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POINCARÉ DUALITY FOR HOPF ALGEBROIDS

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ABSTRACT. We prove a twisted Poincaré duality for (full) Hopf algebroids with bijective antipode. As an application, we recover the Hochschild twisted Poincaré duality of van den Bergh. We also get a Poisson twisted Poincaré duality, which was already stated for oriented Poisson manifolds by Chen et al.

1. Introduction

Left bialgebroids over a (possibly) non-commutative basis A generalize bialgebras. If U is a left bialgebroid, there is a natural U-module structure on A and the category of left modules over a left bialgebroid U is monoidal. Nevertheless, A is generally not a right U-module. Left Hopf left bialgebroids (or $\times_A$-Hopf algebras [24]) generalize Hopf algebras. In a left Hopf left bialgebroid U, the existence of an antipode is not required but, for any element $u \in U$, there exists an element $u_+ \otimes u_-$ corresponding to $u_{(1)} \otimes S(u_{(2)})$. The more restrictive structure of *full Hopf algebroids* ([3]) ensures the existence of an antipode. If L is a Lie–Rinehart algebra (or Lie algebroid) over a commutative k-algebra A ([23]), there exists a standard left bialgebroid structure on its enveloping algebra $V(L)$. This structure is left Hopf. Kowalzig [17] showed that $V(L)$ is a full Hopf algebroid if and only if there exists a right $V(L)$-module structure on A. If X is a $\mathcal{C}^\infty$ Poisson manifold and $A = \mathcal{C}^\infty(X)$, the A-module of global differential one forms $\Omega^1(X)$ is endowed with a natural Lie–Rinehart structure over A, which is of much interest ([6], [10], [12], [15], [22], [26], etc.). In particular, Huebschmann ([12]) exhibited a right $V(\Omega^1(X))$-module structure on A (denoted by A_P) that makes $V(\Omega^1(X))$ a full Hopf algebroid. He also interpreted the Lichnerowicz–Poisson cohomology $H^i_{\mathrm{Pois}}(X)$ as $\mathrm{Ext}^i_{V(\Omega^1(X))}(A, A)$ and the Poisson homology $H_i^{\mathrm{Pois}}(X)$ ([5], [16]) of X as $\mathrm{Tor}_{V(\Omega^1(X))}(A_P, A)$.

A Poincaré duality theorem was proved in [6] for Lie–Rinehart algebras and then extended to left Hopf left bialgebroids in [18]. It asserts, under some conditions, that if $\mathrm{Ext}^i_U(A, U) = 0$ for $i \neq d$, then, for all left U-modules M and all $n \in \mathbb{N}$, there is an isomorphism

$$\mathrm{Ext}^n_U(A, M) \simeq \mathrm{Tor}^U_{d-n}(M \otimes_A \Lambda, A),$$

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where $\Lambda := \mathrm{Ext}^d_U(A,U)$ is endowed with the right U-module structure given by right multiplication in U. If $U = V(L)$ is the enveloping algebra of a finitely generated projective Lie–Rinehart algebra L, it is shown in [6] that $\mathrm{Ext}^n_{V(L)}(A, V(L)) = 0$ if $n \neq \dim L$. Moreover, $\mathrm{Ext}^{\dim L}_{V(L)}(A, V(L)) \simeq \Lambda^{\dim L}_A(L^*)$.

We give a new formulation of the Poincaré duality in the case where U as well as its coopposite U_{coop} is left Hopf and A is endowed with a right U-module structure (denoted by A_R) such that the A^e-module $_\blacktriangleright A_{R\,\blacktriangleleft}$ is invertible.

Theorem 3.5 *Let U be a left and right Hopf left bialgebroid over A. Assume the following:*

(i) $\mathrm{Ext}^i_U(A,U) = \{0\}$ *if $i \neq d$, and set* $\Lambda = \mathrm{Ext}^d_U(A,U)$.
(ii) *The left U-module A admits a finitely generated projective resolution of finite length.*
(iii) *A is endowed with a right U-module structure (denoted by A_R) such that the A^e-module $_\blacktriangleright A_{R\,\blacktriangleleft}$ is invertible.*
(iv) *Let $\mathcal{T}$ be the left U-module* $\mathrm{Hom}_A(A_{R\,\blacktriangleleft}, \Lambda_\blacktriangleleft)$ *(see Proposition 2.7). The A-module $_\triangleright\mathcal{T}$ and the A^{op}-module $\mathcal{T}_\triangleleft$ are projective.*

Then, for all left U-modules M and all $i \in \mathbb{N}$, there is an isomorphism

$$\mathrm{Ext}^i_U(A,M) \simeq \mathrm{Tor}^U_{d-i}\left(A_R, \mathcal{T}_\triangleleft \otimes_A {}_\triangleright M\right).$$

Assume now that H is a full Hopf algebroid. The antipode allows us to transform any left (resp., right) H-module M (resp., N) into a right (resp., left) H-module denoted by M_S (resp., $_SN$). Thus from the left H-module structure on A, we can construct a right H-module structure A_S. From the right H-module structure on Λ, we can make a left H-module structure denoted by $_S\Lambda$. The duality states the following:

$$\mathrm{Ext}^i_H(A,M) \simeq \mathrm{Tor}^H_{d-i}\left(A_S, {}_S\Lambda \otimes_A M\right).$$

