REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 67, No. 1, 2024, Pages 65–80
Published online: March 12, 2024
https://doi.org/10.33044/revuma.2932

# SPECTRALITY OF PLANAR MORAN–SIERPINSKI-TYPE MEASURES

QIAN LI AND MIN-MIN ZHANG

ABSTRACT. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of expanding positive integral matrices with $M_n = \begin{pmatrix} p_n & 0 \\ 0 & q_n \end{pmatrix}$ for each $n \geq 1$, and let $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be a finite digit set in $\mathbb{Z}^2$. The associated Borel probability measure obtained by an infinite convolution of atomic measures

$$\mu_{\{M_n\},D} = \delta_{M_1^{-1}D} * \delta_{(M_2M_1)^{-1}D} * \cdots * \delta_{(M_n \cdots M_2 M_1)^{-1}D} * \cdots$$

is called a Moran–Sierpinski-type measure. We prove that, under certain conditions, $\mu_{\{M_n\},D}$ is a spectral measure if and only if $3 \mid p_n$ and $3 \mid q_n$ for each $n \geq 2$.

## 1. Introduction

Let $\mu$ be a Borel probability measure with compact support on $\mathbb{R}^n$. We say that $\mu$ is a *spectral measure* if there exists a countable discrete set $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \left\{ e^{-2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda \right\}$ forms an orthonormal basis for $L^2(\mu)$. In this case, we call $\Lambda$ a *spectrum* of $\mu$. For the special case that a spectral measure is the restriction of the Lebesgue measure on a bounded Borel subset $\Omega$ of $\mathbb{R}^n$, we call $\Omega$ a *spectral set*. The existence of a spectrum is closely related to the famous Fuglede conjecture, which asserts that $\chi_\Omega \mathrm{d}x$ is a spectral measure if and only if $\Omega$ is a translational tile [17]. This conjecture has been proved to be false by Tao and others in both directions on $\mathbb{R}^n$ for $n \geq 3$ [24, 23, 29, 30]. But it is still open for $n = 1$ and $n = 2$.

Jorgensen and Pedersen initiated an investigation of spectral properties of fractal measures [22]. They showed that the Cantor-typed measure $\mu_{1/k}$, which is the invariant measure of the iterated function system $\{\phi_0(x) = x/k, \phi_1(x) = (x+1)/k\}$, with natural weight, is a spectral measure if $k$ is even, but not a spectral one if $k$ is odd. Since then, the study of the spectral properties of fractal measures became an active research topic, where, for example, self-similar measures, self-affine measures and Moran measures were considered and are still objects of study. The readers may see [4, 5, 1, 3, 2, 7, 9, 6, 8, 12, 11, 10, 13, 16, 18, 20, 25, 28, 27, 32, 26, 19, 14] and

2020 *Mathematics Subject Classification.* Primary 46C05; Secondary 42A65, 28A80.
*Key words and phrases.* Spectral measure, Moran measure, bi-zero set, Fourier transform.

the references therein for recent advances. In particular, Hu and Lau [20] showed a necessary and sufficient condition so that $L^2(\mu_\rho)$ contains an infinite orthogonal set for the more general Bernoulli convolution $\mu_\rho$, $0 < \rho < 1$. Recently, Dai [6] completely settled the problem that the only spectral Bernoulli convolution is $\mu_{1/2k}$. The more general $N$-Bernoulli convolution was completely characterized by Dai et al. in [9]. Let $0 < \rho < 1$ and $D = \{0, 1, \ldots, N-1\}$ with $N > 1$; they showed that $\mu_{\rho,D}$ is a spectral measure if and only if $N \mid \rho^{-1}$. Unlike the one-dimensional situation, the study on the spectral properties of measures in higher dimensions is seldom addressed. See e.g. [14, 11, 10, 8, 12, 26, 28, 27]. We note that the most widely studied are the self-affine measures generated by an expanding matrix and a finite digit set.

A Sierpinski-type measure $\mu_{M,D}$ is defined by

$$\mu_{M,D}(\cdot) = \frac{1}{\#D} \sum_{d \in D} \mu_{M,D}(M(\cdot) - d),$$

where $M = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ is an expanding matrix and $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. The Sierpinski-type measure plays an important role in fractal geometry and in geometric measure theory [15, 21]. Deng and Lau [12], and Dai, Fu, and Yan [8] completely characterized the spectrality of the self-affine measure $\mu_{M,D}$. They proved that $\mu_{M,D}$ is a spectral measure if and only if $3 \mid p$ and $3 \mid q$.

