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STABLE QUASI-PERIODIC ORBITS OF A CLASS OF QUINTIC DUFFING SYSTEMS

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ABSTRACT. For a Duffing-type oscillator with constant damping, a unique odd nonlinearity, and time-dependent coefficients which are quasi-periodic, we prove existence and stability conditions of quasi-periodic solutions. We thus generalize some results for periodic coefficients and quintic nonlinearity. We use the classical theory of perturbations and present some numerical examples for the quintic case to illustrate our findings.

1. INTRODUCTION

Traditionally a Duffing oscillator reads as follows:

$$x'' + cx' + a_0 x + e_0 x^3 = \lambda \cos \omega t,$$

where the cubic nonlinearity yields interesting properties such as existence and stability of periodic solutions under certain conditions. Parameter c is the damping constant, λ is the external excitation amplitude with frequency ω. For more general time-dependent coefficients $a(t), e(t)$ there is no general exact solving method, except for a few cases (see, for instance, [8] and [1]). Nevertheless, time-dependent periodic coefficients appear in many important problems; for instance, in the damped externally excited sinusoidal pendulum with oscillating support,

$$x'' + cx' + \omega_0(1 + \varepsilon h''(\omega_1 t))x + e_0 x^3 = \lambda \cos \omega_2 t,$$

where cubic nonlinearity appears in the approximation of the sine function, here $h(t) = h(t + 2\pi)$ (see, for instance, [10]). In [11] (see also [3]), averaging methods are used to show existence and local stability of some generalizations for periodic time-dependent coefficients,

$$x'' + cx' + a(t)x + e(t)x^3 = \lambda h(t). \tag{1.1}$$

Our aim is to generalize the results on the existence and stability of oscillating solutions for equations (1.1) to equations of the form

$$x'' + cx' + a(t)x + e(t)x^{2k+1} = \lambda h(t), \qquad k = 1, 2, 3, \ldots,$$

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where the parameters are time-dependent continuous *almost periodic* functions. These results are contained in Theorems 2.1 and 3.1. Their proofs use averaging techniques which remain valid in the quasi-periodic case. Some other recent approaches to the almost periodic forcing also appear in the literature (see, for instance, [16]). We use the classical averaging method approach. This general approach to forced oscillations, which goes from the periodic forcing towards the quasi-periodic and almost periodic case, is well known as the Krylov–Bogoliubov–Mitropolski perturbation method. Nevertheless, historically, there has been some misunderstanding of the basic techniques; see, for instance, historical remarks in [13], as well as further discussion and subtle differences between quasi-periodic and almost periodic oscillations in [12]. The formalization of the results can be consulted, for instance, in [10] and [13]. For the readers' convenience, these techniques are briefly summarized in the Appendix.

2. Duffing equation with quintic nonlinearity

We generalize the results in [11], from a cubic to a quintic Duffing equation without cubic nonlinearity,

$$\begin{aligned} x' &= y \\ y' &= -cy - a(t)x - e(t)x^5 + \lambda h(t), \end{aligned} \tag{2.1}$$

with $e \neq 0$. We also consider a generalization of the coefficients in the sense that we will consider *almost periodic (a.p.)* functions $a(t)$, $e(t)$, $h(t)$, instead of periodic functions. See the Appendix for the proper definitions.

By a change of coordinates $y = \varepsilon^{3/2}Y$, $x = \varepsilon^{1/2}X$, $\lambda = \varepsilon^{5/2}\Lambda$, $c = \varepsilon C$, $a = \varepsilon^2 A$, we get

$$\begin{aligned} X' &= \varepsilon Y, \\ Y' &= \varepsilon \left(-Cy - AX - eX^5 + \Lambda h(X)\right). \end{aligned}$$

Notice that this change of coordinates is different from the ones proposed in [3], in the sense that we allow fractional exponents in ε.

The averaged system then becomes

$$\begin{aligned} \overline{x}' &= \varepsilon \overline{y}, \\ \overline{y}' &= \varepsilon \left(-C\overline{y} - A_0\overline{x} - e_0\overline{x}^5 + \Lambda h_0\right), \end{aligned} \tag{2.2}$$

where $A_0 = M(A)$, $e_0 = M(e)$, and $h_0 = M(h)$ are the mean of the corresponding almost periodic functions (see Appendix).

