then the set $D_1^1(x,y)$ is empty. Hence, there exist vertices $u \in D_2^1(x,y)$ and $v \in D_1^2(x,y)$ which are neighbors of w. We thus have $\{u,v,w,x,y\} \subseteq W_{w,z}$, contradicting $\gamma = 4$.

Assume next that $k = 5$. Note that in this case $|D_1^1(x,y)| = 1$. Let us denote the unique vertex of $D_1^1(x,y)$ by u. If w and u are not adjacent, then a similar argument as in the previous paragraph shows that $|W_{w,z}| \ge 5$, a contradiction. Therefore, w and u are adjacent, and so $W_{w,z} = \{x,y,u,w\}$. It follows that the

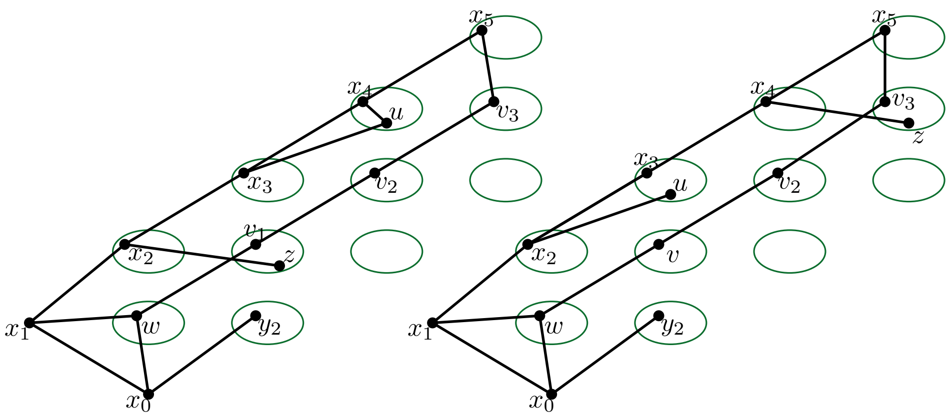

FIGURE 2. (a) Case $d = 5$, $k = 3$, and $\ell = 4$ (left). (b) Case $d = 5$, $k = 3$, and $\ell = 3$ (right).

remaining three neighbors of w (let us denote these neighbors by v_1, v_2, v_3) are also adjacent to z. As $\{u, w, z\} \subseteq W_{u,x}$, at least two of these three common neighbors (say v_1 and v_2) are in D_2^2 (recall D_3^2 and D_2^3 are empty). By the same argument as above (that is $\Gamma(v_1) \cap (D_2^1 \cup D_1^2) = \emptyset$ and $\Gamma(v_2) \cap (D_2^1 \cup D_1^2) = \emptyset$), v_1 and v_2 are adjacent to u, and so $\{u, w, v_1, v_2, z\} \subseteq W_{u,x}$, a contradiction. This shows that $D_3^2(x, y)$ and $D_2^3(x, y)$ are both nonempty. □

5. Case $k = 3$

Let Γ be a regular NDB graph with valency $k = 3$, diameter $d \geq 3$, and $\gamma = \gamma(\Gamma) = d + 1$. Recall that by Theorem 4.4 (i) we have $d \in \{3, 4, 5\}$. In this section we first show that in fact $d = 4$ or $d = 5$ is not possible, and then classify NDB graphs with $k = d = 3$. We start with a proposition which claims that $d \neq 5$. Although the proof of this proposition is rather tedious and lengthy, it is in fact pretty straightforward.

Proposition 5.1. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d \geq 3$, and $\gamma = \gamma(\Gamma) = d + 1$. Then $d \neq 5$.*

Proof. Assume to the contrary that $d = 5$. Pick vertices x_0, x_5 of Γ such that $d(x_0, x_5) = 5$. Pick also a shortest path $x_0, x_1, x_2, x_3, x_4, x_5$ from x_0 to x_5 in Γ. Let $D_j^i = D_j^i(x_1, x_0)$, let $\ell = \ell(x_0, x_1)$ and recall that $2 \leq \ell \leq 5$. Observe that if $\ell \geq 3$, then there is a unique vertex $w \in D_1^1$ and a unique vertex $y_2 \in D_1^2$. In this case x_2 and w are not adjacent, otherwise edge wx_1 is not balanced. Similarly we could prove that w and y_2 are not adjacent, and so w has a neighbor v in D_2^2.

Assume first that $\ell = 5$. Then by Proposition 3.1 (i) vertex x_3 has exactly one neighbor in D_3^3. Now vertex x_2 has a neighbor in $D_1^2 \cup D_2^2$, contradicting Proposition 3.2 (ii).

Assume $\ell = 4$. As x_2 has a neighbor in $D_1^2 \cup D_2^2$, Propositions 3.1 (i) and 3.2 (ii) imply that x_4 is adjacent to u. If x_5 is adjacent to u, then $W_{u,x_4} = \{u\}$, a contradiction. Therefore, x_5 and u are not adjacent, and so $W_{x_4,x_5} = \{x_4, x_3, x_2, x_1, x_0, u\}$. Consequently, $w \notin W_{x_4,x_5}$, which implies $d(x_5, w) = 4$. It follows that there exists a path w, v_1, v_2, v_3, x_5 of length 4, and it is easy to see that $v_1 = v$, $v_2 \in D_3^3$ and $v_3 \in D_4^4$ (see Figure 2 (a)).

If x_2 is adjacent with y_2, then $y_2 \in W_{x_4,x_5}$, a contradiction. Therefore, x_2 has a neighbor $z \in D_2^2$. If $z = v$, then $\{x_2, x_3, x_4, x_5, u, v, v_2, v_3\} \subseteq W_{x_2,x_1}$, a contradiction. Therefore $z \neq v$, $W_{x_2,x_1} = \{x_2, x_3, x_4, x_5, u, z\}$, and z is adjacent to y_2 (recall that z must be at distance 2 from x_0 and that y is not adjacent with x_1 and v). If z has a neighbor in $D_2^3 \cup D_3^3$, then this neighbor would be another vertex in W_{x_2,x_1}, which is not possible. The only other possible neighbor of z is v, and so z and v are adjacent. It is now clear that $W_{w,v} = \{w, x_0, x_1\}$, contradicting $\gamma = 6$.