In the special case of the (full) Hopf algebroid $A \otimes A^{\mathrm{op}}$, we recover the Hochschild twisted Poincaré duality of [1]. In the special case where X is a Poisson manifold and $H = V(\Omega^1(X))$, the duality above can be rewritten in terms of Poisson cohomology and homology. Let M be a left H-module. The coproduct on H allows us to endow $_S\Lambda \otimes_A M$ with a left H-module structure. Denote by $H^i_{\mathrm{Pois}}(M)$ the Poisson cohomology with coefficients in M, and let $H^{\mathrm{Pois}}_i({}_S\Lambda \otimes_A M)$ denote the Poisson homology with coefficients in $_S\Lambda \otimes_A M$. There is an isomorphism

$$H^i_{\mathrm{Pois}}(M) \simeq H^{\mathrm{Pois}}_{d-i}\left({}_S\Lambda \otimes_A M\right).$$

This formula was stated in [9] for oriented Poisson manifolds; see also [20] for polynomial algebras with quadratic Poisson structures, [28] for linear Poisson structures, and [21] for general polynomial Poisson algebras.

Notation. Fix an (associative, unital, commutative) ground ring k. Unadorned tensor products will always be meant over k. All other algebras, modules, etc. will have an underlying structure of a k-module. Secondly, fix an associative and unital k-algebra A, i.e., a ring with a ring homomorphism $\eta_A : k \to Z(A)$ to its centre.

Denote by A^{op} the opposite algebra and by $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ the enveloping algebra of A, and by A-Mod the category of left A-modules.

The notions of *A-ring* and *A-coring* are direct generalizations of the notions of algebra and coalgebra over a commutative ring.

Definition 1.1. An A-coring is a triple (C, Δ, ϵ), where C is an A^e-module (with left action L_A and right action R_A), $\Delta : C \longrightarrow C \otimes_A C$ and $\epsilon : C \longrightarrow A$ are A^e-module morphisms such that

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta, \quad L_A \circ (\epsilon \otimes \mathrm{id}_C) \circ \Delta = \mathrm{id}_C = R_A \circ (\mathrm{id}_C \otimes \epsilon) \circ \Delta.$$

As usual, we adopt Sweedler's Σ-notation $\Delta(c) = c_{(1)} \otimes c_{(2)}$ or $\Delta(c) = c^{(1)} \otimes c^{(2)}$ for $c \in C$.

The notion of A-ring is dual to that of A-coring. It is well known (see [2]) that A-rings H correspond bijectively to k-algebra homomorphisms $\iota : A \longrightarrow H$. An A-ring H is endowed with the following A^e-module structure:

$$\forall h \in H,\ a, b \in H, \quad a \cdot h \cdot b = \iota(a) h \iota(b).$$

2. Preliminaries

We recall the notions and results with respect to bialgebroids that are needed to make this article self-contained; see, e.g., [17] and references therein for an overview on this subject.

2.1. **Bialgebroids.** For an A^{e}-ring U given by the k-algebra map $\eta : A^{\mathrm{e}} \to U$, consider the restrictions $s := \eta(- \otimes 1_U)$ and $t := \eta(1_U \otimes -)$, called *source* and *target* map, respectively. Thus an A^{e}-ring U carries two A-module structures from the left and two from the right, namely

$$a \triangleright u \triangleleft b := s(a)t(b)u, \qquad a \blacktriangleright u \blacktriangleleft b := ut(a)s(b), \qquad \forall\, a, b \in A,\ u \in U.$$

If we let $U_{\triangleleft} \otimes_{A\,\triangleright} U$ be the corresponding tensor product of U (as an A^{e}-module) with itself, we define the *(left) Takeuchi–Sweedler product* as

$$U_{\triangleleft} \times_{A\,\triangleright} U := \big\{ \textstyle\sum_i u_i \otimes u_i' \in U_{\triangleleft} \otimes_{A\,\triangleright} U \mid \sum_i (a \blacktriangleright u_i) \otimes u_i' = \sum_i u_i \otimes (u_i' \blacktriangleleft a)\ \forall a \in A \big\}.$$

By construction, $U_{\triangleleft} \times_{A\,\triangleright} U$ is an A^{e}-submodule of $U_{\triangleleft} \otimes_{A\,\triangleright} U$; it is also an A^{e}-ring via factorwise multiplication, with unit $1_U \otimes 1_U$ and $\eta_{U_{\triangleleft} \times_{A\,\triangleright} U}(a \otimes \tilde{a}) := s(a) \otimes t(\tilde{a})$.

Symmetrically, one can consider the tensor product $U_{\blacktriangleleft} \otimes_{A\,\blacktriangleright} U$ and define the *(right) Takeuchi–Sweedler product* as $U_{\blacktriangleleft} \times_{A\,\blacktriangleright} U$, which is an A^{e}-ring inside $U_{\blacktriangleleft} \otimes_{A\,\blacktriangleright} U$.

Definition 2.1 ([25])**.** A *left bialgebroid* (U, A) is a k-module U with the structure of an A^{e}-ring (U, s^ℓ, t^ℓ) and an A-coring $(U, \Delta_\ell, \epsilon)$ subject to the following compatibility relations:

(i) The A^{e}-module structure on the A-coring U is that of ${}_{\triangleright} U_{\triangleleft}$.
(ii) The coproduct Δ_ℓ is a unital k-algebra morphism taking values in $U_{\triangleleft} \times_{A\,\triangleright} U$.

(iii) For all $a, b \in A$ and $u, u' \in U$, one has

$$\epsilon(1_U) = 1_A, \quad \epsilon(a \triangleright u \triangleleft b) = a\epsilon(u)b, \quad \epsilon(uu') = \epsilon\big(u \blacktriangleleft \epsilon(u')\big) = \epsilon\big(\epsilon(u') \blacktriangleright u\big).$$

A *morphism* between left bialgebroids (U, A) and (U', A') is a pair (F, f) of maps $F : U \to U'$, $f : A \to A'$ that commute with all structure maps in an obvious way.