If we denote by $x_0, x_1, \dots$ the roots of the polynomial $-e_0 q(x)$ that stand for equilibria $(x_0, 0), (x_1, 0), \dots$ of (2.2), where

$$q(x) = x^5 + rx + s, \quad r = A_0/e_0, \quad s = -\Lambda h_0/e_0,$$

then

$$q'(x) = 5x^4 + s$$

has either two real roots, one real root of multiplicity 2, or no real root at all. This means that we will have at most three a.p. orbits emerging from the averaging theory applied to (2.1). Moreover, if x_0 is a root of multiplicity 2, then $q(x_0) = q'(x_0) = 0$. Therefore, there is a multiple root condition which can be written as

$$4^4\left(-\frac{A_0^5}{e_0}\right) = 5^5\left(\Lambda h_0\right)^4. \tag{2.3}$$

We apply averaging theory to the system (see [10]). Early uses of this theory go back to Bogolyubov's averaging principle, originally described in [5]. Thus we get the following assertion, which generalizes the results exposed in [11].

Theorem 2.1. *Suppose that the quintic Duffing system* (2.1) *has* quasi-periodic *coefficients and averaged equations* (2.2)*; if $e_0 \neq 0$, then the following assertions hold:*

- (i) *Suppose that $5^5\Lambda^4/h_0^4 \geq 4^4(-A_0^5/e_0) \geq 0$ and $e_0 > 0$.*
  - (a) *If $c > 0$, then there exists 1 stable a.p. orbit.*
  - (b) *If $c < 0$, then there exists 1 unstable a.p. orbit.*
- (ii) *Suppose that $5^5\Lambda^4/h_0^4 < 4^4(-A_0^5/e_0)$.*
  - (a) *If $e_0 > 0$ and $c > 0$, then there exist 2 stable a.p. solutions.*
  - (b) *If $e_0 > 0$ and $c < 0$, then there exist 3 unstable a.p. solutions.*
  - (c) *If $e_0 < 0$ and $c > 0$, then there exist 3 unstable a.p. solutions.*
  - (d) *If $e_0 < 0$ and $c < 0$, then there exists 1 stable a.p. solution and with 2 unstable a.p. solutions.*
- (iii) *Suppose that equality* (2.3) *holds.*
  - (a) *If $e_0 > 0$ and $c > 0$, then there exists 1 stable a.p. solution.*
  - (b) *If $e_0 > 0$ and $c < 0$, then there exists 1 unstable a.p. solution.*
- (iv) *Suppose that $\Lambda = 0 = A_0$.*
  - (a) *If $c > 0$ and $a > 0$, then the origin is a stable equilibrium (hence a.p.) solution.*
  - (b) *If $c < 0$ and $a > 0$, then the origin is an unstable equilibrium (hence a.p.) solution.*
  - (c) *If $a < 0$, then the origin is an equilibrium (hence a.p.) solution with mixed sign Lyapunov exponents.*

We provide a proof for Theorem 2.1.

According to (2.3) and also to [15], the classification of the roots and their multiplicity can be expressed in terms of the following discriminants:

$$D_4 = 160r^3$$
$$D_5 = 256r^5 + 3125s^4.$$

(I) One real root of multiplicity 1. $D_5 > 0$, $D_4 \leq 0$:

$$3125\Lambda^4 h_0^4 > 256\left(\frac{-A_0^5}{e_0}\right) \geq 0.$$

(II) Three real roots of multiplicity 1. $D_5 < 0$:

$$3125\Lambda^4 h_0^4 < 256\left(\frac{-A_0^5}{e_0}\right).$$

(III) One real root of multiplicity 1 and one of multiplicity 2. $D_5 = 0$, $D_4 < 0$:

$$3125\Lambda^4 h_0^4 = 256\left(\frac{-A_0^5}{e_0}\right) > 0.$$

(IV) One real root of multiplicity 5. $D_4 = D_5 = 0$:

$$A_0 = h_0 = 0.$$

(V) Three real roots of multiplicity 1 and one real root of multiplicity 2. $D_5 = 0$, $D_4 > 0$:

$$3125\Lambda^4 h_0^4 = 256\left(\frac{-A_0^5}{e_0}\right) < 0.$$

This case is not possible.

