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THE FULL GROUP OF ISOMETRIES OF SOME COMPACT LIE GROUPS ENDOWED WITH A BI-INVARIANT METRIC

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Abstract. We describe the full group of isometries of absolutely simple, compact, connected real Lie groups, of $SO(4)$, and of $U(n)$, endowed with suitable bi-invariant Riemannian metrics.

Introduction

In this paper we describe the full group of isometries of some classes of *real Lie groups*, endowed with suitable bi-invariant Riemannian metrics: the *Killing metric* both on any *absolutely simple*[1], *compact, connected* Lie group and on the *special orthogonal group* $SO(4)$, and also the metric induced on the *unitary group* $U(n)$ by the flat Frobenius metric of $M_n(\mathbb{C})$.

In [5] and in [6] we already studied another relevant example of (semi-Riemannian) metric: the so-called trace metric, which is bi-invariant on $GL_n(\mathbb{R})$ and on its Lie subgroups. Some of the techniques used in the present work were developed in those papers and in [7], [8], [9].

Given any Lie group G, the Killing form of its Lie algebra extends, on the whole G, to a bi-invariant symmetric $(0,2)$-tensor, denoted by $\mathcal{K}$ and called the *Killing tensor* of G.

Further properties of G have some relevant consequences. For instance, as is well known, G is semi-simple if and only if $\mathcal{K}$ (and also $-\mathcal{K}$) is a semi-Riemannian metric on G (Cartan's criterion); and if G is semi-simple and compact, then the tensor $-\mathcal{K}$ is a Riemannian metric on G, which we call the *Killing metric* of G. Furthermore, if G is connected, compact, and simple, then $(G, -\mathcal{K})$ is a globally symmetric Riemannian manifold with non-negative sectional curvature and, moreover, if G is also *absolutely simple*, then $(G, -\mathcal{K})$ is an Einstein manifold. The Killing tensor of G is more than just an example of a bi-invariant tensor on G. In fact, if G is connected and absolutely simple, then every bi-invariant real $(0,2)$-tensor on G is a constant multiple of $\mathcal{K}$. These results are discussed in Section 1.

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[1]In the present paper, a real Lie group is said to be *absolutely simple* if the complexification of its Lie algebra is a simple Lie algebra.

Section 2 is devoted to the general result of this paper:

Theorem 2.3 *Let G be an absolutely simple, compact, connected real Lie group and let $-\mathcal{K}$ be its Killing metric. Then $F : (G, -\mathcal{K}) \to (G, -\mathcal{K})$ is an isometry if and only if there exist an element $a \in G$ and an automorphism Φ of the Lie group G such that either $F = L_a \circ \Phi$ or $F = L_a \circ \Phi \circ j$, where L_a is the left translation associated to a and j is the inversion map.*

Many classical groups satisfy all the conditions of the above Theorem, namely the *special orthogonal groups* $S\mathcal{O}(n)$, with $n \geq 3$ and $n \neq 4$, the *special unitary groups* $SU(n)$, with $n \geq 2$, and the *compact symplectic groups* $Sp(n)$, with $n \geq 1$.

A careful analysis of the automorphisms of each group allows us to deduce the complete list of the isometries of $(G, -\mathcal{K})$, where G is one of the previous classical groups (Theorem 2.5).

The manifold $(S\mathcal{O}(4), -\mathcal{K})$ is not included in the previous result: indeed, $S\mathcal{O}(4)$ is semi-simple but not simple. However, $-\mathcal{K}$ is still a Riemannian metric on it. Section 3 is devoted to this particular case. The key points are the following: $(S\mathcal{O}(4), -\mathcal{K})$ is isometric to the Lie group $\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}}$ (endowed with its Killing metric), and the natural covering projection of $SU(2) \times SU(2)$ (endowed with the product of the Killing metrics) onto the previous quotient is clearly a local isometry. All isometries of $SU(2) \times SU(2)$ are obtained by means of the analysis presented in Section 2 via a classical result of de Rham. Since these ones project as isometries of the quotient, we can obtain the main result of Section 3:

Theorem 3.5. *The isometries of $(S\mathcal{O}(4), -\mathcal{K})$ are precisely the following maps:*

$$X \mapsto AXB, \quad X \mapsto AX^TB, \quad X \mapsto A\,\tau(X)B, \quad X \mapsto A\,\tau(X)^TB,$$

where A, B are matrices both in $S\mathcal{O}(4)$ or both in $\mathcal{O}(4) \setminus S\mathcal{O}(4)$ (and τ is a suitable map constructed by means of the Cayley factorization *of $S\mathcal{O}(4)$).*

Finally, Section 4 is devoted to $U(n)$, endowed with the bi-invariant Riemannian metric ϕ, which is the restriction to $U(n)$ of the flat Frobenius metric of $M_n(\mathbb{C})$. This metric is not a multiple of the Killing tensor, because $U(n)$ is not semi-simple (and so its Killing tensor is degenerate). Analogously to Section 3, we get a covering map (which is also a local isometry) from $SU(n) \times \mathbb{R}$ (endowed with a suitable product metric) onto $(U(n), \phi)$. This allows us to get the main result of Section 4:

Theorem 4.7. *The isometries of $(U(n), \phi)$, with $n \geq 2$, are precisely the following maps:*

$$X \mapsto AXB, \quad X \mapsto AX^*B, \quad X \mapsto A\overline{X}B, \quad X \mapsto AX^TB,$$

with $A, B \in U(n)$.

We point out that our arguments are different from [17], where the author determines the group of isometries of simply connected homogeneous spaces of a simple, compact, connected Lie group. In fact, we also analyze $S\mathcal{O}(n)$ and $U(n)$, which are not simply connected.

1. Notations and preliminary facts

Notations 1.1. In this paper we will use many standard notations from matrix theory, which should be clear from the context, such as: $M_n(\mathbb{R})$ for the vector space of real square matrices, $\mathcal{O}(n)$ for the group of real orthogonal matrices, $S\mathcal{O}(n)$ for the group of real special orthogonal matrices, $Sp(n)$ for the compact symplectic group, $M_n(\mathbb{C})$ for the vector space of complex square matrices, $U(n)$ for the group of unitary matrices, $SU(n)$ for the group of special unitary matrices (all matrices are of order n). If A is a matrix, then A^T, A^{-1}, $\overline{A}$, and $A^* := \overline{A}^T$ denote its transpose, its inverse (when it exists), its conjugate, and its transpose conjugate, respectively. I_n is the identity matrix of order n and $\mathbf{i} \in \mathbb{C}$ is the unit imaginary number.

The basic notations and notions on real Lie groups and algebras are the following:

- G is a real Lie group with identity e, $T_P(G)$ is the tangent space to G at any point $P \in G$, $j : x \mapsto x^{-1}$ is the *inversion map* of G, $\mathfrak{g}$ is the Lie algebra of G (identified with the tangent space $T_e(G)$), $\exp : \mathfrak{g} \to G$ is the *exponential map* and $Aut(G)$ denotes the Lie group of all (smooth) automorphisms of G;
- if $\mathfrak{g}$ is a real Lie algebra, $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus \mathbf{i}\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ will denote its *complexification*, which turns to be a complex Lie algebra, having $\mathfrak{g}$ as real subalgebra;
- if $\mathfrak{h}$ is a complex Lie algebra, $\mathfrak{h}^{\mathbb{R}}$ will denote its *realification*, i.e., $\mathfrak{h}^{\mathbb{R}}$ is simply $\mathfrak{h}$ regarded as a real Lie algebra;
- for every $a \in G$, L_a and R_a are, respectively, the *left and right translations* in G associated to a, and $C_a := L_a \circ R_{a^{-1}}$ is the *inner automorphism* of G associated to a;
- for every $a \in G$, Ad_a is the automorphism of $\mathfrak{g}$, defined as the differential at e of C_a. It is well known that $\exp \circ Ad_a = C_a \circ \exp$;
- $\mathcal{K}$ is the left-invariant symmetric $(0,2)$-tensor on the whole G, extending the *Killing form* of $\mathfrak{g}$, and therefore the Killing form of $\mathfrak{g}$ agrees with $\mathcal{K}_e$. We call $\mathcal{K}$ the *Killing tensor* of the Lie group G.