Assume $\ell = 3$. By Proposition 3.1 (i), we have that either x_4 is adjacent to u, or that x_4 has a neighbor in D_4^4. Let us first consider the case when x_4 and u are adjacent. If also x_3 and u are adjacent, then ux_3 is clearly not balanced, and so Propositions 3.1 (i) and 3.2 (ii) imply that u and x_3 have a common neighbor v_2 in D_3^3. Since x_4x_5 is balanced, v_2 must be at distance 2 from x_5, which implies that v_2 and x_5 have a common neighbor $v_3 \in D_4^4$. But now $\{x_2, x_3, x_4, x_5, u, v_2, v_3\} \subseteq W_{x_2,x_1}$, a contradiction. Therefore x_4 is not adjacent to u, and so x_4 has a neighbor z in D_4^4. Propositions 3.1 (i) and 3.2 (ii) imply that x_3 has no neighbors in $D_2^2 \cup D_2^3 \cup D_3^3$, and so x_3 is adjacent to u. This implies that z and x_5 are adjacent, as otherwise x_4x_5 is not balanced. Similarly, by Proposition 3.2 (ii) u has no neighbors in $D_2^3 \cup D_3^3$, and so u is adjacent to v (note that v is the unique vertex of D_2^2). As in the previous paragraph (since $w \notin W_{x_4,x_5} = \{x_4, x_3, x_2, x_1, x_0, u\}$) we obtain that there exists a path w, v, v_2, v_3, x_5 of length 4, and that $v_2 \in D_3^3$, $v_3 \in D_4^4$ (note that it could happen that $z = v_3$). Note that u and x_3 have no neighbors in D_3^3, and that the only neighbor of v in D_3^3 is v_2. Therefore, as $k = 3$, this implies that v_2 is the unique vertex of D_3^3. Let us now examine the cardinality of D_4^4. By Proposition 3.2 (ii), both neighbors of x_5, different from x_4, are in D_4^4, and so $|D_4^4| \geq 2$. On the other hand, if v_2 has two neighbors in D_4^4, then wx_0 is not balanced, and so v_3 is the unique neighbor of v_2 in D_4^4. As x_4 has exactly one neighbor in D_4^4 (namely z), this shows that $|D_4^4| = 2$ and that $v_3 \neq z$. But as Γ is a cubic graph, it must have an even order. Then, there exists a vertex t in D_5^5. Note that t is not adjacent to x_5, and so it must be adjacent to at least one of z, v_3. However, if t is adjacent to z, then x_2x_1 is not balanced, while if it is adjacent to v_3, then wx_0 is not balanced. This shows that $\ell \neq 3$

Assume finally that $\ell = 2$. By Proposition 3.1 (i), vertex x_4 has a neighbor $z \in D_4^4$. Also by Proposition 3.1 (i), vertex x_3 either has a neighbor in D_3^3, or is adjacent with u. Assume first that x_3 is adjacent with u. Note that in this case $x_2 \not\sim u$ (otherwise edge x_2u is not balanced) and $\{x_4, x_3, x_2, x_1, x_0, u\} = W_{x_4,x_5}$. It follows that x_2 cannot have a neighbor in D_1^2 (otherwise the edge x_4x_5 is not balanced) and so x_2 has a neighbor $v \in D_2^2$. Now if v has a neighbor $v_2 \in D_3^3$, then $\{x_2, x_3, x_4, x_5, z, v, v_2\} \subseteq W_{x_2,x_1}$, a contradiction. Therefore v has no neighbors in

D_3^3, implying that $d(x_5, v) = 4$. But this forces $v \in W_{x_4,x_5}$, a contradiction. Thus $x_3 \not\sim u$, and it follows that x_3 has a neighbor $v_2 \in D_3^3$. As $\{x_2, x_3, x_4, x_5, v_2, z\} = W_{x_2,x_1}$, vertex x_2 has no neighbors in $D_1^2 \cup D_2^2$, implying that x_2 is adjacent to u. Since $W_{x_4,x_5} = \{x_4, x_3, x_2, x_1, x_0, u\}$, vertex z is adjacent to x_5, and vertices v_2 and x_5 have a common neighbor in D_4^4. Now, since x_1x_2 is balanced we have that this common neighbor is in fact z, and so z is adjacent to v_2. Now consider the edge v_2z. Note that $\{x_1, x_2, x_3, v_2\} \subseteq W_{v_2,z}$. As $d(x_0, v_2) = 3$, there exist vertices y_1, y_2 such that x_0, y_1, y_2, v_2 is a path of length 3 between x_0 and v_2. Observe that $\{x_0, y_1, y_2, v_2\} \subseteq W_{v_2,z}$. As $\{x_1, x_2, x_3\} \cap \{x_0, y_1, y_2\} = \emptyset$, we have that $|W_{v_2,z}| \geq 7$, a contradiction. □

5.1. **Case $d = 4$ is not possible.** Let Γ be a regular NDB graph with valency $k = 3$, diameter $d \geq 3$, and $\gamma = \gamma(\Gamma) = d + 1$. We now consider the case $d = 4$. Our main result in this subsection is to prove that this case is not possible. For the rest of this subsection pick arbitrary vertices x_0, x_4 of Γ such that $d(x_0, x_4) = 4$. Pick a shortest path x_0, x_1, x_2, x_3, x_4 between x_0 and x_4. Let $D_j^i = D_j^i(x_1, x_0)$ and let $\ell = \ell(x_0, x_1)$. Let u denote the unique vertex of $D_\ell^{\ell-1} \setminus \{x_\ell\}$.

Proposition 5.2. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d = 4$, and $\gamma = \gamma(\Gamma) = d + 1 = 5$. With the notation above, we have that $\ell \neq 4$.*

Proof. Assume to the contrary that $\ell = 4$. Note that in this case, since $k = 3$ and $|D_2^1| = |D_1^2| = 1$, we have $|D_1^1| = 1$. Let w denote the unique vertex of D_1^1, and let z denote the neighbor of x_2, different from x_1 and x_3. Observe that $z \neq w$, as otherwise x_1w is not balanced. Similarly, w is not adjacent to the unique vertex y_2 of D_1^2. Observe also that $\{x_0, x_1, x_2, x_3\} \subseteq W_{x_3,u}$. We claim that $u \in \Gamma(x_4)$. To prove this, suppose that x_4 and u are not adjacent. Then $x_4 \in W_{x_3,u}$, and so z is contained in D_2^2. Observe that $d(z, u) = 2$, otherwise x_3u is not balanced. Therefore, u and z must have a common neighbor z_1 and it is clear that $z_1 \in D_3^3$. But now $\{x_2, x_3, x_4, u, z, z_1\} \subseteq W_{x_2,x_1}$, a contradiction. This proves our claim that $u \sim z$.

Suppose now that $z = y_2$. Then $D_2^3 \cup D_3^4 \cup \{u, x_2, x_3, x_4, y_2\} \subseteq W_{x_2,x_1}$. Note that by the NDB condition we have $|D_2^3 \cup D_3^4| = 3$, and so x_2x_1 is not balanced, a contradiction. We therefore have that $z \in D_2^2$.

