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CHARACTERIZATION OF ESSENTIAL SPECTRA BY QUASI-COMPACT PERTURBATIONS

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Abstract. We are interested in the concept of quasi-compact operators allowing us to provide some advances on the theory of operators acting in Banach spaces. More precisely, our main objective is to exhibit the importance of the use of this notion to outline a new approach in the analysis of the stability problems of upper and lower semi-Fredholm, upper and lower semi-Weyl, and upper and lower semi-Browder operators, and to provide a fine description and characterization of some Browder's essential spectra involving this kind of operators.

1. Introduction

In recent years, there has been significant progress in the theory of semi-Browder and semi-Fredholm operators. These classes of operators have gained recognition for their importance, motivating further exploration of generalizations and the study of various notions of essential spectra under (additive) perturbations. These perturbations can belong to any two-sided ideal of the set of bounded linear operators, including compact, weakly compact, strictly singular or cosingular operators. Additionally, other perturbations such as Riesz operators, polynomially of strict singularity perturbation, or measures of non-strict singularity are also considered. This kind of study plays a really strong and fruitful role across different areas of mathematics, essentially in the theory of stability and characterization. The interested reader can find basic information on this topic in works by F. Abdmouleh and I. Walha [1, 2], P. Aiena [3], S. Grabiner [9], R. Harte [10], V. Müller [19], V. Rakočević [20, 21], and M. Schechter [22].

On the other hand, since 1937, one of the novel classes of perturbations emerged as a prominent generalization of Riesz operators. This class, known as quasi-compact operators, encompasses all types of operators including finite rank, compact, or weakly compact operators, and was introduced by N. Kryloff and N. Bogoliouboff [16]. This kind of notion remains a forceful tool in the study of the ergodic

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properties of the Markov chains and the probabilities theory. Parallel to the contributions of Kryloff and Bogoliouboff, other mathematicians, namely A. Brunel and D. Revuz [4] and R. F. Taylor [23], successfully formulated alternative definitions for this type of operators and proved some important spectral properties via the notion of quasi-compact operators. Later, this class of operator attracted considerable attention of many researchers aiming to make substantial progress in mathematical studies. For more details, the interested reader can see [7, 15, 18] and their referenced theorems.

Note that the notion of quasi-compact operator perturbation turned out to be a useful tool in operator theory, especially in semi-Browder and Browder operators theory. Consequently, given the above argument, the attempt to address the concept of perturbations in operators becomes highly significant in order to develop new findings regarding the invariance of perturbed linear operators, resolve characterization problems associated with linear operators, and extend existing well-studied results [20, 21]. More precisely, our central interest in this paper is to exhibit the importance of the use of this kind of notion to outline a new approach in the analysis of the stability problems of upper and lower semi-Fredholm, upper and lower semi-Weyl and upper and lower semi-Browder operators. Particularly, we gather some conditions that we must impose on linear operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{QK}(X)$ which assert their stability results formulated as follows:

$$T \in \mathcal{P}(X) \iff T + S \in \mathcal{P}(X),$$

where $\mathcal{P}(X) := \{\mathcal{W}_+(X), \mathcal{W}_-(X), \mathcal{B}_+(X), \mathcal{B}_-(X)\}$ (see Subsection 3.1 for more details). Based on these key results, it becomes possible to consequently derive some stability results about their corresponding essential spectra in the second subsection of this paper. This derivation aims to improve and generalize some results that the author recently obtained [2]. In fact, by means of the quasi-compact operator perturbation, we prove that

$$\sigma_*(T) = \sigma_*(T + S),$$

for $\sigma_*(.) := \{\sigma_{\mathcal{F}_+}(.), \sigma_{\mathcal{F}_-}(.), \sigma_{\mathcal{W}_-}(.), \sigma_{\mathcal{W}_+}(.), \sigma_{\mathcal{B}_+}(.), \sigma_{\mathcal{B}_-}(.), \sigma_{\mathcal{B}}(.)\}$.

Our main purpose in the third part of Section 3 is to point out how the quasi-compactness concept allows us to reach a new characterization of the Browder's essential spectrum of linear operator T. Specifically, we define the set

$$\begin{aligned}\mathcal{E}(X) := \{K \in \mathcal{L}(X) : KT = TK,\ K(\mu - T - K)^{-1} \in \mathcal{QK}(X), \text{ and there exists} \\ \varepsilon > 0 \text{ s.t. } \operatorname{dist}(K(\mu - T - K)^{-1}, \mathcal{K}(X)) < \varepsilon,\ \forall \mu \in \rho(T + K)\},\end{aligned}$$

and we establish the Browder's essential spectrum of T as

$$\sigma_{\mathcal{B}}(T) = \bigcap_{K \in\ \mathcal{E}(X)} \sigma(T + K).$$

The remainder of the paper is organized as follows. Some definitions and properties of linear operators are introduced in Section 2; in particular, we present in detail the notion of quasi-compact operator according to its properties and some illustrative examples. In Section 3 we state the main results of this work in three subsections: the first subsection is dedicated to developing some advances on spectral theory of semi-Fredholm and semi-Browder's operators involving the notion of quasi-compact operators perturbations; the second subsection gives practical criteria that guarantee the invariance of various Browder's and Weyl's essential spectra for perturbed linear operators, improves a refinement description of upper and lower Browder essential spectra, and presents our main results, already mentioned above. Finally, we state the characterization of Browder's essential spectrum by means of quasi-compact operators.

