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ON THE PLANARITY, GENUS, AND CROSSCAP OF THE WEAKLY ZERO-DIVISOR GRAPH OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring and $Z(R)$ its zero-divisors set. The weakly zero-divisor graph of R, denoted by $W\Gamma(R)$, is an undirected graph with the nonzero zero-divisors $Z(R)^*$ as vertex set and two distinct vertices x and y are adjacent if and only if there exist $a \in \mathrm{Ann}(x)$ and $b \in \mathrm{Ann}(y)$ such that $ab = 0$. In this paper, we characterize finite rings R for which the weakly zero-divisor graph $W\Gamma(R)$ belongs to some well-known families of graphs. Further, we classify the finite rings R for which $W\Gamma(R)$ is planar, toroidal or double toroidal. Finally, we classify the finite rings R for which the graph $W\Gamma(R)$ has crosscap at most two.

1. Introduction

All rings R considered in this paper will be commutative with unit element $1 \neq 0$. For $x \in R$, the set $\mathrm{Ann}(x) = \{y \in R^* : xy = 0\}$ is the *annihilator* of x. The set of all zero-divisors, nilpotent elements, minimal prime ideals and unit elements of a ring R are denoted by $Z(R)$, $\mathrm{Nil}(R)$, $\mathrm{Min}(R)$ and $U(R)$, respectively. We write $S^* = S \setminus \{0\}$ for any subset S of R. We refer the reader to [6] for any ambiguous notation or vocabulary in ring theory.

Algebraic combinatorics is an area of mathematics which employs methods of abstract algebra in various combinatorial contexts and vice versa. Lately, linking a graph to the algebraic structure has received a lot of attention.

A variety of graphs attached to rings or other algebraic structures can be found in the literature. In [7], Beck introduced for the first time a graph associated to a commutative ring R with the elements of R as its vertices, and was mainly interested in the coloring of commutative rings. In [3], Anderson and Livingston introduced the zero-divisor graph of R, denoted by $\Gamma(R)$, with vertex set $Z(R)^*$ (the set of nonzero zero-divisors of R), and where two vertices $x \neq y \in Z(R)^*$

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are adjacent if and only if $xy = 0$. See [1, 2, 4, 8, 9, 16, 19] for more details. Several authors have looked at the zero-divisor graphs of commutative rings. Later, Redmond [15] established the zero-divisor graph of a noncommutative ring which corresponds to the concept introduced by Demeyer et al. in [10] for semigroups.

In [14], Nikmehr et al. introduced and studied the *weakly zero-divisor graph* of a commutative ring R, denoted by $W\Gamma(R)$. It is an undirected graph with vertex set $Z(R)^*$, where two distinct vertices x and y are adjacent if and only if there exist $a \in \mathrm{Ann}(x)$ and $b \in \mathrm{Ann}(y)$ such that $ab = 0$. The authors in [14] discussed some basic properties of the weakly zero-divisor graph and studied the similarities between $W\Gamma(R)$ and $\Gamma(R)$.

In this paper, we characterize the finite rings R for which $W\Gamma(R)$ is a tree, a unicycle or a split graph. Then we classify the finite rings R for which $W\Gamma(R)$ is a planar, ring, outerplanar, toroidal or double toroidal graph. Finally, we classify the finite rings R for which the graph $W\Gamma(R)$ has crosscap at most two.

2. Preliminaries

Let G be a graph with vertex set $V(G)$. The *distance* between two vertices u and v of G, denoted by $d(u, v)$, is the smallest path from u to v. If there is no such path, then $d(u, v) = \infty$. The *diameter* of G is defined as $\mathrm{diam}(G) = \sup\{d(u, v) : u, v \in V(G)\}$. A *cycle* is a closed path in G. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G. Note that $\mathrm{gr}(G) = \infty$ whenever G contains no cycle. A graph is said to be a *complete graph* if all its vertices are adjacent to each other. A complete graph with n vertices is denoted by K_n. A *bipartite graph* is a graph G whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge in G has one end in V_1 and the other end in V_2. Further, if each vertex of V_1 is adjacent to every vertex of V_2, then G is called a *complete bipartite graph*. The complete bipartite graph with partition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. We write $K_{m,\infty}$ (respectively, $K_{\infty,\infty}$) if one (respectively, both) of the disjoint vertex sets is infinite. A complete bipartite graph of the form $K_{1,n}$ is called a *star graph*. A connected graph is said to be a *tree* if it does not contain cycles. A graph is said to be a *unicycle* whenever it contains a unique cycle. A graph is said to be a *split graph* if its vertex set can be partitioned into a clique and an independent set. We say that a graph is *planar* whenever it can be drawn in the plane in such a way that its edges intersect only at their ends. A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. An undirected graph is said to be *outerplanar* if it can be embedded in the plane in such a way that all the vertices lie on the unbounded face of the drawing. For more details on graph theory, we refer the reader to [17, 18].

The following observation proved by Nikmehr et al. [14] is used frequently in this article and hence given below.

Lemma 2.1 ([14, Lemma 2.1])**.** *If R is a commutative ring, then the following statements hold:*

(1) *If $p-q$ is an edge of $\Gamma(R)$ for some distinct elements $x, y \in Z(R)^*$, then $p-q$ is an edge of $W\Gamma(R)$.*
(2) *If $p \in \mathrm{Nil}(R)^*$, then p is adjacent to all other vertices.*
(3) *$\mathrm{Nil}(R)^*$ is a complete subgraph of $W\Gamma(R)$.*

Theorem 2.2. *If R is a local ring, then $W\Gamma(R)$ is a complete graph.*

Proof. It is clear from Lemma 2.1 (3). □

Theorem 2.3 ([14, Theorem 3.1])**.** *If R is a reduced ring which is not an integral domain, then $W\Gamma(R) = \Gamma(R)$ if and only if $|\operatorname{Min}(R)| = 2$.*

In the following examples, we calculate the weakly zero-divisor graph of some rings.

Example 2.4. If $R = \mathbb{Z}_8$, then $Z(R) = \{0, 2, 4, 6\}$. Also, $\mathrm{Ann}(2) = \{4\}$, $\mathrm{Ann}(4) = \{2, 4, 6\}$ and $\mathrm{Ann}(6) = \{4\}$. Since $2 \cdot 4 = 0$, $4 \cdot 6 = 0$ and $4 \in \mathrm{Ann}(2) \cap \mathrm{Ann}(6)$ such that $4 \cdot 4 = 0$, we have that the graph $W\Gamma(R)$ is K_3.

Example 2.5. If $R = \mathbb{Z}_{25}$, then $Z(R) = \{0, 5, 10, 15, 20\}$. Also, since $5 \cdot 10 = 0$, $5 \cdot 15 = 0$, $5 \cdot 20 = 0$, $10 \cdot 15 = 0$, $10 \cdot 20 = 0$ and $15 \cdot 20 = 0$, we have that the graph $W\Gamma(R)$ is K_4.

Some finite local rings and their weakly zero-divisor graphs are given in Table 1.

3. Basic properties of $W\Gamma(R)$

In this section, we classify the finite rings for which the weakly zero-divisor graph is a unicycle, a tree or a split graph. The following results will play an important role in the characterization of commutative rings whose weakly zero-divisor graph is a unicycle, a tree or a split graph.