Motivated by the above results, in this paper we consider the spectral properties of a class of planar Moran–Sierpinski-type measures. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of expanding positive integral matrices (that is, all the eigenvalues of $M_n$ are strictly greater than 1 in module) with

$$M_n = \begin{pmatrix} p_n & 0 \\ 0 & q_n \end{pmatrix} \in M_2(\mathbb{Z}),$$

and let $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be a finite digit set in $\mathbb{Z}^2$. Write $\delta_D = \frac{1}{\#D} \sum_{d \in D} \delta_d$, where $\#D$ is the cardinality of $D$ and $\delta_d$ is the Dirac measure at point $d$. Then there exists a Borel probability measure with compact support generalized by the infinite convolution

$$\mu_{\{M_n\},D} = \delta_{M_1^{-1}D} * \delta_{(M_2M_1)^{-1}D} * \cdots * \delta_{(M_n \cdots M_2M_1)^{-1}D} * \cdots . \tag{1.1}$$

Here the sign $*$ denotes the convolution of two measures, and the convergence is in the weak sense. The measure $\mu_{\{M_n\},D}$ is called a *Moran–Sierpinski-type measure*, and its support is the *Moran set*

$$T(\{M_n\}, D) = \left\{ \sum_{n=1}^{\infty} (M_n \cdots M_1)^{-1} d_n : d_n \in D \right\} := \sum_{n=1}^{\infty} (M_n \cdots M_1)^{-1} D.$$

Motivated by the above works, we extend the characterization of the spectrality of the Sierpinski-type measure to the Moran measure $\mu_{\{M_n\},D}$ in (1.1). Note that in case $M_n = M = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, the measures $\mu_{\{M_n\},D}$ and $\mu_{M,D}$ coincide. The main result of this paper is as follows.

**Theorem 1.1.** *Let $\mu_{\{M_n\},D}$ be the Moran–Sierpinski-type measure defined as in* (1.1) *and $p_n \equiv \pm q_n \pmod 3$ for all $n \geq 2$. Then $\mu_{\{M_n\},D}$ is a spectral measure if and only if $3 \mid p_n$ and $3 \mid q_n$ for each $n \geq 2$.*

The most subtle part is proving the necessity. We note that convolution plays an important role in the study of the spectrality of the Moran measure $\mu_{\{M_n\},D}$. The following technical theorem gives a connection between two convolution measures, which will be used to prove the necessity.

**Theorem 1.2.** *Let $\mathcal{B} \subset \mathbb{Z}^2$ be a finite set and let $\nu$ be a Borel probability measure with compact support on $\mathbb{R}^2$. Suppose that $\mu := \delta_{\mathcal{B}} * \nu$ is a spectral measure. Further, suppose the following:*

- (i) *Let $\{\lambda_1, \lambda_2\}$ be any bi-zero set of $\mu$. If $\lambda_1 \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$ and $\lambda_2 \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$, then $\lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$.*
- (ii) *$\mathcal{Z}(\hat{\mu}) \subset A^{-1}\mathbb{Z}^2$ for some integral invertible matrix $A$.*

*Then both $\delta_{\mathcal{B}}$ and $\nu$ are spectral measures.*

**Remark 1.3.** Recently, An and Wang [5] proved the above theorem in dimension one, which is a special case of our conclusion.

We organize this paper as follows. In Section 2, we introduce some basic definitions and properties of spectral measures. In Section 3, we will give the proof of Theorem 1.2. We devote Sections 4 and 5 to prove Theorem 1.1.

## 2. Preliminaries

Let $\mu$ be a Borel probability measure with compact support on $\mathbb{R}^2$. The Fourier transform of $\mu$ is defined as usual,

$$\hat{\mu}(\xi) = \int e^{-2\pi i \langle \xi, x \rangle} \, \mathrm{d}\mu(x)$$

for any $\xi \in \mathbb{R}^2$. We will denote by $\mathcal{Z}(\hat{\mu}) = \{\xi : \hat{\mu}(\xi) = 0\}$ the zero set of $\hat{\mu}$. In what follows, $e_\lambda$ stands for the exponential function $e^{-2\pi i \langle \lambda, x \rangle}$. Then for a discrete set $\Lambda \subset \mathbb{R}^2$, $E(\Lambda) = \{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal set of $L^2(\mu)$ if and only if $\hat{\mu}(\lambda - \lambda') = 0$ for $\lambda \neq \lambda' \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}(\hat{\mu}). \tag{2.1}$$

In this case, we say that $\Lambda$ is a *bi-zero set* of $\mu$. Since bi-zero sets (or spectra) are invariant under translation, without loss of generality we always assume that $0 \in \Lambda$ in this paper.