Degeneracies of the Jacobian $J(x)$ of the averaged system (2.2) occur at $x = x_0$ if

$$\det J(x_0) = \det\begin{pmatrix} 0 & 1 \\ -e_0 q'(x_0) & -C \end{pmatrix} = 0,$$

that is, when $-e_0 q'(x_0) = -e_0 q(x_0) = 0$. This implies that x_0 is a real root of multiplicity at least 2. Since $q'(x) = 5x^4 + A_0/e_0$, we have

$$5x_0^4 = -A_0/e_0, \qquad 3125\Lambda^4 h_0^4 = 256\left(\frac{-A_0^5}{e_0}\right).$$

We will now discuss stability criteria for non-degenerate equilibria of the averaged system (2.2). See the Appendix, where the main results on averaging theory are described.

i. In case **(I)**, A_0, e_0 have opposing signs. We have a stable or unstable a.p. orbit for the unique equilibrium, $-e_0 q(x_0) = 0$, depending on the signs: $-C = \operatorname{tr} J(x) > 0$ and $\det J(x) = e_0 q'(x_0) > 0$, or $C < 0$ and $e_0 q'(x_0) > 0$, respectively. The sign of $e_0 q'(x_0)$ coincides with the sign of the leading term of $e_0 q'(x)$ since the distance of the symmetric roots of $q'(x)$ to the origin is less than $|x_0|$.

   Therefore, if $e_0 > 0$ then there would exist a stable or an unstable equilibrium, depending on the sign, $C > 0$ or $C < 0$, respectively.

ii. In case **(II)**, A_0 and e_0 also have opposing signs. The sign of $e_0 q'(x_i)$ at the three different zeroes $x_0 < x_1 < x_2$ of $-e_0 q(x)$ is the sign of the leading term of $e_0 q'(x)$ for the extrema of the interval, x_0 and x_2. Meanwhile, this sign is opposite for the intermediate zero x_1. Therefore, we have the following subcases:
   (a) If $e_0 > 0$ and $C > 0$, then there are 2 stable equilibria and 1 semi-stable equilibrium.
   (b) If $e_0 > 0$ and $C < 0$, then there are 1 semi-stable equilibrium and 2 unstable equilibria.
   (c) If $e_0 < 0$ and $C > 0$, then there are 1 unstable equilibrium and 2 semi-stable equilibria.

(d) If $e_0 < 0$ and $C < 0$, then there are 1 stable equilibrium and 2 semi-stable equilibria.

iii. In case **(III)**, A_0, e_0 have opposing signs. Thus the real root x_1 of multiplicity 1, $-e_0 q(x_1) = 0$, is not a root of $-q'(x)$. By Rolle's Theorem the second real root of $-q'(x)$ would appear in the interval between x_0 and x_1, where $e_0 q(x_0) = e_0 q'(x_0) = 0$. Recall that $q'(x)$ is an even function; therefore, $e_0 q'(x_1) = 5e_0 x_1^4 + A_0$ would have the same sign as its leading term. Thus, stability holds whenever $e_0 > 0$ and $C > 0$, while unstable equilibria would take place when $e_0 > 0$ and $C < 0$.

iv. In case **(IV)**, we have $A_0 = 0 = \Lambda$. For the degenerate cases **(III)** and **(IV)**, another approach is needed to analyze roots of multiplicity greater than 1. Here we have a fixed point at the origin, whose Jacobian

$$\begin{pmatrix} 0 & 1 \\ -a(t) & -c \end{pmatrix}$$

is not degenerate for $c \neq 0$. By a suitable Lyapunov analysis, this equilibrium is locally stable under certain assumptions.