Lemma 3.1 ([12, Theorem VI-2])**.** *Let R be a finite commutative ring. Then R decomposes uniquely (up to order of summands) as a direct sum of local rings.*

Lemma 3.2. *If $m \geq 3$ and $R = R_1 \times R_2 \times \cdots \times R_m$ for some commutative rings R_i, then $W\Gamma(R)$ contains K_5 as a subgraph.*

Proof. Let $p_1 = e_1$, $p_2 = e_1 + e_2$, $p_3 = e_2$, $p_4 = e_2 + e_3$, $p_5 = e_3 \in Z(R)^*$, where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$. Since $p_3 \in \mathrm{Ann}(p_1)$ and $p_5 \in \mathrm{Ann}(p_2)$ such that $p_3p_5 = 0$, $p_1p_3 = 0$, $p_1p_4 = 0$, $p_1p_5 = 0$, $p_5 \in \mathrm{Ann}(p_2)$ and $p_1 \in \mathrm{Ann}(p_3)$ such that $p_1p_5 = 0$, $p_5 \in \mathrm{Ann}(p_2)$ and $p_1 \in \mathrm{Ann}(p_4)$ such that $p_1p_5 = 0$, $p_2p_5 = 0$, $p_5 \in \mathrm{Ann}(p_3)$ and $p_1 \in \mathrm{Ann}(p_4)$ such that $p_1p_5 = 0$, $p_3p_5 = 0$, $p_1 \in \mathrm{Ann}(p_4)$ and $p_3 \in \mathrm{Ann}(p_5)$ such that $p_1p_3 = 0$, we see that the vertices $\{p_1, p_2, p_3, p_4, p_5\}$ induce a complete graph with five vertices. □

TABLE 1. Weakly zero-divisor graphs of some finite local commutative rings

$\lvert Z(R)^*\rvert$	Local ring R	$W\Gamma(R)$
1	$\mathbb{Z}_4,\ \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$	K_1
2	$\mathbb{Z}_9,\ \frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$	K_2
3	$\mathbb{Z}_8,\ \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle},\ \frac{\mathbb{F}_4[x]}{\langle x^2\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$	K_3
4	$\mathbb{Z}_{25},\ \frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$	K_4
6	$\mathbb{Z}_{49},\ \frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}$	K_6
7	$\mathbb{Z}_{16},\ \frac{\mathbb{Z}_2[x]}{\langle x^4\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^2-2,x^4\rangle},\ \frac{\mathbb{Z}_2[x]}{\langle x^3-2,x^4\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3+x^2-2,x^4\rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x^3,xy,y^2-x^2\rangle},$ $\frac{\mathbb{Z}_2[x]}{\langle x^3,x^2-2x\rangle},\ \frac{\mathbb{Z}_8[x]}{\langle x^2-4,2x\rangle},\ \frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2-2,y^3\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^2\rangle},$ $\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy-2\rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2\rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2,xy\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3,2x\rangle},\ \frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2\rangle},$ $\frac{\mathbb{Z}_8[x]}{\langle x^2,2x\rangle},\ \frac{\mathbb{F}_8[x]}{\langle x^2\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle},\ \frac{\mathbb{Z}_4[x,y]}{\langle 2x,2y,x^2,y^2,xy\rangle},\ \frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2}$	K_7
8	$\mathbb{Z}_{27},\ \frac{\mathbb{Z}_9[x]}{\langle 3x,x^2-3\rangle},\ \frac{\mathbb{Z}_9[x]}{\langle 3x,x^2-6\rangle}\ \frac{\mathbb{Z}_3[x]}{\langle x^3\rangle},\ \frac{\mathbb{Z}_3[x]}{\langle x,y\rangle^2},\ \frac{\mathbb{Z}_9[x]}{\langle 3,x\rangle^2},\ \frac{\mathbb{F}_9[x]}{\langle x^2\rangle},\ \frac{\mathbb{Z}_9[x]}{\langle x^2+1\rangle}$	K_8

Lemma 3.3. *Let R_1 and R_2 be local commutative rings. If either R_1 or R_2 is not a field, then $W\Gamma(R_1 \times R_2)$ contains K_4 as a subgraph.*

Proof. Suppose without loss of generality that R_1 is not a field with nonzero maximal ideal $\Im_1$. Then there exists $\alpha \in \Im_1^*$ such that $\mathrm{Ann}(\alpha) = \Im_1$. Let $q_1 = (1,0)$, $q_2 = (\alpha,0)$, $q_3 = (\alpha,1)$ and $q_4 = (0,1) \in Z(R)^*$. Since $q_4 \in \mathrm{Ann}(q_1)$ and $q_2 \in \mathrm{Ann}(q_2)$ with $q_2q_4 = 0$, $q_4 \in \mathrm{Ann}(q_1)$ and $q_2 \in \mathrm{Ann}(q_3)$ with $q_2q_4 = 0$, $q_1q_4 = 0$, $q_4 \in \mathrm{Ann}(q_2)$ and $q_2 \in \mathrm{Ann}(q_3)$ with $q_2q_4 = 0$, $q_2q_4 = 0$, $q_2 \in \mathrm{Ann}(q_3)$ and $q_2 \in \mathrm{Ann}(q_4)$ with $q_2q_2 = 0$, we get that $\{q_1,q_2,q_3,q_4\}$ induces a K_4 in $W\Gamma(R_1 \times R_2)$. □

Now we are ready to characterize the finite commutative rings such that their weakly zero-divisor graph is a unicycle, a tree or a split graph.

Theorem 3.4. *If R is a finite commutative ring, then $W\Gamma(R)$ is a unicycle if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. Assume that $W\Gamma(R)$ is a unicycle. Since R is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \Im_i)$ is a local ring for each i and $m \geq 1$. If $m \geq 3$, then by Lemma 3.2, $W\Gamma(R)$ contains two different cycles, which contradicts our assumption.

If $m = 2$ and either R_1 or R_2 is not a field, then by Lemma 3.3, $W\Gamma(R)$ contains two different cycles, a contradiction. Hence R_1 and R_2 are both fields.

This implies that $W\Gamma(R) \cong K_{|R_1^*|,|R_2^*|}$. Since we are assuming that $W\Gamma(R)$ is a unicycle, $|R_1^*| = 2$ and $|R_2^*| = 2$. Therefore, $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Finally, if $m = 1$, then R is a local ring. Thus by Theorem 2.2, $W\Gamma(R)$ is complete. Since we are assuming that $W\Gamma(R)$ is a unicycle, $|Z(R)^*| = 3$. Therefore, by Table 1, $R \cong \mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$. □

Theorem 3.5. *If R is a finite commutative ring, then $W\Gamma(R)$ is a tree if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$ or $\mathbb{Z}_2 \times \mathbb{F}$.*

Proof. Assume that $W\Gamma(R)$ is a tree. Since R is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \Im_i)$ is a local ring for each i and $m \geq 1$. If $m \geq 3$, then by Lemma 3.2, $W\Gamma(R)$ contains a cycle, a contradiction. Hence $m \leq 2$.

Now, if $m = 2$ and either R_1 or R_2 is not a field, then by Lemma 3.3, $W\Gamma(R)$ contains a cycle, a contradiction. Hence R_1 and R_2 are both fields. Thus, $W\Gamma(R) \cong K_{|R_1^*|,|R_2^*|}$. Since we are assuming that $W\Gamma(R)$ is a tree, $|R_1^*| = 1$ or $|R_2^*| = 1$. Hence $R_1 \cong \mathbb{Z}_2$ or $R_2 \cong \mathbb{Z}_2$.

If $n = 1$, then $W\Gamma(R)$ is a complete graph by Theorem 2.2, because R is a local ring. Also, we are assuming that $W\Gamma(R)$ is a tree, then $1 \leq |Z(R)^*| \leq 2$. Therefore by Table 1, $R \cong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$. □

Theorem 3.6 ([17])**.** *If G is a connected graph, then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$, C_4 or C_5.*

Theorem 3.7. *If R is a finite commutative ring with $|Z(R)^*| \geq 2$, then $W\Gamma(R)$ is a split graph if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$ or $\mathbb{Z}_2 \times \mathbb{F}$.*

Proof. Assume that $W\Gamma(R)$ is a split graph. Since R is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \Im_i)$ is a local ring for each i and $m \geq 1$. If $m \geq 3$, then by Lemma 3.2, $W\Gamma(R)$ contains C_4, a contradiction by Theorem 3.6.