2. Preliminary results

In this section, we recall some definitions and give some preliminary results relevant to linear bounded operators. Let X be a Banach space. We denote by $\mathcal{L}(X)$ the set of all bounded linear operators on X. We denote by $\mathcal{K}(X)$, $\mathcal{F}_0(X)$ the subsets of $\mathcal{L}(X)$ formed, respectively, by all compact and by all finite rank operators on X. For $T \in \mathcal{L}(X)$, we write $\mathcal{N}(T) = \{x \in X : Tx = 0\} \subset X$ for the null space and $\mathcal{R}(T) \subset X$ for the range of T. The nullity of T, denoted by $\alpha(T)$, is defined as the dimension of $\mathcal{N}(T)$, and the deficiency of T, denoted by $\beta(T)$, is defined as the codimension of $\mathcal{R}(T)$ in X. The spectrum of T will be denoted by $\sigma(T)$. The resolvent set of T, denoted by $\rho(T)$, is the complement of $\sigma(T)$ in the complex plane and is defined as

$$\rho(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ has a bounded inverse}\}.$$

An operator $T \in \mathcal{L}(X)$ is called a *semi-Fredholm* operator on X if its range $\mathcal{R}(T)$ is a closed subspace of X and at least one of $\alpha(T)$ and $\beta(T)$ is finite. For such an operator, we define an index $i(T)$ by $i(T) = \alpha(T) - \beta(T)$. Let $\Phi_+(X)$ (resp., $\Phi_-(X)$) denote the set of upper (resp., lower) semi-Fredholm operators on X, that is, the set of semi-Fredholm operators with closed range $\mathcal{R}(T)$ and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$). An operator T is said to be a *Fredholm* operator on X if it is both an upper semi-Fredholm and a lower semi-Fredholm operator on X.

The set of upper (resp., lower) Weyl operators is defined by

$$\mathcal{W}_+(X) := \{T \in \mathcal{L}(X) : T \in \Phi_+(X) \text{ and } i(T) \leq 0\}$$
$$(\text{resp., } \mathcal{W}_-(X) := \{T \in \mathcal{L}(X) : T \in \Phi_-(X) \text{ and } i(T) \geq 0\}).$$

Consequently, the set of Weyl operators is defined as $\mathcal{W}(X) := \mathcal{W}_+(X) \cap \mathcal{W}_-(X)$.

We define the *ascent* of T by

$$a(T) := \min\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\},$$

whenever this minimum exists. If no such number exists, the ascent of T is defined to be ∞. Likewise, this statement leads to the introduction of the *descent* of T by

$$d(T) := \min\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\},$$

where the minimum over the empty set is taken to be ∞.

In order to define other sets of linear operators, we introduce the set of upper (resp., lower) *semi-Browder operators* on X as follows:

$$\mathcal{B}_+(X) = \{T \in \mathcal{L}(X) : T \in \Phi_+(X),\ i(T) \leq 0, \text{ and } a(T) < +\infty\}$$

$$(\text{resp., } \mathcal{B}_-(X) = \{T \in \mathcal{L}(X) : T \in \Phi_-(X),\ i(T) \geq 0, \text{ and } d(T) < +\infty\}).$$

In what follows, we list some classical definitions of linear bounded operators used in our formulation.

Definition 2.1 ([19])**.** Let X be Banach space and $T \in \mathcal{L}(X)$. The *reduced minimum modulus* of T is defined by

$$\gamma(T) := \inf\big\{\|Tx\| : x \in X,\ \text{dist}(x, \mathcal{N}(T)) = 1\big\},$$

where $\text{dist}(x, \mathcal{N}(T))$ is the distance between an element x and the subspace $\mathcal{N}(T)$ of X.

Definition 2.2. Let $T \in \mathcal{L}(X)$.

(i) T is called a *weakly compact* operator on X if $T(M)$ is relatively weakly compact for every bounded subset $M \in X$.

The set of weakly compact operators on X will be denoted by $\mathcal{WC}(X)$.

(ii) T is called a *Riesz* operator if $\lambda - T$ is a Fredholm operator for all non-zero complex numbers $\lambda \in \mathbb{C}\backslash\{0\}$.

The class of Riesz operators on X will be denoted by $\mathcal{R}(X)$.

In 1937, N. Kryloff and N. Bogoliouboff [16] introduced the concept of quasi-compact operators as a generalization of sets of finite rank, compact, weakly compact, and Riesz operators, which were frequently used in many areas of mathematics, particularly in the study of ergodic properties of Markov chains. In 1939, K. Yosida provided an alternative definition for quasi-compact operators.

Definition 2.3 ([11])**.** An operator $T \in \mathcal{L}(X)$ is called *quasi-compact* if there exists a compact operator K and $n \in \mathbb{N}$ such that $\|T^n - K\| < 1$.

We denote the class of quasi-compact operators by $\mathcal{QK}(X)$.

Let us recall some important examples of quasi-compact operators.

Examples of quasi-compact operators.

(i) A compact operator $T : X \longrightarrow X$ is quasi-compact.

(ii) Operators with compact power (or weakly compact operators) are quasi-compact.

(iii) Note that, for $T \in \mathcal{L}(X)$, we may characterize the quasi-compact operator T by its essential spectral radius, denoted by $r_{\text{ess}}(T)$; that is,

$$T \in \mathcal{QK}(X) \quad \text{if and only if} \quad r_{\text{ess}}(T) < 1,$$

where $r_{\text{ess}}(T)$ is defined as

$$r_{\text{ess}}(T) := \lim_{n \to +\infty} (\text{dist}(T^n, \mathcal{K}(X)))^{\frac{1}{n}},$$

for $\text{dist}(T^n, \mathcal{K}(X)) = \inf_{K \in \mathcal{K}(X)} \|T^n - K\|$. Hence, we can regard a Riesz operator as an example of a quasi-compact operator when $r_{\text{ess}}(T) = 0$ for

$T \in \mathcal{R}(X)$. Consequently, we order by inclusion the last various classes of linear operators as follows:

$$\mathcal{F}_0(X) \subset \mathcal{K}(X) \subset \mathcal{WC}(X) \subset \mathcal{R}(X) \subset \mathcal{QK}(X).$$

Some interesting properties of this kind are developed by J. Martínez and J. M. Mazón [18, Proposition 2.2] as follows:

Proposition 2.4. *For any operator $T \in \mathcal{L}(X)$, the following assertions are equivalent:*

(i) T *is quasi-compact.*
(ii) $\lim\limits_{n\to+\infty} \|T^n + \mathcal{K}(X)\| = 0$.
(iii) The spectrum radius $r(\hat{T}, \hat{\mathcal{L}}(X)) < 1$, *where* $\hat{T} := T + \mathcal{K}(X)$ *in the Calkin algebra* $\hat{\mathcal{L}}(X) := \mathcal{L}(X)/\mathcal{K}(X)$.
(iv) $T = U + K$, *where* $K \in \mathcal{K}(X)$ *is of finite rank and* $U \in \mathcal{L}(X)$ *has spectral radius* $r(U) < 1$.