By Proposition 3.2 (ii) it follows that u and x_4 have a neighbor z_1 and z_2 in D_3^3, respectively. We observe that $z_1 \neq z_2$, as otherwise x_4u is not balanced. Note that z has no neighbors in D_3^3, as otherwise x_2x_1 is not balanced. Therefore, z is not adjacent to any of z_1, z_2, which gives us $W_{x_3,x_4} = W_{x_3,u} = \{x_3, x_2, x_1, x_0, z\}$. Consequently, $d(w, u) = d(w, x_4) = 3$, and so the (unique) neighbor of w in D_2^2 is adjacent to both z_1 and z_2. But this implies that wx_0 is not balanced, a contradiction. □

Proposition 5.3. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d = 4$, and $\gamma = \gamma(\Gamma) = d + 1 = 5$. With the notation above, we have that $\ell \neq 3$.*

Proof. Suppose that $\ell = 3$. By Lemma 2.2 we have $|D_1^2| = 1$, and since $k = 3$ also $|D_1^1| = 1$. Let w and y_2 denote the unique vertex of D_1^1 and D_1^2, respectively.

Since $\gamma = 5$, y_2 has at least one neighbor y_3 in D_2^3, and $|D_3^4| \leq 2$. If $D_3^4 = \emptyset$, then there are three vertices in D_2^3, which are all adjacent to y_2, contradicting $k = 3$. By Proposition 5.2 we have that $|D_3^4| \neq 2$, and so $|D_3^4| = 1$, $|D_2^3| = 2$. Let y_4 denote the unique element of D_3^4 and let u_1 denote the unique element of $D_2^3 \setminus \{y_3\}$. Without loss of generality assume that y_4 and y_3 are adjacent. Observe that $\Gamma(y_2) = \{x_0, y_3, u_1\}$, and so w has a neighbor $v \in D_2^2$, and it is easy to see that v is the unique vertex of D_2^2 (see Figure 3 (a)). By Proposition 3.1 (i) we find that either $x_3 \in \Gamma(u)$, or x_3 has a neighbor in D_3^3.

Case 1: there exists $z \in \Gamma(x_3) \cap D_3^3$. Note that in this case we have $W_{x_2,x_1} = \{x_2, x_3, x_4, u, z\}$. We split our analysis into two subcases.

Subcase 1.1: vertices u and x_4 are not adjacent. As x_2x_1 is balanced and as v is the unique vertex of D_2^2, this forces u to be adjacent with v and z. As every vertex in D_3^3 is at distance 3 from x_1 and as vertices u, x_3 already have three neighbors each, this implies that beside z there is at most one more vertex in D_3^3 (which must be adjacent with v). But this shows that x_4 could have at most one neighbor in D_3^3 (observe that z could not be adjacent with x_4, as otherwise z is not at distance 3 from x_0), and consequently x_4 has at least one neighbor in $D_4^4 \cup D_3^4$. But now x_2x_1 is not balanced, a contradiction.

Subcase 1.2: vertices u and x_4 are adjacent. By Proposition 3.2 (ii), vertex u is either adjacent to $v \in D_2^2$ or to $z \in D_3^3$. If u is adjacent to v, then $\{x_0, x_1, x_2, u, v, w\} \subseteq W_{u,x_4}$, a contradiction. This shows that $u \sim z$. Note that the third neighbor of z is one of the vertices v, y_3, u_1, and so z and x_4 are not adjacent. Consequently, $W_{x_3,x_4} = \{x_3, x_2, x_1, x_0, z\}$, and so w must be at distance 3 from x_4. Therefore, v and x_4 have a common neighbor $v_1 \in D_3^3$. Note that $v_1 \neq z$ as z and x_4 are not adjacent. Every vertex in D_3^3, different from z and v_1, must be adjacent with v in order to be at distance 3 from x_1. This shows that $|D_3^3| \leq 3$. If there exists vertex $v_2 \in D_3^3$, which is different from z and v_1, then there must be a vertex $t \in D_4^4$ (recall that Γ is of even order). As t could not be adjacent with x_4, it must be adjacent with at least one of v_1, v_2. However, this is not possible (note that in this case $\{w, v, v_1, v_2, x_4, t\} \subseteq W_{w,x_0}$, a contradiction). Therefore, $D_3^3 = \{z, v_1\}$ and $D_4^4 = \emptyset$. It follows that y_4 is adjacent with v_1 and u_1. If z and v are adjacent, then $W_{x_1,w} = \{x_1, x_2, u, x_3\}$, contradicting $\gamma = 5$. Therefore, z is adjacent to either y_3 or u_1. This shows that either y_3 or u_1 is contained in $W_{x_3,x_4} = \{x_3, x_2, x_1, x_0, z\}$, a contradiction.

Case 2: x_3 and u are adjacent. Observe that $x_4 \notin \Gamma(u)$, otherwise ux_3 is not balanced. It follows that $W_{x_3,x_4} = \{x_3, x_2, x_1, x_0, u\}$, and so $d(w, x_4) = 3$. Therefore there exists a common neighbor z of x_4 and v, and note that $z \in D_3^3$. Reversing the roles of the paths x_0, x_1, x_2, x_3, x_4 and x_1, x_0, y_2, y_3, y_4, we get that u_1 and y_3 are adjacent, and that $y_4 \notin \Gamma(u_1)$. As $|W_{x_1,w}| = 5$, vertex u must have a neighbor, which is at distance 3 from x_1 and at distance 4 from w. As x_4, y_3 and u_1 are all at distance 3 from w, this implies that u has a neighbor $z_1 \in D_3^3$, which is not adjacent with v (and is therefore different from z). Note that since z_1 is at distance 3 from x_0, it is adjacent with u_1. As $k = 3$, v has a neighbor $z_2 \neq z$ in D_3^3. Pick now a vertex $t \in D_4^4$ (observe that $D_4^4 \neq \emptyset$ as Γ has even order). If t is