Now, if $m = 2$ and either R_1 or R_2 is a field, then by Lemma 3.3, $W\Gamma(R)$ contains C_4, a contradiction by Theorem 3.6. Hence R_1 and R_2 are both fields. Thus, $W\Gamma(R) \cong K_{|R_1^*|,|R_2^*|}$. Since we are assuming that $W\Gamma(R)$ is a split graph, $|R_1^*| = 1$ or $|R_2^*| = 1$. Hence $R_1 \cong \mathbb{Z}_2$ or $R_2 \cong \mathbb{Z}_2$.

Finally, if $m = 1$, then $W\Gamma(R)$ is a complete graph, because R is local. Also, we are assuming that $W\Gamma(R)$ is a split graph, then $2 \leq |Z(R)^*| \leq 3$. Therefore by Table 1, $R \cong \mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$. □

4. Planar, outerplanar, and ring graph $W\Gamma(R)$

In this section, we characterize the finite commutative rings R for which $W\Gamma(R)$ is a planar, a ring or an outerplanar graph. We recall the characterization of planar graphs given by Kuratowski, which will play an important role in the characterization of commutative rings whose weakly zero-divisor graph is planar.

Theorem 4.1 (Kuratowski's Theorem, [17])**.** *A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Theorem 4.2. *If R is a finite commutative ring, then $W\Gamma(R)$ is a planar graph if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, $\mathbb{Z}_{25}$, $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$, $\mathbb{Z}_2\times\mathbb{F}$, $\mathbb{Z}_3\times\mathbb{F}$, $\mathbb{Z}_4\times\mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}\times\mathbb{Z}_2$.*

Proof. Assume that $W\Gamma(R)$ is planar. Since R is finite, by Lemma 3.1, $R\cong R_1\times R_2\times\cdots\times R_m$, where $(R_i,\Im_i)$ is a local ring for each i and $m\geq 1$. If $m\geq 3$, then by Lemma 3.2, $W\Gamma(R)$ contains K_5 as a subgraph, a contradiction by Theorem 4.1.

If $m=2$ and $\Im_i\neq(0)$ for each $i=1,2$, then by [14, Theorem 2.6], $W\Gamma(R)$ contains K_5 induced by the set $\{(1,0),(\alpha_1,0),(0,\alpha_2),(\alpha_1,\alpha_2),(0,1)\}$, where $\alpha_i\in\Im_i^*$ for each i, a contradiction by Theorem 4.1. Hence one of the R_i must be a field. Consider the following cases:

Case (i) If R_1 and R_2 both are fields, then $W\Gamma(R)\cong K_{|R_1^*|,|R_2^*|}$. Since we are assuming that $W\Gamma(R)$ is planar, $|R_1^*|\leq 2$ or $|R_2^*|\leq 2$ by Theorem 4.1. Hence $R\cong\mathbb{Z}_2\times\mathbb{F}$ or $\mathbb{Z}_3\times\mathbb{F}$.

Case (ii) If R_1 is not a field with $\Im_1\neq(0)$ and R_2 is a field, then there is $\alpha\in\Im_1^*$ such that $\operatorname{Ann}(\alpha)=\Im_1$. Suppose $|\Im_1^*|\geq 2$. Let $q_1=(0,1)$, $q_2=(\alpha,0)$, $q_3=(\beta,0)$, $r_1=(1,0)$, $r_2=(\gamma,0)$, $r_3=(\delta,0)$, where $\alpha\neq\beta\in\Im_1^*$ and $1\neq\gamma,\delta\in U(R_1)$. Since $q_1r_i=0$, $q_2\in\operatorname{Ann}(q_2)$ and $q_1\in\operatorname{Ann}(r_i)$ such that $q_1q_2=0$, $q_2\in\operatorname{Ann}(q_3)$ and $q_1\in\operatorname{Ann}(r_i)$ such that $q_1q_2=0$ for each $i=1,2,3$, we get that $\{q_1,q_2,q_3,r_1,r_2,r_3\}$ induces $K_{3,3}$ in $W\Gamma(R)$, a contradiction by Theorem 4.1. Hence $|\Im_1^*|=1$, which shows that $R_1\cong\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$.

Suppose $|R_2^*|\geq 2$ and let $\alpha\in\Im_1^*$ such that $\alpha^2=0$. Let $s_1=(1,0)$, $s_2=(\alpha,0)$, $s_3=(\alpha_1,0)$, $t_1=(0,1)$, $t_2=(0,\alpha_2)$, $t_3=(\alpha,1)\in Z(R)^*$, where $1\neq\alpha_1\in U(R_1)$ and $1\neq\alpha_2\in R_2^*$. Since $s_it_j=0$ for each $j=1,2$, $t_1\in\operatorname{Ann}(s_i)$ and $s_2\in\operatorname{Ann}(t_3)$ such that $s_2t_1=0$ for each $i=1,2,3$, we get that $\{s_1,s_2,s_3,t_1,t_2,t_3\}$ induces $K_{3,3}$ in $W\Gamma(R)$, a contradiction by Theorem 4.1. Hence $|R_2^*|=1$, which shows that $R_2\cong\mathbb{Z}_2$.

Finally, if $m=1$, then $W\Gamma(R)$ is a complete graph by Theorem 2.2, because R is a local ring. Also, we are assuming that $W\Gamma(R)$ is a planar graph, then $1\leq|Z(R)^*|\leq 4$ by Theorem 4.1. Therefore by Table 1, $R\cong\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$.

Conversely, if $R\cong\mathbb{Z}_2\times\mathbb{F}$ or $\mathbb{Z}_3\times\mathbb{F}$, then $W\Gamma(R)\cong K_{1,n}$ or $K_{2,n}$, where $n\geq 1$ is a positive integer. Hence $W\Gamma(R)$ is planar by Theorem 4.1. If $R\cong\mathbb{Z}_4\times\mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}\times\mathbb{Z}_2$, the planar embedding of $W\Gamma(R)$ is shown in Figure 1. Also, if $R\cong\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$, then the result follows from Table 1 and Theorem 4.1. □

Let C be a cycle of G. Any edge in G that connects two nonadjacent vertices in C is called a *chord*. A *primitive* cycle is one that has no chords. Furthermore, we claim that G has the *primitive cycle property* (PCP) if any two primitive cycles

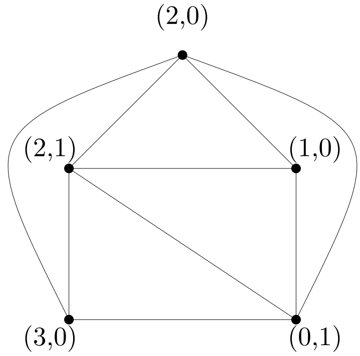

FIGURE 1. Planar embedding of $W\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_2) \cong W\Gamma(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2)$.

intersect in at most one edge. The *frank* of G, denoted by $\text{frank}(G)$, equals the number of primitive cycles of G. Also, $\text{rank}(G) = q - n + r$, where q, n and r denote the number of edges, vertices and connected components of G, respectively. Section 2 of [11] contains a detailed definition of a ring graph. The authors in [11] also demonstrated the following equivalence.