Lemma 1.2. *The Killing tensor $\mathcal{K}$ of the Lie group G is bi-invariant on G and it is preserved by every $\phi \in Aut(G)$ and by the inversion map j (i.e., $\phi^*(\mathcal{K}) = \mathcal{K}$ and $j^*(\mathcal{K}) = \mathcal{K}$).*

Proof. $\mathcal{K}_e$ is invariant with respect to all automorphisms of $\mathfrak{g}$, hence the left-invariant tensor $\mathcal{K}$ is preserved by all smooth automorphisms of G (in particular by all inner automorphisms) and so $\mathcal{K}$ is right-invariant too. For the assertion on j see, for instance, [12, pp. 147–148]. ☐

Remarks-Definitions 1.3. We say that a (finite dimensional) Lie algebra $\mathfrak{g}$ is *simple* if it is non-abelian and has no ideals except 0 and $\mathfrak{g}$; while we say that $\mathfrak{g}$ is *semi-simple* if it splits into the direct sum of simple Lie algebras; by the well-known Cartan criterion, $\mathfrak{g}$ is semi-simple if and only if its Killing form is non-degenerate (see, for instance, [2]).

A Lie group is said to be *simple* (respectively, *semi-simple*) if its Lie algebra is simple (respectively, semi-simple). Hence a simple Lie group is semi-simple too.

Note that if G is a semi-simple Lie group, then $(G, \mathcal{K})$ and $(G, -\mathcal{K})$ are *semi-Riemannian* manifolds. We refer to $-\mathcal{K}$ (the opposite of the Killing tensor $\mathcal{K}$) as the *Killing metric* of the (semi-simple) Lie group.

Proposition 1.4. *Let G be a semi-simple connected Lie group. Then*

(a) *the geodesics of the semi-Riemannian manifold $(G, -\mathcal{K})$ are precisely the curves of the form $t \mapsto x \exp(tv)$ for every $t \in \mathbb{R}$, with x arbitrary in G and v arbitrary in the Lie algebra $\mathfrak{g}$ of G (so $(G, -\mathcal{K})$ is* geodesically complete*);*

(b) *the Levi-Civita connection ∇ of $(G, -\mathcal{K})$ is the 0-connection of Cartan–Schouten, defined by*

$$\nabla_X(Y) := \frac{1}{2}[X, Y],$$

where X, Y are left-invariant vector fields on G;

(c) *the curvature tensor of type $(1,3)$ of $(G, -\mathcal{K})$ is*

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z = \frac{1}{4}[[X, Y], Z],$$

where X, Y, Z are left-invariant vector fields on G;

(d) *the curvature tensor of type $(0,4)$ of $(G, -\mathcal{K})$ is the bi-invariant tensor, defined by*

$$R_{XYZW} := -\mathcal{K}(R_{XY}Z, W) = -\frac{1}{4}\mathcal{K}([X, Y], [Z, W]),$$

where X, Y, Z, W are left-invariant vector fields on G.

Proof. Parts (a), (b), and (c) follow directly from the results contained in [12, p. 148 and pp. 548–550] (our tensor R is the opposite of the corresponding tensor of [12]). Part (c) implies that $R_{XYZW} = -\frac{1}{4}\mathcal{K}([[X, Y], Z], W)$. By the skew-symmetry, with respect to the Killing form, of every operator $ad_v : x \mapsto [v, x]$ (see, for instance, [1]), we have $\mathcal{K}([[X, Y], Z], W) = \mathcal{K}([X, Y], [Z, W])$, and this concludes (d). □

Remark-Definition 1.5. We say that a real Lie group G is a *complex Lie group* if it possesses a complex analytic structure, compatible with the real one, such that multiplication and inversion are holomorphic. It is known that a real Lie group G with Lie algebra $\mathfrak{g}$ is complex if and only if there exists a complex Lie algebra $\mathfrak{h}$ such that $\mathfrak{h}^{\mathbb{R}} = \mathfrak{g}$ (see [14, Prop. 1.110, p. 95]).

Lemma 1.6. *Let G be a real Lie group and let $\mathfrak{g}$ be its Lie algebra with $\mathfrak{g}^{\mathbb{C}}$ as its complexification. Then the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is simple if and only if G is simple and not complex.*

Proof. It follows from [14, Thm. 6.94, p. 407], remembering that if $\mathfrak{g}^{\mathbb{C}}$ is a simple complex Lie algebra, then $\mathfrak{g}$ is a simple real Lie algebra. □

Definition 1.7. We say that a real Lie group is *absolutely simple* if it is simple and not complex or, equivalently by Lemma 1.6, if the complexification of its Lie algebra is a simple, complex Lie algebra.

A standard consequence of Schur's lemma is the following.

Proposition 1.8. *Let G be a real Lie group and assume that G is connected and absolutely simple. Then every bi-invariant real $(0,2)$-tensor on G is a constant multiple of the Killing metric $-\mathcal{K}$ of G.*

Lemma 1.9. *Let G be a real Lie group and assume that G is semi-simple and compact. Then the Killing tensor $\mathcal{K}$ of G is negative definite at every point (i.e., the Killing metric $-\mathcal{K}$ is a Riemannian metric on G).*

Proof. It follows from [12, Prop. 6.6 (i), p. 132; Cor. 6.7, p. 133]. □

Remark 1.10. Let G be a simple, compact, connected real Lie group and let $\mathfrak{g}$ be its Lie algebra; denote by Δ the *diagonal* of $G \times G$ and by Z the *center* of G. Z is a closed subgroup of G and it is finite. Indeed, the center of $\mathfrak{g}$ is zero (since G is simple, see [12, Cor. 6.2, p. 132]). Since the Lie algebra of Z agrees with the center of $\mathfrak{g}$, then Z is a discrete subgroup of the compact group G, and therefore Z is finite.

Now we denote by $\mathcal{U}$ the semisimple compact connected Lie group defined by $\mathcal{U} := \frac{G\times G}{(Z\times Z)\cap\Delta}$, and consider the map

$$T : \mathcal{U} \times G \to G, \quad T(\{(g,h)\}, x) = gxh^{-1},$$

where $\{(g,h)\}$ is the class of (g,h) in $\frac{G\times G}{(Z\times Z)\cap\Delta}$. T is an effective and transitive left action of $\mathcal{U}$ on G and its isotropy subgroup at the identity is $\widehat{\Delta} := \frac{\Delta}{(Z\times Z)\cap\Delta}$. Therefore G is diffeomorphic to the homogeneous space $\frac{\mathcal{U}}{\widehat{\Delta}}$. Moreover, for every $\{(g,h)\} \in \mathcal{U}$, the map $x \mapsto T(\{(g,h)\}, x)$ is an isometry with respect to $-\mathcal{K}$ (and to $\mathcal{K}$). Finally, the pair $(\mathcal{U}, \widehat{\Delta})$ is a *Riemannian symmetric pair* (in the sense of [12, p. 209]) with *involutive automorphism* given by $\sigma(\{(g,h)\}) = \{(h,g)\}$.

Proposition 1.11. *Let G be a simple, compact, connected real Lie group and let $-\mathcal{K}$ be its Killing metric. Then $(G, -\mathcal{K})$ is a globally symmetric Riemannian manifold with non-negative sectional curvature; furthermore, every connected component of the Lie group of its isometries is diffeomorphic to $\frac{G\times G}{(Z\times Z)\cap\Delta}$, where Z is the center of G and Δ is the diagonal of $G \times G$. Moreover, if G is absolutely simple too, then $(G, -\mathcal{K})$ is an Einstein manifold.*

Proof. By [12, Prop. 3.4, p. 209], $(G, -\mathcal{K})$ is a globally symmetric Riemannian manifold, via Remark 1.10. By Proposition 1.4 (d), the sectional curvature of the space generated by two left-invariant and $\mathbb{R}$-independent vector fields X, Y of G agrees with $-\dfrac{1}{4}\mathcal{K}([X,Y],[X,Y])$, which is non-negative and equal to 0 if and only if $[X,Y] = 0$. The assertion about the connected components of the Lie group of the isometries follows from [12, Thm. 4.1 (i), p. 243] and from the fact that in a Lie group all connected components are diffeomorphic to the component containing the identity.

