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THE CORTEX OF A CLASS OF SEMIDIRECT PRODUCT EXPONENTIAL LIE GROUPS

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Abstract. In the present paper, we are concerned with the determination of the cortex of semidirect product exponential Lie groups. More precisely, we consider a finite dimensional real vector space V and some abelian matrix group $H = \exp\left(\sum_{i=1}^{n} \mathbb{R}A_i\right)$, where $\{A_1, \dots, A_n\}$ is a set of pairwise commuting non-singular matrices acting on V. We first investigate the cortex of the action of the group H on V. As an application, we investigate the cortex of the group semidirect product $G := V \rtimes \mathbb{R}^n$.

1. Introduction

1.1. **State of the art.** A. M. Vershik and S. I. Karpushev define in [17] the cortex of a locally compact group G as

$$\operatorname{cor}(G) = \{\pi \in \widehat{G} : \pi \text{ cannot be separated from the identity representation of } G\},$$

where $\widehat{G}$ is the dual of G (set of class of unitary irreducible representations of G), that is, $\pi \in \operatorname{cor}(G)$ if and only if, for any neighborhood V of $\mathbf{1}_G$ (identity representation of G) and for each neighborhood U of π, one has $V \cap U$ is a non-empty set. Note that $\widehat{G}$ is equipped with the topology which can be described in terms of weak containment (see [14]) and which is in general not separated. However, if G is abelian, $\widehat{G}$ is separated, and hence $\operatorname{cor}(G) = \{\mathbf{1}_G\}$.

Suppose now that $G = \exp \mathfrak{g}$ is an exponential Lie group, with Lie algebra $\mathfrak{g}$. Then Kirillov's theory says that $\widehat{G}$ is homeomorphic to the set $\mathfrak{g}^\star/\operatorname{Ad}^\star G$ of coadjoint orbits in the dual $\mathfrak{g}^\star$ of $\mathfrak{g}$, equipped with the quotient topology. Using this identification, we can see the cortex of G as the set of orbits which are not separated to the trivial orbit $\{0\}$. For simplicity, in [5] the authors define the cortex of $\mathfrak{g}^\star$ as the union of these orbits. In other words, the cortex of $\mathfrak{g}^\star$ is the set of points y of $\mathfrak{g}^\star$ which are limit of a sequence $x^{(p)} = \operatorname{Ad}^\star s_p \ell^{(p)}$, where, for each p, s_p

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belongs to $G = \exp \mathfrak{g}$, $\ell^{(p)}$ to $\mathfrak{g}^\star$, and $\lim_p \ell^{(p)} = 0$:

$$\mathrm{Cor}(\mathfrak{g}^\star) = \Big\{ y \in \mathfrak{g}^\star : y = \lim_p \mathrm{Ad}^\star_{s_p} \ell^{(p)},\ \lim_p \ell^{(p)} = 0 \Big\},$$

and we have $\pi_\ell \in \mathrm{cor}(G)$ if and only if $\ell \in \mathrm{Cor}(\mathfrak{g}^\star)$. In this context, the cortex of some Lie algebras has been studied in [4, 10, 11, 12]. In [7] the authors generalize this notion and define the cortex $\mathrm{Cor}(V) = \mathrm{C}_V(G)$ of a representation of a locally compact group G on a finite-dimensional vector space V as the set of all $v \in V$ for which $G.v$ and $\{0\}$ cannot be Hausdorff-separated in the orbit-space V/G. They give a precise description of $\mathrm{C}_V(G)$ in the case $G = \mathbb{R}A$, where A is a real nilpotent matrix acting on V.

In fact, the cortex of V (or $\mathfrak{g}^\star$) is generally not easy to determine and describe, even if $G = V \rtimes H$ is a nilpotent, connected and simply connected Lie group.

Given a set of pairwise commuting matrices $A_1, \ldots, A_n \in \mathbb{R}^{m\times m}$, one has a natural distribution $x \mapsto D(x) = \mathbb{R}\text{-span}\{A_1 x, \ldots, A_n x\}$; under some considerations of regularity (see [3, 2]), $D(x)$ corresponds to the tangent space at x to the submanifold M_x of $\mathbb{R}^m$ given by $M_x = \{e^{t_1 A_1} \cdots e^{t_n A_n} x,\ t_1, \ldots, t_n \in \mathbb{R}\}$. A natural question is to seek the behavior of $(M_x)_x$ when x tends to zero.

For the setting studied here, we consider a class of Lie groups given by the semidirect product of abelian groups $G = V \rtimes_\pi \mathbb{R}^n$ (in [2] G is called an inhomogeneous vector group), where V is an m-dimensional real vector space and π is the continuous representation of the topological additive group $\mathbb{R}^n$ in V given by

$$\pi : \mathbb{R}^n \to GL(V), \quad t = (t_1, \ldots, t_n) \mapsto \pi(t) = e^{\left(\sum_{i=1}^n t_i A_i\right)}, \qquad (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

where e^A denotes the matrix exponential of $A \in \mathbb{R}^{m\times m}$. The representation $\pi^\star$ on the dual $V^\star$ of V derives from π as

$$\pi^\star(t) = (\pi(-t))^T, \quad t \in \mathbb{R}^n,$$

where the superscript T denotes the transpose matrix operator. The orbit under $\pi^\star$ of $\xi \in V^\star$ is given by

$$\mathcal{O}_\xi^{\pi^\star} = \{\pi^\star(t)\xi,\ t \in \mathbb{R}^n\}.$$

In our setting, $\{A_1, \ldots, A_n\}$ is a set of pairwise commuting non-singular matrices in $\mathbb{R}^{m\times m}$. Under some considerations on the eigenvalues of the matrices $(A_i)_{1\le i\le n}$, G turns out to be a solvable exponential Lie group.

On the other hand, if $\mathfrak{g}$ is the Lie algebra of G, then $\mathfrak{g} = V \times \mathfrak{h}$ (with $\mathfrak{h} = \sum_{i=1}^n \mathbb{R}A_i$) and the coadjoint orbit of any $(\xi, \lambda) \in \mathfrak{g}^\star$ is given by $\mathrm{Ad}^\star(G)(\xi, \lambda) = (\exp(\mathfrak{h}^T)\xi) \times (\lambda + \mathfrak{h}_\xi^\perp)$, where $(\lambda + \mathfrak{h}_\xi^\perp)$ is an affine subvariety in $\mathfrak{h}^\star$ (for more details, see [8]); besides these considerations a description of $\mathrm{Cor}(\mathfrak{g}^\star)$ is derived.