Remark 2.5. Following [17], we infer that the adjoint of the quasi-compact operator T on a reflexive Banach space X is still also a quasi-compact operator on X^*.

3. Main results

Quasi-compact operators perturbation is important in operator theory, particularly for the study of invariant and characterization problems of linear bounded operators. In this section, our focus will be to articulate this objective through three distinct subsections.

3.1. **Semi-Fredholm theory via quasi-compact perturbations.** To discuss the stability results of semi-Fredholm, semi-Weyl and semi-Browder linear bounded operators via the concept of quasi-compact perturbation, the following result of semi-Fredholm operators may be essential. We will discuss it firstly for upper and lower semi-Fredholm operators as these sets are not equal in general. Before moving to the desired results, we will introduce an illustrative example showing the difference between semi-Fredholm operators.

Example 3.1 **($\Phi_+(X) \neq \Phi_-(X)$).** Let us define on $l^2(\mathbb{N})$ the bounded operator T as

$$Tx = (x_1, 0, x_2, 0x_3, 0..), \quad \text{where } x = (x_i) \in l^2(\mathbb{N}).$$

Obviously, the operator T is injective with closed range $R(T)$ such that $\operatorname{codim} R(T) = \infty$.

Therefore, T is an upper semi-Fredholm but not a lower semi-Fredholm operator on $l^2(\mathbb{N})$, which asserts that $T \in \Phi_+(l^2(\mathbb{N}))$ but $T \notin \Phi_-(l^2(\mathbb{N}))$.

Our first fundamental result is formulated as follows:

Proposition 3.2. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{QK}(X)$. Assume that there exists $\varepsilon > 0$ such that* $\operatorname{dist}(S, \mathcal{K}(X)) < \varepsilon$. *Then we have*

(i) $T \in \Phi_+(X) \;\Rightarrow\; T + S \in \Phi_+(X)$ *with* $i(T + S) = i(T)$.

Assume further that X is a reflexive Banach space; then we have *(ii)* $T \in \Phi_-(X) \;\Rightarrow\; T+S \in \Phi_-(X)$ *with* $i(T+S) = i(T)$.

Proof. (i) Let $T \in \Phi_+(X)$ and $S \in \mathcal{QK}(X)$. Hence, based on Proposition 2.4 (iv), we infer that there exists a finite rank operator F and $R \in \mathcal{L}(X)$ with $r(R) < 1$ such that $S = F + R$. Thus, we get

$$T + S = T + F + R.$$

By using [22, Theorem 5.22] under the fact that $T \in \Phi_+(X)$ and $F \in \mathcal{F}_0(X)$, it can be concluded that

$$T + F \in \Phi_+(X) \quad \text{with } i(T+F) = i(T).$$

On the other hand, one has $\mathcal{R}(T+F)$ is a closed subspace of X; we infer from [19, Theorem 10.2] that $\gamma(T+F) > 0$. Thus, we deduce that there exists $\varepsilon = \gamma(T+F) > 0$ such that

$$\|R\| = \|S - F\| < \gamma(T+F), \quad \text{while } F \in \mathcal{F}_0(X) \subseteq \mathcal{K}(X).$$

So, in what follows, we conclude from the use of [19, Theorem 18.4], that $T + S \in \Phi_+(X)$ with $i(T+S) = i(T)$.

(ii) The proof of this item may be checked in the same way as the previous one.

In fact, consider T as a lower semi-Fredholm operator in reflexive Banach space X and $S \in \mathcal{QK}(X)$. Thus, we deduce from the use of [19, Theorem 16.4] with Remark 2.5 that $T^* \in \Phi_+(X^*)$ and $S^* \in \mathcal{QK}(X^*)$. Arguing in the same way as the previous item, while $\gamma(T+F) = \gamma((T+F)^*)$ (see [19, Theorem 10.3]), we conclude that $(T+S)^* \in \Phi_+(X^*)$ with $i((T+S)^*) = i(T^*)$. Hence, [19, Theorem 16.4] leads to

$$T + S \in \Phi_-(X) \quad \text{with } i(T+S) = i(T). \qquad \square$$

For the stability results of semi-Weyl operators involving the concept of quasi-compact perturbations, we can readily deduce the following results:

Corollary 3.3. *Let X be a reflexive Banach space, $T \in \mathcal{L}(X)$ and $S \in \mathcal{QK}(X)$. Assume that there exists $\varepsilon > 0$ such that* $\operatorname{dist}(S, \mathcal{K}(X)) < \varepsilon$. *Then we have*

$$T \in \mathcal{W}_*(X) \iff T + S \in \mathcal{W}_*(X)$$

for $\mathcal{W}_*(X) := \{\mathcal{W}_-(X), \mathcal{W}_+(X)\}$.

Note that, in general, a Weyl operator is not a Browder operator. For this, we will extend the previous results of Weyl operators to Browder operators. Below, we introduce an example which explains the strict inclusion or the distinction between these two operators.

Example 3.4 (A Weyl operator isn't a Browder operator)**.** Consider the unilateral Shift operator $V \in \mathcal{L}(l^2(\mathbb{N}))$ defined as

$$\begin{cases} V : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N}) \\ x \longmapsto Vx := (0, x_1, x_2, \dots), \quad \text{where } x = (x_i) \in l^2(\mathbb{N}). \end{cases}$$

Thus, V and V^* are two Fredholm operators with $i(V) = -1$ and $i(V^*) = 1$.

Let us define the linear operator $T : l^2(\mathbb{N}) \times l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N}) \times l^2(\mathbb{N})$ by

$$T := \begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix}.$$

As V and V^* are two Fredholm operators and T is a diagonal operator matrix, we infer that T is also a Fredholm operator on $l^2(\mathbb{N}) \times l^2(\mathbb{N})$ with index null, that is, $i(T) = i(V) + i(V^*) = 0$. So, we conclude that T is a Weyl operator. Moreover, we have

$$\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

This asserts that 0 is not isolated in $\sigma(T)$, which makes us conclude that T is not a Browder operator.