The last statement is a consequence of Proposition 1.8, taking into account that the Ricci tensor of $(G, -\mathcal{K})$ is bi-invariant. ☐

Remark 1.12. For further details and information on Lie groups with bi-invariant metrics, we refer the reader to [4, Ch. 2].

2. Isometries of a compact Lie group

Lemma 2.1. *Let $\mathfrak{g}$ be a real Lie algebra, whose complexification $\mathfrak{g}^{\mathbb{C}}$ is a simple, complex Lie algebra, and let L be an isometry with respect to the Killing form $\mathcal{B}$ of $\mathfrak{g}$ such that $L([v,w]) = [v, L(w)]$ for every $v, w \in \mathfrak{g}$. Then $L = \pm Id_{\mathfrak{g}}$.*

Proof. The killing form $\mathcal{B}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ is the extension by $\mathbb{C}$-linearity of the Killing form $\mathcal{B}$ of $\mathfrak{g}$; by $\mathbb{C}$-linearity too, L can be extended to a map $L^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, which is an isometry with respect to the Killing form $\mathcal{B}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$, satisfying again the analogous condition $L^{\mathbb{C}}([v,w]) = [v, L^{\mathbb{C}}(w)]$ for every $v, w \in \mathfrak{g}^{\mathbb{C}}$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $L^{\mathbb{C}}$ and let $V_\lambda \neq \{0\}$ be the corresponding eigenspace. If $v \in \mathfrak{g}^{\mathbb{C}}$ and $w \in V_\lambda$, then $L^{\mathbb{C}}([v,w]) = [v, \lambda w] = \lambda[v,w]$, so $[v,w] \in V_\lambda$, which turns out to be a non-zero ideal of $\mathfrak{g}^{\mathbb{C}}$, and therefore $V_\lambda = \mathfrak{g}^{\mathbb{C}}$, i.e., $L^{\mathbb{C}} = \lambda Id_{\mathfrak{g}^{\mathbb{C}}}$. Since $L^{\mathbb{C}}$ is an isometry with respect to the Killing form $\mathcal{B}^{\mathbb{C}}$, which is non-degenerate by Cartan's criterion, the map $L^{\mathbb{C}}$ agrees with $\pm Id_{\mathfrak{g}^{\mathbb{C}}}$, and therefore $L = \pm Id_{\mathfrak{g}}$. ☐

Proposition 2.2. *Let G be an absolutely simple, compact, connected real Lie group, and let $-\mathcal{K}$ be its Killing metric. Then $F : (G, -\mathcal{K}) \to (G, -\mathcal{K})$ is an isometry fixing the identity $e \in G$ if and only if there exists an automorphism Φ of the Lie group G such that either $F = \Phi$ or $F = \Phi \circ j$, where j is the inversion map.*

Proof. Lemma 1.2 implies that the automorphisms and the inversion map of the Lie group G are isometries with respect to $-\mathcal{K}$ fixing e.

For the converse, let $\mathcal{J}$ be the group of isometries of $(G, -\mathcal{K})$, let $\mathcal{J}_e$ be the corresponding subgroup of isotropy at e and let $\mathcal{J}^0$, $\mathcal{J}_e^0$ be their connected components containing the identity. In Remark 1.10, we observed that $(\mathcal{U}, \widehat{\Delta})$ is a Riemannian symmetric pair, and so by [12, Thm. 4.1 (i), p. 243], we have $\mathcal{J}^0 \simeq \mathcal{U}$ (as Lie groups). From this we get that $\dim(\mathcal{J}) = \dim(\mathcal{J}^0) = \dim(\mathcal{U}) = 2\dim(G)$, and therefore $\dim(\mathcal{J}_e^0) = \dim(\mathcal{J}_e) = \dim(\mathcal{J}) - \dim(G) = \dim(G)$.

Let us consider the adjoint representations of G and of its Lie algebra $\mathfrak{g}$, denoted by $Ad : G \to GL(\mathfrak{g})$ and by $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, respectively; we indicate with $Ad(G)$ and with $ad(\mathfrak{g})$ their images. Note that $Ad(G)$ is a closed Lie subgroup of $GL(\mathfrak{g})$ and $ad(\mathfrak{g})$ is its Lie algebra; moreover, since the kernel of the map ad agrees with the center of $\mathfrak{g}$, which is zero, we get that $ad : \mathfrak{g} \to ad(\mathfrak{g})$ is an isomorphism of Lie algebras; this implies that $Ad(G)$ and G have the same dimension.

Let us also consider the representation $d : \mathcal{J}_e \to GL(\mathfrak{g})$, defined as the differential at e of every element of $\mathcal{J}_e$. By [16, Prop. 62, p. 91], d is a faithful representation and so $d(\mathcal{J}_e^0) = (d(\mathcal{J}_e))^0$ (the component of the image $d(\mathcal{J}_e)$ containing the identity).

Hence $\dim(d(\mathcal{J}_e))^0 = \dim(\mathcal{J}_e^0) = \dim(G)$. Since G is connected, we have the inclusion $Ad(G) \subseteq (d(\mathcal{J}_e))^0$. Now these manifolds have the same dimension; hence, by the domain invariance theorem, $Ad(G)$ is open in $(d(\mathcal{J}_e))^0$; moreover, $Ad(G)$ is compact and $(d(\mathcal{J}_e))^0$ is connected, and this allows us to get that $Ad(G) = (d(\mathcal{J}_e))^0$.

For any fixed $F \in \mathcal{J}_e$, the previous equality gives $dF\ Ad(G)\, dF^{-1} = Ad(G)$. Hence there exists a unique automorphism α of $Ad(G)$ such that

$$dF \circ Ad_X \circ dF^{-1} = \alpha(Ad_X) \quad \text{for every } X \in G. \tag{2.1}$$

We denote by $\exp : \mathfrak{g} \to G$ and by $\widehat{\exp} : ad(\mathfrak{g}) \to Ad(G)$ the two usual exponential maps. It is well known that $Ad \circ \exp = \widehat{\exp} \circ ad$ (see, for instance, [10, Thm. 3.28, p. 60]).

For every $t \in \mathbb{R}$ and every $v \in \mathfrak{g}$, equation (2.1) implies

$$dF \circ Ad_{\exp(tv)} \circ dF^{-1} = \alpha(Ad_{\exp(tv)}). \tag{2.2}$$

Now let $\widetilde{\alpha}$ be the unique automorphism of $ad(\mathfrak{g})$ such that $\alpha \circ \widehat{\exp} = \widehat{\exp} \circ \widetilde{\alpha}$. The map $\overline{\alpha} := ad^{-1} \circ \widetilde{\alpha} \circ ad$ is an automorphism of the Lie algebra $\mathfrak{g}$, satisfying $Ad \circ \exp \circ \overline{\alpha} = \alpha \circ Ad \circ \exp$. Hence, for every $t \in \mathbb{R}$ and every $v \in \mathfrak{g}$, equation (2.2) implies

$$dF \circ Ad_{\exp(tv)} \circ dF^{-1} = Ad_{\exp(t\overline{\alpha}(v))}\,. \tag{2.3}$$

Now, if we differentiate the identity (2.3) with respect to t, for $t = 0$, we get

$$dF \circ ad_v \circ dF^{-1} = ad_{\overline{\alpha}(v)}.$$

Since $ad_v(w) = [v, w]$ for every $v, w \in \mathfrak{g}$ and remembering that $\overline{\alpha}$ is an automorphism of the Lie algebra $\mathfrak{g}$, we get $dF([v, w]) = [\overline{\alpha}(v), dF(w)] = \overline{\alpha}([v, \overline{\alpha}^{-1}(dF(w))])$, and so $(\overline{\alpha}^{-1} \circ dF)([v, w]) = [v, (\overline{\alpha}^{-1} \circ dF)(w)]$ for every $v, w \in \mathfrak{g}$. Note that dF and $\overline{\alpha}$ are both isometries of $\mathfrak{g}$ with respect to its Killing form; moreover, since G is absolutely simple, its Lie algebra $\mathfrak{g}$ satisfies the hypotheses of Lemma 2.1; thus we obtain $dF = \pm\overline{\alpha}$.