1.2. **Structure of the paper.** The paper is organized as follows. In section 2, we give some essential tools which will be useful for the remaining sections, namely the notations and a summary of the results of [8] concerning the structure of commuting matrices. In section 3, we are concerned with the characterization of the cortex of the abelian matrix group $H = \exp\left(\sum_{i=1}^n \mathbb{R}A_i\right)$ ($A_1, \ldots, A_n$ are pairwise commuting non-singular matrices in $\mathbb{R}^{m\times m}$). We first describe explicitly the cortex of the representation $\pi^\star$ on the dual space $V^\star$ of V under some considerations on the spectra of $(A_i)_{1\leq i\leq n}$. We consider the Lie group semidirect product $G = V \rtimes_\pi \mathbb{R}^n$ with Lie algebra $\mathfrak{g} = V \times_{\mathrm{d}\pi} \mathfrak{h}$, where $\mathfrak{h} = \sum_{i=1}^n \mathbb{R}A_i$, and we illustrate the results of [1, 9] to describe the adjoint and coadjoint actions of G on $\mathfrak{g}$ and $\mathfrak{g}^\star$, respectively. As an application of the results of section 3, a description of the cortex of $\mathfrak{g}^\star$ is given.

2. Notations and preliminaries

Let $\mathfrak{h} = \sum_{j=1}^n \mathbb{R}A_j$ be a Lie subalgebra in $\mathfrak{gl}(m,\mathbb{R})$, where $\{A_1,\dots,A_n\}$ is a set of pairwise commuting matrices in $\mathbb{R}^{m\times m}$, and let $H = \exp\mathfrak{h}$ be the corresponding matrix group, where

$$\exp : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}, \quad A \mapsto e^A := \exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is the exponential matrix mapping. Observe that H is solvable, simply connected, but not necessarily closed or exponential or even type 1. Let V be an m-dimensional real vector space; then H acts on V via

$$H \times V \to V, \quad (e^A, v) \mapsto e^A v.$$

Equivalently, we have a continuous finite dimensional representation of the topological group $\mathbb{R}^n$:

$$\pi : \mathbb{R}^n \to GL(V), \quad t = (t_1,\dots,t_n) \mapsto \pi(t) = e^{t\cdot\mathbf{A}},$$

where

$$t\cdot\mathbf{A} := \sum_{j=1}^n t_j A_j, \quad t = (t_1,\dots,t_n) \in \mathbb{R}^n,\ \mathbf{A} = (A_1,\dots,A_n).$$

The orbit of $v \in V$ under π is denoted by $\mathcal{O}_v^\pi$ and is given by

$$\mathcal{O}_v^\pi = \left\{e^{t\cdot\mathbf{A}}v,\ t = (t_1,\dots,t_n) \in \mathbb{R}^n\right\}.$$

The representation π induces a semidirect product group $G = V \rtimes_\pi \mathbb{R}^n$ with law

$$(v,t)(w,s) = (v + \pi(t)w, t+s), \quad t,s \in \mathbb{R}^n,\ v,w \in V.$$

The representation $\pi^\star$ on the dual $V^\star$ of V derives from π as

$$\pi^\star(t) = (\pi(-t))^T, \quad t \in \mathbb{R}^n.$$

The orbit under $\pi^\star$ of $x \in V^\star$ is given by

$$\mathcal{O}_x^{\pi^\star} = \{\pi^\star(t)x,\ t \in \mathbb{R}^n\}.$$

In this paper, we first concentrate on the study of the cortex of the representation $\pi^\star$ on $V^\star$. To this end, recall the following definition.

Definition 2.1 ([7])**.** Let G be a locally compact group and σ be a continuous representation of G on a finite dimensional real vector space W. The cortex of G is defined as

$$\mathrm{C}_W(\sigma) = \Big\{ \lim_{k\to\infty} \sigma(g_k)w^{(k)} : (g_k)_k \subset G,\ (w^{(k)})_k \subset W,\ \text{with } \lim_{k\to\infty} w^{(k)} = 0 \Big\}.$$

Remark 2.2. Let G be a locally compact group and σ be a continuous representation of G on a finite dimensional (real) vector space W. If $\mathcal{U}$ is a dense subset in W, then we can verify that

$$\mathrm{C}_W(\sigma) = \Big\{ \lim_{k\to\infty} \sigma(g_k) w^{(k)} : (g_k)_k \subset G,\ (w^{(k)})_k \subset \mathcal{U} \text{ with } \lim_{k\to\infty} w^{(k)} = 0 \Big\}.$$

Lemma 2.3. *Let σ_1 and σ_2 be the continuous representations on the m-dimensional real vector space W given by*

$$\sigma_i(t) = e^{tM_i}, \quad t \in \mathbb{R},\ i = 1, 2,$$

where $M_1, M_2 \in \mathbb{R}^{m\times m}$. If there exists a non-singular matrix B such that $M_1 = BM_2B^{-1}$, then

$$\mathrm{C}_W(\sigma_1) = B\mathrm{C}_W(\sigma_2).$$

Proof. If $M_1 = BM_2B^{-1}$, then

$$e^{tM_1} = Be^{tM_2}B^{-1} \qquad \text{for all } t \in \mathbb{R}.$$

On the other hand, for any $w \in C_W(\sigma_1)$, there exist $(w^{(k)})_k \subset W$ with $\lim\limits_{k\to\infty} w^{(k)} = 0$ and $(t^{(k)})_k \subset \mathbb{R}$ such that

$$w = \lim_{k\to\infty} e^{t^{(k)}M_1} w^{(k)} = B\Big(\lim_{k\to\infty} e^{t^{(k)}M_2} B^{-1} w^{(k)} \Big) \in B\mathrm{C}_W(\sigma_2),$$

since $\lim_{k\to\infty} B^{-1}v^{(k)} = 0$, and thus $C_W(\sigma_1) \subset B\mathrm{C}_W(\sigma_2)$. The inclusion $\mathrm{C}_W(\sigma_2) \subset B^{-1}\mathrm{C}_W(\sigma_2)$ derives from the rule $M_2 = B^{-1}M_1B$, and therefore

$$\mathrm{C}_W(\sigma_1) = B\mathrm{C}_W(\sigma_2). \qquad \square$$

2.1. **Structure of commuting matrices.** It is well known that, given a set of commuting matrices over the complex numbers, there exists a basis with respect to which all matrices have upper triangular form. Let $\mathcal{N}(m, \mathbb{K})$ denote the subspace of proper upper triangular matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. On the other hand, each complex number a is identified with the 2×2 real matrix

$$\begin{pmatrix} \mathfrak{Re}(a) & -\mathfrak{Im}(a) \\ \mathfrak{Im}(a) & \mathfrak{Re}(a) \end{pmatrix}$$

and hence we can identify $\mathfrak{gl}(m, \mathbb{C})$ with a subspace of $\mathfrak{gl}(2m, \mathbb{R})$. The following structure result will be useful for the study of the cortex of $\pi^\star$.