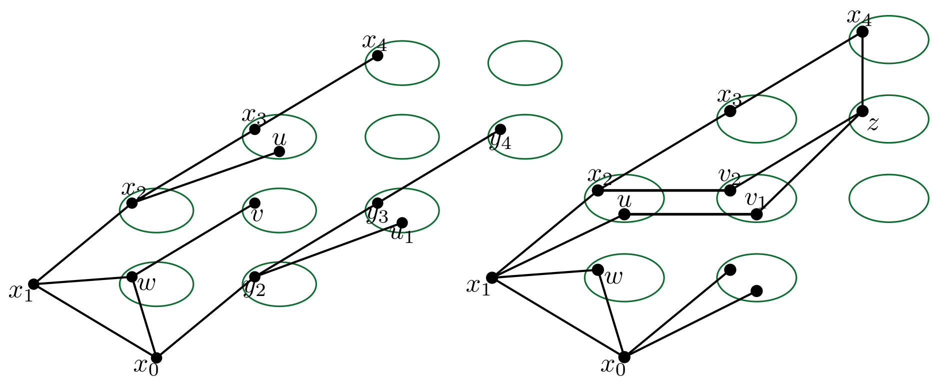

FIGURE 3. (a) Case $d = 4$, $k = 3$, and $\ell = 3$ (left). (b) Case $d = 4$, $k = 4$, and $\ell = 2$ (right).

adjacent with x_4 or with z_1, then $t \in W_{x_2,x_1} = \{x_2, x_3, x_4, u, z_1\}$, a contradiction. If t is adjacent with z or z_2, then $t \in W_{w,x_0} = \{w, v, z, z_2, x_4\}$, a contradiction. This finally proves that $\ell \neq 3$. □

Proposition 5.4. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d = 4$, and $\gamma = \gamma(\Gamma) = d + 1 = 5$. With the notation above, Γ is triangle-free.*

Proof. Pick an edge $xy \in E(\Gamma)$ and let $D^i_j = D^i_j(x, y)$. If either D^4_3 or D^3_4 is nonempty, then Propositions 5.2 and 5.3 together with Lemma 2.2 imply that $|D^1_2| = |D^2_1| = 2$. As Γ is 3-regular, the set D^1_1 is empty, and so xy is not contained in any triangle.

Assume next that $D^4_3 = D^3_4 = \emptyset$. If the edge xy is contained in a triangle, then D^1_2 and D^2_1 both contain at most one vertex, and so D^2_3 and D^3_2 could contain at most two vertices as Γ is 3-regular. We thus have $|W_{x,y}| \leq 4$, contradicting $\gamma = 5$. The result follows. □

Proposition 5.5. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d \geq 3$, and $\gamma = \gamma(\Gamma) = d + 1$. Then $d \neq 4$.*

Proof. Suppose, towards a contradiction, that $d = 4$, and so $\gamma = 5$. Assume the notation from the first paragraph of this subsection, and note that Propositions 5.2 and 5.3 imply that $\ell = 2$. By Lemma 2.2 we have $|D^2_1| = 2$. Let u_1, y_2 denote the vertices of D^2_1. Note that D^1_1 is empty. We also observe that by Proposition 3.1 (i) either $u \in \Gamma(x_3)$, or x_3 has a neighbor in D^3_3. We consider these two cases separately.

CASE 1: u and x_3 are adjacent. Then $\{x_0, x_1, x_2, x_3, u\} = W_{x_3,x_4}$, and so neither x_2 nor u have neighbors in D^2_1. Since Γ is triangle-free, there exists $w \in \Gamma(x_2) \cap D^2_2$, and w has a neighbor in D^2_1 (by definition of the set D^2_2). We may assume without loss of generality that $w \in \Gamma(y_2)$. Note that $d(w, x_3) = 2$, and so $d(w, x_4) = 2$

as well, as otherwise x_3x_4 is not balanced. It follows that there exists a common neighbor z of w and x_4, and it is clear that $z \in D_3^3$.

Similarly we find that u has a neighbor $w_1 \in D_2^2$, and as $k = 3$, we have that $w_1 \neq w$. Note that $\{x_2, x_1, x_0, w, y_2\} = W_{x_2,x_3}$, and so $d(x_3, u_1) = 3$ (otherwise $u_1 \in W_{x_2,x_3}$, a contradiction). Note, however, that $d(x_3, u_1) = 3$ is only possible if w_1 and u_1 are adjacent. A similar argument as above shows that w_1 and x_4 must have a common neighbor $z_1 \in D_3^3$. If $z_1 = z$, then $\{z, w, w_1, y_2, u_1, x_0\} \subseteq W_{z,x_4}$, a contradiction. Therefore $z_1 \neq z$, and it is now clear that $D_2^2 = \{w, w_1\}$, $D_3^3 = \{z, z_1\}$. If there exists $t \in D_4^4$, then t is adjacent to either z or z_1, but none of these two possible edges is balanced, and so $D_4^4 = \emptyset$. If z (resp., z_1) has a neighbor in D_3^4, then x_2x_1 (resp., ux_1) is not balanced, a contradiction. As Γ is triangle-free, z and z_1 both have a neighbor in D_2^3. Assume now for a moment that there exists a vertex $y_4 \in D_3^4$. In this case $\gamma = 5$ forces that there is a unique vertex in D_2^3, which is therefore adjacent to both z and z_1, to y_4, and to at least one of y_2, u_1, contradicting $k = 3$. It follows that $D_3^4 = \emptyset$. Let us denote the neighbors of z and z_1 in D_2^3 by v and v_1, respectively. Note that as zx_4 and z_1x_4 are balanced, we have that $W_{z,x_4} = \{z, w, v, y_2, x_0\}$ and $W_{z_1,x_4} = \{z_1, w_1, v_1, u_2, x_0\}$. It follows that v and v_1 must be adjacent to y_2 and u_1, respectively, and so $v \neq v_1$. As $k = 3$, also v and v_1 are adjacent. It is now easy to see that Γ is not NDB with $\gamma = 5$ (for example, edge x_1u is not balanced). This shows that u and x_3 are not adjacent.

Case 2: x_3 has a neighbor w in D_3^3. As Γ is triangle-free, x_2 has a neighbor z in $D_1^2 \cup D_2^2$, and $w \not\sim x_4$. If $z \in D_1^2$, then $\{x_0, x_1, x_2, x_3, z, w\} \subseteq W_{x_3,x_4}$, a contradiction. This yields that $z \in D_2^2$. If $d(z, x_4) \geq 3$, then again $\{x_0, x_1, x_2, x_3, z, w\} \subseteq W_{x_3,x_4}$, a contradiction. Therefore, z and x_4 have a common neighbor $w_1 \in D_3^3$, and $w_1 \neq w$ as $w \not\sim x_4$. But now $\{x_2, x_3, x_4, z, w, w_1\} \subseteq W_{x_2,x_1}$, a contradiction. This finishes the proof. □

5.2. **Case $d = 3$.** In this subsection we consider the case $d = 3$. We start with the following proposition.