Let $\pi : \widetilde{G} \to G$ be the universal covering group of G and let $\widetilde{F} : \widetilde{G} \to \widetilde{G}$ be such that $F \circ \pi = \pi \circ \widetilde{F}$, with $\widetilde{F}(\widetilde{e}) = \widetilde{e}$, where $\widetilde{e}$ is the identity of $\widetilde{G}$; from this we get $\widetilde{F}_* = \pi_*^{-1} \circ dF \circ \pi_* = \pi_*^{-1} \circ (\pm\overline{\alpha}) \circ \pi_*$, where $\widetilde{F}_*$, π_* denote the differentials at the identity $\widetilde{e}$ of $\widetilde{F}$ and π, respectively. If we denote by β the automorphism of the Lie algebra $\widetilde{\mathfrak{g}}$ of $\widetilde{G}$, given by $\beta = \pi_*^{-1} \circ \overline{\alpha} \circ \pi_*$, we can write $\widetilde{F}_* = \pm\beta$.

By [23, Thm. 3.27, p. 101], there exists a unique automorphism Ψ of the simply connected Lie group $\widetilde{G}$, whose differential at the identity $\widetilde{e}$, Ψ_*, agrees with β. Hence $\widetilde{F}_* = \pm\Psi_*$.

Since Ψ is an automorphism of $\widetilde{G}$, it is an isometry of $(\widetilde{G}, -\widetilde{\mathcal{K}})$, where $-\widetilde{\mathcal{K}}$ is the Killing metric of $\widetilde{G}$ (remember Lemma 1.2).

It is easy to check that $\pi : (\widetilde{G}, -\widetilde{\mathcal{K}}) \to (G, -\mathcal{K})$ is a local isometry and this implies that $\widetilde{F} : (\widetilde{G}, -\widetilde{\mathcal{K}}) \to (\widetilde{G}, -\widetilde{\mathcal{K}})$ is an isometry too.

If $\widetilde{F}_* = \Psi_*$, then $\widetilde{F} = \Psi$ (see, for instance, [16, Prop. 62, p. 91]) and hence $F \circ \pi = \pi \circ \Psi$. The surjectivity of π, together with the fact that π and Ψ are Lie group homomorphisms, implies that F is a (bijective) endomorphism of G. This allows us to conclude that $F \in Aut(G)$.

Suppose now that $\widetilde{F}_* = -\Psi_*$. We denote by $\widetilde{j}$ the inversion map of $\widetilde{G}$ and by $\widetilde{j}_*$ its differential at the identity $\widetilde{e}$. By Lemma 1.2, $\widetilde{j}$ is an isometry of $(\widetilde{G}, -\widetilde{\mathcal{K}})$; furthermore, $\widetilde{j}_*$ agrees with the opposite of the identity map (see, for instance, [12, p. 147]).

Now $\widetilde{F}_* = \widetilde{j}_* \circ \Psi_* = (\widetilde{j} \circ \Psi)_*$ and, arguing as in the previous case, we get that $\widetilde{F} = \widetilde{j} \circ \Psi$, and so $F \circ \pi = \pi \circ \widetilde{j} \circ \Psi = j \circ \pi \circ \Psi$. Hence $j \circ F \circ \pi = \pi \circ \Psi$ and, as above, we obtain that $\Phi := j \circ F \in Aut(G)$; therefore we conclude that $F = j \circ \Phi = \Phi \circ j$, with $\Phi \in Aut(G)$. ☐

Theorem 2.3. *Let G be an absolutely simple, compact, connected real Lie group and let $-\mathcal{K}$ be its Killing metric. Then $F : (G, -\mathcal{K}) \to (G, -\mathcal{K})$ is an isometry if and only if there exist an element $a \in G$ and an automorphism Φ of the Lie group G such that either $F = L_a \circ \Phi$ or $F = L_a \circ \Phi \circ j$, where L_a is the left translation associated to a and j is the inversion map.*

Proof. Note that $L_a \circ \Phi$ and $L_a \circ \Phi \circ j$ are both isometries, because they are compositions of isometries (remember again Lemma 1.2).

The converse follows from Proposition 2.2, because, for $a = F(e)$, $L_{a^{-1}} \circ F$ is an isometry fixing the identity $e \in G$. ☐

Remark 2.4. As is well known, relevant examples of absolutely simple, compact, connected real Lie groups are

- the special orthogonal group $S\mathcal{O}(n)$, $n \geq 3$, $n \neq 4$;
- the special unitary group $SU(n)$, $n \geq 2$;
- the compact symplectic group $Sp(n)$, $n \geq 1$.

The automorphisms of $S\mathcal{O}(n)$, with $n \geq 3$ odd, of $SU(2)$ and of $Sp(n)$, with $n \geq 1$, are precisely the *inner automorphisms* of the corresponding group.

Furthermore, the automorphisms of $S\mathcal{O}(n)$, with $n \geq 6$ even, are precisely the maps $X \mapsto AXA^T$, with $A \in \mathcal{O}(n)$.

Finally, the automorphisms of $SU(n)$, with $n \geq 3$, are the inner automorphisms and all the maps $X \mapsto C\overline{X}C^*$, where $C \in SU(n)$.

From these facts and from Theorem 2.3 we can easily get the following.

Theorem 2.5.

(a) *The isometries of $(S\mathcal{O}(n), -\mathcal{K})$, with $n \geq 3$ odd, are precisely the maps*
$$X \mapsto AXB \quad \text{and} \quad X \mapsto AX^TB,$$
with $A, B \in S\mathcal{O}(n)$.

(b) *The isometries of $(S\mathcal{O}(n), -\mathcal{K})$, with $n \geq 6$ even, are precisely the maps*
$$X \mapsto AXB \quad \text{and} \quad X \mapsto AX^TB,$$
with A, B both in $S\mathcal{O}(n)$ or both in $\mathcal{O}(n) \setminus S\mathcal{O}(n)$.

(c) *The isometries of $(SU(2), -\mathcal{K})$ are precisely the maps*
$$X \mapsto AXB \quad \text{and} \quad X \mapsto AX^*B,$$
with $A, B \in SU(2)$.

(d) *The isometries of* $(SU(n), -\mathcal{K})$*, with* $n \geq 3$*, are precisely the maps*

$$X \mapsto AXB, \quad X \mapsto AX^*B, \quad X \mapsto A\overline{X}B, \quad \textit{and} \quad X \mapsto AX^TB,$$

with $A, B \in SU(n)$*.*

(e) *The isometries of* $(Sp(n), -\mathcal{K})$*, with* $n \geq 1$*, are precisely the maps*

$$X \mapsto AXB \quad \textit{and} \quad X \mapsto AX^*B,$$

with $A, B \in Sp(n)$*.*

Remark 2.6. The Lie groups of isometries of $(SO(n), -\mathcal{K})$, with $n \geq 3$ odd, of isometries of $(SU(2), -\mathcal{K})$, and of isometries of $(Sp(n), -\mathcal{K})$, with $n \geq 1$, have two connected components, while the Lie groups of isometries of $(SO(n), -\mathcal{K})$, with $n \geq 6$ even, and of isometries of $(SU(n), -\mathcal{K})$, with $n \geq 3$, have four connected components.

Remark 2.7. If G is one of the groups $SO(n)$, $n \geq 3$ and $n \neq 4$, $SU(n)$, $n \geq 2$, or $Sp(n)$, $n \geq 1$, then $\mathcal{K}_A(X, Y) = c \cdot \mathrm{tr}(A^*XA^*Y)$ for some strictly positive constant c, for every $A \in G$, and for every $X, Y \in T_A(G)$ (as we can deduce, for instance, from [20, Ex. 6.19, p. 129]).