Theorem 2.4 ([8])**.** *Let $A_1, \dots, A_n \in \mathbb{R}^{m\times m}$ be commuting matrices. Then there exist $B \in GL(m, \mathbb{R})$, $d_s \in \mathbb{N}$, and $\mathbb{K}_s \in \{\mathbb{R}, \mathbb{C}\}$ (for $s = 1, \dots, l$) such that*

$$\sum_{s=1}^{l} d_s \dim_{\mathbb{R}} \mathbb{K}_s = m,$$

and, for $j = 1, \dots, k$,

$$T_j = BA_jB^{-1} = \begin{pmatrix} T_{j,1} & 0 & \dots & 0 \\ 0 & T_{j,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & T_{j,l} \end{pmatrix}$$

with blocks $T_{j,s} \in \mathbb{K}_s \mathbf{1}_{d_s} + \mathcal{N}(d_s, \mathbb{K}_s)$. If the spectra of $A_1, \dots, A_n$ are known, B is explicitly computable by repeated applications of Gaussian elimination. One has $\mathbb{K}_1 = \dots = \mathbb{K}_l = \mathbb{R}$ if and only if spectra$(A_s) \subset \mathbb{R}$ *for all $1 \le s \le l$.*

Fix a basis $(v_1, \dots, v_m)$ in the complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V = V \oplus iV$ (where $i^2 = -1$) so that the matrices $A_1, \dots, A_n$ take the form $T_1, \dots, T_n$, respectively, of Theorem 2.4. Note that there is a natural extension of the representation π of $\mathbb{R}^n$ on $V_{\mathbb{C}}$ and likewise for the representation $\pi^\star$ of $\mathbb{R}^n$ on $V^\star_{\mathbb{C}}$. Alternatively, and from now on, we shall consider the matrices $(T_j)_{1\le j\le n}$ instead of $(A_j)_{1\le j\le n}$ so that the matrix of each $\pi(t)$ is an upper triangular matrix. On the other hand, if $\mathcal{B} = (e_1, \dots, e_m)$ is the dual basis in $V^\star_{\mathbb{C}}$, then with respect to $\mathcal{B}$ the representation $\pi^\star$ acts on $V^\star_{\mathbb{C}}$ by lower triangular non-singular matrices.

A complex form λ is a root for the action of $\mathfrak{h}$ on $V^\star$ if, for each $A \in \mathfrak{h}$, $\lambda(A)$ is an eigenvalue of A. If λ is a root, the corresponding generalized eigenspace for λ is

$$V^\star_\lambda = \bigcap_{A\in\mathfrak{h}} \ker_{\mathbb{C}}(A - \lambda(A)I_m)^m.$$

For any A commuting with $A_1, \dots, A_n$, the space $V^\star_\lambda$ is A-invariant and hence $\pi^\star$-invariant and there is a finite set of linear complex functionals $\mathcal{R} = \{\lambda_1, \dots, \lambda_s\}$ such that

$$F_\lambda \neq \{0\}, \quad \lambda \in \mathcal{R} \qquad \text{and} \qquad V^\star_{\mathbb{C}} = \oplus_{\lambda\in\mathcal{R}} F_\lambda. \tag{2.1}$$

Since $A_1, \dots, A_n \in \mathbb{R}^{m\times m}$, the set $\mathcal{R}$ is invariant under complex conjugation and the mapping $V^\star_{\mathbb{C}} \ni \lambda \mapsto \overline{\lambda}$ (componentwise complex conjugation) induces a bijection $F_\lambda \to \overline{F_\lambda}$; more precisely, one has

$$F_{\overline{\lambda}} = \overline{F_\lambda}, \quad \overline{F_\lambda} = \{\overline{\xi} : \xi \in F_\lambda\}, \qquad \lambda \in \mathcal{R}.$$

It then further follows that there exist real-valued linear functionals $\alpha_j = \mathfrak{Re}(\lambda_j)$, $\beta_j = \mathfrak{Im}(\lambda_j)$ satisfying

$$\lambda_j(A) = \alpha_j(A) + i\beta_j(A), \quad A \in \mathfrak{h},\ j = 1, \dots, s.$$

Denote by Λ_j the character of H defined by

$$\Lambda_j(e^A) = e^{\lambda_j(A)} = e^{\alpha_j(A)}e^{i\beta_j(A)}, \quad A \in \mathfrak{h}.$$

From now on, choose an ordering for the roots such that $\lambda_1,\dots,\lambda_r$ are real and $\lambda_{r+1},\dots,\lambda_s$ are not real. If there are no real roots, then $r=0$. On the other hand, since $\mathcal{R}$ is stable under complex conjugation, $s-r=2p$ is even and the roots $\lambda_{r+1},\dots,\lambda_s$ are pairwise conjugated, that is, one can write

$$\lambda_{r+j}=\overline{\lambda_{r+j-p}},\quad j=p+1,\dots,s.$$

As in [2, 3, 8], we identify $V^\star$ with a real vector subspace in $V^\star_{\mathbb{C}}$, since

$$V^\star_{\mathbb{C}}=\left(\oplus_{j=1}^{r}F_{\lambda_j}\right)\oplus\left(\oplus_{j=r+1}^{p}F_{\lambda_j}\right)\oplus\left(\oplus_{j=p+1}^{s}F_{\lambda_j}\right). \tag{2.2}$$

We choose only one term from each pair $(\lambda,\overline{\lambda})$ in $\mathcal{R}$, and we thereby obtain a subset of $\mathcal{R}$, which we write as $\{\lambda_1,\dots,\lambda_p\}$. The space $V^\star$ is the following real subspace in $V^\star_{\mathbb{C}}$:

$$V^\star=\left(\bigoplus_{j=1}^{r}V^\star_{\lambda_j}\cap V\right)\oplus\left(\bigoplus_{j=r+1}^{p}(V^\star_{\lambda_j}+\overline{V^\star_{\lambda_j}})\cap V^\star\right). \tag{2.3}$$

Therefore, if $k\in\{1,\dots,r\}$, then λ_k is real and we put $W_k=F_{\lambda_k}\cap V^\star$, and if $k\in\{r+1,\dots,p\}$, then we put $W_k=F_{\lambda_k}$; finally, we let

$$W=\bigoplus_{j=1}^{p}W_j. \tag{2.4}$$

On the other hand, according to the decomposition (2.2), each $\xi\in V^\star_{\mathbb{C}}$ is written as $\xi=\sum_{j=1}^{s}\xi^{(j)}$, where $\xi^{(j)}\in F_{\lambda_j}, j=1,\dots,s$. We define the $\mathbb{R}$-linear mapping

$$V^\star\to W,\quad \xi=\sum_{j=1}^{s}\xi^{(j)}\mapsto\xi'=\sum_{j=1}^{p}\xi^{(j)}.$$

This mapping is an isomorphism. With this in place, we have the identification

$$V^\star=\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_r}\times\mathbb{C}^{m_{r+1}}\times\cdots\times\mathbb{C}^{m_p}.$$

Accordingly, we write

$$\xi=\left[\xi^{(1)},\dots,\xi^{(p)}\right]^T=\left[\xi^{(1)}_1,\dots,\xi^{(1)}_{m_1},\xi^{(2)}_1,\dots,\xi^{(2)}_{m_2},\dots,\xi^{(p)}_1,\dots,\xi^{(p)}_{m_p}\right]^T.$$