Proposition 5.6. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d = 3$, and $\gamma = 4$. Then for every edge x_0x_1 of Γ we have that $|D_2^1(x_1, x_0)| = |D_1^2(x_1, x_0)| = 2$.*

Proof. Pick an edge x_0x_1 of Γ and let $D_j^i = D_j^i(x_1, x_0)$. Observe first that $|D_2^1| \leq 2$ as $k = 3$. By Proposition 4.5 we have that $D_3^2 \neq \emptyset$, and so pick $x_3 \in D_3^2$. Note that x_1 and x_3 have a common neighbor $x_2 \in D_2^1$. Assume to the contrary that $|D_2^1| = 1$, and so $|D_3^2| = 2$, $|D_1^1| = 1 = |D_1^2|$. Let us denote the unique vertex of D_1^2 by y_2 (note that y_2 has two neighbors, say y_3 and u_1 in D_2^3), the unique vertex of D_1^1 by w, and the unique vertex of $D_3^2 \setminus \{x_3\}$ by u (note that $\Gamma(x_2) = \{x_1, x_3, u\}$). Note that w has a neighbor v in D_2^2, and that $D_2^2 = \{v\}$.

Assume first that u and x_3 are not adjacent. Then $W_{x_2,x_3} = \{x_2, u, x_1, x_0\}$, and so w is at distance 2 from x_3 (otherwise $w \in W_{x_2,x_3}$). It follows that x_3 is adjacent with v. Similarly we show that u is adjacent with v. As none of the neighbors of v is contained in D_3^3, every vertex from D_3^3 must be adjacent to either u or x_3, and so $D_3^3 \cup \{x_2, x_3, u\} \subseteq W_{x_2,x_1}$. It follows that $|D_3^3| \leq 1$. As Γ is a cubic graph, it must have an even order, which gives us $D_3^3 = \emptyset$. This shows that both u and x_3 have a neighbor in D_2^3. But now $\{y_2, y_3, u_1, x_3, u\} \cup D_2^3 \subseteq W_{y_2,x_0}$, a contradiction.

Therefore, u and x_3 must be adjacent, and they have a common neighbor x_2. Let z_1 and z_2 denote the third neighbor of u and x_3, respectively. If $z_1 = z_2$ then ux_3 is not balanced, and so we have that $z_1 \neq z_2$. Furthermore, as $\{x_2, x_3, u\} \subseteq W_{x_2,x_1}$, not both of z_1, z_2 are contained in $D_3^3 \cup D_2^3$. Therefore, either z_1 or z_2 is equal to v. Without loss of generality assume that $z_1 = v$. But then $d = 3$ forces $W_{x_2,u} = \{x_2, x_1, x_0\}$, a contradiction. This shows that $|D_2^1| = 2$, and by Lemma 2.2 also $|D_1^2| = 2$. □

Corollary 5.7. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d = 3$, and $\gamma = 4$. Then Γ is triangle-free and $D_3^3(x, y) = \emptyset$ for every edge xy of Γ.*

Proof. Pick an arbitrary edge xy of Γ and let $D_j^i = D_j^i(x, y)$. By Proposition 4.5 we get that the sets D_2^1, D_1^2, D_3^2, and D_2^3 are all nonempty. Furthermore, by Proposition 5.6 and Lemma 2.2 we have that $|D_2^1| = |D_1^2| = 2$ and $|D_2^3| = |D_3^2| = 1$ (recall that $\gamma = 4$). Since $k = 3$, it follows that $D_1^1 = \emptyset$. This shows that Γ is triangle-free.

We next assert the set D_3^3 is empty. Suppose to the contrary there exists $z \in D_3^3$ and let w denote a neighbor of z. Assume first that $w \in D_2^2$. Since $D_1^1 = \emptyset$, there exist vertices $u \in D_2^1$ and $v \in D_1^2$ which are neighbors of w. We thus have $\{u, v, w, x, y\} \subseteq W_{w,z}$, contradicting $\gamma = 4$. This shows that $w \notin D_2^2$. Therefore z is adjacent to both vertices which are in D_2^3 and D_3^2. As z has three neighbors, none of which is in D_2^2, and as $|D_3^2| = |D_2^3| = 1$, it follows that z has a neighbor $w' \in D_3^3$. But by the same argument as above, w' must be adjacent to both vertices in D_2^3 and D_3^2, contradicting the fact that Γ is triangle-free. □

Theorem 5.8. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d \geq 3$, and $\gamma = d + 1$. Then Γ is isomorphic to the 3-dimensional hypercube Q_3.*

Proof. By Theorem 4.4 (i), Proposition 5.1 and Proposition 5.5 we have that $d = 3$. Pick an edge xy of Γ and let $D_j^i = D_j^i(x, y)$. Observe that Γ is triangle-free and $D_3^3 = \emptyset$ by Corollary 5.7. We first show that $D_2^2 = \emptyset$ as well. Observe that as $D_1^1 = \emptyset$, every vertex of D_2^2 must have a neighbor in both D_2^1 and D_1^2. This shows that $\left|D_2^2\right| \in \{1, 2, 3\}$, and so $|V(\Gamma)| \in \{9, 10, 11\}$. However, since Γ is regular with $k = 3$, we have $|V(\Gamma)| = 10$ and $\left|D_2^2\right| = 2$. In [5], it is shown that the number of connected 3-regular graphs with 10 vertices is 19, but only five of them have diameter $d = 3$ and girth $g \geq 4$. Out of these five graphs, only four have all vertices with eccentricity 3 (see Figure 4). It is easy to see that none of these graphs is NDB with $\gamma = 4$. This shows that $D_2^2 = \emptyset$, and so $|V(\Gamma)| = 8$. But it is well known (and also easy to see) that Q_3 is the only cubic triangle-free graph with eight vertices and diameter $d = 3$. □

6. Case $k = 4$

Let Γ be a regular NDB graph with valency $k = 4$, diameter $d \geq 3$, and $\gamma = \gamma(\Gamma) = d + 1$. Recall that by Theorem 4.4 (ii) we have $d \in \{3, 4\}$. In this section

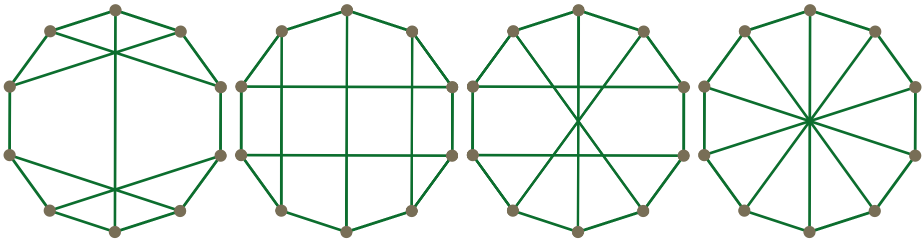

FIGURE 4. Connected 3-regular graphs of order 10 with diameter $d = 3$, girth $g \geq 4$, and with all vertices with eccentricity 3.

we first show that the case $d = 4$ is not possible, and then classify regular NDB graphs with $k = 4$ and $d = 3$. We start with the following lemma.