We denote by ϕ the (flat) *Frobenius hermitian metric* of $M_m(\mathbb{C})$ $(m \geq 2)$, defined by $\phi(A, B) = \mathrm{Re}(\mathrm{tr}(AB^*))$ for every $A, B \in M_m(\mathbb{C})$. To simplify the notation, we denote also by ϕ its restriction to each submanifold N of $M_m(\mathbb{C})$ and we call it the *Frobenius metric* of N. It is just a computation that, if $A \in U(m)$, then the maps L_A and R_A are isometries of $(M_m(\mathbb{C}), \phi)$, and therefore the Frobenius metric of $U(m)$ is bi-invariant. Moreover, arguing as in [6, Recall 4.1], it is simple to verify that the expression of the Frobenius metric ϕ of $U(m)$ is as follows: $\phi_A(X, Y) = -\mathrm{tr}(A^*XA^*Y)$ for every $A \in U(m)$ and every $X, Y \in T_A(U(m))$.

In each of the above cases, G is a (closed) Lie subgroup of $U(n)$ or of $U(2n)$, i.e., G is a submanifold of some $U(m)$ $(m \geq 2)$; hence, on G, the metric ϕ is bi-invariant and $\phi = -\frac{1}{c}\mathcal{K}$ (with $c > 0$). Therefore, if G is one of the above groups, then Proposition 1.4, Proposition 1.11, and Theorem 2.5 also hold with ϕ instead of $-\mathcal{K}$.

Remark 2.8. Parts (a) and (b) of Theorem 2.5 can be compared with an analogous result, obtained in [3, Thm. 1], where the distance on $SO(n)$ is induced by the so-called *c-spectral norm*, which is different from the distance induced by the Killing metric.

3. Isometries of $SO(4)$

Remark 3.1. By Lemma 1.9, the Killing metric of the semi-simple compact Lie group $SO(4)$ is a Riemannian metric on $SO(4)$. It is easy to check that the Killing form of the *special orthogonal Lie algebra* $\mathfrak{so}(4)$, evaluated at U, V, agrees with $2\,\mathrm{tr}(U, V)$ (this extends to the case $n = 2$ the formula (3) of [20, Ex. 6.19, p. 129]). Hence the Killing metric $-\mathcal{K}$ of $SO(4)$ agrees with the double of the Frobenius metric ϕ of $SO(4)$. Therefore, for the Lie group $SO(4)$, Proposition 1.4 holds for ϕ as well as for $-\mathcal{K}$. However, in [6, Prop. 4.3], we already proved that $(SO(4), \phi)$

(and so also $(\mathcal{SO}(4), -\mathcal{K})$) is an Einstein globally symmetric Riemannian manifold with non-negative sectional curvature.

Remarks-Definitions 3.2.

(a) The map $\rho : \mathbb{C} \to M_2(\mathbb{R})$, given by
$$\rho(z) := \begin{pmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{pmatrix},$$
is a monomorphism of $\mathbb{R}$-algebras between $\mathbb{C}$ and $M_2(\mathbb{R})$.

More generally, for any $h \geq 1$, we still denote by ρ the monomorphism of $\mathbb{R}$-algebras $M_h(\mathbb{C}) \to M_{2h}(\mathbb{R})$, which maps the $h \times h$ complex matrix $Z = (z_{ij})$ to the $(2h) \times (2h)$ block real matrix $(\rho(z_{ij}))$, having h^2 blocks of order 2×2. We refer to ρ as the *decomplexification map* of $M_h(\mathbb{C})$ into $M_{2h}(\mathbb{R})$.

It is known that, for every $Z \in M_h(\mathbb{C})$, the map ρ satisfies
$$\mathrm{tr}(\rho(Z)) = 2\,\mathrm{Re}(\mathrm{tr}(Z)), \quad \det(\rho(Z)) = |\det(Z)|^2, \quad \text{and} \quad \rho(Z^*) = \rho(Z)^T.$$

For simplicity, we still denote by ρ all its restrictions to any subset of $M_h(\mathbb{C})$. Hence, for instance, $\rho(U(h)) = \rho(M_h(\mathbb{C})) \cap \mathcal{SO}(2h)$ is a Lie subgroup of $\mathcal{SO}(2h)$ (isomorphic to $U(h)$) and, in particular, $\rho(SU(2))$ is a Lie subgroup of $\mathcal{SO}(4)$, isomorphic to $SU(2)$.

(b) We consider the matrix $J = J^T = J^{-1} \in \mathcal{O}(4)$, defined by
$$J := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
and the Lie subgroup of $\mathcal{SO}(4)$, conjugate to $\rho(SU(2))$ in $\mathcal{O}(4)$, defined by $J\rho(SU(2))J$. It is easy to check that $\rho(SU(2)) \cap \big(J\rho(SU(2))J\big) = \{\pm I_4\}$, and that X commutes with JYJ for every $X, Y \in \rho(SU(2))$. Moreover, it is known that every matrix of $\mathcal{SO}(4)$ has a *Cayley factorization* as commutative product of a matrix of $\rho(SU(2))$ and of a matrix of $J\rho(SU(2))J$, and that such factorization is unique up to the sign of both matrices (see, for instance, [13, Thm. 3.2] and also [15], [22], [18]).

(c) Let us consider $X = \rho(X_1)\,[J\rho(X_2)J]$, with $X_1, X_2 \in SU(2)$, a matrix in $\mathcal{SO}(4)$, together with its Cayley's factorization. The map $\tau : \mathcal{SO}(4) \to \mathcal{SO}(4)$, given by $X = \rho(X_1)\,[J\rho(X_2)J] \mapsto \tau(X) := \rho(X_1)\,[J\rho(X_2)J]^T = \rho(X_1)\,[J\rho(X_2^*)J]$, is well defined and bijective; moreover, $\tau^2 = \mathit{Id}$ and $\tau \circ j = j \circ \tau$ (where j is the inversion map (i.e., the transposition map) of $\mathcal{SO}(4)$).

(d) The map $\widehat{\chi} : SU(2) \times SU(2) \to \mathcal{SO}(4)$, defined by $\widehat{\chi}(X, Y) = \rho(X)J\rho(Y)J$, is an epimorphism of Lie groups, whose kernel is $\{\pm(I_2, I_2)\}$. Then $\widehat{\chi}$ induces a Lie group isomorphism $\chi : \frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}} \to \mathcal{SO}(4)$. Therefore $(\mathcal{SO}(4), -\mathcal{K})$ is a Riemannian manifold isometric to $\Big(\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}}, -\mathcal{K}'\Big)$, where $-\mathcal{K}'$ is the Killing metric of the Lie group $\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}}$.

(e) The Killing tensor of $SU(2)\times SU(2)$ is $\mathcal{K}_2\times\mathcal{K}_2$, where $\mathcal{K}_2$ denotes the Killing tensor of $SU(2)$. We denote by $\sigma : SU(2)\times SU(2)\to SU(2)\times SU(2)$ the map which interchanges the two factors of $SU(2)\times SU(2)$.

By a classical result due to de Rham (see [19, Thm. III, p. 341]), the isometries of $(SU(2)\times SU(2), -(\mathcal{K}_2\times\mathcal{K}_2))$ are precisely the maps

$$\psi_1\times\psi_2 : (X,Y)\mapsto(\psi_1(X),\psi_2(Y))$$

and

$$(\psi_1\times\psi_2)\circ\sigma : (X,Y)\mapsto(\psi_1(Y),\psi_2(X)),$$

where ψ_1,ψ_2 are isometries of $(SU(2),-\mathcal{K}_2)$. In particular, the map σ is an isometry of $(SU(2)\times SU(2), -(\mathcal{K}_2\times\mathcal{K}_2))$.

From these facts and from Theorem 2.5 (c), if we denote by j the inversion map of $SU(2)$, we get the following.