Fix j, $1\le j\le p$ and according to Theorem 2.4, then if $l_j=\alpha_j$ is real-valued, choose an ordered basis for $\mathbb{R}^{m_j}$ over $\mathbb{R}$ so that, for each $A\in\mathfrak{h}$, the matrix for $A|_{\mathbb{R}^{m_j}}$ is upper triangular with real entries. Otherwise, choose an ordered basis for $\mathbb{C}^{m_j}$ over $\mathbb{C}$ so that the matrix for $A|_{\mathbb{C}^{m_j}}$ is upper triangular with complex entries. Therefore each $A\in\mathfrak{h}$ is identified with an upper triangular matrix consisting of p blocks:

$$A=\begin{bmatrix}A^{(1)} & & & \\ & A^{(2)} & & \\ & & \ddots & \\ & & & A^{(p)}\end{bmatrix} \tag{2.5}$$

so that $A\xi=(A^{(1)}\xi^{(1)},\dots,A^{(p)}\xi^{(p)})^T,\xi\in V^\star$ and each $A^{(j)}$ has the form $l_j(A)\mathrm{Id}+n(A^{(j)})$ with $n(A^{(j)})$ strictly upper triangular. Each $A\in\mathfrak{h}$ has the Jordan–Chevalley decomposition $A=d(A)+n(A)$, where $d(A)$ (respectively, $n(A)$) is the diagonal part of A (respectively, the nilpotent part of A) with $d(A)n(A) = n(A)d(A)$ and hence we can write

$$e^A\xi = e^{d(A)+n(A)}\xi = \left(e^{l_1(A^{(1)})}e^{n(A^{(1)})}\xi^{(1)}, \dots, e^{\lambda_p(A^{(p)})}e^{n(A^{(p)})}\xi^{(p)}\right).$$

Example 2.5. Define an action of $\mathbb{R}^2$ on $V^\star = \mathbb{R}^3$ by

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Here $p = 2$, and the roots are $\lambda_1, \lambda_2, \overline{\lambda_2}$ with

$$\begin{cases} \lambda_1(A_1) = 1 \\ \lambda_1(A_2) = -1 \end{cases} \qquad \begin{cases} \lambda_2(A_1) = 1 + i \\ \lambda_2(A_2) = 2 + i. \end{cases}$$

With this identification, the matrices become

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1+i \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 2+i \end{pmatrix}.$$

3. The cortex of the representation $\pi^\star$

In this section we are concerned with the characterization of the cortex of the action of the abelian matrix group $H = \exp\left(\sum_{i=1}^{n} \mathbb{R}A_i\right)$ on the vector space V. By (2.5) one has

$$A_j = \begin{bmatrix} A_j^{(1)} & & & \\ & A_j^{(2)} & & \\ & & \ddots & \\ & & & A_j^{(p)} \end{bmatrix}, \quad j = 1, \dots, n,$$

where, for each $k = 1, \dots, p$, one has

$$A_j^{(k)} = \lambda_k^{(j)} I_{m_k} + N_j^{(k)}, \tag{3.1}$$

where $N_j^{(k)}$ is a strictly upper triangular matrix and I_{m_k} is the identity matrix in $\mathbb{R}^{m_k \times m_k}$. Therefore

$$e^{t_j A_j} = \begin{bmatrix} e^{t_j A_j^{(1)}} & & & \\ & e^{t_j A_j^{(2)}} & & \\ & & \ddots & \\ & & & e^{t_j A_j^{(p)}} \end{bmatrix}$$

and

$$e^{t_j A_j^{(k)}} = e^{t_j \lambda_k^{(j)}} e^{t_j N_j^{(k)}} = e^{t_j \lambda_k^{(j)}} \sum_{\ell=1}^{m_k - 1} \frac{t_j^\ell}{\ell!} (N_j^{(k)})^\ell.$$

The orbit of $\xi \in V^\star$ (under $\pi^\star$) is given by

$$\mathcal{O}_\xi = \left\{e^{t_1 A_1^T + \cdots + t_n A_n^T}\xi, \ t = (t_1, \dots, t_n) \in \mathbb{R}^n\right\}.$$

For $x \in \mathcal{O}_\xi$, we can write $x = \left(x^{(1)}, \dots, x^{(p)}\right)$, where, for $k = 1, \dots, p$ we have

$$\begin{cases} x_1^{(k)} = e^{\sum_{j=1}^{n} t_j \lambda_k^{(j)}} \xi_1^{(k)} \\ x_2^{(k)} = e^{\sum_{j=1}^{n} t_j \lambda_k^{(j)}} \left(a_{2,1}^{(k)}(t)\xi_2^{(k)} + \xi_1^{(k)}\right) \\ \qquad \vdots \\ x_{m_k}^{(k)} = e^{\sum_{j=1}^{n} t_j \lambda_k^{(j)}} \left(\displaystyle\sum_{i=1}^{m_k - 1} a_{m_k,i}^{(k)}(t)\xi_i^{(k)} + \xi_{m_k}^{(k)}\right), \end{cases}$$

where $a_{j,i}^{k}(t)$ are complex-valued polynomials in the variables $t_1, \dots, t_n$. For ease of notation, we write

$$L_k(t) = \sum_{j=1}^{n} \lambda_k^{(j)} t_j, \quad t = (t_1, \dots, t_n) \in \mathbb{R}^n.$$

Recall that some of the roots $(\lambda_k)_{1 \le k \le p}$ are real while others are complex. Put

$$\lambda_k^{(j)} = \lambda_k(A_j^{(k)}) = \alpha_k^{(j)} + i\beta_k^{(j)}, \quad j = 1, \dots, n, \ k = 1, \dots, p,$$

with

$$\alpha_k^{(j)} = \mathfrak{Re}(\lambda_k(A_j)), \quad \beta_k^{(j)} = \mathfrak{Im}(\lambda_k(A_j)).$$

Then, each complex-valued functional L_k can be written as

$$L_k(t) = \mathfrak{Re}(L_k(t)) + i\mathfrak{Im}(L_k(t) = \sum_{j=1}^{n} \alpha_k^{(j)} t_j + i \sum_{j=1}^{n} \beta_k^{(j)} t_j.$$

With these notations in place, we can write

$$\begin{cases} x_1^{(k)} = e^{\mathfrak{Re}(L_k(t))} e^{i\mathfrak{Im}(L_k(t))} \xi_1^{(k)}, \\ x_2^{(k)} = e^{\mathfrak{Re}(L_k(t))} e^{i\mathfrak{Im}(L_k(t))} \left(a_{2,1}^{(k)}(t)\xi_1^{(k)} + \xi_2^{(k)}\right) \\ \qquad \vdots \\ x_{m_k}^{(k)} = e^{\mathfrak{Re}(L_k(t))} e^{i\mathfrak{Im}(L_k(t))} \left(\displaystyle\sum_{i=1}^{m_k - 1} a_{m_k,i}^{(k)}(t)\xi_i^{(k)} + \xi_{m_k}^{(k)}\right). \end{cases} \tag{3.2}$$