Lemma 6.1. *Let Γ be a regular NDB graph with valency $k = 4$, diameter $d = 4$, and $\gamma = \gamma(\Gamma) = d + 1$. Pick vertices x_0, x_4 of Γ such that $d(x_0, x_4) = 4$, and pick a shortest path x_0, x_1, x_2, x_3, x_4 between x_0 and x_4. Let $\ell = \ell(x_0, x_1)$, $D^i_j = D^i_j(x_1, x_0)$, and $D^{\ell-1}_\ell = \{x_\ell, u\}$. Then $\ell = 2$. Moreover, $u \sim x_2$ and $u \sim x_3$.*

Proof. Assume first that $\ell = 4$. By Proposition 3.1 (i), vertex x_3 has a neighbor z in D^3_3. Now $W_{x_2,x_1} = \{x_2, x_3, x_4, u, z\}$, and so x_2 has no neighbors in $D^2_2 \cup D^2_1$. Consequently, x_2 has two neighbors in D^1_1, contradicting Proposition 3.1 (ii).

Assume now that $\ell = 3$. By Proposition 3.1 (i) x_3 does not have neighbors in $D^3_2 \cup D^2_2$, and so by Proposition 3.2 (ii) we get that x_3 and u are adjacent, and that x_3 has a neighbor z in D^3_3. By Proposition 3.2 (ii) vertex x_2 has no neighbors in $D^2_2 \cup D^2_1$, and so x_2 has a neighbor w in D^1_1. Now $\{x_3, x_2, x_1, x_0, w\} \subseteq W_{x_3,x_4}$, implying that x_4 is adjacent to both u and z. Similarly, $\{u, x_2, x_1, x_0, w\} \subseteq W_{u,x_4}$, and so u has no neighbors in $D^2_2 \cup D^3_2$. It follows that u has a neighbor in D^3_3, and by Proposition 3.2 (ii), this neighbor is z. But now the edge x_3u is not balanced, a contradiction.

This shows that $\ell = 2$. By Proposition 3.1 (i), vertex x_3 has either one or two neighbors in D^3_3. If x_3 has two neighbors in D^3_3, then by Proposition 3.2 (i) vertex x_2 has no neighbors in $D^2_2 \cup D^2_1$. Therefore, x_2 is adjacent to the unique vertex $w \in D^1_1$, and is also adjacent to u. But now we have that $\{x_3, x_2, x_1, x_0, u, w\} \subseteq W_{x_3,x_4}$, a contradiction.

Therefore, x_3 has exactly one neighbor in D^3_3. As by Proposition 3.1 (i) vertex x_3 has no neighbors in $D^2_2 \cup D^3_2$, we have that $x_3 \sim u$. Consequently $\{x_3, x_2, x_1, x_0, u\} \subseteq W_{x_3,x_4}$, and so x_2 and u have no neighbors in $D^1_1 \cup D^2_1$. Since $k = 4$ and since edges x_2x_1 and ux_1 are balanced, it follows both of x_2 and u have exactly one neighbor in D^2_2, and that $x_2 \sim u$. □

Proposition 6.2. *Let Γ be a regular NDB graph with valency $k = 4$, diameter $d \geq 3$, and $\gamma = \gamma(\Gamma) = d + 1$. Then $d \neq 4$.*

Proof. Assume to the contrary that $d = 4$. Pick vertices x_0, x_4 of Γ such that $d(x_0, x_4) = 4$. Pick a shortest path x_0, x_1, x_2, x_3, x_4 between x_0 and x_4. Let $D^i_j =$

$D^i_j(x_1, x_0)$, let $\ell = \ell(x_0, x_1)$ and let $D^{\ell-1}_\ell = \{x_\ell, u\}$. Recall that by Lemma 6.1 we have that $\ell = 2$ and that vertex u is adjacent with x_2 and x_3. Let z denote a neighbor of x_3 in D^3_3 (note that by Proposition 3.1 (i) vertex x_3 has no neighbors in $D^2_2 \cup D^3_2$).

Since $W_{x_3,x_4} = \{x_3, x_2, x_1, x_0, u\}$, vertices x_2 and u have no neighbors in $D^1_1 \cup D^2_1$. Let us denote the neighbors of u and x_2 in D^2_2 by v_1, v_2, respectively. Note that $v_1 \neq v_2$, otherwise edge ux_2 is not balanced. Furthermore, $\{x_3, x_2, x_1, x_0, u\} = W_{x_3,x_4}$ implies that x_4 and z are adjacent, and that x_4 is at distance 2 from both v_1 and v_2. Consequently, v_1 and v_2 both have a common neighbor, say z_1 and z_2, respectively, with x_4, and these common neighbors must be in D^3_3. But as edges x_2x_1 and ux_1 are balanced, this implies that $z_1 = z = z_2$ (see Figure 3 (b)).

Note that v_1 and v_2 both have at least one neighbor in $D^1_1 \cup D^2_1$. Let us denote a neighbor of v_1 (resp., v_2) in $D^1_1 \cup D^2_1$ by w_1 (resp., w_2). If $w_1 \neq w_2$, then $\{z, v_1, v_2, w_1, w_2, x_0\} \subseteq W_{z,x_4}$, contradicting $\gamma = 5$. Therefore $w_1 = w_2$ and by applying Lemma 6.1 to the path x_0, w_1, v_1, z, x_4 we get that vertices v_1 and v_2 are adjacent. But now it is easy to see that $W_{u,x_2} = \{u, v_1\}$, a contradiction. This finishes the proof. □

Proposition 6.3. *Let Γ be a regular NDB graph with valency $k = 4$, diameter $d = 3$, and $\gamma = \gamma(\Gamma) = 4$. Then for every edge x_0x_1 of Γ we have that $|D^1_2(x_1, x_0)| = |D^2_1(x_1, x_0)| = 2$.*

Proof. Pick an edge x_0x_1 of Γ and let $D^i_j = D^i_j(x_1, x_0)$. By Proposition 4.5 we have that $D^2_3 \neq \emptyset$, and so $\gamma = 4$ implies $|D^1_2| \leq 2$. Assume to the contrary that $|D^1_2| = 1$, and so $|D^2_3| = 2$, $|D^1_1| = 2$, and $|D^2_1| = 1$. Let x_3, u be vertices of D^2_3, and let x_2 be the unique vertex of D^1_2. Let z denote the neighbor of x_2, different from x_1, x_3, u, and note that $z \in D^2_2 \cup D^2_1 \cup D^1_1$. In each of these three cases we derive a contradiction.