2.1. **Examples of quasi-periodic solutions with quintic nonlinearities.** We consider the following set of parameters: $h_0 = 4/5$, $e_0 \in \{\pm 1\}$, $A_0 \in \{0, \pm 1\}$, $\Lambda \geq 0$, $C = 0.01$, with time dependent almost periodic coefficients

$$\begin{aligned} A &= A_0 + 0.5\cos(3.1416\, t), \qquad e = e_0 + 0.05\cos(\sqrt{2}t), \\ h &= h_0 + 0.001\sin(3.1416t), \end{aligned}$$

Example for case (I): Take $A_0 = -1$, $e_0 = 1$, $\Lambda = 1$, and $\varepsilon = 1$. We have a stable almost periodic orbit as a limit for the initial condition $x \approx 0.055$, $y \approx 0.873$ and also for the initial condition $x = 1.13$, $y = 0$ (see Fig. 1). The a.p. solution lies near the equilibrium $x_0 \approx 1.1$, $y_0 = 0$ of (2.2) as $\varepsilon \to 0$. Numerical evidence shows that a different recurrence behaviour may occur which is not related to a.p. solutions. Take for instance the initial condition $x \approx -0.115$, $y \approx -2.249$ as in Fig. 2. For the complementary case $C = -0.01$, we have an unstable a.p. solution. Numerical evidence also shows that for some initial conditions solutions diverge towards infinity as $t \to \infty$. Meanwhile, some other solutions diverge in finite time.

Example for case (II): Take $A_0 = -1$, $e_0 = 1$, $\Lambda = 1/2$, $\varepsilon = 0.1$, $C = 0.01$. For the initial conditions $(-0.95, 0)$ and $(0.85, 0)$, we have that solutions converge towards two different stable a.p. orbits (see Fig. 3). As $\varepsilon \to 0$, these a.p. orbits converge towards x_0 and x_2, respectively, with $x_0 \approx -0.85$ and $x_2 \approx 1.08$. Here the middle equilibrium is $x_1 \approx -0.41$. Numerical evidence shows that some initial data solutions go towards infinity in finite time.

Example for case (III): Take $A_0 = -1$, $e_0 = 1$, $\Lambda = 5^{-1/4}$, and $\varepsilon = 0.1$. For the initial condition $(0, 0)$ the solution approaches a stable a.p. which tends towards the simple root $x_1 \approx 1.1$ as $\varepsilon \to 0$ (see Fig. 4). Numerical evidence shows no other limit as $t \to \infty$ for other initial conditions.

Example for case (IV): Take $A_0 = \lambda = 0$, $c = 1$, and $e_0 = 1$. Here the origin is a local equilibrium which behaves like a local attractor.

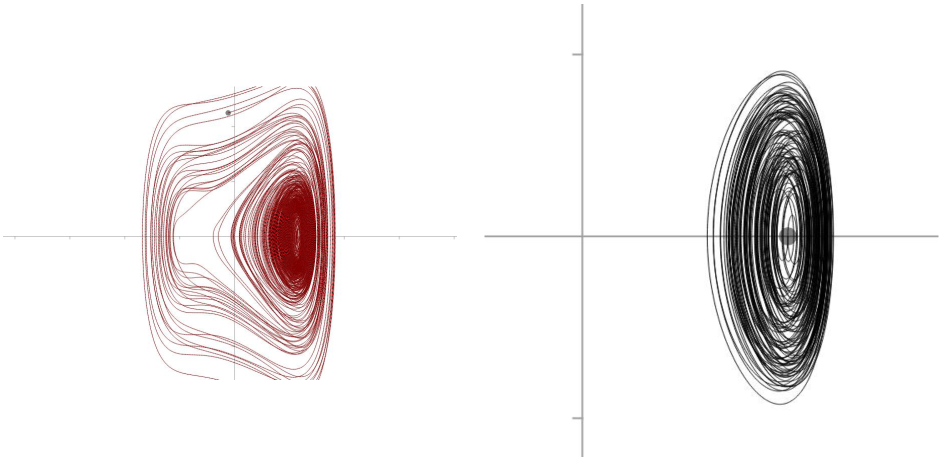

FIGURE 1. Convergence towards an almost periodic solution near $x_0 \approx 1.1$, from two different initial conditions.