Theorem 4.3 ([11])**.** *If G is a connected graph, then following are equivalent:*
1. *G is a ring graph,*
2. $\text{rank}(G) = \text{frank}(G)$,
3. *G satisfies PCP and G does not contain a subdivision of K_4 as a subgraph.*

As a result, each ring graph is planar. In the following theorem, we characterize all finite commutative rings R for which $W\Gamma(R)$ is a ring graph.

Theorem 4.4. *If R is a finite commutative ring, then $W\Gamma(R)$ is a ring graph if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\mathbb{Z}_2 \times \mathbb{F}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. Since every ring graph is a planar graph, it is enough to deal with rings whose weakly zero-divisor graphs are planar. If $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, then $W\Gamma(R)$ contains K_4 induced by the set $\{(1,0),(0,1),(0,2),(2,1)\}$ as shown in Figure 1. Hence by Theorem 4.3, $W\Gamma(R)$ is not a ring graph. Also, if $R \cong \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$, then $W\Gamma(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2) \cong W\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_2)$, which implies that $W\Gamma\big(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2\big)$ is not a ring graph.

If $R \cong \mathbb{Z}_2 \times \mathbb{F}$, then $W\Gamma(R) \cong K_{1,n}$, where $n \geq 1$ is a positive integer. Thus, by Theorem 4.3, $W\Gamma(R)$ is a ring graph. Also, if $R \cong \mathbb{Z}_3 \times \mathbb{F}$, then $W\Gamma(R) \cong K_{2,n-1}$, where $n = |\mathbb{F}|$. Thus, $\text{rank}(W\Gamma(R)) = n-2$ and $\text{frank}(W\Gamma(R)) = \frac{(n-1)(n-2)}{2}$. Hence $W\Gamma(R)$ is a ring graph if and only if $n - 2 = \frac{(n-1)(n-2)}{2}$, which implies that $n = 2$ or $n = 3$. Hence $\mathbb{F} \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

If $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $W\Gamma(R) \cong K_1$ by Table 1, which is a ring graph. If $R \cong \mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, then $W\Gamma(R) \cong K_2$, again a ring graph. If $R \cong \mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$,

$\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, then $W\Gamma(R) \cong K_3$ by Table 1; this is also a ring graph. If $R \cong \mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$, then $W\Gamma(R) \cong K_4$, which is not a ring graph by Theorem 4.3. □

Theorem 4.5 ([17]). *A graph G is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.*

In the next theorem, we determine all finite commutative rings with outerplanar weakly zero-divisor graphs.

Theorem 4.6. *If R is a finite commutative ring, then $W\Gamma(R)$ is an outerplanar graph if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, $\mathbb{Z}_2 \times \mathbb{F}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. In view of Theorems 4.3 and 4.5, one can say that every outerplanar graph is a ring graph. Thus it is enough to deal with the rings R for which $W\Gamma(R)$ is a ring graph. Hence the result follows from Theorem 4.4. □

5. Genus of $W\Gamma(R)$

In this section, we classify the finite commutative rings R for which $W\Gamma(R)$ has genus at most two.

The minimal integer k such that a graph G can be drawn without crossing itself on a sphere with k handles (i.e. an oriented surface of genus k) is called the *genus* of G, denoted by $\gamma(G)$. A planar graph has genus 0 because it can be drawn on a sphere without self-crossing. The following results deal with genus features of complete and complete bipartite graphs.

Lemma 5.1 ([18]). $\gamma(K_m) = \lceil \frac{(m-3)(m-4)}{12} \rceil$ *if $m \geq 3$. In particular, $\gamma(K_m) = 1$ if $m = 5, 6, 7$.*

Lemma 5.2 ([18]). $\gamma(K_{n,m}) = \lceil \frac{(n-2)(m-2)}{4} \rceil$ *if $n, m \geq 2$. In particular, $\gamma(K_{4,4}) = \gamma(K_{3,m}) = 1$ if $m = 3, 4, 5, 6$. Also $\gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{m,3}) = 2$ if $m = 7, 8, 9, 10$.*

Lemma 5.3 ([13]). *If G is a connected graph with q edges and $m \geq 3$ vertices, then*

$$\gamma(G) \geq \left\lceil \frac{q}{6} - \frac{m}{2} + 1 \right\rceil .$$

Lemma 5.4. *If $R \cong R_1 \times R_2 \times \cdots \times R_m$ is a commutative ring, where $(R_i, \Im_i)$ is a commutative ring for each i and $m \geq 4$, then $W\Gamma(R)$ contains K_9 as a subgraph.*

Proof. Let $p_i = e_i$ for $1 \leq i \leq 4$ and $p_5 = e_3 + e_4$, $p_6 = e_2 + e_3$, $p_7 = e_2 + e_4$, $p_8 = e_2 + e_3 + e_4$, $p_9 = e_1 + e_2 \in Z(R)^*$, where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$. Since $p_i p_j = 0$ for each $1 \leq i, j \leq 4$, the subgraph induced by the set $\{p_1, p_2, p_3, p_4\}$ is K_4 in $W\Gamma(R)$. Since $p_1 p_5 = 0$, $p_2 p_5 = 0$, $p_1 \in \mathrm{Ann}(p_i)$ and $p_2 \in \mathrm{Ann}(p_5)$ such that $p_1 p_2 = 0$ for $i = 1, 2$, the subgraph induced by the set $\{p_1, p_2, p_3, p_4, p_5\}$ is K_5 in $W\Gamma(R)$. Since $p_1 p_6 = 0$, $p_1 \in \mathrm{Ann}(p_i)$ and $p_4 \in \mathrm{Ann}(p_6)$ such that $p_1 p_4 = 0$ for $i = 2, 3, 5$, $p_4 p_6 = 0$, we have that $\{p_1, p_2, \ldots, p_6\}$ induces K_6 in $W\Gamma(R)$. Since

$p_1p_7 = 0$, $p_1 \in \mathrm{Ann}(p_i)$ and $p_3 \in \mathrm{Ann}(p_7)$ such that $p_1p_3 = 0$ for $i = 2, 4, 5, 6$, $p_3p_7 = 0$, we have that $\{p_1, p_2, \dots, p_7\}$ induces K_7 in $W\Gamma(R)$. Since $p_1p_8 = 0$, $p_3 \in \mathrm{Ann}(p_i)$ and $p_1 \in \mathrm{Ann}(p_8)$ such that $p_1p_3 = 0$ for $i = 2, 7$, $p_2 \in \mathrm{Ann}(p_i)$ and $p_1 \in \mathrm{Ann}(p_8)$ such that $p_1p_2 = 0$ for $i = 3, 4, 5$, $p_4 \in \mathrm{Ann}(p_6)$ and $p_1 \in \mathrm{Ann}(p_8)$ such that $p_1p_4 = 0$, we have that $\{p_1, p_2, \dots, p_8\}$ induces K_8 in $W\Gamma(R)$. Since $p_3 \in \mathrm{Ann}(p_i)$ and $p_1 \in \mathrm{Ann}(p_9)$ such that $p_1p_3 = 0$ for $i = 1, 2, 4, 7$, $p_1 \in \mathrm{Ann}(p_i)$ and $p_4 \in \mathrm{Ann}(p_9)$ such that $p_1p_4 = 0$ for $i = 3, 5, 6, 8$, we have that $\{p_1, p_2, \dots, p_9\}$ induces K_9 in $W\Gamma(R)$. $\square$

Lemma 5.5. *Let $R \cong R_1 \times R_2 \times R_3$ be a commutative ring, where R_i is a local ring for each $i = 1, 2, 3$. If R_i is not a field for at least one $i = 1, 2, 3$, then $W\Gamma(R)$ contains $K_{4,7}$ as a subgraph.*