As for any ring, we can define the categories U-Mod and Mod-U of left and right modules over U. Note that U-Mod forms a monoidal category but Mod-U usually does not. However, in both cases there is a forgetful functor $U\text{-Mod} \to A^e\text{-Mod}$ (respectively $\text{Mod-}U \to A^e\text{-Mod}$) given by the following formulas: For $m \in M$, $n \in N$, and $a, b \in A$,

$$a \triangleright m \triangleleft b := s^\ell(a)t^\ell(b)m, \qquad a \blacktriangleright m \blacktriangleleft b := ns^\ell(b)t^\ell(a).$$

For example, the base algebra A itself is a left U-module via the left action

$$u(a) := \epsilon(u \blacktriangleleft a) = \epsilon(a \blacktriangleright u) \quad \forall u \in U,\ \forall a \in A,$$

but in general there is no right U-action on A.

Example 2.2. Let A be a commutative k-algebra and $\mathrm{Der}_k(A)$ the A-module of k-derivations of A. Let L be a Lie–Rinehart algebra ([23]) over A with anchor $\rho : L \to \mathrm{Der}_k(A)$. Its enveloping algebra $V(L)$ is endowed with a standard left bialgebroid ([26]) described as follows: For all $a \in A$, $D \in L$, and $u \in V(L)$,

(i) s^ℓ and t^ℓ are equal to the natural injection $\iota : A \to V(L)$;
(ii) $\Delta_\ell : V(L) \to V(L) \otimes_A V(L)$, $\quad \Delta_\ell(a) = a \otimes_A 1$, $\ \Delta_\ell(D) = D \otimes_A 1 + 1 \otimes_A D$;
(iii) $\epsilon(u) = \rho(u)(1)$.

In this example, the left action of $V(L)$ on A coincides with the anchor extended to $V(L)$.

2.2. **Left and right Hopf left bialgebroids.** For any left bialgebroid U, define the *Hopf–Galois maps*

$$\alpha_\ell : {}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} U_\triangleleft \to U_\triangleleft \otimes_A {}_\triangleright U, \quad u \otimes_{A^{\mathrm{op}}} v \mapsto u_{(1)} \otimes_A u_{(2)} v,$$
$$\alpha_r : U_\blacktriangleleft \otimes^A {}_\triangleright U \to U_\triangleleft \otimes_A {}_\triangleright U, \qquad u \otimes^A v \mapsto u_{(1)} v \otimes_A u_{(2)}.$$

With the help of these maps, we make the following definition due to Schauenburg [24]:

Definition 2.3. A left bialgebroid U is called a *left Hopf left bialgebroid* or $\times_A$ *Hopf algebra* if α_ℓ is a bijection. Likewise, it is called a *right Hopf left bialgebroid* if α_r is a bijection. In either case, we adopt for all $u \in U$ the following (Sweedler-like) notation

$$u_+ \otimes_{A^{\mathrm{op}}} u_- := \alpha_\ell^{-1}(u \otimes_A 1), \quad u_{[+]} \otimes^A u_{[-]} := \alpha_r^{-1}(1 \otimes_A u), \tag{2.1}$$

and call both maps $u \mapsto u_+ \otimes_{A^{\mathrm{op}}} u_-$ and $u \mapsto u_{[+]} \otimes^A u_{[-]}$ *translation maps.*

Remarks 2.4. Let $(U, A, s^\ell, t^\ell, \Delta, \epsilon)$ be a left bialgebroid.

(i) In case $A = k$ is central in U, one can show that α_ℓ is invertible if and only if U is a Hopf algebra, and the translation map reads $u_+ \otimes u_- := u_{(1)} \otimes S(u_{(2)})$, where S is the antipode of U. On the other hand, U is a Hopf algebra with invertible antipode if and only if both α_ℓ and α_r are invertible, and then $u_{[+]} \otimes u_{[-]} := u_{(2)} \otimes S^{-1}(u_{(1)})$.

(ii) The underlying left bialgebroid in a *full* Hopf algebroid with bijective antipode is both a left and right Hopf left bialgebroid (but not necessarily vice versa); see [3, Proposition 4.2] for the details of this construction recalled below in Section 4

Example 2.5. If L is a Lie–Rinehart algebra over a commutative k-algebra A with anchor ρ, then its enveloping algebra $V(L)$, endowed with its standard bialgebroid structure, is a left Hopf left bialgebroid. The translation map is described as follows (see Proposition 2.6; in this case, $A = A^{\mathrm{op}}$ and $s^\ell = t^\ell$): If $a \in A$ and $D \in L$,

$$a_+ \otimes_{A^{\mathrm{op}}} a_- = a \otimes_{A^{\mathrm{op}}} 1, \quad D_+ \otimes_{A^{\mathrm{op}}} D_- = D \otimes_{A^{\mathrm{op}}} 1 - 1 \otimes_{A^{\mathrm{op}}} D.$$

It is also a right Hopf left bialgebroid as it is cocommutative.