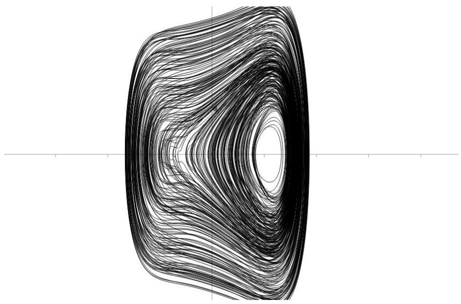

FIGURE 2. Recurrent behaviour apparently not related to an almost periodic solution near $x_0 \approx 1.1$.

3. Duffing equation with odd nonlinearity

Condition (2.3) can be generalized for the odd Duffing system for $k \in \mathbb{N}$,

$$\begin{aligned} x' &= y, \\ y' &= -cy - a(t)x - e(t)x^{2k+1} + \lambda h(t), \end{aligned} \tag{3.1}$$

as

$$(2k)^{2k} \left(-\frac{A_0^{2k+1}}{e_0} \right) = (2k+1)^{2k+1} \left(\Lambda h_0 \right)^{2k}. \tag{3.2}$$

Thus, Theorem 2.1 can be generalized as the following assertion.

Theorem 3.1. *For the quintic Duffing system* (3.1), *if* $e_0 \neq 0$ *then the following assertions hold:*

(i) *Suppose that* $(2k+1)^{2k+1} \left(\Lambda h_0 \right)^{2k} > (2k)^{2k} \left(-\frac{A_0^{2k+1}}{e_0} \right) \geq 0$ *and* $e_0 > 0$.

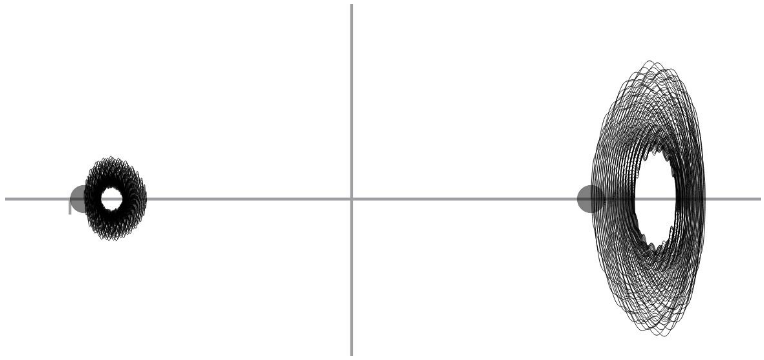

FIGURE 3. Two solutions converging towards two stable almost periodic solutions near $x_0 \approx -0.85$ and $x_2 \approx 1.08$, respectively.

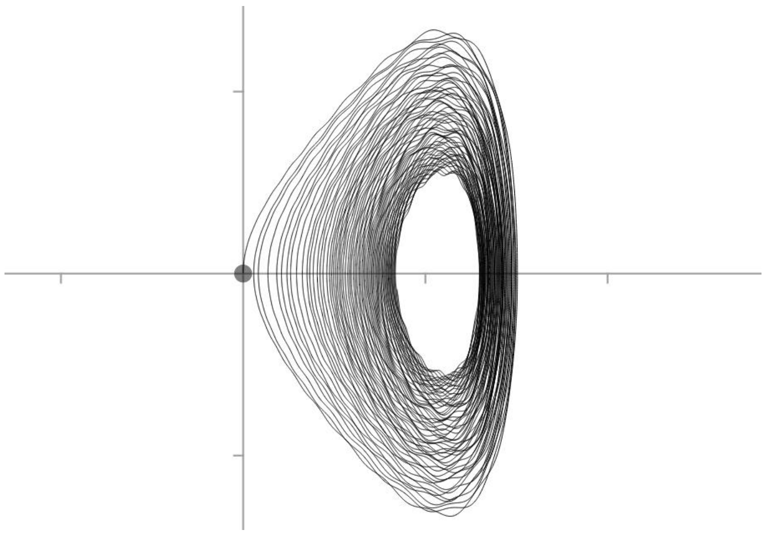

FIGURE 4. A solution converging towards a stable almost periodic solution near $x_0 \approx 1.1$.