In what follows, let us write, of $T \in \mathcal{L}(X)$, the commutant of the operator T as well:

$$\operatorname{com}(T) := \{S \in \mathcal{L}(X) : ST = TS\}.$$

When we are interested in the stability in terms of semi-Browder operators, we need to discuss their ascent and deficiency. Thus, we formulate the following result:

Lemma 3.5. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{QK}(X)$ such that $S \in \operatorname{com}(T)$. Assume that there exists $\delta > 0$ such that $\operatorname{dist}(S, \mathcal{K}(X)) < \delta$ with $T \in \Phi_+(X)$. Then, we obtain*

$$a(T) < \infty \quad \Longleftrightarrow \quad a(T + S) < \infty.$$

Proof. To prove the first implication, assume that $T \in \Phi_+(X)$, $S \in \mathcal{QK}(X)$ such that $ST = TS$ and $\operatorname{dist}(S, \mathcal{K}(X)) < \delta$ with $a(T) < \infty$ and we claim that $a(T+S) < \infty$.

Indeed, consider, for $\xi \in [0, 1]$, the following operator T_ξ having the form

$$T_\xi = T + \xi S.$$

One has $S \in \mathcal{QK}(X)$ and $\xi \in [0, 1]$; we prove that $\xi S \in \mathcal{QK}(X)$.

Therefore, we have $\operatorname{dist}(\xi S, \mathcal{K}(X)) < \delta$. Using Proposition 3.2 (i), we infer that

$$T_\xi = T + \xi S \in \Phi_+(X) \quad \text{with } i(T_\xi) = i(T) \leq 0 \text{ for each } \xi \in [0, 1].$$

According to the fact that T and S are commuting (see Theorem 3 in [8]), we can deduce that there exists $\tau = \tau(\xi) > 0$ such that

$$\overline{\mathcal{N}^\infty(T_\xi)} \cap \mathcal{R}^\infty(T_\xi) = \overline{\mathcal{N}^\infty(T_\nu)} \cap \mathcal{R}^\infty(T_\nu), \quad \forall \nu \in \mathcal{D}(\xi, \tau). \tag{3.1}$$

That is, the function of ξ, $\overline{\mathcal{N}^\infty(T_\xi)} \cap \mathcal{R}^\infty(T_\xi)$ is locally constant on the connected set $[0, 1]$. Since every locally constant function on a connected set is constant, we conclude that

$$\overline{\mathcal{N}^\infty(T_\xi)} \cap \mathcal{R}^\infty(T_\xi) = \overline{\mathcal{N}^\infty(T)} \cap \mathcal{R}^\infty(T), \quad \forall \xi \in [0, 1].$$

On the other hand, the operator T^p is bounded for all $p \in \mathbb{N}$, since $T \in \mathcal{L}(X)$. Consequently, we derive by Chapter 3, Problem 5.9 in [14], that $\mathcal{N}(T^p)$ is a closed subspace of X. Therefore, in view of the fact that $a(T) < \infty$, [24, Proposition 1.6,(i)] leads to

$$\overline{\mathcal{N}^\infty(T)} \cap \mathcal{R}^\infty(T) = \mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}.$$

Consequently, from Eq. (3.1), we can conclude that $\overline{\mathcal{N}^\infty(T+S)} \cap \mathcal{R}^\infty(T+S) = 0$. Therefore, we obtain $a(T+S) < \infty$, by [24, Proposition 1.6].

The proof of the reverse implication may be checked in the same way as the previous one. It is sufficient to replace $a(T)$ with $a(T+S)$, the operators T and S with $T+S$ and $-S$, and the perturbed function T_ξ with $(T+S)_\xi$ with the function $(T+S)_\xi := T + (\xi+1)S$ for every constant ξ belonging to the connected set $[-1, 0]$. □

A derivative consequence from the last result and Proposition 3.2 is formulated below.

Theorem 3.6. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{QK}(X)$ such that $S \in \mathrm{com}(T)$. Assume that there exists $\delta > 0$ such that $\mathrm{dist}(S, \mathcal{K}(X)) < \delta$; then we obtain*

$$T \in \mathcal{B}_+(X) \iff T+S \in \mathcal{B}_+(X).$$

Theorem 3.6 is a generalization of [9, Theorem 2 (a)]. As a consequence of Theorem 3.6 we enrich the stability results for lower semi-Browder operators involving the notion of quasi-compact operator:

Corollary 3.7. *Let X be a reflexive Banach space, $T \in \mathcal{L}(X)$ and $S \in \mathcal{QK}(X)$ such that $S \in \mathrm{com}(T)$. Assume that there exists $\delta > 0$ such that $\mathrm{dist}(S, \mathcal{K}(X)) < \delta$. Then, we get*

$$T \in \mathcal{B}_-(X) \iff T+S \in \mathcal{B}_-(X).$$

Proof. By a duality argument, we prove, similarly to Theorem 3.6 and Lemma 3.5, the closeness of $\mathcal{B}_-(X)$ under perturbation via commuting quasi-compact operators. □

Remark 3.8. According to Theorem 3.6 and Corollary 3.7, we can easily derive the stability result for Browder operators, that is, for a reflexive Banach space X, $T \in \mathcal{L}(X), S \in \mathcal{QK}(X)$ such that $ST = TS$, $\exists \delta > 0,\ \mathrm{dist}(S, \mathcal{K}(X)) < \delta$, we have

$$T \in \mathcal{B}(X) \iff T+S \in \mathcal{B}(X).$$

The results of Theorems 3.6 and Corollary 3.7 are an extension and an improvement of [21, Theorem 1 and Corollary 2] to a large class of quasi-compact operator perturbations.