Proof. Suppose without loss of generality that R_1 is not a field with nonzero maximal ideal $\Im_1$. Then there is $\alpha \in \Im_1^*$ such that $\mathrm{Ann}(\alpha) = \Im_1$. Consider $q_1 = (1, 0, 0)$, $q_2 = (\alpha, 0, 0)$, $q_3 = (1, 1, 0)$, $q_4 = (0, 1, 0)$, $r_1 = (0, 0, 1)$, $r_2 = (1, 0, 1)$, $r_3 = (0, 1, 1)$, $r_4 = (\alpha, 1, 0)$, $r_5 = (\alpha, 0, 1)$, $r_6 = (\alpha, 1, 1)$, $r_7 = (u, 0, 0) \in Z(R)^*$, where $1 \neq u \in U(R_1)$. Since $q_1r_1 = 0$, $r_1 \in \mathrm{Ann}(q_1)$ and $q_4 \in \mathrm{Ann}(r_2)$ such that $q_4r_1 = 0$, $q_1r_3 = 0$, $q_4 \in \mathrm{Ann}(q_1)$ and $r_1 \in \mathrm{Ann}(r_4)$ such that $q_4r_1 = 0$, $r_1 \in \mathrm{Ann}(q_1)$ and $q_4 \in \mathrm{Ann}(r_5)$ such that $q_4r_1 = 0$, $q_4 \in \mathrm{Ann}(q_1)$ and $q_2 \in \mathrm{Ann}(r_6)$ such that $q_2q_4 = 0$, $q_4 \in \mathrm{Ann}(q_1)$ and $r_1 \in \mathrm{Ann}(r_7)$ such that $q_4r_1 = 0$, $q_2r_1 = 0$, $r_1 \in \mathrm{Ann}(q_2)$ and $q_4 \in \mathrm{Ann}(r_2)$ such that $q_4r_1 = 0$, $q_2r_3 = 0$, $r_1 \in \mathrm{Ann}(r_4)$ and $q_4 \in \mathrm{Ann}(q_2)$ such that $q_4r_1 = 0$, $r_1 \in \mathrm{Ann}(q_2)$ and $q_4 \in \mathrm{Ann}(r_5)$ such that $q_4r_1 = 0$, $q_4 \in \mathrm{Ann}(q_2)$ and $q_2 \in \mathrm{Ann}(r_6)$ such that $q_2q_4 = 0$, $r_1 \in \mathrm{Ann}(q_2)$ and $q_4 \in \mathrm{Ann}(r_7)$ such that $q_4r_1 = 0$, $q_3r_1 = 0$, $r_1 \in \mathrm{Ann}(q_3)$ and $q_4 \in \mathrm{Ann}(r_2)$ such that $q_4r_1 = 0$, $r_1 \in \mathrm{Ann}(q_3)$ and $q_1 \in \mathrm{Ann}(r_3)$ such that $q_1r_1 = 0$, $r_1 \in \mathrm{Ann}(q_3)$ and $q_2 \in \mathrm{Ann}(r_4)$ such that $q_2r_1 = 0$, $r_1 \in \mathrm{Ann}(q_3)$ and $q_4 \in \mathrm{Ann}(r_5)$ such that $q_4r_1 = 0$, $r_1 \in \mathrm{Ann}(q_3)$ and $q_2 \in \mathrm{Ann}(r_6)$ such that $q_2r_1 = 0$, $r_1 \in \mathrm{Ann}(q_3)$ and $q_4 \in \mathrm{Ann}(r_7)$ such that $q_4r_1 = 0$, $q_4r_1 = 0$, $q_4r_2 = 0$, $r_1 \in \mathrm{Ann}(q_4)$ and $q_1 \in \mathrm{Ann}(r_3)$ such that $q_1r_1 = 0$, $q_1 \in \mathrm{Ann}(q_4)$ and $r_1 \in \mathrm{Ann}(r_4)$ such that $q_1r_1 = 0$, $q_4r_5 = 0$, $r_1 \in \mathrm{Ann}(q_4)$ and $q_2 \in \mathrm{Ann}(r_6)$ such that $q_2r_1 = 0$, $q_4r_7 = 0$, we get that $\{q_1, q_2, q_3, q_4, r_1, r_2, \dots, r_7\}$ induces $K_{4,7}$ in $W\Gamma(R)$. $\square$

Lemma 5.6. *Let $R \cong F_1 \times F_2 \times F_3$ be a commutative ring, where F_i is a field for each $i = 1, 2, 3$. If $|F_i| \geq 3$ for some $i = 1, 2, 3$, then $W\Gamma(R)$ contains K_9 as a subgraph.*

Proof. Suppose without loss of generality that $|F_1| \geq 3$. Let $s_1 = (1, 0, 0)$, $s_2 = (\alpha, 0, 0)$, $s_3 = (0, 1, 0)$, $s_4 = (0, 0, 1)$, $s_5 = (1, 1, 0)$, $s_6 = (\alpha, 1, 0)$, $s_7 = (1, 0, 1)$, $s_8 = (\alpha, 0, 1)$, $s_9 = (0, 1, 1) \in Z(R)^*$, where $1 \neq \alpha \in F_1^*$. Since s_i is adjacent with s_j for each i and j, $\{s_1, s_2, \dots, s_9\}$ induces K_9 in $W\Gamma(R)$. $\square$

Lemma 5.7. *Let $R \cong R_1 \times F$ be a commutative ring, where $(R_1, \Im_1)$ is a local ring with $\Im_1 \neq (0)$ and F is a field. If $|\Im_1|^* = 2$, then $W\Gamma(R)$ contains $K_{6,5}$ as a subgraph.*

Proof. Since $|\Im_1^*| = 2$, it follows that $R_1 \cong \mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$ and hence $|U(R_1)| = 6$. Let $\alpha, \beta \in \Im_1^*$ be such that $\alpha\beta = 0$ and $\mathrm{Ann}(\alpha) = \Im_1$. Let $w_1 = (\delta_1, 0)$, $w_2 = (\delta_2, 0)$,

$w_3 = (\delta_3, 0)$, $w_4 = (\delta_4, 0)$, $w_5 = (\delta_5, 0)$, $w_6 = (\delta_6, 0)$, $z_1 = (\alpha, 0)$, $z_2 = (\beta, 0)$, $z_3 = (0, 1)$, $z_4 = (\alpha, 1)$, $z_5 = (\beta, 1) \in Z(R)^*$, where $\delta_i \in U(R_1)$ for each i. Since $z_3 \in \mathrm{Ann}(w_i)$ and $z_1 \in \mathrm{Ann}(z_j)$ such that $z_1 z_3 = 0$ for $1 \le i \le 6$ and $1 \le j \le 5$, we have that $\{w_1, w_2, \dots, w_6, z_1, z_2, \dots, z_5\}$ induces $K_{6,5}$ in $W\Gamma(R)$. □

Lemma 5.8. *Let $R \cong R_1 \times F$ be a commutative ring, where $(R_1, \Im_1)$ is a local ring with $\Im_1 \neq (0)$ and F is a field. If $|\Im_1|^* \ge 3$, then $W\Gamma(R)$ contains $K_{4,7}$ as a subgraph.*

Proof. Suppose $\alpha, \beta, \delta \in \Im_1^*$ are such that $\alpha\beta = \alpha\delta = 0$ and $\mathrm{Ann}(\alpha) = \Im_1$. Let $e_1 = (1, 0)$, $e_2 = (u, 0)$, $e_3 = (v, 0)$, $e_4 = (w, 0)$, $f_1 = (\alpha, 0)$, $f_2 = (\beta, 0)$, $f_3 = (0, 1)$, $f_4 = (\alpha, 1)$, $f_5 = (\beta, 1)$, $f_6 = (\delta, 0)$, $f_7 = (\delta, 1) \in Z(R)^*$, where $1 \neq u, v, w \in U(R_1)$. Since $f_3 \in \mathrm{Ann}(e_i)$ and $f_1 \in \mathrm{Ann}(f_j)$ such that $e_1 f_3 = 0$ for $1 \le i \le 4$ and $1 \le j \le 7$, we have that $\{e_1, e_2, e_3, e_4, f_1, f_2, \dots, f_7\}$ induces $K_{4,7}$ in $W\Gamma(R)$. □