(a) *If* $c > 0$*, then there exists 1 stable a.p. orbit.*
(b) *If* $c < 0$*, then there exists 1 unstable a.p. orbit.*

(ii) *Suppose that* $(2k+1)^{2k+1} (\Lambda h_0)^{2k} < (2k)^{2k} \left(-\frac{A_0^{2k+1}}{e_0}\right)$.
(a) *If* $e_0 > 0$ *and* $c > 0$*, then there exist 2 stable a.p. solutions.*
(b) *If* $e_0 > 0$ *and* $c < 0$*, then there exist 3 unstable a.p. solutions.*
(c) *If* $e_0 < 0$ *and* $c > 0$*, then there exist 3 unstable a.p. solutions.*
(d) *If* $e_0 < 0$ *and* $c < 0$*, then there exists 1 stable a.p. solution and 2 unstable a.p. solutions.*

(iii) *Suppose that equality* (3.2) *holds.*
  (a) *If* $e_0 > 0$ *and* $c > 0$*, then there exists 1 stable a.p. solution.*
  (b) *If* $e_0 > 0$ *and* $c < 0$*, then there exists 1 unstable a.p. solution.*

(iv) *Suppose that* $\Lambda = 0 = A_0$.
  (a) *If* $c > 0$ *and* $a > 0$*, then the origin is a stable equilibrium (hence a.p.) solution.*
  (b) *If* $c < 0$ *and* $a > 0$*, then the origin is an unstable equilibrium (hence a.p.) solution.*
  (c) *If* $a < 0$*, then the origin is an equilibrium (hence a.p.) solution with mixed sign Liapunov exponents.*

4. Conclusions and further problems

If we consider a quintic Duffing equation with cubic-quintic nonlinearity,

$$
\begin{aligned}
x' &= y \\
y' &= -cy - ax - bx^3 - ex^5 + \lambda h(x)
\end{aligned}
$$

with $e \neq 0$, then the corresponding classification of roots and their multiplicities depend on certain conditions given in terms of certain discriminants. Such conditions appear for instance in [15]. This will be described elsewhere.

On the other hand, we consider the equation with quadratic and cubic nonlinearities subjected to combined parametric and external excitation having incommensurate frequencies ω_i,

$$x'' + cx' + \omega_0^2(1 + k_1 \cos \omega_1 t)x + k_2 x^2 + k_3 x^3 = \lambda \cos(\omega_2 t),$$

which models a one-mode vibration of a heavy elastic structure suspended between two fixed supports at the same level and excited by a quasi-periodic forcing (see [2] and references therein); the quadratic term arises from curvature considerations. The oscillation in the parameter may be due to a harmonic axial load. Thus we have also motivations to study systems of the form

$$x'' + cx' + a(t)x + b(t)x^2 + e(t)x^3 = \lambda h(t),$$

as well as some other models, such as special cases of van der Pol–Mathieu (see [9] and [14]).

5. Appendix: Averaging method for almost periodic functions

We refer the reader to [6], [4], and [7] for more detailed information on almost periodic functions.

Definition 5.1. According to Bohr's definition, a function $\varphi \in \mathrm{C}(\mathbb{R})$ is *almost periodic (a.p.)* if, for all $\epsilon > 0$, there exists a set of real numbers $T(\varphi, \epsilon)$ such that there is $l(\varphi, \epsilon) > 0$ such that in any interval of length $l(\varphi, \epsilon)$ there is at least one point of $T(\varphi, \epsilon)$. The set $T(\varphi, \epsilon)$ satisfies

$$|\varphi(t + \tau(\epsilon)) - \varphi(t)| < \epsilon$$

for each $t \in \mathbb{R}$ and $\tau(\epsilon) \in T(\varphi, \epsilon)$. We will call each number $\tau(\epsilon)$ a *translation number*. A *length* of $T(\varphi, \epsilon)$ will be a number $l(\varphi, \epsilon)$ satisfying that any interval of length $l(\varphi, \epsilon)$ intersects $T(\varphi, \epsilon)$.