A practical use of the previous results leads to the stability results of semi-Browder and Browder operators via the quasi compact notion as follows:

Corollary 3.9. *Let X be a reflexive Banach space, $T, S \in \mathcal{L}(X)$ such that T is invertible and $ST = TS$. Then, we get*

$$ST^{-1} \in \mathcal{QK}(X), \exists \delta > 0 : \mathrm{dist}(ST^{-1}, \mathcal{K}(X)) < \delta \implies T+S \in \mathcal{B}_*(X)$$

for $\mathcal{B}_*(X) := \{\mathcal{B}_+(X), \mathcal{B}_-(X), \mathcal{B}(X)\}$.

Proof. Assuming that $T \in \mathcal{L}(X)$ is invertible, $ST = TS$, and $ST^{-1} \in \mathcal{QK}(X)$ such that $\mathrm{dist}(ST^{-1}, \mathcal{K}(X)) < \delta$, we will show that $T+S \in \mathcal{B}_+(X)$.

In fact: As $0 \in \rho(T)$, we infer that the operator $T+S$ may be written as

$$T+S=(I+ST^{-1})T.$$

Using Remark 3.8, we infer that $I+ST^{-1} \in \Phi(X)$ with $i(I+ST^{-1})=0$. Thus, if $T \in \Phi_+(X)$, we get by [19, Theorems 16.5 and 16.12] that

$$T+S=(I+ST^{-1})T \in \Phi_+(X) \quad \text{with } i(T+S)=i(I+ST^{-1})+i(T)=0.$$

It remains to show that

$$a(T+S)<\infty \quad \text{and} \quad d(T+K)<\infty.$$

Note that, for every $n \in \mathbb{N}$, we have

$$(T+S)^n=(I+ST^{-1})^n T^n.$$

As T^n is one-to-one for every $n \in \mathbb{N}$, we infer that

$$\mathcal{N}((T+S)^n) \subset \mathcal{N}((I+ST^{-1})^n).$$

Consequently,

$$\alpha((T+S)^n) \leq \alpha((I+ST^{-1})^n).$$

Using [5, Lemma 1] and Theorem 3.6, we obtain

$$\alpha((T+S)^n) \leq a(I+ST^{-1})\alpha(I+ST^{-1})<\infty.$$

Thus, we find that $T+S \in \mathcal{B}_+(X)$.

The previous result also holds for lower semi-Browder operators. The desired result for Browder operators follows immediately from the two previous results. □

3.2. **Perturbation results of some essential spectra.** The aim of this subsection is to translate the formulation from the previous subsection into the context of essential spectra. Also, we are interested in the study of the characterization problems associated with these spectra. In the following, let us introduce some essential spectra that are relevant to our purpose.

Definition 3.10. Let $T \in \mathcal{L}(X)$. We define

(i) the Browder's essential approximate point spectrum of T, denoted by $\sigma_{\mathcal{B}_+}(T)$, as

$$\sigma_{\mathcal{B}_+}(T):=\{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_+(X)\};$$

(ii) the Browder's essential defect spectrum of T, denoted by $\sigma_{\mathcal{B}_-}(T)$, as

$$\sigma_{\mathcal{B}_-}(T):=\{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_-(X)\};$$

(iii) the Browder's essential spectrum of T, denoted by $\sigma_{\mathcal{B}}(T)$, as

$$\sigma_{\mathcal{B}}(T):=\{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}(X)\}.$$

A useful characterization of Browder's essential approximate point and Browder's essential defect spectrum by quasi-compact operator perturbations is formulated below.

Theorem 3.11. *Let $T \in \mathcal{L}(X)$ and denote by $E(X)$ the following set:*

$$E(X) := \{S \in \mathcal{L}(X) : S \in \operatorname{com}(T), \exists\, \delta > 0 : \ \operatorname{dist}(S, \mathcal{K}(X)) < \delta\}.$$

Then, we get

$$\sigma_{\mathcal{B}_+}(T) := \bigcap_{S \in \mathcal{QK}(X) \cap \mathbf{E}(X)} \sigma_a(T+S).$$

Assume further that X is a reflexive Banach space, then we obtain

$$\sigma_{\mathcal{B}_-}(T) := \bigcap_{S \in \mathcal{QK}(X) \ \cap \ \mathbf{E}(X)} \sigma_d(T+S),$$

where the approximate point (resp., the defect) spectrum of T is denoted by $\sigma_a(T)$ (resp., $\sigma_d(T)$) and defined as $\sigma_a(T) := \{\lambda \in \mathbb{C} : \lambda - T$ is not bounded below$\}$ (resp., $\sigma_d(T) := \{\lambda \in \mathbb{C} : \lambda - T$ is not onto$\}$).

Proof. It is trivial to see that all compact operators belong to the subset $\mathcal{QK}(X) \cap \mathbf{E}(X)$; thus we conclude from the works of V. Rakočević [20, 21] that

$$\bigcap_{S \in \mathcal{QK}(X) \cap \mathbf{E}(X)} \sigma_*(T+S) \subset \bigcap_{S \in \mathcal{K}(X) ST=TS} \sigma_*(T+S) := \sigma_{\mathcal{B}_j}(T), \tag{3.2}$$

where $(\sigma_*(.), \sigma_{\mathcal{B}_j}(.)) = \{(\sigma_a(.), \sigma_{\mathcal{B}_+}(.)), (\sigma_d(.), \sigma_{\mathcal{B}_-}(.))\}$.