From now on, we suppose that ξ lies in the open dense subset $\Omega \subset V^\star$, defined as

$$\Omega = \left\{ \xi = (\xi_1, \dots, \xi_m) \in V^\star : \prod_{i=1}^{m} \xi_i \neq 0 \right\}.$$

Let $1 \le k \le p$ and assume that

$$\mathfrak{Re}(\lambda_j^{(k)}) = \alpha_k^{(j)} \neq 0 \quad \text{for any } j = 1, \dots, n.$$

Our goal is to seek the limits of each $x_i^{(k)}$, $i = 1, \dots, m_k$, $k = 1, \dots, p$, of (3.2) when ξ tends to zero and $\|t\|$ is non-bounded. To this end, note that

$$|x_i^{(k)}|^2 = x_i^{(k)}\overline{x_i^{(k)}} = e^{2\mathfrak{Re}(L_k(t))}\left|\sum_{i=1}^{m_k-1}\left(a_{m_k,i}^{(k)}(t)\xi_i^{(k)} + \xi_i^{(k)}\right)\right|^2, \quad i = 1, \dots, m_k.$$

Then if we denote

$$f_{i,k}(t) = \left|\sum_{i=1}^{m_k-1}\left(a_{m_k,i}^{(k)}(t)\xi_i^{(k)} + \xi_i^{(k)}\right)\right|^2, \quad i = 1, \dots, m_p,$$

these functions are real-valued polynomials, and hence, if $\|t\|$ is bounded, we get

$$\lim_{v\to 0,\, \|t\|<\infty} x_i^{(k)} = 0, \quad k = 1, \dots, p,\ i = 1, \dots, m_k.$$

Thus if we are seeking non-trivial solutions of the cortex of π, we necessarily have to consider all limits when $v \to 0$ and $\|t\| \to \infty$. Now, since the functions $f_{i,k}$ are real-valued polynomials and $\mathfrak{Re}(L_k)$ is a real-valued functional in the same variable t, we have

$$\lim_{\mathfrak{Re}(L_k(t))\to-\infty} e^{2\mathfrak{Re}(L_k(t))} f_{i,k}(t) = 0, \qquad \lim_{\mathfrak{Re}(L_k(t))\to\infty} e^{2\mathfrak{Re}(L_k(t))} f_{i,k}(t) = \infty.$$

With this in mind, let $(w_1^{(k)}, \dots, w_{m_k}^{(k)}) \in W_k$ (see (2.4)) with $\prod_{i=1}^{m_k} w_i^{(k)} \neq 0$. The first equation of (3.2) gives

$$x_1^{(k)} = e^{L_k(t)}\xi_1^{(k)} = e^{\mathfrak{Re}(L_k(t))}|\xi_1^{(k)}|e^{i\mathfrak{Im}(L_k(t))}e^{i\arg(\xi_1^{(k)})}.$$

By assumption, $\xi_1^{(k)}$ converges to zero (with $|\xi_1^{(k)}| \neq 0$, $k = 1, 2, \dots$), thus we can choose $(t^{(j)})_j \in \mathbb{R}^n$ such that

$$\lim_{j\to\infty} \mathfrak{Re}(L_k(t^{(j)})) = \ln\left(\frac{|w_1^{(k)}|}{|\xi_1^{(k)}|}\right).$$

On the other hand, $(\xi_1^{(k)})$ can be chosen such that

$$\arg(\xi_1^{(k)}) + \mathfrak{Im}(L_k(t^{(j)})) = \arg(w_1^{(k)}) \mod 2\pi.$$

Finally, we get

$$\lim_{\xi_1^{(k)}\to 0} x_1^{(k)} = w_1^{(k)}.$$

Now we focus on the remaining coordinates $w_j^{(k)}$ of $w^{(k)}$ for $j = 2, \dots, m_k$. Recall that

$$x_j^{(k)} = e^{L_k(t)}\left(\sum_{i=1}^{j-1} a_{j,i}^{(k)}(t)\xi_i^{(k)} + \xi_j^{(k)}\right).$$

Hence, we choose

$$\xi_i^{(k)} = \frac{1}{1 + \|t\|^{n_i^{(k)}}}, \quad i = 1, \dots, j-1,$$

where $n_i^{(k)}$ is large enough such that

$$\lim_{\|t\|\to\infty} \xi_1^{(k)} = \cdots = \lim_{\|t\|\to\infty} \xi_i^{(k)} = 0, \quad i = 1, \dots, j,$$

and

$$\xi_j^{(k)} \equiv \xi_1^{(k)} \left(\frac{w_j^{(k)}}{w_1^{(k)}} - \sum_{i=1}^{j-1} a_{j,i}^{(k)}(t) \frac{\xi_i^{(k)}}{\xi_1^{(k)}} \right).$$

Thus $\lim_{\|t\|\to\infty} x_j^{(k)} = w_j^{(k)}$, and if $\pi^{\star(k)}$ denotes the restriction of the representation $\pi^\star$ on W_k, one concludes that

$$C_{W_k}(\pi^{\star(k)}) = W_k.$$

Hence one has the following result.

Proposition 3.1. *Let $\pi^{\star(k)}$ be the subrepresentation of $\pi^\star$ in W_k. If $\mathfrak{Re}(L_k)$ is non-zero, then*

$$C_{W_k}(\pi^{\star(k)}) = W_k.$$

Now we consider all the blocks of $\pi^\star$; then, for $x \in \mathcal{O}_\xi^{\pi^\star}$, we can write

$$\begin{cases} \begin{cases} x_1^{(1)} = e^{\mathfrak{Re}(L_1(t))} e^{i\mathfrak{Im}(L_1(t))} \xi_1^{(1)} \\ x_2^{(1)} = e^{\mathfrak{Re}(L_1(t))} e^{i\mathfrak{Im}(L_1(t))} \left(a_{2,1}^{(1)}(t)\xi_1^{(1)} + \xi_2^{(1)} \right) \\ \quad \vdots \\ x_{m_1}^{(1)} = e^{\mathfrak{Re}(L_1(t))} e^{i\mathfrak{Im}(L_1(t))} \left(\sum_{i=1}^{m_1-1} a_{m_1,i}^{(1)}(t)\xi_i^{(1)} + \xi_{m_1}^{(1)} \right) \end{cases} \\ \qquad\qquad \vdots \\ \begin{cases} x_1^{(p)} = e^{\mathfrak{Re}(L_p(t))} e^{i\mathfrak{Im}(L_p(t))} \xi_1^{(p)} \\ x_2^{(p)} = e^{\mathfrak{Re}(L_p(t))} e^{i\mathfrak{Im}(L_p(t))} \left(a_{2,1}^{(p)}(t)\xi_1^{(p)} + \xi_2^{(p)} \right) \\ \quad \vdots \\ x_{m_p}^{(p)} = e^{\mathfrak{Re}(L_p(t))} e^{i\mathfrak{Im}(L_p(t))} \left(\sum_{i=1}^{m_p-1} a_{m_p,i}^{(p)}(t)\xi_i^{(p)} + \xi_{m_p}^{(p)} \right). \end{cases} \end{cases}$$