Notice that a periodic function φ with period τ is actually an a.p. function, with translation numbers $T(\varphi, \epsilon) = \{n\tau \mid n \in \mathbb{Z}\}$ for any $\epsilon > 0$. In particular, our results also apply to the periodic case. Since the sum of a.p. functions and the multiplication of an a.p. function by constant values both result in a.p. functions, we can conclude that trigonometric polynomials are a.p. functions. A linear combination

$$\varphi(t) = a_1 \cos(\lambda_1 t) + b_1(\lambda_1 t) + a_2 \cos(\lambda_2 t) + b_2 \sin(\lambda_2 t)$$

is an a.p. function even if λ_1 and λ_2 are rationally independent. In such a case φ is not periodic. Interestingly, every function φ which can be approximated uniformly by trigonometric polynomials is an a.p. function. A consequence of these properties is that a.p. functions form a Banach space, $\mathrm{C_{ap}}(\mathbb{R})$. Furthermore, we have the following known result.

A fundamental fact of the theory of almost periodic functions is that, for every almost periodic function φ, there exists its *mean value*

$$\varphi_0 = M(\varphi) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(t)\, dt,$$

which is a continuous linear functional denoted by $M : \mathrm{C_{ap}}(\mathbb{R}) \to \mathbb{R}$. It is also possible to associate to an almost periodic function φ its unique *generalized Fourier expansion*,

$$\varphi(t) \sim \varphi_0 + \sum_{n\in\mathbb{N}} a(\varphi, \lambda_n) \cos(\lambda_n t) + b(\varphi, \lambda_n) \sin(\lambda_n t).$$

Recall that the module

$$\widehat{\varphi} := \langle \{\lambda_n\}_{n\in\mathbb{N}} \rangle = \left\{ \sum_{k=1}^{N} r_k \lambda_{n_k} \mid r_k \in \mathbb{Z} \right\} \subset \mathbb{R}$$

of an almost periodic function φ is the $\mathbb{Z}$-module generated by the generalized Fourier spectrum $\{\lambda_n > 0\}_{n\in\mathbb{N}}$.

Definition 5.2. An almost periodic function $\varphi \in \mathrm{C_{ap}}(\mathbb{R})$ is a *quasi-periodic* function if its module of frequencies $\widehat{\varphi}$ is finitely generated.

Thus a typical generalized Fourier expansion for a quasi-periodic function reads

$$\varphi_0 + \sum_{k=1}^{M} \varphi_k(t), \qquad \varphi_k(t) = \sum_{n\in\mathbb{N}} a(\varphi, k, n) \cos(n\beta_k t) + b(\varphi, k, n) \sin(n\beta_k t),$$

where $\{\beta_1, \ldots, \beta_k\}$ is a basis of rationally independent generators for $\widehat{\varphi}$. Thus, a quasi-periodic function φ can also be described as a finite sum of periodic functions φ_k, $k = 1, \ldots, N$.

Our aim is to consider the initial value problem

$$\begin{aligned} x' &= \varepsilon f(x,y,t), \qquad x(0) = x_0, \\ y' &= \varepsilon g(x,y,t), \qquad y(0) = y_0, \end{aligned} \tag{5.1}$$

where $(x,y),(x_0,y_0) \in D \subset \mathbb{R}^2$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$. Suppose that f,g are almost periodic with respect to t and C^1 with respect to (x,y). A more adequate consideration requires f,g to be *uniformly almost periodic* with respect to (x,y) in the compact set $D \subset \mathbb{R}^2$, which means that

$$|f(x,y,t+\tau(\epsilon)) - f(x,y,t)| < \epsilon \qquad \forall t \in \mathbb{R},\ \forall (x,y) \in D$$

for each translation number $\tau(\epsilon) \in T(\epsilon,f)$ and length $l(\epsilon,f) > 0$, not depending on a particular choice (x,y) but remaining the same throughout the entire compact domain D. More specifically, if f,g have generalized Fourier expansions

$$\begin{aligned} f(x,y,t) &\sim f_0(x,y) + \sum_{n\in\mathbb{N}} a(f,\lambda_n,x)\cos(\lambda_n t) + b(f,\lambda_n,x)\sin(\lambda_n t), \\ g(x,y,t) &\sim g_0(x,y) + \sum_{n\in\mathbb{N}} a(g,\lambda_n,x)\cos(\lambda_n t) + b(g,\lambda_n,x)\sin(\lambda_n t), \end{aligned} \tag{5.2}$$

then f,g are uniformly almost periodic, whenever the exponents λ_n do not depend on (x,y) (see [7, Chapter VI]).