(i) We reach the result of this item by double inclusion. The first one follows from Eq. (3.2). For the reverse inclusion, assume that $\lambda \notin \bigcap\{\sigma_a(T+S),\ S \in \mathcal{QK}(X) \cap \mathbf{E}(X)\}$; then there exists $S \in \mathcal{QK}(X) \cap \mathbf{E}(X)$ for which $\lambda - T - S$ is bounded below. Thus, [19, Theorem 9.4] asserts that $\lambda - T - S$ is one-to-one with closed range $\mathcal{R}(\lambda - T - S)$. Therefore, $\lambda - T - S \in \Phi_+(X)$ with $i(\lambda - T - S) = -\beta(\lambda - T - S) \leq 0$ and $a(\lambda - T - S) = 0 < \infty$. Following Theorem 3.6, one has $S \in \mathcal{QK}(X) \cap \mathbf{E}(X)$, and we infer that $\lambda - T \in \mathcal{B}_+(X)$. Consequently, we find

$$\sigma_{\mathcal{B}_+}(T) \subset \bigcap\{\sigma_a(T+S), \ \ S \in \mathcal{QK}(X) \cap \mathbf{E}(X)\}.$$

(ii) Assume that $\lambda \notin \bigcap\{\sigma_d(T+S), \ \ S \in \mathcal{QK}(X) \cap \mathbf{E}(X)\}$; then $\exists S \in \mathcal{QK}(X) \cap \mathbf{E}(X)$ such that $\lambda - T - S$ is surjective, and so $\lambda - T - S \in \Phi_-(X)$ with $i(\lambda - T - S) = \alpha(\lambda - T - S) \geq 0$ and $d(\lambda - T - S) = 0 < \infty$. That is, $\lambda - T - S \in \mathcal{B}_-(X)$. According to the fact that $S \in \mathcal{QK}(X) \cap \mathbf{E}(X)$, we deduce from Corollary 3.7 that $\lambda - T \in \mathcal{B}_-(X)$. This with Eq. (3.2) ends the proof. ☐

In [21], V. Rakočević showed the invariance of the approximate point and defect spectrum under commuting Riesz operators perturbations. So, we generalize this results as follows.

Theorem 3.12. *Let X be a reflexive Banach space, $T \in \mathcal{L}(X)$. Then, we have*

$$S \in \mathcal{QK}(X) : S \in \operatorname{com}(T), \ \exists\ \varepsilon > 0, \ \operatorname{dist}(S, \mathcal{K}(X)) < \varepsilon \implies \sigma_*(T) = \sigma_*(T+S),$$

where $\sigma_*(.) := \{\sigma_{\mathcal{B}_+}(.), \sigma_{\mathcal{B}_-}(.), \sigma_{\mathcal{B}}(.)\}$.

Proof. It is enough to prove the result for the approximate point spectrum. The result for the defect spectrum may be checked in the same way: it is sufficient to use Corollary 3.7.

Indeed, consider $T \in \mathcal{L}(X)$ and $S \in \mathcal{QK}(X)$ such that $S \in \mathrm{com}(T)$ and there exists $\varepsilon > 0$ for which we have $\mathrm{dist}(S, \mathcal{K}(X)) < \varepsilon$. Assume that $\lambda \notin \sigma_{\mathcal{B}_+}(T)$; we will show that $\lambda \notin \sigma_{\mathcal{B}_+}(T+S)$. Since $S \in \mathcal{QK}(X)$, we infer that $-S \in \mathcal{QK}(X)$. Consequently, it is seen that $-S \in \mathrm{com}(\lambda - T)$ while $S \in \mathrm{com}(T)$, satisfying that $\mathrm{dist}(-S, \mathcal{K}(X)) < \varepsilon$. Keeping in mind the fact that $\lambda - T \in \mathcal{B}_+(X)$, Theorem 3.6 asserts that $\lambda - T - S \in \mathcal{B}_+(X)$. Hence, we find $\sigma_{\mathcal{B}_+}(T+S) \subset \sigma_{\mathcal{B}_+}(T)$.

Conversely, suppose that $\lambda \notin \sigma_{\mathcal{B}_+}(T+S)$. Hence, $\lambda - T - S \in \mathcal{B}_+(X)$. It is easy to see that S commutes with $\lambda - T - S$, while S commutes with T. Therefore, using again Theorem 3.6, we deduce that $\lambda - T - S + S = \lambda - T \in \mathcal{B}_+(X)$. This implies that $\lambda \notin \sigma_{\mathcal{B}_+}(T)$.

Consequently, we derive the result for Browder's essential spectrum while

$$\sigma_{\mathcal{B}}(T) := \sigma_{\mathcal{B}_+}(T) \cup \sigma_{\mathcal{B}_-}(T). \qquad \square$$

Before moving to the stability results of semi-Fredholm and semi-Weyl operators via the concept of quasi-compact operators, we start to define some essential spectra.

Definition 3.13. Let $T \in \mathcal{L}(X)$. We define

(i) the *upper* (resp., *lower*) *Fredholm essential spectrum* of T, denoted by $\sigma_{\mathcal{F}_+}(T)$ (resp., $\sigma_{\mathcal{F}_-}(T)$), as

$$\sigma_{\mathcal{F}_+}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_+(X)\}$$
$$(\text{resp., } \sigma_{\mathcal{F}_-}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_-(X)\});$$

(ii) the *upper* (resp., *lower*) *Weyl essential spectrum* of T, denoted by $\sigma_{\mathcal{W}_+}(T)$ (resp., $\sigma_{\mathcal{W}_-}(T)$), as

$$\sigma_{\mathcal{W}_+}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{W}_+(X)\}$$
$$(\text{resp. } \sigma_{\mathcal{W}_-}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{W}_-(X)\}).$$

Corollary 3.14. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{QK}(X)$. Assume that there exists $\varepsilon > 0$ such that $\mathrm{dist}(S, \mathcal{K}(X)) < \varepsilon$. Then we have*

(i) $\sigma_{\mathcal{F}_+}(T) = \sigma_{\mathcal{F}_+}(T+S)$ *and* $\sigma_{\mathcal{W}_+}(T) = \sigma_{\mathcal{W}_+}(T+S)$.

Assume further that X is a reflexive Banach space. Then we obtain

(ii) $\sigma_{\mathcal{F}_-}(T) = \sigma_{\mathcal{F}_-}(T+S)$ *and* $\sigma_{\mathcal{W}_-}(T) = \sigma_{\mathcal{W}_-}(T+S)$.

Proof. (i) As $S \in \mathcal{QK}(X)$ with $\mathrm{dist}(S, \mathcal{K}(X)) < \delta$, we infer that $-S$ is also a quasi-compact operator on X with $\mathrm{dist}(-S, \mathcal{K}(X)) < \delta$. Thus, using Proposition 3.2,(i) and the fact that $\mu \notin \sigma_{\mathcal{W}+}(T)$, we obtain $\mu - T - S \in \Phi+(X)$ with $i(\mu - T - S) + i(\mu - T) \leq 0$.