Let's assume that

$$\mathfrak{Re}(\lambda_j^{(k)}) \text{ is non-zero for all } k = 1, \dots, p \text{ and } j = 1, \dots, n.$$

We see that the real-valued functionals $(\mathfrak{Re}(L_k(t))_{1\le k\le p}$ may have different sign when $\|t\|$ is large enough, and hence accordingly to what has been established for the case of one block, we shall consider the following system of inequalities:

$$\begin{cases} \mathfrak{Re}(L_1(t)) := \sum_{j=1}^{n} t_j\big(\alpha_1^{(j)}\big) > 0, \\ \mathfrak{Re}(L_2(t)) := \sum_{j=1}^{n} t_j\big(\alpha_2^{(j)}\big) > 0, \\ \qquad\vdots \\ \mathfrak{Re}(L_p(t)) := \sum_{j=1}^{n} t_j\big(\alpha_p^{(j)}\big) > 0. \end{cases} \tag{3.3}$$

Let

$$q := \operatorname{rank}(\mathfrak{Re}(L_1), \ldots, \mathfrak{Re}(L_p)) \leq \min(p, n).$$

Without loss of generality, we may assume that the functionals $\mathfrak{Re}(L_1), \ldots, \mathfrak{Re}(L_q)$ are linearly independent. Let

$$u_1 = \mathfrak{Re}(L_1), \ \ldots, \ u_q = \mathfrak{Re}(L_q);$$

for each $i = p+1, \ldots, q$, there exists $(\gamma_{i,j})_{i,j} \subset \mathbb{R}$ such that

$$\mathfrak{Re}(L_i) = \sum_{j=1}^{q} \gamma_{i,j} u_j.$$

Equivalently, the system (3.3) becomes

$$\begin{cases} u_1 > 0, \ldots, u_q > 0, \\ \gamma_{q+1,1} u_1 + \cdots + \gamma_{q+1,q} u_q > 0, \\ \qquad\vdots \\ \gamma_{p,1} u_1 + \cdots + \gamma_{p,q} u_q > 0. \end{cases} \tag{3.4}$$

Case 1: The system (3.4) *is consistent.* In this situation, there exists $(u_1^0 = \mathfrak{Re}(L_1(t^0)), \ldots, u_q^0 = \mathfrak{Re}(L_q(t^0))) \in (0, \infty)^q$ such that

$$\mathfrak{Re}(L_{q+1}(t^0)) > 0, \ \ldots, \ \mathfrak{Re}(L_p(t^0)) > 0.$$

Using this together with Proposition 3.1, we conclude that

$$C_{V^\star}(\pi^\star) = V^\star.$$

Note that if $\mathfrak{Re}(L_1), \ldots, \mathfrak{Re}(L_p)$ are linearly independent, that is, if

$$\operatorname{rank}(\mathfrak{Re}(L_1), \ldots, \mathfrak{Re}(L_p)) = p,$$

then $C_{V^\star}(\pi^\star) = V^\star$.

Case 2: The system (3.4) *is inconsistent.* Let $(F_i)_{1\le i\le p}$ be the functionals on $\mathbb{R}^q$ defined by

$$F_i(u_1,\dots,u_q)=\begin{cases}u_i & \text{if } 1\le i\le q,\\ \sum_{j=1}^{q}\gamma_{i,j}u_j & \text{if } q+1\le i\le p.\end{cases}$$

Each functional F_i $(i=1,\dots,p)$ involves a partition of $\mathbb{R}^q$ into three non-empty disjoint components,

$$\mathbb{R}^q=\ker F_i\sqcup C_i^+\sqcup C_i^-,\quad i=1,\dots,p,$$

where

- $\ker F_i=\{u=(u_1,\dots,u_q)\in\mathbb{R}^q : F_i(u)=0\}$,
- $C_i^+=\{u=(u_1,\dots,u_q)\in\mathbb{R}^q : F_i(u)>0\}$,
- $C_i^-=\{u=(u_1,\dots,u_q)\in\mathbb{R}^q : F_i(u)<0\}$.

Thus, it yields a finite partition of $\mathbb{R}^q\setminus\bigcup_{i=1}^{p}\ker F_i$:

$$\mathbb{R}^q\setminus\bigcup_{i=1}^{p}\ker F_i=\bigsqcup_{j=1}^{N}C_j, \tag{3.5}$$

where each C_j $(j=1,\dots,N)$ is a non-empty open cone in $\mathbb{R}^q$ such that

$$C_j=\left(\bigcap_{i\in I_j^+}C_i^+\right)\cap\left(\bigcap_{i\in I_j^-}C_i^-\right),$$

with I_j^+ and I_j^- non-empty disjoint subsets in $\{1,\dots,p\}$ satisfying

$$\{1,\dots,p\}=I_j^+\cup I_j^-,\quad j=1,\dots,N,\ I_j^-\ne\emptyset,\ I_j^+\ne\emptyset.$$

According to Proposition 3.1 and Case 1, we conclude that

$$C_{V^\star}(\pi^\star)\equiv\bigcup_{j=1}^{N}\mathbb{R}^{|I_j^+|}\times\{0_{|I_j^-|}\}.$$

Thus, we obtain the following theorem.

Theorem 3.2. *Let π be the representation of $\mathbb{R}^n$ in V, and let $\pi^\star$ be its contragredient representation on $V^\star$. Suppose that the real part of each eigenvalue of each matrix A_j $(j=1,\dots,n)$ is non-zero. Then the cortex of $\pi^\star$ is either $V^\star$ or a union of proper non-trivial subspaces in $V^\star$.*

We now deduce the following.

Corollary 3.3. *The interior of the cortex of the representation $\pi^\star$ is either $V^\star$ or empty.*

Example 3.4. We consider the action of $\mathbb{R}^2=\exp(\mathbb{R}A_1+\mathbb{R}A_2)$ on $V^\star=\mathbb{R}^5$, where

$$A_1=\operatorname{diag}(1,0,-1,0,-1),\quad A_1=\operatorname{diag}(0,1,0,-1,-1).$$

Therefore the system (3.4) becomes

$$\begin{cases} F_1(u) = u_1 > 0,\ F_2(u) = u_2 > 0, \\ F_3(u) = -u_1 > 0,\ F_4(u) = -u_2 > 0, \\ F_5(u) = -u_1 - u_2 > 0. \end{cases}$$