Assume first that $z \in D^2_2$. Then $D^1_2(x_2, x_1) = \{x_3, u, z\}$, and $\gamma = 4$ forces $D^2_3(x_2, x_1) = \emptyset$, contradicting Proposition 4.5.

Assume next that $z \in D^2_1$ (note that z is the unique vertex in D^2_1). Then $\{x_2, z, x_3, u\} \cup D^3_2 \subseteq W_{x_2,x_1}$. As $D^3_2 \neq \emptyset$ by Proposition 4.5, this contradicts $\gamma = 4$.

Assume finally that $z \in D^1_1$. Recall that $|D^1_1| = 2$ and denote the other vertex of D^1_1 by w. If z and w are adjacent, then $W_{x_1,z} = \{x_1\}$, a contradiction. If z has a neighbor $v \in D^2_2$, then $\{z, v, x_2, u, x_3\} \subseteq W_{z,x_0}$, a contradiction. This shows that z is adjacent to the unique vertex of D^2_1. Let us denote this vertex by y_2. As $W_{x_2,x_3} = W_{x_2,u} = \{x_2, x_1, x_0, z\}$, vertices x_3 and u are both at distance 2 from y_2. But this shows that $W_{z,y_2} = \{x_1, z, x_2\}$, a contradiction. □

Theorem 6.4. *Let Γ be a regular NDB graph with valency $k = 4$, diameter $d \geq 3$, and $\gamma = \gamma(\Gamma) = d+1$. Then Γ is isomorphic to the line graph of the 3-dimensional hypercube Q_3.*

Proof. By Theorem 4.4 (ii) and Proposition 6.2 we have that $d = 3$. Pick an arbitrary edge xy of Γ. By Proposition 6.3 we have that $|D^1_2(x, y)| = |D^2_1(x, y)| = 2$. Consequently $|D^1_1(x, y)| = 1$, and so Γ is an edge-regular graph with $\lambda = 1$. Observe that $\gamma = 4$ also implies that $|D^2_3(x,y)| = |D^3_2(x,y)| = 1$. Observe that Γ contains $|V(\Gamma)|k/6 = 2|V(\Gamma)|/3$ triangles, and so $|V(\Gamma)|$ is divisible by 3.

Pick vertices x_0, x_3 of Γ at distance 3 and let x_0, x_1, x_2, x_3 be a shortest path from x_0 to x_3. Abbreviate $D^i_j = D^i_j(x_1, x_0)$. Obviously $D^2_3 = \{x_3\}$ and $x_2 \in D^1_2$. Let us denote the other vertex of D^1_2 by u, the vertices of D^2_1 by y_2, v, the vertex of D^3_2 by y_3, and the vertex of D^1_1 by w. Without loss of generality we may assume that y_2 and y_3 are adjacent. Since Γ is edge-regular with $\lambda = 1$, we also obtain that x_2 and u are adjacent, that y_2 and v are adjacent, and that w has two neighbors, say z_1 and z_2, in D^2_2, and that z_1, z_2 are also adjacent. As $W_{x_2,x_3} = \{x_2, x_1, x_0, u\}$, x_3 is at distance 2 from w, and so x_3 is adjacent to exactly one of z_1, z_2. Without loss of generality we could assume that x_3 and z_1 are adjacent.

Note that $\Gamma(w) = \{x_0, x_1, z_1, z_2\}$, and so x_2 and w are not adjacent. Vertex x_2 is also not adjacent to y_2, as otherwise edge x_2y_2 is not contained in a triangle. If $x_2 \sim v$, then $v \sim u$ and the edge ux_2 is contained in two triangles, contradicting $\lambda = 1$. It follows that x_2 has no neighbors in D^2_1. Therefore, x_2 has a neighbor in D^2_2. Consequently, by Proposition 3.2 (i), x_3 could have at most one neighbor in $D^3_3 \cup D^3_2$.

We now show that $D^3_3 = \emptyset$. Assume to the contrary that there exists $t \in D^3_3$. If t is adjacent to z_1 or z_2, then $\{w, z_1, z_2, x_3, t\} \subseteq W_{w,x_0}$, a contradiction. If t is adjacent with $z \in D^2_2 \setminus \{z_1, z_2\}$, then z has a neighbor in D^1_2 and a neighbor in D^2_1, implying that $|W_{z,t}| \geq 5$, a contradiction. It follows that t has no neighbors in D^2_2, and so t is adjacent with x_3 (and with y_3). Now the unique common neighbor of x_3 and t must be contained in $D^3_3 \cup D^3_2$, contradicting the fact that x_3 could have at most one neighbor in $D^3_3 \cup D^3_2$. This shows that $D^3_3 = \emptyset$.

Let us now estimate the cardinality of D^2_2. Observe that each $z \in D^2_2 \setminus \{z_1, z_2\}$ has a neighbor in D^1_2. But u could have at most two neighbors in D^2_2, while x_2 has exactly one neighbor in D^2_2. It follows that $2 \leq |D^2_2| \leq 5$, and so $11 \leq |V(\Gamma)| \leq 14$. As $|V(\Gamma)|$ is divisible by 3, we have that $|V(\Gamma)| = 12$. By [9, Corollary 6], there are just two edge-regular graphs on 12 vertices with $\lambda = 1$, namely the line graph of 3-dimensional hypercube (see Figure 5), and the line graph of the Möbius ladder graph on eight vertices. It is easy to see that the latter one is not even distance-balanced. □

7. Case $k = 5$

Let Γ be a regular NDB graph with valency $k = 5$, diameter $d \geq 3$, and $\gamma = \gamma(\Gamma) = d + 1$. Recall that by Theorem 4.4 we have $d = 3$, and so $\gamma = 4$. In this section we classify such NDB graphs. We first show that in this case we have $|D^1_2(x_1, x_0)| = |D^2_1(x_1, x_0)| = 2$ for every edge x_1x_0 of Γ.