Lemma 5.9. *Let $R \cong R_1 \times F$ be a commutative ring, where $(R_1, \Im_1)$ is a local ring with $\Im_1 \neq (0)$ and F is a field. If $|\Im_1^*| = 1$ and $|F| \ge 4$, then $W\Gamma(R)$ contains $K_9 \setminus \{e\}$ as a subgraph, where e denotes an edge.*

Proof. Let $\alpha \in \Im_1^*$ be such that $\alpha^2 = 0$. Let $k_1 = (0, 1)$, $k_2 = (0, a)$, $k_3 = (0, b)$, $k_4 = (\alpha, 1)$, $k_5 = (\alpha, a)$, $k_6 = (1, 0)$, $k_7 = (\alpha, 0)$, $k_8 = (u, 0)$, $k_9 = (\alpha, b) \in Z(R)^*$, where $1 \neq a, b \in F^*$ and $1 \neq u \in U(R_1)$. It is easy to see that $W\Gamma(R)$ contains $K_9 \setminus \{e\}$ induced by the set $\{k_1, k_2, \dots, k_9\}$. □

Now, we can characterize the finite commutative rings R with genus one $W\Gamma(R)$.

Theorem 5.10. *If R is a finite commutative ring, then $\gamma(W\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}$, $\mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x]}{\langle x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2-2,x^4\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3-2,x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x^2-2,x^4\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^3,xy,y^2-x^2\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3,x^2-2x\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2-4,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2-2,y^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2,xy\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2,2x\rangle}$, $\frac{\mathbb{F}_8[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle 2x,2y,x^2,y^2,xy\rangle}$, $\frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2}$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_7$, $\mathbb{Z}_4 \times \mathbb{Z}_3$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Assume that $\gamma(W\Gamma(R)) = 1$. Since R is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \Im_i)$ is a local ring for each i and $m \ge 1$. If $m \ge 4$, then by Lemma 5.4, $W\Gamma(R)$ contains K_9. Thus by Lemma 5.1, $\gamma(W\Gamma(R)) \ge 3$, a contradiction. Hence $m \le 3$. Consider the following cases:

Case (i) If $m = 3$ and R_i is not a field for some $i = 1, 2, 3$, then by Lemma 5.5, $W\Gamma(R)$ contains $K_{4,7}$ as a subgraph. Thus by Lemma 5.2, $\gamma(W\Gamma(R)) \ge 3$, a contradiction. Hence R_i is a field for each $i = 1, 2, 3$.

If $|R_i| \ge 3$ for some $i = 1, 2, 3$, then by Lemma 5.6, $W\Gamma(R)$ contains K_9 as a subgraph. Thus by Lemma 5.1, $\gamma(W\Gamma(R)) \ge 3$, a contradiction. Hence $|R_i| = 2$ for each $i = 1, 2, 3$. This implies that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case (ii) If $m = 2$ and $\Im_i \neq (0)$ for each $i = 1, 2$, then by [14, Theorem 2.6], $W\Gamma(R)$ contains K_8 induced by the set $\{(1, 0), (\alpha_1, 0), (0, \alpha_2), (0, 1), (\alpha_1, 1), (1, \alpha_2), (\alpha_1, \alpha_2), (u, 0)\}$, where $\alpha_i \in \Im_i$ for $i = 1, 2$ and $1 \neq u \in U(R_1)$. Thus, $\gamma(W\Gamma(R)) > 1$ by

Lemma 5.1, a contradiction. Hence at least one of the R_i is a field. Consider the following subcases:

Subcase (a) If R_1 and R_2 both are fields, then by Theorem 2.3, $W\Gamma(R) = \Gamma(R)$. Hence $R \cong \mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{F}_4 \times \mathbb{Z}_7$ by [19, Theorem 3.1].

Subcase (b) Suppose R_1 is not a field with $\Im_1 \neq (0)$ and R_2 is a field. If $|\Im_1^*| = 2$, then by Lemma 5.7, $W\Gamma(R)$ contains $K_{6,5}$. Thus $\gamma(W\Gamma(R)) \geq 3$ by Lemma 5.2, a contradiction. Also, if $|\Im_1^*| \geq 3$, then by Lemma 5.8, $W\Gamma(R)$ contains $K_{4,7}$, a contradiction by Lemma 5.2. Hence $|\Im_1^*| = 1$, which shows that $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$. Finally, if $|R_2| \geq 4$, then by Lemma 5.9, $W\Gamma(R)$ contains $K_9 \setminus \{e\}$, a contradiction by Lemma 5.1. Hence $|R_2| \leq 3$. It is clear from Theorem 4.2 that $|R_1| \neq 2$. Hence $R_2 \cong \mathbb{Z}_3$.

Case (iii) If $m = 1$, then $W\Gamma(R)$ is a complete graph, because R is local. Also, we are assuming that $\gamma(W\Gamma(R)) = 1$, then $5 \leq |Z(R)^*| \leq 7$. Therefore by Table 1, $R \cong \mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}$, $\mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x]}{\langle x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2-2,x^4\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3-2,x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x^2-2,x^4\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^3,xy,y^2-x^2\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3,x^2-2x\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2-4,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2-2,y^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2,xy\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2,2x\rangle}$, $\frac{\mathbb{F}_8[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle 2x,2y,x^2,y^2,xy\rangle}$ or $\frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2}$.

Conversely, if $R \cong \mathbb{Z}_{49}$ or $\frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}$, then $W\Gamma(R) \cong K_6$. Thus by Lemma 5.1, $\gamma(W\Gamma(R)) = 1$. If $R \cong \mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x]}{\langle x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2-2,x^4\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3-2,x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x^2-2,x^4\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^3,xy,y^2-x^2\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3,x^2-2x\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2-4,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2-2,y^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2,xy\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2,2x\rangle}$, $\frac{\mathbb{F}_8[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle 2x,2y,x^2,y^2,xy\rangle}$ or $\frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2}$, then $W\Gamma(R) \cong K_7$. Thus $\gamma(W\Gamma(R)) = 1$ again by Lemma 5.1. If $R \cong \mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{F}_4 \times \mathbb{Z}_7$, then $\gamma(W\Gamma(R)) = \gamma(\Gamma(R)) = 1$ by [19, Theorem 3.1]. If $R \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3$, the toroidal embedding of $W\Gamma(R)$ is shown in Figure 2. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $W\Gamma(R) \cong K_6$ by [14, Theorem 2.6]. Hence $\gamma(W\Gamma(R)) = 1$ by Lemma 5.1. □

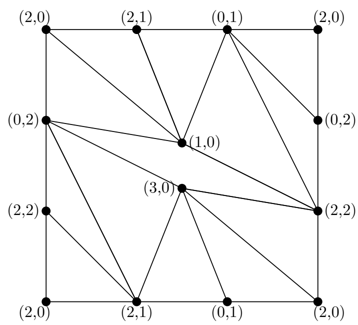

FIGURE 2. Toroidal embedding of $W\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_3) \cong W\Gamma\big(\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3\big)$.

We end this section with the classification of finite commutative rings R with genus two $W\Gamma(R)$.