Hence, we conclude that

$$\sigma_{\mathcal{W}_+}(T+S) \subset \sigma_{\mathcal{W}_+}(T).$$

Evidently, we thus obtain

$$\sigma_{\mathcal{F}_+}(T+S) \subset \sigma_{\mathcal{F}_+}(T).$$

Conversely, for upper semi-Weyl operator $\mu - T - S$, for $\mu \notin \sigma_{\mathcal{W}_+}(T+S)$, a direct translation of the result of Proposition 3.2 (i), in terms of essential spectra, is formulated as

$$\sigma_{\mathcal{W}_+}(T) \subset \sigma_{\mathcal{W}_+}(T+S) \quad \text{and} \quad \sigma_{\mathcal{F}_+}(T) \subset \sigma_{\mathcal{F}_+}(T+S).$$

(ii) A similar reasoning as that of item (i) allows us to reach the result of the lower Fredholm and lower Weyl essential spectra. It is sufficient to use Proposition 3.2 (ii) and Corollary 3.3; the details are therefore omitted. □

A further strong stability result for Browder's essential spectra is given as:

Theorem 3.15. *Let X be a reflexive Banach space and T, $S \in \mathcal{L}(X)$ such that S commutes with T and $\rho(T) \cap \rho(S) \neq \emptyset$.*

Assume, for some $\lambda \in \rho(T) \cap \rho(S)$, that

(i) $\mathcal{Q} := (\lambda - T)^{-1} - (\lambda - S)^{-1} \in \mathcal{QK}(X)$;

(ii) there exists $\varepsilon > 0$ such that $\operatorname{dist}(\mathcal{Q}, \mathcal{K}(X)) < \varepsilon$.

Then, we get

$$\sigma_{\mathcal{B}_*}(T) = \sigma_{\mathcal{B}_*}(S) \quad \textit{for } \sigma_{\mathcal{B}_*}(.) := \{\sigma_{\mathcal{B}_+}(.), \sigma_{\mathcal{B}_-}(.), \sigma_{\mathcal{B}}(.)\}.$$

Proof. Without loss of generality, we assume that $\lambda = 0$, thus $0 \in \rho(T) \cap \rho(S)$. Therefore, we write, for $\xi \in \mathbb{C}\backslash\{0\}$, the following operator:

$$\xi - T := -\xi[\xi^{-1} - T^{-1}]T.$$

Obviously, while T is one-to-one, we observe that $\mathcal{N}(\xi - T) = \mathcal{N}(\xi^{-1} - T^{-1})$. Thus, $\alpha(\xi - T) = \alpha(\xi^{-1} - T^{-1})$. Also, since T is surjective, then $\mathcal{R}(\xi - T) = \mathcal{R}(\xi^{-1} - T^{-1})$. So, $\beta(\xi - T) = \beta(\xi^{-1} - T^{-1})$.

In addition, when T commutes with $\xi^{-1} - T^{-1}$, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{N}((\xi - T)^n) &= \{x,\ (\xi - T)^n x = 0\} \\ &= \{x,\ (\xi^{-1} - T^{-1})^n T^n x = 0\}.\end{aligned}$$

Clearly, as $a(T) = 0$, we can conclude that $\mathcal{N}((\xi - T)^n) = \mathcal{N}((\xi^{-1} - T^{-1})^n)$. Hence, $a(\xi - T) = a(\xi^{-1} - T^{-1})$.

On the other hand, while $d(T) = 0$ and T commutes with $\xi^{-1} - T^{-1}$, we write, for every $n \in \mathbb{N}$,

$$\begin{aligned}\mathcal{R}((\xi - T)^n) &= \{(\xi - T)^n x : x \in \mathcal{D}(T^n)\} \\ &= \{(\xi^{-1} - T^{-1})^n T^n x : x \in \mathcal{D}(T^n)\} \\ &= \{(\xi^{-1} - T^{-1})^n y : y \in X\} \\ &= \mathcal{R}((\xi^{-1} - T^{-1})^n).\end{aligned}$$

Thus, we deduce that $d(\xi - T) = d(\xi^{-1} - T^{-1})$.

This shows that $\xi - T \in \mathcal{B}_+(X)$ (resp., $\xi - T \in \mathcal{B}_-(X)$) if and only if $\xi^{-1} - T^{-1} \in \mathcal{B}_+(X)$ (resp., $\xi^{-1} - T^{-1} \in \mathcal{B}_-(X)$). Using Theorem 3.6 (resp., Corollary 3.7) and the fact that $\mathcal{Q} := T^{-1} - S^{-1} \in \mathcal{QK}(X)$ with $\operatorname{dist}(\mathcal{Q}, \mathcal{K}(X)) < \delta$, we infer that $\xi - T \in \mathcal{B}_+(X)$ (resp., $\xi - T \in \mathcal{B}_-(X)$) if and only if $\xi^{-1} - S^{-1} \in \mathcal{B}_+(X)$ (resp., $\xi^{-1} - S^{-1} \in \mathcal{B}_-(X)$) if and only if $\xi - S \in \mathcal{B}_+(X)$ (resp., $\xi - S \in \mathcal{B}_-(X)$).

This shows the stability result of the upper (resp., lower) Browder's essential spectrum. Consequently, we obtain

$$\sigma_{\mathcal{B}}(T) = \sigma_{\mathcal{B}_+}(T) \cup \sigma_{\mathcal{B}_-}(T) = \sigma_{\mathcal{B}_+}(S) \cup \sigma_{\mathcal{B}_-}(S) = \sigma_{\mathcal{B}}(S). \qquad \square$$

Remark 3.16. Theorem 3.15 is an extension and an improvement of [6, Theorem 2.3] to Browder's essential spectra involving the notion of quasi-compact operator perturbation.