The following proposition collects some properties of the translation maps [24]:

Proposition 2.6. *Let U be a left bialgebroid.*

(i) *If U is a left Hopf left bialgebroid, the following relations hold:*

$$\begin{aligned}
u_+ \otimes_{A^{\mathrm{op}}} u_- &\in U \times_{A^{\mathrm{op}}} U,\\
u_{+(1)} \otimes_A u_{+(2)} u_- &= u \otimes_A 1 \in U_{\lhd} \otimes_{A} {}_{\rhd}U,\\
u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} u_{(2)} &= u \otimes_{A^{\mathrm{op}}} 1 \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\lhd},\\
u_{+(1)} \otimes_A u_{+(2)} \otimes_{A^{\mathrm{op}}} u_- &= u_{(1)} \otimes_A u_{(2)+} \otimes_{A^{\mathrm{op}}} u_{(2)-},\\
u_+ \otimes_{A^{\mathrm{op}}} u_{-(1)} \otimes_A u_{-(2)} &= u_{++} \otimes_{A^{\mathrm{op}}} u_- \otimes_A u_{+-},\\
(uv)_+ \otimes_{A^{\mathrm{op}}} (uv)_- &= u_+ v_+ \otimes_{A^{\mathrm{op}}} v_- u_-,\\
u_+ u_- &= s^\ell(\varepsilon(u)),\\
\varepsilon(u_-) \blacktriangleright u_+ &= u,\\
(s^\ell(a) t^\ell(b))_+ \otimes_{A^{\mathrm{op}}} (s^\ell(a) t^\ell(b))_- &= s^\ell(a) \otimes_{A^{\mathrm{op}}} s^\ell(b),
\end{aligned}$$

where, in the first relation, we mean the Takeuchi–Sweedler product

$$U \times_{A^{\mathrm{op}}} U := \Big\{ \textstyle\sum_i u_i \otimes v_i \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\lhd} \mid \sum_i u_i \lhd a \otimes v_i = \sum_i u_i \otimes a \blacktriangleright v_i \ \forall a \in A \Big\}.$$

(ii) *There are similar relations for $u_{[+]} \otimes_A u_{[-]}$ if U is a right Hopf left bialgebroid (see* [8] *for an exhaustive list).*

The existence of a translation map if U is a left or right Hopf left bialgebroid makes it possible to endow Hom-spaces and tensor products of U-modules with further natural U-module structures. These structures were systematically studied in [8, Proposition 3.1.1]. They generalize the case of $V(L)$ ([7], see [4], [14] for the particular case $L = \mathrm{Der}(A)$)

Proposition 2.7. *Let (U, A) be a left bialgebroid. Let $M, M' \in U$-Mod and $N, N' \in$ Mod-U be left and right U-modules, respectively. We denote the respective actions by juxtaposition.*

(i) *Let (U, A) be additionally a left Hopf left bialgebroid.*
- *The A^{e}-module $\mathrm{Hom}_{A^{\mathrm{op}}}(M, M')$ carries a left U-module structure given by*
$$(u \cdot f)(m) := u_+\big(f(u_- m)\big).$$
- *The A^{e}-module $\mathrm{Hom}_A(N, N')$ carries a left U-module structure via*
$$(u \cdot f)(n) := \big(f(nu_+)\big)u_-.$$
- *The A^{e}-module ${}_\blacktriangleright N \otimes_{A^{\mathrm{op}}} M_\triangleleft$ carries a right U-module structure via*
$$(n \otimes_{A^{\mathrm{op}}} m) \cdot u := nu_+ \otimes_{A^{\mathrm{op}}} u_- m.$$

(ii) *Let (U, A) be a right Hopf left bialgebroid instead.*
- *The A^{e}-module $\mathrm{Hom}_A(M, M')$ carries a left U-module structure given by*
$$(u \cdot f)(m) := u_{[+]}\big(f(u_{[-]} m)\big).$$
- *The A^{e}-module $\mathrm{Hom}_{A^{\mathrm{op}}}(N, N')$ carries a left U-module structure given by*
$$(u \cdot f)(n) := \big(f(nu_{[+]})\big)u_{[-]}.$$
- *The A^{e}-module $N_\blacktriangleleft \otimes^A {}_\triangleright M$ carries a right U-module structure given by*
$$(n \otimes^A m) \cdot u := nu_{[+]} \otimes^A u_{[-]} m.$$

Corollary 2.8 ([8])**.** *Let U be a left and right left bialgebroid. For any $N \in$ Mod-U, the evaluation map*
$$P_\blacktriangleleft \otimes_{A\,\triangleright} \mathrm{Hom}_A({}_\blacktriangleright P, {}_\blacktriangleright N) \to N, \quad p \otimes_A \phi \mapsto \phi(p)$$
is a morphism of right U-modules.

Proof. A very similar result is stated in [8, Proposition 3.2.1]. □

3. Poincaré duality

We start by recalling the definition of an *invertible module* ([11]).

Definition 3.1. Let A be a k-algebra. An $A \otimes A^{\mathrm{op}}$-module X is *invertible* if there exists an $A \otimes A^{\mathrm{op}}$-module Y and isomorphisms of $A \otimes A^{\mathrm{op}}$-modules
$$f : X \otimes_A Y \to A$$
$$g : Y \otimes_A X \to A$$
such that, for all $(x, y) \in X^2$ and all $(x', y') \in Y$,
$$f(x, y')y = xg(y', y) \quad \text{and} \quad x' f(x, y') = g(x', x)y'.$$

Remark 3.2. In [27], Yekutieli classifies invertible $A \otimes A^{\mathrm{op}}$-modules in the case where A is a non-commutative graded algebra.