Recall that

$$f_0(x,y) = \lim_{T\to 0} \frac{1}{T}\int_0^T f(x,y,t)\,dt$$

and

$$g_0(x,y) = \lim_{T\to 0} \frac{1}{T}\int_0^T g(x,y,t)\,dt.$$

Consider also the averaged initial value problem

$$\begin{aligned} \overline{x}' &= \varepsilon f_0(\overline{x},\overline{y}), \qquad \overline{x}(0) = x_0, \\ \overline{y}' &= \varepsilon g_0(\overline{x},\overline{y}), \qquad \overline{y}(0) = y_0. \end{aligned} \tag{5.3}$$

The main problem, roughly speaking, is to establish to what extent the solutions of the averaged problem (5.3) "approximate" the solutions of the original almost periodic system (5.1).

The notion of approximation suitable in such a context comes from perturbation theory. We will clarify the idea of approximation using the averaged system, as outlined below.

The following result summarizes the tools that we use. See also [10, Lemmas V.3.1 and V.3.2], which deal with this issue. Suppose that $f(x,y,t)$, $g(x,y,t)$ are continuous in t and uniformly almost periodic with respect to (x,y) in the compact set $D \subset \mathbb{R}^2$, and have continuous bounded partial derivatives $\partial_x f$, $\partial_y f$, $\partial_x g$, $\partial_y g$ in the domain $D \times [0,\infty)$.

Theorem 5.3. [13, Theorem 4.3.6 and Lemma 4.6.6] *Suppose that the Fourier spectrum* (5.2) *of f and g is such that $\lambda_n \geq \alpha > 0$, with α independent of $n \in \mathbb{N}$. Then, the solution $(\overline{x}(t), \overline{y}(t))$ of the averaged initial value problem* (5.3) *approximates the solution of the quasi-periodic problem* (5.1) *as follows:*

$$(x(y), y(t)) = (\overline{x}(t), \overline{y}(t)) + \mathcal{O}(\varepsilon)$$

as $\varepsilon \to 0$ on the time scale $1/\varepsilon$.

In other words, the asymptotic approximation yields a solution $(x(t), y(t)) \in D$ such that

$$\|(x, y) - (\overline{x}, \overline{y})\|_\infty \leq k\,\varepsilon$$

for an ε-independent constant $k > 0$ and the sup-norm in $D \times [0, L/\varepsilon)$, for an $\varepsilon_0 > 0$ small enough and for every $\varepsilon \in (0, \varepsilon_0]$ and for every $t \in [0, L/\varepsilon)$, where $L > 0$ is a constant.

Notice that the hypotheses of Theorem 5.3 about the Fourier spectrum are fulfilled taking, for instance, quasi-periodic functions f, g.

The stability statement is contained in the following assertion (see also [10, Theorem V.3.1]).

Theorem 5.4. [13, Theorem 5.5.1] *Assuming the same conditions as in Theorem 5.3, the averaged system* (5.3) *is* C^1 *in D. Furthermore, suppose (x_0, y_0) is asymptotically stable with respect to the linearization $J(x_0, y_0) := \frac{\partial(f_0, g_0)}{\partial(\overline{x}, \overline{y})}$ with basin of attraction D^0. If $K \subset D^o$ is compact, $(x_1, y_1) \in K$, and $(\overline{x}(t), \overline{y}(t)) \in D$ is the solution of the averaged system*

$$\overline{x}' = \varepsilon f_0(\overline{x}, \overline{y}), \qquad \overline{x}(0) = x_1,$$
$$\overline{y}' = \varepsilon g_0(\overline{x}, \overline{y}), \qquad \overline{y}(0) = y_1,$$

then there exists a constant $k > 0$ such that

$$\|x(t) - \overline{x}(t)\|_\infty \leq k\,\delta_1(\varepsilon) \qquad \forall t \in [0, \infty),$$

where $\delta_1(\varepsilon)$ satisfies $\lim_{\varepsilon \to 0^+} \delta_1(\varepsilon) = 0$ and is called the order function*. In the periodic case, $\delta_1(\varepsilon) = \varepsilon$.*

For the instability claims, we refer the reader to [10, Theorem V.3.1].

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