3.3. **Characterization of Browder's essential spectrum of linear bounded operators.** The goal of this subsection is to show how the concept of quasi-compact perturbations enables us to characterize the Browder's essential spectrum of linear bounded operators. As a first step let us summarize the basic property of Weyl operators via the concept of quasi-compact operators in the following theorem, which was developed by A. Brunel and D. Revuz [4].

Theorem 3.17 ([4, Theorem I.6])**.** *Let $T \in \mathcal{L}(X)$. If $T \in \mathcal{QK}(X)$, then $\lambda - T \in \Phi(X)$, with $i(\lambda - T) = 0$ for every $\lambda \in \mathbb{C}$ such that $|\lambda| \geq 1$.*

As an accurate generalization of the Browder's essential spectrum characterization of V. Rakočević [21] to quasi-compact perturbations is formulated as follows:

Theorem 3.18. *Let $T \in \mathcal{L}(X)$. Then, we get*

$$\sigma_{\mathcal{B}}(T) = \bigcap \left\{ \sigma(T+K), K \in \mathcal{E}(X) \right\},$$

where

$$\mathcal{E}(X) := \{K \in \mathcal{L}(X) : KT = TK,\ K(\mu - T - K)^{-1} \in \mathcal{QK}(X) \text{ and there exists } \varepsilon > 0 : \operatorname{dist}(K(\mu - T - K)^{-1}, \mathcal{K}(X)) < \varepsilon \ \ \forall \mu \in \rho(T+K)\}.$$

Proof. The proof may be done by double inclusions. For the first inclusion, suppose that $\lambda \notin \bigcap \{\sigma(T+K), K \in \mathcal{E}(X)\}$. Thus, there exists $K \in \mathcal{E}(X)$ such that $\lambda \in \rho(T+K)$. Therefore, for such K, we have

$$\lambda - T - K \in \Phi(X), \quad \text{with } \alpha(\lambda - T - K) = \beta(\lambda - T - K).$$

Hence, the operator $\lambda - T$ may be written as follows for such $K \in \mathcal{E}(X)$:

$$\lambda - T = (I + K(\lambda - T - K)^{-1})(\lambda - T - K).$$

Since $(K(\lambda - T - K)^{-1}) \in \mathcal{QK}(X)$, from Theorem 3.17, we obtain

$$I + K(\lambda - T - K)^{-1} \in \Phi(X), \quad \text{with } i(I + K(\lambda - T - K)^{-1}) = 0.$$

Consequently, using [19, Theorem 16.12], we derive that

$$(I + K(\lambda - T - K)^{-1})(\lambda - T - K) \in \Phi(X),$$

with

$$\begin{aligned} i(\lambda - T) &= i((I + K(\lambda - T - K)^{-1})(\lambda - T - K)) \\ &= i(I + K(\lambda - T - K)^{-1}) + i(\lambda - T - K) \\ &= 0. \end{aligned}$$

Therefore, $\lambda - T$ is a Weyl operator on X. To prove it for Browder operator on X, it remains to show that

$$a(\lambda - T) < +\infty \quad \text{and} \quad d(\lambda - T) < +\infty.$$

According to [13, Lemmas 1.1 and 1.4], we have for such $K \in \mathcal{E}(X)$:

$$\begin{aligned}(\lambda - T)^n &= (I + (K(\lambda - T - K)^{-1}))^n(\lambda - T - K)^n \\ &= (\lambda - T - K)^n(I + (K(\lambda - T - K)^{-1}))^n \quad \text{for every } n \in \mathbb{N}.\end{aligned}$$

Then, as $a(\lambda - T - K) = 0$, we infer that

$$\mathcal{N}((\lambda - T)^n) \subseteq \mathcal{N}((I + (K(\lambda - T - K)^{-1}))^n) \quad \text{for } n \in \mathbb{N}.$$

Therefore, in view of [5, Lemma 1], we conclude that

$$\begin{aligned}\alpha((\lambda - T)^n) &\leq \alpha((I + K(\lambda - T - K)^{-1})^n) \\ &\leq a(I + K(\lambda - T - K)^{-1})\alpha(I + K(\lambda - T - K)^{-1}) \quad \text{for every } n \in \mathbb{N}.\end{aligned}$$

One has $K \in \mathcal{E}(X)$, we infer in view of the use of Theorem 3.9 that

$$I + K(\lambda - T - K)^{-1} \in \mathcal{B}(X).$$

Consequently, $a(I + K(\lambda - T - K)^{-1}) < \infty$. Likewise, one has

$$\alpha(I + K(\lambda - T - K)^{-1}) = \beta(I + K(\lambda - T - K)^{-1}) < \infty.$$

Theorem 4.3 in [12] leads to

$$a(I + K(\lambda - T - K)^{-1}) = d(I + K(\lambda - T - K)^{-1}) < \infty.$$

Hence, we get $\lambda - T \in B(X)$.

Obviously, all compact operator K which commutes with T is still an operator of the subset $\mathcal{E}(X)$. For this argument, we may easily observe that

$$\cap\{\sigma(T+K),\ \ K \in \mathcal{E}(X)\} \subset \cap\{\sigma(T+K),\ \ K \in \mathcal{K}(X), \quad \text{with } KT = TK\} = \sigma_{\mathcal{B}}(T).$$

□

Remark 3.19. (i) Theorem 3.18 may be regarded as an improvement and generalization of the results of V. Rakočević [21] to the class of quasi-compact perturbations.

(ii) The previous result covered the characterization of the Weyl spectrum of T. Also, the classes of finite rank, compact, weakly compact and Riesz operators remain as illustrative examples of the previous characterization of the Browder essential spectrum of T.

Conclusion. Recently, the notion of quasi-compact operator perturbation has played a central role in the development of modern spectral theory. Especially, this kind of perturbation is a new branch in the study of the invariance of perturbed linear operators and the characterization problems of linear operators. The main objective of this paper is to exhibit the importance of the use of this kind of notion to outline a new approach in the analysis of the stability problems of upper and lower semi-Fredholm, upper and lower semi-Weyl and upper and lower semi-Browder operators. Therefore, the obtained stability results of semi-Fredholm, semi-Weyl and semi-Browder operators play a significant role in formulation of some new spectral results in terms of their corresponding essential spectra.

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