The cones $(C_j)_{1\leq j\leq 6}$ of the partition (3.5) are as follows:

$$\begin{aligned}
C_1 &= \big\{u = (u_1, u_2) \in \mathbb{R}^2 : F_1(u) > 0,\, F_2(u) > 0,\, F_3(u) < 0,\, F_4(u) < 0,\, F_5(u) < 0\big\},\\
C_2 &= \big\{u = (u_1, u_2) \in \mathbb{R}^2 : F_1(u) < 0,\, F_2(u) > 0,\, F_3(u) > 0,\, F_4(u) < 0,\, F_5(u) < 0\big\},\\
C_3 &= \big\{u = (u_1, u_2) \in \mathbb{R}^2 : F_1(u) < 0,\, F_2(u) > 0,\, F_3(u) > 0,\, F_4(u) < 0,\, F_5(u) > 0\big\},\\
C_4 &= \big\{u = (u_1, u_2) \in \mathbb{R}^2 : F_1(u) < 0,\, F_2(u) < 0,\, F_3(u) > 0,\, F_4(u) > 0,\, F_5(u) > 0\big\},\\
C_5 &= \big\{u = (u_1, u_2) \in \mathbb{R}^2 : F_1(u) > 0,\, F_2(u) < 0,\, F_3(u) > 0,\, F_4(u) > 0,\, F_5(u) < 0\big\},\\
C_6 &= \big\{u = (u_1, u_2) \in \mathbb{R}^2 : F_1(u) > 0,\, F_2(u) < 0,\, F_3(u) < 0,\, F_4(u) > 0,\, F_5(u) > 0\big\}.
\end{aligned}$$

Accordingly, we get

$$\begin{aligned} C_{V^\star}(\pi^\star) = (\mathbb{R}^2 \times \{0_{\mathbb{R}^3}\}) &\cup (\{0\} \times \mathbb{R}^2 \times \{0\} \times \mathbb{R}) \cup (0_{\mathbb{R}^2} \times \mathbb{R}^3) \\ &\cup (\mathbb{R} \times \{0\} \times \mathbb{R}^2 \times \{0\}) \cup (\mathbb{R} \times \{0_{\mathbb{R}^2}\} \times \mathbb{R}^2). \end{aligned}$$

From Proposition 3.1 and Theorem 3.2, we deduce the following.

Corollary 3.5. *Let $\{A_1, \dots, A_n\}$ be a set of pairwise commuting real non-singular matrices, and let $d(A_1), \dots, d(A_n)$ be the corresponding semisimple part in the Jordan–Chevalley decomposition of the matrices $A_1, \dots, A_n$, respectively. Let π and δ denote the representation of $\mathbb{R}^n$ given by*

$$\pi(t) = e^{t\cdot\mathbf{A}}, \quad \delta(t) = e^{td(\mathbf{A})}, \qquad t \in \mathbb{R}^n,\ d(\mathbf{A}) = (d(A_1), \dots, d(A_n)).$$

If the real part of each eigenvalue of any matrix A_j (for $j = 1, \dots, n$) is non-zero, then

$$C_{V^\star}(\pi^\star) = C_{V^\star}(\delta^\star).$$

Proposition 3.6. *Let π be the representation corresponding to the set of pairwise commuting real matrices $\{A_1, \dots, A_n\}$, and let π^0 be a subrepresentation of π associated to a non-empty subset $(A_i)_{i\in I_0}$, where $I_0 \subset \{1, \dots, n\}$. If $C_{V^\star}((\pi^0)^\star) = V^\star$, then $C_{V^\star}(\pi^\star) = V^\star$.*

Proof. This is due to the fact that $\mathcal{O}_\xi^{(\pi^0)^\star} \subset \mathcal{O}_\xi^{\pi^\star}$ for any $\xi \in V^\star$. □

Now combining Proposition 3.6 and Corollary 3.5 we get the following.

Corollary 3.7. *Let π be the representation of $\mathbb{R}^n$ in V defined as above. Assume that, for some $j = 1, \ldots, n$, one has*

$$\mathfrak{Re}(\lambda_j^{(k)}) > 0 \quad \textit{for all } k = 1, \ldots, p$$

or

$$\mathfrak{Re}(\lambda_j^{(k)}) < 0 \quad \textit{for all } k = 1, \ldots, p.$$

Then

$$C_{V^\star}(\pi^\star) = V^\star.$$

4. The cortex of the semidirect product $G = V \rtimes_\pi \mathbb{R}^n$

Recall that one has the identification of $\mathbb{R}^n$ with the abelian matrix group $H = \exp\left(\sum_{i=1}^n \mathbb{R}A_i\right)$, where $(A_i)_{1\leq i\leq n}$ is a set of pairwise commuting real matrices in $\mathbb{R}^{m\times m}$ fulfilling the conditions of Theorem 3.2. We use the results of section 3 to give a description of the cortex of a class of semidirect product of exponential Lie groups/algebras.

4.1. **Semidirect product of vector groups.** Here we recall some of the results of [1, 9, 16]. Let $G = V \rtimes_\pi \mathbb{R}^n$ be the group endowed with the law

$$(v,t)(w,s) = (v + \pi(t)w, t+s) = (v + e^{t\cdot\mathbf{A}}w, t+s), \quad v, w \in V, \quad t, s \in \mathbb{R}^n,$$

where

$$t\cdot\mathbf{A} = \sum_{i=1}^n t_i A_i, \quad t = (t_1, \ldots, t_n) \in \mathbb{R}^n, \quad \mathbf{A} = (A_1, \ldots, A_n).$$

In [2] the group G is called the semidirect product of the vector groups V and $\mathbb{R}^n$. The Lie algebra of G is $\mathfrak{g} = V \times \mathfrak{h}$ and is equipped with the Lie bracket

$$[(v, t\cdot\mathbf{A}), (w, s\cdot\mathbf{A})] = ((t\cdot\mathbf{A})w - (s\cdot\mathbf{A})v, 0),$$

where $v, w \in V$, $t, s \in \mathbb{R}^n$, $\mathbf{A} = (A_1, \ldots, A_n)$.

Since $\mathfrak{g} = V \times_{\mathrm{d}\pi} \mathfrak{h}$, $\mathrm{ad}_v := \mathrm{ad}_{(v,0)}$ and $\mathrm{ad}_{t\cdot\mathbf{A}} := \mathrm{ad}_{(0,t\cdot\mathbf{A})}$ can be written in 2×2 matrix form:

$$\mathrm{ad}_v = \begin{pmatrix} 0 & N_v \\ 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_{t\cdot\mathbf{A}} = \begin{pmatrix} t\cdot\mathbf{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $N_v : \mathfrak{h} \to V$ is the linear mapping (which we identify with its matrix) given by $N_v(s\cdot\mathbf{A}) = -(s\cdot\mathbf{A})v$. Since $\mathrm{ad}_v^2 = 0$,

$$\mathrm{Ad}_{(v,t)} = \begin{pmatrix} e^{t\cdot\mathbf{A}} & N_v \\ 0 & I_n \end{pmatrix}.$$

Similarly, if $\mathfrak{g}^\star$ denotes the dual space of $\mathfrak{g}$, then $\mathfrak{g}^\star = V^\star \times \mathfrak{h}^\star$, and the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^\star$ is given by

$$\mathrm{ad}^\star_{(v,t\cdot\mathbf{A})} = \begin{pmatrix} \xi \\ \lambda \end{pmatrix} = \begin{pmatrix} -(t\cdot\mathbf{A})^T & 0 \\ -N_v^T & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \lambda \end{pmatrix}.$$