Proposition 3.3 ([13, p. 167]). *Let A be a k-algebra and P an $A \otimes A^{\mathrm{op}}$-module. Then, if M is an A-module, we endow $\mathrm{Hom}_A(P, M)$ with the $A \otimes A^{\mathrm{op}}$-module structure: For all $(a, b) \in A$, $p \in P$, and $\lambda \in \mathrm{Hom}_A(P, M)$,*

$$\langle a \cdot \lambda \cdot b, p\rangle = \langle \lambda, p \cdot a\rangle b.$$

P is an invertible A^e-module if and only if it satisfies the following conditions:

- *The A-module P is a finitely generated projective A-module.*
- *The left $A \otimes A^{\mathrm{op}}$-module morphism*

$$g : A \to \mathrm{Hom}_A(P, P), \quad a \mapsto \{p \mapsto p \cdot a\}$$

  is an isomorphism.
- *The evaluation map*

$$ev : P \otimes_A \mathrm{Hom}_A(P, A) \to A, \quad p \otimes_{A^{\mathrm{op}}} \phi \mapsto \phi(p)$$

  is an isomorphism of $A \otimes A^{\mathrm{op}}$-modules.

Remark 3.4. Let U be a left and right Hopf left bialgebroid over A. If moreover, P is endowed with a right U-module structure such that the $A \otimes A^{\mathrm{op}}$-module structure on P is isomorphic to that given by $\blacktriangleright$ and $\blacktriangleleft$, then the evaluation map is an isomorphism of left U-modules (Corollary 2.8).

We can now state the *twisted Poincaré duality:*

Theorem 3.5. *Let U be a left and right Hopf left bialgebroid over A. Assume the following:*

(i) $\mathrm{Ext}^i_U(A, U) = \{0\}$ *if $i \neq d$, and set $\Lambda = \mathrm{Ext}^d_U(A, U)$ with the right U-module structure given by right multiplication on U.*
(ii) *The left U-module A admits a finitely generated projective resolution of finite length.*
(iii) *A is endowed with a right U-module structure (denoted by A_R) such that the A^e-module $_\blacktriangleright A_{R\,\blacktriangleleft}$ is invertible.*
(iv) *Let $\mathcal{T}$ be the left U-module $\mathrm{Hom}_A(_\blacktriangleright A_R, {}_\blacktriangleright \Lambda)$ (see Proposition 2.7). The A-module $_\triangleright \mathcal{T}$ and the A^{op}-module $\mathcal{T}_\triangleleft$ are projective.*

Then, for all left U-modules M and all $n \in \mathbb{N}$, there is an isomorphism

$$\mathrm{Ext}^i_U(A, M) \simeq \mathrm{Tor}^U_{d-i}\left(A_R, \mathcal{T}_\triangleleft \otimes_A {}_\triangleright M\right).$$

Remark 3.6. In the case where U is the enveloping algebra $V(L)$ of a finitely generated projective Lie–Rinehart algebra L over A with anchor $\rho : L \to \mathrm{Der}_k(A)$, the hypotheses are all satisfied (see [6]). More precisely, if L is a projective A-module with constant rank n, then $\mathrm{Ext}^i_{V(L)}(A, V(L)) = \{0\}$ if $i \neq n$. To describe the right $V(L)$-module $\mathrm{Ext}^n_{V(L)}(A, V(L))$, we make use of the *Lie derivative* $\mathcal{L}$ over the Lie–Rinehart algebra L, which we briefly recall.

The k-Lie algebra L acts on $L^* = \mathrm{Hom}_A(L, A)$ as follows: For all $D, \Delta \in L$ and $\lambda \in L^*$,

$$\mathcal{L}_D(\lambda)(\Delta) = \rho(D)\,[\lambda(\Delta)] - \lambda([D, \Delta]).$$

By extension, the Lie derivative $\mathcal{L}_D$ is also defined on $\Lambda^n_A(L^*)$. This allows us to endow $\Lambda^n_A(L^*)$ with a natural right $V(L)$-module structure determined as follows:

$$\forall a \in A,\ \forall D \in L,\ \forall \omega \in \Lambda^n_A(L^*), \quad \omega \cdot a = a\omega, \quad \omega \cdot D = -\mathcal{L}_D(\omega).$$

The right $V(L)$-modules $\mathrm{Ext}^n_{V(L)}(A, V(L))$ and $\Lambda^n_A(L^*)$ are isomorphic ([6]; see [4] or [14] for the special case $L = \mathrm{Der}_k(A)$).

In the particular case where X is an n-dimensional Poisson manifold, $A = \mathcal{C}^\infty(X)$, $L = \Omega^1(X)$ and $L^* = \mathrm{Der}(A)$, the Lie derivative $\mathcal{L}_{df}$ over $\Lambda^n_A(L^*) = \Lambda^n_A(\mathrm{Der}(A))$ is the Lie derivative with respect to the Hamiltonian vector field $H_f = \{f, -\}$.