Proposition 3.3. *The isometries of $(SU(2)\times SU(2), -(\mathcal{K}_2\times\mathcal{K}_2))$ are precisely the maps of the form*

$$\begin{aligned}
&(L_{A_1}\circ R_{A_2})\times(L_{B_1}\circ R_{B_2}),\\
&(L_{A_1}\circ R_{A_2}\circ j)\times(L_{B_1}\circ R_{B_2}),\\
&(L_{A_1}\circ R_{A_2})\times(L_{B_1}\circ R_{B_2}\circ j),\\
&(L_{A_1}\circ R_{A_2}\circ j)\times(L_{B_1}\circ R_{B_2}\circ j),\\
&\big((L_{A_1}\circ R_{A_2})\times(L_{B_1}\circ R_{B_2})\big)\circ\sigma,\\
&\big((L_{A_1}\circ R_{A_2}\circ j)\times(L_{B_1}\circ R_{B_2})\big)\circ\sigma,\\
&\big((L_{A_1}\circ R_{A_2})\times(L_{B_1}\circ R_{B_2}\circ j)\big)\circ\sigma,\\
&\big((L_{A_1}\circ R_{A_2}\circ j)\times(L_{B_1}\circ R_{B_2}\circ j)\big)\circ\sigma,
\end{aligned}$$

where A_1,A_2,B_1,B_2 are arbitrary elements of $SU(2)$.

In particular the isometries of $(SU(2)\times SU(2), -(\mathcal{K}_2\times\mathcal{K}_2))$ fixing the identity (I_2,I_2) are the previous ones, with $A_1^*=A_2$ and $B_1^*=B_2$.

Proposition 3.4. *Let $\pi : SU(2)\times SU(2)\to\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}}$ be the natural covering projection. If Ψ is an isometry of $\left(\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}},-\mathcal{K}'\right)$ fixing the identity of the group, then there exists a unique isometry $\widetilde{\Psi}$ of $((SU(2)\times SU(2)),-(\mathcal{K}_2\times\mathcal{K}_2))$ fixing the identity (I_2,I_2) such that $\Psi\circ\pi=\pi\circ\widetilde{\Psi}$.*

Conversely, if $\widetilde{\Psi}$ is an isometry of $((SU(2)\times SU(2)),-(\mathcal{K}_2\times\mathcal{K}_2))$ fixing the identity (I_2,I_2), then there exists a unique isometry Ψ of $\left(\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}},-\mathcal{K}'\right)$ fixing the identity of the group such that $\Psi\circ\pi=\pi\circ\widetilde{\Psi}$.

Proof. Let Ψ be an isometry of $\left(\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}},-\mathcal{K}'\right)$ fixing the identity of the group. Since $SU(2)\times SU(2)$ is simply connected, there exists a unique homeomorphism $\widetilde{\Psi} : SU(2)\times SU(2)\to SU(2)\times SU(2)$ fixing the identity (I_2,I_2) such that

$\Psi \circ \pi = \pi \circ \widetilde{\Psi}$. Since π is a local isometry from $(SU(2) \times SU(2), -(\mathcal{K}_2 \times \mathcal{K}_2))$ onto $\left(\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}}, -\mathcal{K}'\right)$, the map $\widetilde{\Psi}$ is an isometry of $(SU(2) \times SU(2), -(\mathcal{K}_2 \times \mathcal{K}_2))$.

For the converse, we denote by μ the isometry of $(SU(2) \times SU(2), -(\mathcal{K}_2 \times \mathcal{K}_2))$ defined by $\mu(A, B) = (-A, -B)$. From Theorem 2.5 (c) and from Remarks-Definitions 3.2 (e), the map μ commutes with all isometries $\widetilde{\Psi}$ of $(SU(2) \times SU(2), -(\mathcal{K}_2 \times \mathcal{K}_2))$ fixing the identity of the group, and so all these last project as isometries of the quotient. □

Theorem 3.5. *The isometries of* $(S\mathcal{O}(4), -\mathcal{K})$ *are precisely the following maps:*

$$X \mapsto AXB, \quad X \mapsto AX^TB, \quad X \mapsto A\,\tau(X)B, \quad X \mapsto A\,\tau(X)^TB,$$

where A, B *are matrices both in* $S\mathcal{O}(4)$ *or both in* $\mathcal{O}(4) \setminus S\mathcal{O}(4)$.

Proof. By Propositions 3.3 and 3.4, all isometries of $\left(\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}}, -\mathcal{K}'\right)$ fixing the identity are obtained by projecting onto the quotient the following isometries of $((SU(2) \times SU(2)), -(\mathcal{K}_2 \times \mathcal{K}_2))$:

$$\begin{array}{ll} C_A \times C_B, & (C_A \times C_B) \circ \sigma, \\ (C_A \times C_B) \circ (j \times id), & (C_A \times C_B) \circ (j \times id) \circ \sigma, \\ (C_A \times C_B) \circ (id \times j), & (C_A \times C_B) \circ (id \times j) \circ \sigma, \\ (C_A \times C_B) \circ (j \times j), & (C_A \times C_B) \circ (j \times j) \circ \sigma, \end{array}$$

with $A, B \in SU(2)$. Here id and j denote, respectively, the identity and the inversion map of $SU(2)$, whereas C_X denotes, as usual, the inner automorphism of $SU(2)$ associated to any element X of $SU(2)$.

By Remarks-Definitions 3.2 (d), the isometries of $(S\mathcal{O}(4), -\mathcal{K})$ fixing the identity I_4 are of the form $\chi \circ \Phi \circ \chi^{-1}$, where Φ is one of the above isometries of $\left(\frac{SU(2)\times SU(2)}{\{\pm(I_2,I_2)\}}, -\mathcal{K}'\right)$.

Standard computations show that $\chi \circ (C_A \times C_B) \circ \chi^{-1} = C_{\widehat{\chi}(A,B)}$ for every $A, B \in SU(2)$; $\chi \circ (id \times j) \circ \chi^{-1} = \tau$ (and so τ is an isometry of $(S\mathcal{O}(4), -\mathcal{K})$); $\chi \circ (j \times id) \circ \chi^{-1} = \tau \circ \widehat{j} = \widehat{j} \circ \tau$; $\chi \circ (j \times j) \circ \chi^{-1} = \widehat{j}$, where $\widehat{j}$ denotes the inversion map of $S\mathcal{O}(4)$ and $\chi \circ \sigma \circ \chi^{-1} = C_J$, where J is the matrix of $\mathcal{O}(4) \setminus S\mathcal{O}(4)$, defined in Remarks-Definitions 3.2 (b). From this, we get that the complete list of the isometries of $(S\mathcal{O}(4), -\mathcal{K})$ fixing the identity I_4 is the following:

$$C_M, \quad C_M \circ \widehat{j} \circ \tau, \quad C_M \circ \tau, \quad C_M \circ \widehat{j},$$

where M is an arbitrary matrix of $\mathcal{O}(4)$.

To get the full group of isometries of $(S\mathcal{O}(4), -\mathcal{K})$, it suffices to compose these isometries with a left translation L_A, where $A \in S\mathcal{O}(4)$. This allows us to conclude the proof. □

Remark 3.6. The full group of isometries of $(S\mathcal{O}(4), -\mathcal{K})$ has 8 connected components, all diffeomorphic to $\frac{S\mathcal{O}(4)\times S\mathcal{O}(4)}{\{\pm(I_4,I_4)\}}$.

Remark 3.7. Also Theorem 3.5 can be compared with the analogous result obtained in [3, Thm. 1] for $n = 4$. In this case as well, the distance on $S\mathcal{O}(4)$ is different from the distance induced by the Killing metric.

4. Isometries of $U(n)$

In this section we describe the full group of isometries of the Riemannian manifold $(U(n), \phi)$ $(n \geq 2)$, where ϕ is the Frobenius metric of $U(n)$, defined by $\phi_A(X,Y) = -\operatorname{tr}(A^*XA^*Y)$ for every $A \in U(n)$ and for every $X, Y \in T_A(U(n))$. By the way, note that ϕ can also be obtained by the Frobenius metric ϕ_0 of $S\mathcal{O}(2n)$ as $\phi = \dfrac{1}{2}\rho^*(\phi_0)$, where ρ is the decomplexification map of $U(n)$ into $S\mathcal{O}(2n)$.

Remarks-Definitions 4.1.

(a) The pair $(SU(n) \times \mathbb{R}, p)$, where p is the map $SU(n) \times \mathbb{R} \to U(n)$ defined by $p(B,x) = e^{\mathbf{i}x}B$, is the (analytic) universal covering group of $U(n)$. Indeed, p is clearly an analytic homomorphism of Lie groups, whose differential at the point $(B,x) \in SU(n) \times \mathbb{R}$ maps the tangent vector (W,λ) to $e^{\mathbf{i}x}(W + \mathbf{i}\lambda B)$. At the identity $(I_n, 0)$, this map has kernel zero and so it is an isomorphism; hence, by [23, Prop. 3.26, p. 100], it is a covering map.