Proposition 7.1. *Let Γ be a regular NDB graph with valency $k = 5$, diameter $d = 3$, and $\gamma = 4$. Then for every edge x_0x_1 of Γ we have that $|D^1_2(x_1, x_0)| = |D^2_1(x_1, x_0)| = 2$.*

Proof. Pick an edge x_0x_1 of Γ and let $D^i_j = D^i_j(x_1, x_0)$. By Proposition 4.5 we have that $D^2_3 \neq \emptyset$, and so $\gamma = 4$ implies $|D^1_2| \leq 2$. Assume to the contrary that

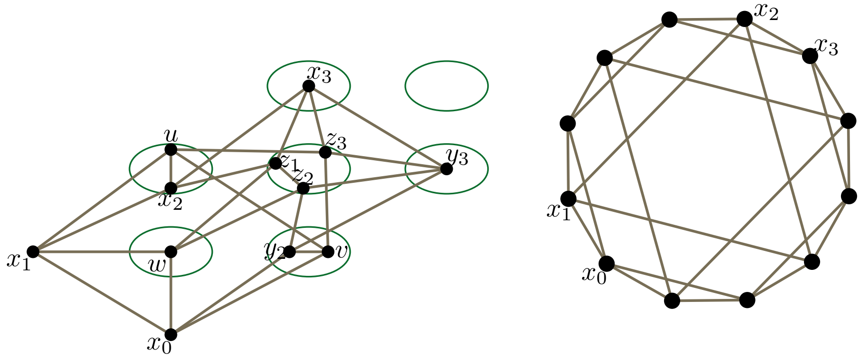

FIGURE 5. The line graph of Q_3, drawn in two different ways.

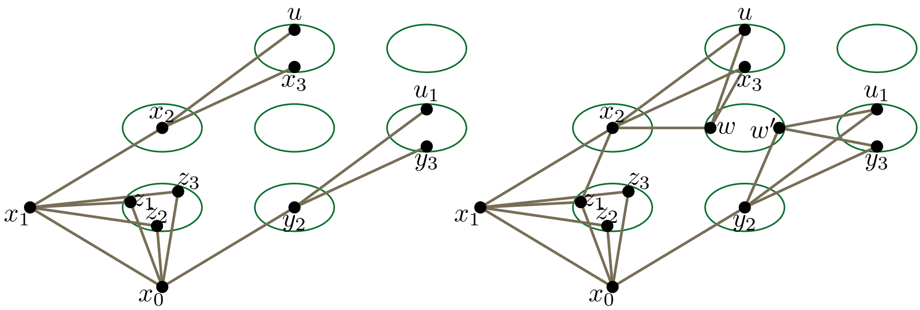

FIGURE 6. Graph Γ from Proposition 7.1.

$|D_2^1| = 1$, and so $|D_3^2| = 2$, $|D_1^1| = 3$, and $|D_1^2| = 1$. Let x_3, u be vertices of D_3^2, and let x_2 be the unique vertex of D_2^1. Let us denote the unique vertex of D_1^2 by y_2, and the vertices of D_1^1 by z_1, z_2, z_3. Note that also $|D_2^3| = 2$, and let us denote these two vertices by y_3, u_1. Clearly we have that x_2 is adjacent to both x_3 and u, and y_2 is adjacent to both y_3 and u_1 (see the diagram on the left side of Figure 6). Observe that each edge xy of Γ is contained in at least one triangle; otherwise $|W_{x,y}| \geq 5 > \gamma$, a contradiction. Therefore, x_2 and y_2 both have at least one neighbor in D_1^1. On the other hand, these two vertices could not have more than one neighbor in D_1^1, as otherwise $|W_{x_2,x_3}| \geq 5$ (resp., $|W_{y_2,y_3}| \geq 5$), a contradiction. Without loss of generality we could assume that z_1 is the unique neighbor of x_2 in D_1^1. Note that it follows from Proposition 3.1 (ii) that x_2 and y_2 are not adjacent. This shows that x_2 has a unique neighbor (say w) in D_2^2. As $W_{x_2,x_3} = W_{x_2,u} = \{x_2, x_1, x_0, z_1\}$, vertex w is adjacent to both u and x_3. Similarly we prove that also y_2 has a unique neighbor in D_2^2, say w', and that w' is adjacent to both u_1 and y_3.

If $w = w'$, then the degree of w is at least 6, a contradiction. Therefore, $w \neq w'$ (see the diagram on the right side of Figure 6).

Note that $W_{x_2,x_1} = \{x_2, x_3, u, w\}$, and so both y_3 and u_1 are at distance 3 from x_2. Similarly, $W_{x_1,x_2} = \{x_1, x_0, z_2, z_3\}$, and so y_2 is at distance 2 from x_2. Therefore y_2 and x_2 have a common neighbor, and by the comments above the only possible common neighbor is z_1. It follows that z_1 and y_2 are adjacent. But now $\{y_2, x_0, x_1, z_1, x_2\} \subseteq W_{y_2,y_3}$ (recall that $d(x_2, y_3) = 3$), a contradiction. This shows that $|D_2^1| = 2$. By Lemma 2.2 we obtain that $|D_1^2| = 2$ as well. □

Theorem 7.2. *Let* Γ *be a regular NDB graph with valency* $k = 5$*, diameter* $d \geq 3$*, and* $\gamma = d + 1$*. Then* Γ *is isomorphic to the icosahedron.*

Proof. First recall that by Theorem 4.4 we have $d = 3$, and so $\gamma = 4$. We will first show that Γ is edge-regular with $\lambda = 2$. Pick an arbitrary edge xy and observe that by Proposition 7.1 we obtain $|D_2^1(x, y)| = 2$, which forces $|D_1^1(x, y)| = 2$. This shows that Γ is edge-regular with $\lambda = 2$. It follows that for every vertex x of Γ, the subgraph of Γ which is induced on $\Gamma(x)$ is isomorphic to the five-cycle C_5. By [4, Proposition 1.1.4], Γ is isomorphic to the icosahedron. □

Proof of Theorem 1.1. It is straightforward to see that all graphs from Theorem 1.1 are regular NDB graphs with $\gamma = d+1$. Assume now that Γ is a regular NDB graph with valency k, diameter d, and $\gamma = d+1$. If $d = 2$, then it follows from Remark 2.3 that Γ is isomorphic either to the Petersen graph, the complement of the Petersen graph, the complete multipartite graph $K_{t\times 3}$ with t parts of cardinality 3 ($t \geq 2$), the Möbius ladder graph on eight vertices, or the Paley graph on 9 vertices. If $d \geq 3$, then it follows from Theorem 4.4 that $k \in \{3, 4, 5\}$. If $k = 3$, then Γ is isomorphic to the 3-dimensional hypercube Q_3 by Theorem 5.8. If $k = 4$ then Γ is isomorphic to the line graph of Q_3 by Theorem 6.4. If $k = 5$, then Γ is isomorphic to the icosahedron by Theorem 7.2. □

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