To prove Theorem 3.5, we will make use of the following lemma, where the U-module structures are given by Proposition 2.7:

Lemma 3.7 ([18, Lemma 16]). *Let U be a right Hopf left bialgebroid. Let N be a right U-module and let M and $\mathcal{T}$ be two left U-modules. Then there is an isomorphism of k-modules:*

$$(N_{\blacktriangleleft} \otimes_{A\,\triangleright} \mathcal{T}) \otimes_U M \simeq N \otimes_U (\mathcal{T}_{\triangleleft} \otimes_{A\,\triangleright} M).$$

Let $P^\bullet$ be a bounded finitely generated projective resolution of the left U-module A and let $Q^\bullet$ be a projective resolution of the left U-module M. The following computation holds in $D^b(k\text{-Mod})$, the bounded derived category of k-modules:

$$\begin{aligned}
\mathrm{RHom}_U(A, M) &\simeq \mathrm{Hom}_U(P^\bullet, M) \\
&\simeq \mathrm{Hom}_U(P^\bullet, U) \otimes_U M \\
&\simeq \Lambda[-d] \otimes_U Q^\bullet \\
&\simeq [A_{R\,\blacktriangleleft} \otimes_{A\,\triangleright} \mathcal{T}] \otimes_U Q^\bullet\ [-d] \quad \text{(Remark 3.4)} \\
&\simeq A_R \otimes_U (\mathcal{T}_{\triangleleft} \otimes_{A\,\triangleright} Q^\bullet) \qquad \text{(Lemma 3.7)} \\
&\simeq A_R \otimes^L_U (\mathcal{T}_{\triangleleft} \otimes_{A\,\triangleright} M).
\end{aligned}$$

The last isomorphism follows from the fact that the A-module ${}_{\triangleright}\mathcal{T}$ is projective and from the next lemma.

Lemma 3.8. *Denote by ${}^{\ell}U$ the left U-module structure on U given by left multiplication. The map*

$$\begin{aligned}
\alpha_r(\mathcal{T}) : {}^{\ell}U_{\blacktriangleleft} \otimes_{A\,\triangleright} \mathcal{T} &\to \mathcal{T}_{\triangleleft} \otimes_{A\,\triangleright} U \\
u \otimes t &\mapsto u_{(1)} t \otimes u_{(2)}
\end{aligned}$$

is an isomorphism. One has $\alpha_r^{-1}(t \otimes u) = u_{[+]} \otimes u_{[-]} t$. Thus the U-module $\mathcal{T}_{\triangleleft} \otimes_{A\,\triangleright} U$ is projective if the A-module ${}_{\triangleright}\mathcal{T}$ is projective.

Theorem 3.5 is thus proved.

Remark 3.9.

(i) In the case where $U = A \otimes A^{\mathrm{op}}$ (see Examples 4.2), $\mathrm{Ext}^i_U(A, M)$ is the Hochschild cohomology and we recover Van den Berg's Hochschild twisted Poincaré duality. Moreover, the beginning of the proof is similar to that of [1, Theorem 1].

(ii) The isomorphism $\mathrm{Ext}^n_U(A, M) \simeq \mathrm{Tor}^U_{d-n}(M_{\triangleleft} \otimes_{A} {}_{\blacktriangleright}\Lambda, A)$ is proved in [18]. But one can show that if the A-module $\Lambda_{\blacktriangleleft}$ is projective, one has an isomorphism $\mathrm{Tor}^U_{d-n}(M_{\triangleleft} \otimes_{A} {}_{\blacktriangleright}\Lambda, A) \simeq \mathrm{Tor}^U_{d-n}(\Lambda, M)$.

In the case of full Hopf algebroids, there is a natural choice of right U-module structure on A.

4. The case of a full Hopf algebroid

Recall that a *full Hopf algebroid* structure ([2], [3]) on a k-module H consists of the following data:

(i) a left bialgebroid structure $H^\ell := (H, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ over a k-algebra A;
(ii) a right bialgebroid structure $H^r := (H, B, s^r, t^r, \Delta_r, \partial)$ over a k-algebra B;
(iii) the assumption that the k-algebra structures for H in (i) and in (ii) be the same;
(iv) a k-module map $S : H \to H$;
(v) some compatibility relations between the previously listed data, for which we refer the reader to [2], [3].

The detailed definition with the same notation can be found in [19]. We shall denote by lower Sweedler indices the left coproduct Δ_ℓ and by upper indices the right coproduct Δ_r, that is, $\Delta_\ell(h) =: h_{(1)} \otimes_A h_{(2)}$ and $\Delta_r(h) =: h^{(1)} \otimes_B h^{(2)}$ for any $h \in H$. A full Hopf algebroid (with bijective antipode) is both a left and right Hopf left bialgebroid but not necessarily vice versa. In this case, the translation maps in (2.1) are given by

$$h_+ \otimes_{A^{\mathrm{op}}} h_- = h^{(1)} \otimes_{A^{\mathrm{op}}} S(h^{(2)}) \quad \text{and} \quad h_{[+]} \otimes_{B^{\mathrm{op}}} h_{[-]} = h^{(2)} \otimes_{B^{\mathrm{op}}} S^{-1}(h^{(1)}),$$

formally similar as for Hopf algebras.

The following proposition ([2, 3]) will be needed to prove the main result in this section.

Proposition 4.1. *Let $H = (H^\ell, H^r)$ be a (full) Hopf algebroid over A with bijective antipode S. Then the following statements hold:*

(i) *The maps $\nu := \partial s^\ell : A \to B^{\mathrm{op}}$ and $\mu := \epsilon s^r : B \to A^{\mathrm{op}}$ are isomorphisms of k-algebras.*
(ii) *One has $\nu^{-1} = \epsilon t^r$ and $\mu^{-1} = \partial t^\ell$.*
(iii) *The pair of maps $(S, \nu) : H^\ell \to (H^r)^{\mathrm{op}}_{\mathrm{coop}}$ gives an isomorphism of left bialgebroids.*
(iv) *The pair of maps $(S, \mu) : H^r \to (H^\ell)^{\mathrm{op}}_{\mathrm{coop}}$ gives an isomorphism of right bialgebroids.*

Examples 4.2.