We next turn to the coadjoint action of G on $\mathfrak{g}^\star$. We get

$$\mathrm{Ad}^\star_{(v,t)}\begin{pmatrix}\xi\\ \lambda\end{pmatrix}=\begin{pmatrix}(e^{-t\cdot\mathbf{A}})^T & 0\\ -N_v^T & I_n\end{pmatrix}\begin{pmatrix}\xi\\ \lambda\end{pmatrix}.$$

From these formulae, we derive that $\mathrm{spec}(\mathrm{ad}_{(v,t\cdot\mathbf{A})})\subset\{0\}\cup\mathrm{spec}(t\cdot\mathbf{A})$ (see [9]). For instance, we can choose the matrices $(A_j)_{1\le j\le n}$ so that, for each $j=1,\dots,n$, one has $\mathrm{spec}(A_j)\subset\mathbb{C}\setminus i\mathbb{R}$ ($i^2=-1$); thus $\mathfrak{g}=V\times_{\mathrm{d}\pi}\mathfrak{h}$ is a solvable exponential Lie algebra (see [6]).

4.2. **Coadjoint orbits.** Recall that $\mathfrak{g}=V\times_{\mathrm{d}\pi}\mathfrak{h}$, and, for $\xi\in V^\star$, let

$$\mathfrak{h}_\xi=\left\{\mathfrak{h}\ni A=\sum_{i=1}^n\mathbb{R}A_i:A^T\xi=0\right\}:=\ker\left[A\mapsto A^T\xi\right]$$

and

$$\mathfrak{h}_\xi^\perp=\{\lambda\in\mathfrak{h}^\star:\langle\lambda,\mathfrak{h}_\xi\rangle=0\}.$$

By [9, Lemma 15], one has

$$\mathrm{Ad}^\star(G)(\xi,\lambda)=\mathrm{Ad}^\star(H)\xi\times(\lambda+\mathfrak{h}_\xi^\perp),\tag{4.1}$$

where

$$H=\left\{e^{\sum_{i=1}^n t_iA_i},\ t_1,\dots,t_n\in\mathbb{R}\right\}.$$

4.3. **The cortex of $\mathfrak{g}^\star$.** The Lie group $G=V\rtimes_\pi\mathbb{R}^n$ and hence the Lie algebra $\mathfrak{g}$, under the considerations of Theorem 3.2, turn out to be exponential [6]. Thus $\widehat{G}$ is homeomorphic to the coadjoint orbit space of G, and there exists a canonical bijection $\kappa:\mathfrak{g}^\star/\mathrm{Ad}^\star(G)\to\widehat{G}$, the Kirillov–Bernat correspondence. Furthermore, this bijection is a homeomorphism, when we endow the orbit space with the quotient topology and $\widehat{G}$ with the Fell–Jacobson topology (see [15] for details). Therefore one has that $\sigma_{(\xi,\lambda)}$ is the cortex of G if and only if $(\xi,\lambda)\in\mathrm{Cor}(\mathfrak{g}^\star)$, where

$$\mathrm{Cor}(\mathfrak{g}^\star)=\left\{\lim_{\|(\xi,\lambda)\|\to0}\mathrm{Ad}^\star_{(v,t)}(\xi,\lambda),\ (v,t)\in G\right\}.$$

Consequently to the rule (4.1) we obtain the following.

Theorem 4.1. *Let G be the semidirect exponential Lie group $G=V\rtimes_\pi\mathbb{R}^n$ with Lie algebra $\mathfrak{g}=V\times_{\mathrm{d}\pi}\mathfrak{h}$.*

(a) *The cortex of the dual $\mathfrak{g}^\star$ of $\mathfrak{g}$ satisfies*

$$\mathrm{Cor}(\mathfrak{g}^\star)\subset\mathrm{C}_{V^\star}(\pi^\star)\times\mathfrak{h}_0^\perp,$$

where

$$\mathfrak{h}_0^\perp=\left\{\lambda:=\lim_{\xi\to0}\lambda_\xi,\ \lambda_\xi\in\mathfrak{h}_\xi^\perp,\ \xi\in V^\star\right\}.$$

(b) *If pr_1 is the projection given by*

$$\mathrm{pr}_1:\mathfrak{g}^\star\to V^\star,\quad(\xi,\lambda)\mapsto\xi,$$

then

$$\mathrm{pr}_1(\mathrm{Cor}(\mathfrak{g}^\star))=\mathrm{C}_{V^\star}(\pi^\star).$$

Remark 4.2. (i) Note that, for each $\xi \in V^\star$, $\mathfrak{h}_\xi$ (respectively, $\mathfrak{h}_\xi^\perp$) is a vector subspace in $\mathfrak{h}$ (respectively, $\mathfrak{h}^\star$).

(ii) For any $\xi \in V^\star$ and $a \in \mathbb{R} \setminus \{0\}$, one has

$$\mathfrak{h}_{a\xi} = \mathfrak{h}_\xi, \quad \mathfrak{h}_{a\xi}^\perp = \mathfrak{h}_\xi^\perp.$$

(iii) It is shown in [13] that

$$\mathfrak{h}_0^\perp = \overline{\bigcup_{\xi \in \mathcal{U}} \mathfrak{h}_\xi^\perp},$$

where $\mathcal{U}$ is the Zariski open layer of the generic H-orbits in $V^\star$.

Finally, let $(\lambda_j)_{1\le j\le p}$ be the set of roots of $\mathfrak{h} = \sum_{i=1}^n \mathbb{R}A_i$ corresponding to the decomposition (2.3). We give the following theorem.

Theorem 4.3. *Let π be the representation of $\mathbb{R}^n \equiv \exp\left(\sum_{i=1}^n \mathbb{R}A_i\right)$ in V and let G be the semidirect product $G = V \rtimes_\pi \mathbb{R}^n$ with Lie algebra $\mathfrak{g} = V \times \mathfrak{h}$. Let $(\lambda_j)_{1\le j\le n}$ be a set of roots of $\mathfrak{h} = \sum_{i=1}^n \mathbb{R}A_i$ given in (2.1). If $\bigcap_{j=1}^p \ker \lambda_j = \{0\}$, then*

$$\mathfrak{h}_0^\perp = \mathfrak{h}^\star.$$

Proof. Let $\xi = (\xi_1, \dots, \xi_n) = (\xi^{(1)}, \dots, \xi^{(p)}) \in V^\star$ with $\prod_{k=1}^p \xi_1^{(k)} \neq 0$, and let $A \in \mathfrak{h}$ be such that $A^T\xi = 0$. By (3.1) one obtains

$$\lambda_1(A)\xi_1^{(1)} = \cdots = \lambda_p(A)\xi_1^{(p)} = 0,$$

that is, $A \in \bigcap_{j=1}^p \ker \lambda_j = \{0\}$. Therefore, for any generic $\xi \in V^\star$, one has $\mathfrak{h}_\xi = 0$ and $\mathfrak{h}_0^\perp = \mathfrak{h}^\star$. □

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