Theorem 5.11. *If R is a finite commutative ring, then $\gamma(W\Gamma(R)) = 2$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_{27}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x, x^2-3\rangle}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x, x^2-6\rangle}$ $\frac{\mathbb{Z}_3[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_3[x]}{\langle x,y\rangle^2}$, $\frac{\mathbb{Z}_9[x]}{\langle 3,x\rangle^2}$, $\frac{\mathbb{F}_9[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_9[x]}{\langle x^2+1\rangle}$, $\mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$ or $\mathbb{Z}_5 \times \mathbb{Z}_7$.*

Proof. Assume that $\gamma(W\Gamma(R)) = 2$. Since R is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \Im_i)$ is a local ring for each i and $m \geq 1$. If $m \geq 4$, then by Lemma 5.4, $W\Gamma(R)$ contains K_9. Thus by Lemma 5.1, $\gamma(W\Gamma(R)) \geq 3$, a contradiction. Hence $m \leq 3$. Consider the following cases:

Case (i) If $m = 3$ and R_i is not a field for some $i = 1, 2, 3$, then by Lemma 5.5, $W\Gamma(R)$ contains $K_{4,7}$ as a subgraph. Thus by Lemma 5.2, $\gamma(W\Gamma(R)) \geq 3$, a contradiction. Hence R_i is a field for each $i = 1, 2, 3$.

If $|R_i| \geq 3$ for some $i = 1, 2, 3$, then by Lemma 5.6, $W\Gamma(R)$ contains K_9 as a subgraph. Thus by Lemma 5.1, $\gamma(W\Gamma(R)) \geq 3$, a contradiction. Hence $|R_i| = 2$ for each $i = 1, 2, 3$. This implies that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus $\gamma(W\Gamma(R)) = 1$ by Theorem 5.10, again a contradiction.

Case (ii) If $m = 2$ and $\Im_i \neq (0)$ for each $i = 1, 2$, then by [14, Theorem 2.6], $W\Gamma(R)$ contains K_9 induced by the set $\{(1,0), (\alpha_1, 0), (0, \alpha_2), (0,1), (\alpha_1, 1), (1, \alpha_2), (\alpha_1, \alpha_2), (u, 0), (0, v)\}$, where $\alpha_i \in \Im_i$ for $i = 1, 2$, $1 \neq u \in U(R_1)$ and $1 \neq v \in U(R_2)$. Thus, $\gamma(W\Gamma(R)) > 2$ by Lemma 5.1, a contradiction. Hence at least one of the R_i is a field. Consider the following subcases:

Subcase (a) If R_1 and R_2 both are fields, then by Theorem 2.3, $W\Gamma(R) = \Gamma(R)$. Hence $R \cong \mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$ or $\mathbb{Z}_5 \times \mathbb{Z}_7$ by [5, Theorem 4].

Subcase (b) Suppose R_1 is not a field with $\Im_1 \neq (0)$ and R_2 is a field. If $|\Im_1^*| = 2$, then by Lemma 5.7, $W\Gamma(R)$ contains $K_{6,5}$, a contradiction by Lemma 5.2. Also, if $|\Im_1^*| \geq 3$, then by Lemma 5.8, $W\Gamma(R)$ contains $K_{4,7}$. Thus by Lemma 5.2, $\gamma(W\Gamma(R)) \geq 3$, a contradiction. Hence $|\Im_1^*| = 1$, which shows that $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$. Finally, if $|R_2| \geq 4$, then by Lemma 5.9, $W\Gamma(R)$ contains $K_9 \setminus \{e\}$. Thus, $\gamma(W\Gamma(R)) > 3$ by Lemma 5.3, a contradiction. Hence $|R_2| \leq 3$. If $|R_2| = 2$, then by Theorem 4.2, $\gamma(W\Gamma(R)) = 0$. Also, if $|R_2| = 3$, then by Theorem 5.10, $\gamma(W\Gamma(R)) = 1$. Hence in this case $\gamma(W\Gamma(R)) \neq 2$.

Case (iii) If $m = 1$, then $W\Gamma(R)$ is a complete graph, because R is local. Also, we are assuming that $\gamma(W\Gamma(R)) = 2$, then $|Z(R)^*| = 8$. Therefore by Table 1, $R \cong \mathbb{Z}_{27}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x, x^2-3\rangle}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x, x^2-6\rangle}$ $\frac{\mathbb{Z}_3[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_3[x]}{\langle x,y\rangle^2}$, $\frac{\mathbb{Z}_9[x]}{\langle 3,x\rangle^2}$, $\frac{\mathbb{F}_9[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_9[x]}{\langle x^2+1\rangle}$.

Conversely, if $R \cong \mathbb{Z}_{27}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x, x^2-3\rangle}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x, x^2-6\rangle}$ $\frac{\mathbb{Z}_3[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_3[x]}{\langle x,y\rangle^2}$, $\frac{\mathbb{Z}_9[x]}{\langle 3,x\rangle^2}$, $\frac{\mathbb{F}_9[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_9[x]}{\langle x^2+1\rangle}$, then $W\Gamma(R) \cong K_8$, which implies that $\gamma(W\Gamma(R)) = 2$ by Lemma 5.1. Also, if $R \cong \mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$ or $\mathbb{Z}_5 \times \mathbb{Z}_7$, then by Theorem 2.3 and [5, Theorem 4], $\gamma(W\Gamma(R)) = \gamma(\Gamma(R)) = 2$. ☐

6. Crosscap of $W\Gamma(R)$

In this section, we characterize the finite commutative rings R for which $W\Gamma(R)$ has crosscap at most two.

Let N_k denote the sphere with k crosscaps, where k is a non-negative integer, that is, N_k is a non-oriented surface with k crosscaps. The *crosscap number* of a graph G, denoted by $\overline{\gamma}(G)$, is the minimal integer k such that G can be embedded in N_k. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. It is easy to see that $\overline{\gamma}(H) \leq \overline{\gamma}(G)$ for all subgraphs H of G. The crosscap of various particular types of graphs are given in the following lemmas, which are useful for proving the results of this section.

Lemma 6.1 ([18]). *If $m \geq 3$, then*

$$\overline{\gamma}(K_m) = \begin{cases} \left\lceil \frac{(m-3)(m-4)}{6} \right\rceil & \text{if } m \geq 3 \text{ and } m \neq 7; \\ 3 & \text{if } m = 7. \end{cases}$$

Lemma 6.2 ([18]). *If $n, m \geq 2$, then*

$$\overline{\gamma}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil.$$

Lemma 6.3 ([13]). *If G is a connected graph with q edges and $m \geq 3$ vertices, then*

$$\overline{\gamma}(G) \geq \left\lceil \frac{q}{3} - m + 2 \right\rceil.$$

Now, we can characterize the finite commutative rings R with crosscap at most two $W\Gamma(R)$.