(i) Let A be a k-algebra; then $A^e = A \otimes_k A^{\mathrm{op}}$ is a A-Hopf algebroid described as follows: For all $a, b \in A$,
- $s^\ell(a) = a \otimes_k 1, \quad t^\ell(b) = 1 \otimes_k b$;
- $\Delta_\ell : A^e \to A^e \otimes_A A^e, \quad a \otimes b \mapsto (a \otimes_k 1) \otimes_A (1 \otimes_k b)$;
- $\epsilon : A^e \to A, \quad a \otimes b \mapsto ab$;
- $s^r(a) = 1 \otimes_k a, \quad t^r(b) = b \otimes_k 1$;
- $\Delta_r : A^e \to A^e \otimes_{A^{\mathrm{op}}} A^e, \quad a \otimes b \mapsto (1 \otimes_k a) \otimes_A (b \otimes_k 1)$;
- $\partial : A^e \to A, \quad a \otimes b \mapsto ba$.

(ii) Let A be a commutative k-algebra and L be a Lie–Rinehart algebra over A. Its enveloping algebra $V(L)$ is endowed with a standard left bialgebroid structure (see Example 2.2). Kowalzig [17] showed that the left bialgebroid $V(L)$ can be endowed with a Hopf algebroid structure if and only if there exists a right $V(L)$-module structure on A. Then the right bialgebroid structure $V(L)_r$ is described as follows: For any $a \in A$, $D \in L$, and $u \in V(L)$,
- $\partial(u) = 1 \cdot u$;
- $\Delta_r : V(L) \to V(L)_{\blacktriangleleft} \otimes_{A\,\blacktriangleright} V(L), \quad \Delta_r(D) = D \otimes_A 1 + 1 \otimes_A D - \partial(X) \otimes_A 1$ and $\Delta_r(a) = a \otimes 1$;
- $S(a) = a$, $S(D) = -D + \partial(D)$.

It is in particular the case if X is a $\mathcal{C}^\infty$ Poisson manifold, $A = \mathcal{C}^\infty(X)$, and $L = \Omega^1(X)$ is the A-module of global differential 1-forms on X. Huebschmann [12] showed that there is a right $V(\Omega^1(X))$-module structure on A determined as follows: For all $(a, u, v) \in A^3$,

$$a \cdot u = au \quad \text{and} \quad a \cdot udv = \{au, v\}.$$

Thus, $V(\Omega^1(X))$ is endowed with a (full) Hopf algebroid structure.

Notation. Let (H^ℓ, H^r, S) be a full Hopf algebroid over A.

(i) If N is a right H^ℓ-module, we will denote by ${}_SN$ the left H^ℓ-module defined by

$$\forall h \in H,\ \forall n \in N, \quad h \cdot_S n = n \cdot S(h).$$

(ii) If M is a left H^ℓ-module, we will denote by M_S the right H^ℓ-module defined by

$$\forall h \in H,\ \forall m \in M, \quad m \cdot_S h = S(h) \cdot m.$$

Remark 4.3. If $H = (H^\ell, H^r, S)$ is a Hopf algebroid over a k-algebra A, we have the following module structures:

- a left H^ℓ-module structure given by $h \cdot_\ell a = \epsilon(hs^\ell(a)) = \epsilon(ht^\ell(a))$;
- a right H^r-module structure given by $\alpha \cdot_r h = \partial(s^r(\alpha)h) = \partial(t^r(a)h)$.

Thanks to Proposition 4.1, these two structures are linked by the relation

$$S(h) \cdot_\ell \mu(\alpha) = \mu[\alpha \cdot_r h].$$

Theorem 4.4. *Let (H^ℓ, H^r) be a full Hopf algebroid over A with bijective antipode S. Consider A with its left H-module structure (as in Remark 4.3). We keep the notation of Proposition 4.1; in particular, $\mu = \epsilon s^r$ and $\nu = \partial s^\ell$.*

(i) *If $a \in A$, then $1 \cdot_S t^\ell(a) = a$. Thus the A-module ${}_{\blacktriangleright}(A_S)$ is free with basis 1.*
(ii) *If $a \in A$, then $\alpha \cdot_S s^\ell(a) = \mu\nu(a)\alpha$. Thus the A^{op}-module $A_{S\,\blacktriangleleft}$ is free with basis 1.*
(iii) *If N is a right H^ℓ-module, the left H^ℓ-module $\mathrm{Hom}_A({}_{\blacktriangleright}(A_S), {}_{\blacktriangleright}N)$ is isomorphic to ${}_SN$.*
(iv) *The A^e-module ${}_{\blacktriangleright}A_{S\,\blacktriangleleft}$ (defined from the right H^ℓ-module structure on A_S) is invertible.*

Proof. (i) Using Proposition 4.1, we have

$$1 \cdot_S t^\ell(a) = S(t^\ell(a))[1] \underset{\text{Prop. 4.1}}{=} t^r\nu(a)[1] = \epsilon\,[t^r\nu(a)] = a.$$

(ii) Similarly, one has: $1 \cdot_S s^\ell(a) = S(s^\ell(a))(1) = \epsilon s^r \nu(a) = \mu\nu(a)$.