(b) From (a), we easily get that, if $\mathcal{K}$ and $\widehat{\mathcal{K}}$ are the Killing tensors of $U(n)$ and of $SU(n) \times \mathbb{R}$, respectively, then we have $p^*(\mathcal{K}) = \widehat{\mathcal{K}}$. Since $\widehat{\mathcal{K}}$ is the product of the Killing tensors of $SU(n)$ and of $\mathbb{R}$ (and this last is zero), and remembering again [20, Ex. 6.19, p. 129], we have $\widehat{\mathcal{K}}_{(B,x)}((W,\lambda),(W',\lambda')) = 2n\operatorname{tr}(B^*WB^*W')$ for every $B \in SU(n)$, every $W, W' \in T_B(SU(n))$, and every $x, \lambda, \lambda' \in \mathbb{R}$.

Let $A := e^{\mathbf{i}x}B = p(B,x)$ (with $B \in SU(n)$ and $x \in \mathbb{R}$). If $Y, Z \in T_A(U(n))$, then, by (a), Y and Z are the images, through the tangent map of p, of $\Big(e^{-\mathbf{i}x}Y - \frac{\operatorname{tr}(A^*Y)}{n}B, -\frac{\mathbf{i}}{n}\operatorname{tr}(A^*Y)\Big)$ and of $\Big(e^{-\mathbf{i}x}Z - \frac{\operatorname{tr}(A^*Z)}{n}B, -\frac{\mathbf{i}}{n}\operatorname{tr}(A^*Z)\Big)$, respectively (note that $\operatorname{tr}(A^*Y)$ and $\operatorname{tr}(A^*Z)$ are purely imaginary, because A^*Y and A^*Z are skew-hermitian matrices).

Since $p^*(\mathcal{K}) = \widehat{\mathcal{K}}$, we get that

$$\begin{aligned}\mathcal{K}_A(Y,Z) &= \widehat{\mathcal{K}}_{(B,x)}\Bigg(\Big(e^{-\mathbf{i}x}Y - \frac{\operatorname{tr}(A^*Y)}{n}B, -\frac{\mathbf{i}}{n}\operatorname{tr}(A^*Y)\Big),\\ &\qquad\qquad \Big(e^{-\mathbf{i}x}Z - \frac{\operatorname{tr}(A^*Z)}{n}B, -\frac{\mathbf{i}}{n}\operatorname{tr}(A^*Z)\Big)\Bigg)\\ &= 2n\operatorname{tr}\left(B^*\big(e^{-\mathbf{i}x}Y - \frac{\operatorname{tr}(A^*Y)}{n}B\big)\ B^*\big(e^{-\mathbf{i}x}Z - \frac{\operatorname{tr}(A^*Z)}{n}B\big)\right)\\ &= 2n\left(\operatorname{tr}(A^*YA^*Z) - \frac{1}{n}\operatorname{tr}(A^*Y)\operatorname{tr}(A^*Z)\right)\\ &= 2n\operatorname{tr}(A^*YA^*Z) - 2\operatorname{tr}(A^*Y)\operatorname{tr}(A^*Z)\\ &= -2n\,\phi_A(Y,Z) - 2\operatorname{tr}(A^*Y)\operatorname{tr}(A^*Z).\end{aligned}$$

Therefore we can state the following.

Lemma 4.2. *The Killing tensor $\mathcal{K}$ of $U(n)$ has the expression*

$$\begin{aligned}\mathcal{K}_A(Y,Z) &= 2n\ \mathrm{tr}(A^*YA^*Z) - 2\ \mathrm{tr}(A^*Y)\,\mathrm{tr}(A^*Z)\\ &= -2n\,\phi_A(Y,Z) - 2\ \mathrm{tr}(A^*Y)\,\mathrm{tr}(A^*Z)\end{aligned}$$

for every $A \in U(n)$ and every $Y, Z \in T_A(U(n))$.

Remark 4.3. The Killing tensor $\mathcal{K}$ of $U(n)$ is a (degenerate) negative semi-definite tensor (and so $U(n)$ is not semi-simple). It suffices to check it at the identity $I_n \in U(n)$. By Lemma 4.2, we have $\mathcal{K}_{I_n}(\mathbf{i}I_n, \mathbf{i}I_n) = 0$; furthermore, if Y is a skew-hermitian matrix with purely imaginary eigenvalues $\mathbf{i}y_1, \ldots, \mathbf{i}y_n$, then

$$\mathcal{K}_{I_n}(Y,Y) = -2n\sum_{j=1}^{n} y_j^2 + \sum_{h,j=1}^{n} 2y_h y_j \leq -2n\sum_{j=1}^{n} y_j^2 + \sum_{h,j=1}^{n} (y_h^2 + y_j^2) = 0.$$

Remark-Definition 4.4. On the product manifold $SU(n) \times \mathbb{R}$, we consider the metric $\mathcal{H}$ defined as follows:

$$\mathcal{H}_{(B,x)}\big((W,\lambda),(W',\lambda')\big) = -\,\mathrm{tr}(B^*WB^*W') + n\lambda\,\lambda'$$

for every $B \in SU(n)$, every $W, W' \in T_B(SU(n))$, and every $x, \lambda, \lambda' \in \mathbb{R}$. Note that the metric $\mathcal{H}$ is the product of a constant positive multiple of the Killing metric of $SU(n)$ and of a constant positive multiple of the euclidean metric of $\mathbb{R}$. By [19, Thm. III, p. 341], the isometries of $(SU(n) \times \mathbb{R}, \mathcal{H})$ are precisely the maps of the form $\Phi \times \alpha$, where Φ is an isometry of $SU(n)$, endowed with its Killing metric, and α is an euclidean isometry of $\mathbb{R}$.

Lemma 4.5.

(a) *The map $p : (SU(n) \times \mathbb{R}, \mathcal{H}) \to (U(n), \phi)$ is a local isometry.*
(b) *For every isometry F of $(U(n), \phi)$ fixing the identity I_n of $U(n)$, there is a unique isometry $\widehat{F}$ of $(SU(n) \times \mathbb{R}, \mathcal{H})$ fixing the identity $(I_n, 0)$ of $SU(n) \times \mathbb{R}$ such that $p \circ \widehat{F} = F \circ p$.*

Proof. If $x, \lambda, \lambda' \in \mathbb{R}$, $B \in SU(n)$, $W, W' \in T_B(SU(n))$ (so $\mathrm{tr}(B^*W) = \mathrm{tr}(B^*W') = 0$), by Remarks-Definitions 4.1 (a), we have

$$\begin{aligned}p^*(\phi)_{(B,x)}\big((W,\lambda),(W',\lambda')\big) &= \phi_{(e^{\mathbf{i}x}B)}\big(e^{\mathbf{i}x}(W + \mathbf{i}\lambda B), e^{\mathbf{i}x}(W' + \mathbf{i}\lambda' B)\big)\\ &= -\,\mathrm{tr}\,\big((B^*W + \mathbf{i}\lambda I_n)(B^*W' + \mathbf{i}\lambda' I_n)\big)\\ &= -\,\mathrm{tr}(B^*WB^*W') + n\lambda\lambda'\\ &= \mathcal{H}_{(B,x)}\big((W,\lambda),(W',\lambda')\big),\end{aligned}$$

i.e., $p^*(\phi) = \mathcal{H}$ and the proof of (a) is complete. Part (b) follows from part (a) and from the fact that $(SU(n) \times \mathbb{R}, p)$ is the universal covering of $U(n)$. □

Proposition 4.6.