Theorem 6.4. *If R is a finite commutative ring, then $\overline{\gamma}(W\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{\langle x^2 \rangle}$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Since $\gamma(W\Gamma(R)) \leq \overline{\gamma}(W\Gamma(R))$, it is enough to deal with the rings R for which $\gamma(W\Gamma(R)) = 1$. If $R \cong \mathbb{Z}_{49}$ or $\frac{\mathbb{Z}_7[x]}{\langle x^2 \rangle}$, then $W\Gamma(R) \cong K_6$. Thus by Lemma 6.1, $\overline{\gamma}(W\Gamma(R)) = 1$. If $R \cong \mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x]}{\langle x^4 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2-2,x^4 \rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3-2,x^4 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x^2-2,x^4 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^3,xy,y^2-x^2 \rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3,x^2-2x \rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2-4,2x \rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2-2,y^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy-2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2,xy \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,2x \rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2 \rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2,2x \rangle}$, $\frac{\mathbb{F}_8[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle 2x,2y,x^2,y^2,xy \rangle}$ or $\frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z \rangle^2}$, then $W\Gamma(R) \cong K_7$. Thus $\overline{\gamma}(W\Gamma(R)) = 3$ by Lemma 6.1. If $R \cong \mathbb{F}_4 \times \mathbb{F}_4$ or $\mathbb{F}_4 \times \mathbb{Z}_5$, then $W\Gamma(R) \cong K_{3,3}$ or $K_{3,4}$. Thus by Lemma 6.2, $\overline{\gamma}(W\Gamma(R)) = 1$. If $R \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{F}_4 \times \mathbb{Z}_7$, then $W\Gamma(R) \cong K_{4,4}$ or $K_{3,6}$. Thus by Lemma 6.2, $\overline{\gamma}(W\Gamma(R)) = 2$. If $R \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_3$, then $W\Gamma(R)$ contains $K_7 \setminus \{e\}$ induced by the set $\{(1,0),(2,0),(3,0),(0,1),(0,2),(2,1),(2,2)\}$. Thus by Lemma 6.3, $\overline{\gamma}(W\Gamma(R)) > 1$.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $W\Gamma(R) \cong K_6$ by [14, Theorem 2.6]. Hence $\overline{\gamma}(W\Gamma(R)) = 1$ by Lemma 6.1. □

Theorem 6.5. *If R is a finite commutative ring, then $\overline{\gamma}(W\Gamma(R)) = 2$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_7$, $\mathbb{Z}_4 \times \mathbb{Z}_3$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3$.*

Proof. Since $\gamma(W\Gamma(R)) \leq \overline{\gamma}(W\Gamma(R))$, it is enough to deal with the rings R for which $W\Gamma(R)$ has genus at most two. It is clear from Theorem 6.4 that if $R \cong \mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\overline{\gamma}(W\Gamma(R)) = 1$. If $R \cong \mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x]}{\langle x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2-2,x^4\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3-2,x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x^2-2,x^4\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^3,xy,y^2-x^2\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3,x^2-2x\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2-4,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2-2,y^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2,xy\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2,2x\rangle}$, $\frac{\mathbb{F}_8[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle 2x,2y,x^2,y^2,xy\rangle}$ or $\frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2}$, then $W\Gamma(R) \cong K_7$. Thus $\overline{\gamma}(W\Gamma(R)) = 3$ by Lemma 6.1. If $R \cong \mathbb{Z}_{27}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x,x^2-3\rangle}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x,x^2-6\rangle}$ $\frac{\mathbb{Z}_3[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_3[x]}{\langle x,y\rangle^2}$, $\frac{\mathbb{Z}_9[x]}{\langle 3,x\rangle^2}$, $\frac{\mathbb{F}_9[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_9[x]}{\langle x^2+1\rangle}$, then $W\Gamma(R) \cong K_8$. Thus by Lemma 6.2, $\overline{\gamma}(W\Gamma(R)) = 4$. If $R \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{F}_4 \times \mathbb{Z}_7$, then $W\Gamma(R) \cong K_{4,4}$ or $K_{3,6}$ and hence by Lemma 6.2, $\overline{\gamma}(W\Gamma(R)) = 2$. If $R \cong \mathbb{F}_4 \times \mathbb{F}_8$, then $W\Gamma(R) \cong K_{3,7}$ and hence $\overline{\gamma}(W\Gamma(R)) = 3$. If $R \cong \mathbb{F}_4 \times \mathbb{F}_4$, then $W\Gamma(R) \cong K_{3,8}$ and hence $\overline{\gamma}(W\Gamma(R)) = 3$. If $R \cong \mathbb{F}_4 \times \mathbb{F}_{11}$, then $W\Gamma(R) \cong K_{3,10}$ and hence $\overline{\gamma}(W\Gamma(R)) = 4$. If $R \cong \mathbb{Z}_5 \times \mathbb{Z}_7$, then $W\Gamma(R) \cong K_{4,6}$ and hence $\overline{\gamma}(W\Gamma(R)) = 4$. If $R \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3$, the embedding of $W\Gamma(R)$ onto N_2 is shown in Figure 3. □

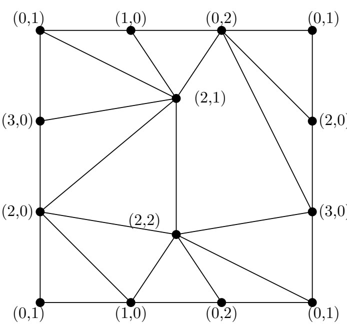

FIGURE 3. Embedding of $W\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_3) \cong W\Gamma\big(\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3\big)$ on N_2.

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References

[1] S. Akbari and A. Mohammadian, On the zero-divisor graph of a commutative ring, *J. Algebra* **274** no. 2 (2004), 847–855. DOI MR Zbl

[2] D. D. Anderson and M. Naseer, Beck's coloring of a commutative ring, *J. Algebra* **159** no. 2 (1993), 500–514. DOI MR Zbl

[3] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** no. 2 (1999), 434–447. DOI MR Zbl

[4] T. Asir and K. Mano, Classification of rings with crosscap two class of graphs, *Discrete Appl. Math.* **265** (2019), 13–21. DOI MR Zbl

[5] T. Asir and K. Mano, Classification of non-local rings with genus two zero-divisor graphs, *Soft Comput.* **24** no. 1 (2020), 237–245, correction ibid. **25**, no. 4, 3355–3356 (2021). DOI Zbl

[6] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass.-London-Don Mills, Ont., 1969. MR Zbl

[7] I. Beck, Coloring of commutative rings, *J. Algebra* **116** no. 1 (1988), 208–226. DOI MR Zbl

[8] N. Bloomfield and C. Wickham, Local rings with genus two zero divisor graph, *Comm. Algebra* **38** no. 8 (2010), 2965–2980. DOI MR Zbl

[9] H.-J. Chiang-Hsieh, Classification of rings with projective zero-divisor graphs, *J. Algebra* **319** no. 7 (2008), 2789–2802. DOI MR Zbl

[10] F. R. DeMeyer, T. McKenzie, and K. Schneider, The zero-divisor graph of a commutative semigroup, *Semigroup Forum* **65** no. 2 (2002), 206–214. DOI MR Zbl

[11] I. Gitler, E. Reyes, and R. H. Villarreal, Ring graphs and complete intersection toric ideals, *Discrete Math.* **310** no. 3 (2010), 430–441. DOI MR Zbl

[12] B. R. McDonald, *Finite rings with identity*, Pure and Applied Mathematics, Vol. 28, Marcel Dekker, New York, 1974. MR Zbl

[13] B. Mohar and C. Thomassen, *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001. MR Zbl

[14] M. J. Nikmehr, A. Azadi, and R. Nikandish, The weakly zero-divisor graph of a commutative ring, *Rev. Un. Mat. Argentina* **62** no. 1 (2021), 105–116. DOI MR Zbl

[15] S. P. Redmond, The zero-divisor graph of a non-commutative ring, *Int. J. Commut. Rings* **1** no. 4 (2002), 203–211. Zbl

[16] H.-J. Wang, Zero-divisor graphs of genus one, *J. Algebra* **304** no. 2 (2006), 666–678. DOI MR Zbl

[17] D. B. West, *Introduction to graph theory*, 2nd ed., New Delhi: Prentice-Hall of India, 2005. Zbl

[18] A. T. White, *Graphs, groups and surfaces*, North-Holland Mathematics Studies, No. 8, North-Holland, Amsterdam-London; American Elsevier, New York, 1973. MR Zbl

[19] C. Wickham, Classification of rings with genus one zero-divisor graphs, *Comm. Algebra* **36** no. 2 (2008), 325–345. DOI MR Zbl

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