(iii) The map

$$\begin{aligned}\mathrm{Hom}_A({}_{\blacktriangleright}A_S, {}_{\blacktriangleright}N) &\to {}_SN\\ \lambda &\mapsto \lambda(1)\end{aligned}$$

is an isomorphism of left H^ℓ-modules, as the following computation shows. Let $\alpha \in A_S$, $h \in H^\ell$, and $\lambda \in \mathrm{Hom}_A({}_{\blacktriangleright}A_S, {}_{\blacktriangleright}N)$. Using assertion (i) and Proposition 4.1, we have

$$\begin{aligned}(h \cdot \lambda)(1) &= \lambda(1 \cdot_S h^{(1)})S(h^{(2)})\\ &= \lambda\big[S(h^{(1)})(1)\big]S(h^{(2)})\\ &= \lambda\big[\epsilon S(h^{(1)})\big]S(h^{(2)})\\ &= \lambda\big[1 \cdot_S t^\ell \epsilon S(h^{(1)})\big]S(h^{(2)})\\ &= \lambda(1)t^\ell\epsilon[S(h^{(1)})]S(h^{(2)})\\ &= \lambda(1)t^\ell\epsilon[S(h)_{(2)}]S(h)_{(1)}\\ &= \lambda(1)S(h).\end{aligned}$$

(iv) Let N be a right H^ℓ-module and let $n \in N$. Denote by λ_n the element of $\mathrm{Hom}_A({}_{\blacktriangleright}A_S, {}_{\blacktriangleright}N)$ determined by $\lambda_n(1) = n$. By assertions (i) and (ii), the map $(A_S)_{\blacktriangleleft} \otimes_{A\,\triangleright} \mathrm{Hom}_A({}_{\blacktriangleright}A_S, {}_{\blacktriangleright}N) \to N$, $p \otimes_{A^{\mathrm{op}}} \phi \mapsto \phi(p)$ is an isomorphism with inverse $n \mapsto 1 \otimes \lambda_n$.

We need now to check that the map $A \to \mathrm{Hom}_A({}_{\blacktriangleright}A_S, {}_{\blacktriangleright}A_S)$, $a \mapsto \{p \mapsto p \blacktriangleleft a\}$ is an isomorphism. By assertion (iii), this boils down to showing that $A \to {}_S(A_S)$, $a \mapsto 1 \blacktriangleleft a$ is an isomorphism. But this is true as $1 \blacktriangleleft a = \mu\nu(a)$. Indeed,

$$1 \blacktriangleleft a = S^2(s^\ell(a))(1) = \epsilon S^2\left[s^\ell(a)\right] = \mu\partial\left[S(s^\ell(a))\right] = \mu\nu\epsilon(s^\ell(a)) = \mu\nu(a). \qquad \square$$

We can now state the *twisted Poincaré duality for full Hopf algebroids.*

Theorem 4.5. *Let (A, H^ℓ, H^r) be a Hopf algebroid over A with bijective antipode S. As in Proposition 4.1, we set $\mu = \epsilon s^r$ and $\nu = \partial s^\ell$. Assume the following:*

(i) $\mathrm{Ext}^i_{H^\ell}(A, H^\ell) = \{0\}$ *if $i \neq d$, and set $\Lambda = \mathrm{Ext}^d_{H^\ell}(A, H^\ell)$.*

(ii) ${}_{\blacktriangleright}\mathrm{Ext}^d_{H^\ell}(A, H^\ell)$ *is a projective A-module and* $\mathrm{Ext}^d_{H^\ell}(A, H^\ell)_{\blacktriangleleft}$ *is a projective A^{op}-module.*

(iii) *The left H^ℓ-module A admits a finitely generated projective resolution of finite length.*

Then for all left H-modules M and all $i \in \mathbb{N}$, there is an isomorphism

$$\mathrm{Ext}^i_{H^\ell}(A, M) \simeq \mathrm{Tor}^{H^\ell}_{d-i}\left(A_S, {}_S\Lambda_{\triangleleft} \otimes_{A\, \triangleright} M\right).$$

As an application, we find a Poincaré duality for smooth Poisson algebras. Assume that X is a $\mathcal{C}^\infty$ Poisson manifold, $L = \Omega^1(X)$, and M is a $V(L)$-module. Huebschmann [12] showed that, for any $i \in \mathbb{N}$, the k-space $\mathrm{Ext}^i_{V(\Omega^1(X))}(A, M)$ coincides with the ith Poisson cohomology space with coefficients in M, $H^i_{\mathrm{Pois}}(A, M)$. Also, the k-space $\mathrm{Tor}_i^{V(\Omega^1(X))}(A_S, M)$ coincides with the ith Poisson homology space with coefficients in M, $H_i^{\mathrm{Pois}}(A, M)$.

Corollary 4.6. *Let X be a $\mathcal{C}^\infty$ n-dimensional Poisson manifold, $A = \mathcal{C}^\infty(X)$, and let M be a left $V(\Omega^1(X))$-module. Let S be the antipode of the (full) Hopf algebroid $V(\Omega^1(X))$ (see Examples 4.2). Then $\mathcal{T}$ is isomorphic to ${}_S\left(\Lambda^n_A \Omega^1(X)^*\right) = {}_S\left[\Lambda^n_A \mathrm{Der}(A)\right]$, where df acts (on the right) on $\Lambda^n_A \mathrm{Der}(A)$ as the opposite of the Lie derivative of the Hamiltonian vector field H_f (see Remark 3.6). For all $i \in \mathbb{N}$, there is an isomorphism*

$$H^i_{\mathrm{Pois}}(A, M) \simeq H^{\mathrm{Pois}}_{n-i}(A, {}_S\left[\Lambda^n_A \mathrm{Der}(A)\right] \otimes_A M).$$

Remark 4.7. This formula is proved in [9] for oriented Poisson manifolds and $M = A$; see also [20] for polynomial algebras with quadratic Poisson structures, [28] for linear Poisson structures, and [21] for general polynomial Poisson algebras.

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