(a) *Every isometry of $(SU(2) \times \mathbb{R}, \mathcal{H})$ fixing the identity $(I_2, 0)$ of $SU(2) \times \mathbb{R}$ projects (through the covering map p) as an isometry of $(U(2), \phi)$ fixing the identity I_2 of $U(2)$.*

(b) *The isometries of* $(U(2),\phi)$ *fixing the identity* I_2 *of* $U(2)$ *are precisely the maps*

$$X \mapsto SXS^*, \quad X \mapsto SX^*S^*, \quad X \mapsto S\overline{X}S^*, \quad X \mapsto SX^TS^*,$$

with $S \in SU(2)$.

Proof. We denote by id the identity map of $\mathbb{R}$, by Id the identity map of $SU(2)$, by j the inversion map of $SU(2)$, and by $C_B = L_B \circ R_{B^*}$ the inner automorphism of $SU(2)$, associated to B. By Remark-Definition 4.4 and by Theorem 2.5 (c), the isometries of $(SU(2)\times\mathbb{R},\mathcal{H})$ fixing the identity $(I_2,0) \in SU(2)\times\mathbb{R}$ are precisely the maps of the form

$$C_B \times (\pm id) = (C_B \times id) \circ (Id \times (\pm id))$$

and

$$(C_B \circ j) \times (\pm id) = (C_B \times id) \circ (j \times (\pm id)),$$

with $B \in SU(2)$. Easy computations show that all the maps $C_B \times id$ (with $B \in SU(2)$), $Id \times id$, $Id \times (-id)$, $j \times id$ and $j \times (-id)$ project as maps of $U(2)$. More precisely, $C_B \times id$ projects as the inner automorphism of $U(2)$ associated to B, $Id \times id$ as the identity map of $U(2)$, $Id \times (-id)$ as the involution of $U(2)$ given by $A \mapsto \dfrac{A}{\det(A)}$, $j \times id$ as the involution of $U(2)$ given by $A \mapsto \det(A)A^*$, and $j \times (-id)$ as the inversion of $U(2)$. By composition, the maps of $U(2)$, obtained in this way, are the following:

$$X \mapsto BXB^*, \quad X \mapsto BX^*B^*, \quad X \mapsto \frac{BXB^*}{\det(X)}, \quad X \mapsto \det(X)BX^*B^*,$$

with $B \in SU(2)$. They are isometries of $(U(2),\phi)$, by Lemma 4.5 (a). Part (b) of the same Lemma implies that there are no other isometries.

To conclude it suffices to remark that, denoted by $W := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SU(2)$, we have $\dfrac{X}{\det(X)} = W^*\overline{X}W$ and $\det(X)X^* = WX^TW^*$ for every $X \in U(2)$. □

Theorem 4.7. *The isometries of* $(U(n),\phi)$, *with* $n \geq 2$, *are precisely the following maps:*

$$X \mapsto AXB, \quad X \mapsto AX^*B, \quad X \mapsto A\overline{X}B, \quad X \mapsto AX^TB,$$

with $A, B \in U(n)$.

Proof. We assume first $n \geq 3$, but we use the same notation as in the proof of Proposition 4.6 (with $n \geq 3$) and, moreover, we write $\mu(X) := \overline{X}$ and $\eta(X) := X^T$ for every $X \in SU(n)$. Again, by Remark-Definition 4.4 and by Theorem 2.5 (d), the isometries of $(SU(n)\times\mathbb{R},\mathcal{H})$ fixing the identity $(I_n,0)$ of $SU(n)\times\mathbb{R}$ are precisely the maps of the form

$$\begin{aligned}
&C_B \times (\pm id) = (C_B \times id) \circ (Id \times (\pm id)),\\
&(C_B \circ j) \times (\pm id) = (C_B \times id) \circ (j \times (\pm id)),\\
&(C_B \circ \mu) \times (\pm id) = (C_B \times id) \circ (\mu \times (\pm id)),\\
&(C_B \circ \eta) \times (\pm id) = (C_B \times id) \circ (\eta \times (\pm id)),
\end{aligned}$$

with $B \in SU(n)$.

Since the isometries of $(SU(n) \times \mathbb{R}, \mathcal{H})$, projecting (throughout p) as maps of $U(n)$, form a group with respect to the composition, it suffices to examine the following isometries:

- $C_B \times id$, which projects as the inner automorphism of $U(n)$, associated to the matrix $B \in SU(n)$,
- $Id \times id$, which projects as the identity map of $U(n)$,
- $j \times (-id)$, which projects as the inversion map of $U(n)$,
- $\mu \times (-id)$, which projects as the (complex) conjugation map of $U(n)$,
- $\eta \times id$, which projects as the transposition map of $U(n)$, and
- $Id \times (-id)$, $j \times id$, $\mu \times id$, $\eta \times (-id)$, which, on the contrary, do not project as maps of $U(n)$.

The proofs of the first five cases are obvious. For the isometries, which do not project as maps of $U(n)$, we consider, as an example, only the case $Id \times (-id)$; the other cases can be treated in the same way.

We have $I_n = p(I_n, 0) = p\left(e^{\frac{2\pi \mathbf{i}}{n}} I_n,\ -\frac{2\pi}{n}\right)$, $p \circ \big(Id \times (-id)\big)\ (I_n, 0) = I_n$, and $p \circ \big(Id \times (-id)\big)\ \left(e^{\frac{2\pi \mathbf{i}}{n}} I_n,\ -\frac{2\pi}{n}\right) = e^{\frac{4\pi \mathbf{i}}{n}} I_n$; these last two are different, because $n \geq 3$, and so the isometry $Id \times (-id)$ does not project as a map of $U(n)$.

Therefore, taking into account Lemma 4.5, the isometries of $(U(n), \phi)$ $(n \geq 3)$ fixing the identity I_n are the following maps: $X \mapsto BXB^*$, $X \mapsto BX^*B^*$, $X \mapsto B\overline{X}B^*$, $X \mapsto BX^TB^*$, with $B \in SU(n)$. Note that such isometries are formally the same as those of the case $n = 2$ in Proposition 4.6 (b). Now, by left (or right) translation with a matrix of $U(n)$, we obtain all the isometries in the statement both for $n = 2$ and for $n \geq 3$. □

Remarks 4.8.

(a) The full group of isometries of $(U(n), \phi)$, for $n \geq 2$, has 4 connected components, all diffeomorphic to $\frac{U(n)\times U(n)}{\{\lambda(I_n, I_n) \,:\, \lambda \in \mathbb{C},\, |\lambda|=1\}}$. Indeed, arguing as in Remark 1.10, the group generated by left and right translations of $U(n)$ is diffeomorphic to $\frac{U(n)\times U(n)}{(Z\times Z)\cap\Delta}$, where Z and Δ are, respectively, the center of $U(n)$ and the diagonal of $U(n) \times U(n)$. We conclude, because the center of $U(n)$ is $\{\lambda I_n : \lambda \in \mathbb{C}, |\lambda| = 1\}$.

(b) For every $n \geq 2$, $(U(n), \phi)$ is a (globally) symmetric Riemannian manifold. Indeed, for every $A \in U(n)$, the map $X \mapsto AX^*A$ is an isometry of $(U(n), \phi)$ fixing A and whose differential at A is the opposite of the identity map of $T_A(U(n))$.

Remark 4.9. Compare Theorem 4.7 with an analogous result of [11, Thm. 8], where the distance considered on $U(n)$ is again different from the distance induced by the Frobenius metric.

Remark 4.10. Following [21, p. 60] (in particular, Thm. 1.5), it is possible to get that, for $n \geq 2$, the group of the automorphisms of $U(n)$ is the semidirect product of its subgroup of inner automorphisms with the subgroup generated by the map $\mu : X \mapsto \overline{X}$. Hence, by Theorem 4.7, we deduce that, for every $n \geq 2$, the isometries of $(U(n), \phi)$ fixing the identity are precisely the automorphisms and the antiautomorphisms of the Lie group $U(n)$.

Note that an analogous result holds in the case of $(G, -\mathcal{K})$ (instead of $(U(n), \phi)$), where G is an absolutely simple, compact, connected real Lie group (see Proposition 2.2), but not in the case of $(S\mathcal{O}(4), -\mathcal{K})$; indeed, the maps $X \mapsto \tau(X)$ and $X \mapsto \tau(X)^T$ of Theorem 3.5 are neither automorphisms nor antiautomorphisms.

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