The following criterion is a universal test to decide whether a countable set $\Lambda \subset \mathbb{R}^2$ is a bi-zero set (a spectrum) of $\mu$ or not. For $\xi \in \mathbb{R}^2$, we write

$$Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2.$$

**Theorem 2.1** ([22])**.** *Let $\mu$ be a Borel probability measure with compact support on $\mathbb{R}^2$, and let $\Lambda \subset \mathbb{R}^2$ be a countable set. Then*

(i) *$\Lambda$ is a bi-zero set of $\mu$ if and only if $Q_\Lambda(\xi) \le 1$ for $\xi \in \mathbb{R}^2$;*
(ii) *$\Lambda$ is a spectrum of $\mu$ if and only if $Q_\Lambda(\xi) \equiv 1$ for $\xi \in \mathbb{R}^2$;*
(iii) *$Q_\Lambda(x)$ has an entire analytic extension to $\mathbb{C}^2$ if $\Lambda$ is a bi-zero set of $\mu$.*

As a simple consequence of Theorem 2.1, the following useful theorem was proved in [9] and will be used to prove our main result.

**Theorem 2.2.** *Let $\mu = \mu_0 * \mu_1$ be the convolution of two probability measures $\mu_i$, $i = 0, 1$, which are not Dirac measures. Suppose that $\Lambda$ is a bi-zero set of $\mu_0$. Then $\Lambda$ is also a bi-zero set of $\mu$, but it cannot be a spectrum of $\mu$.*

## 3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Let $\mathcal{B} \subset \mathbb{Z}^2$ be a finite set and let $\nu$ be a Borel probability measure with compact support on $\mathbb{R}^2$. Write $\mu = \delta_{\mathcal{B}} * \nu$. Then

$$\mathcal{Z}(\hat{\mu}) = \mathcal{Z}(\hat{\delta}_{\mathcal{B}}) \cup \mathcal{Z}(\hat{\nu}).$$

Before proving Theorem 1.2, we need the following lemma.

**Lemma 3.1.** *Suppose that $\{\lambda_1, \lambda_2\}$ is a bi-zero set of $\mu$. Then the following two statements are equivalent:*

(i) *if $\lambda_1 \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$ and $\lambda_2 \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$, then $\lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$;*
(ii) *if $\lambda_1, \lambda_2 \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$, then $\lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$.*

*Proof.* If (i) is true, let $\lambda_1, \lambda_2 \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$. Suppose that

$$\lambda_3 := \lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}}).$$

Then $\lambda_1 = \lambda_3 - (-\lambda_2) \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$, a contradiction. Hence (ii) holds.

Suppose (ii) holds. Let $\lambda_1 \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$ and $\lambda_2 \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$. Suppose that

$$\lambda_3 := \lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}}).$$

Then $\lambda_1 = \lambda_3 - (-\lambda_2) \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$. This is a contradiction, and thus (i) follows. □

We define an equivalence relation $\sim$ on a set $\Lambda \subseteq \mathbb{R}^2$ by

$$\lambda \sim \lambda' \text{ if and only if } \lambda - \lambda' \in \mathbb{Z}^2.$$

Set $[\lambda] = \{\lambda' \in \Lambda : \lambda \sim \lambda'\}$. Then $\Lambda/\sim = \{[\lambda] : \lambda \in \Lambda\}$ is a partition of $\Lambda$.

*Proof of Theorem 1.2.* Let $\Lambda$ be a spectrum of $\mu$. Then

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}(\hat{\mu}) = \mathcal{Z}(\hat{\delta}_{\mathcal{B}}) \cup \mathcal{Z}(\hat{\nu}).$$

We take $0 \in \Lambda' \subset \Lambda$ as a maximal bi-zero set of $\delta_{\mathcal{B}}$. Write

$$\Lambda' = \{\lambda_i\}_{i=1}^t$$

for some $t \in \mathbb{N}$. Then, for any $\lambda \in \Lambda \setminus \Lambda'$, there is a $\lambda_i \in \Lambda'$ such that $\lambda - \lambda_i \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$. And we assert that the $\lambda_i$ is unique. Suppose, on the contrary, that there exist two distinct $\lambda_i, \lambda_j \in \Lambda'$ such that $\lambda - \lambda_i,\ \lambda - \lambda_j \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$. Since $\Lambda'$ is a bi-zero set of $\delta_{\mathcal{B}}$, we know that $\lambda_i - \lambda_j \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$. Write

$$\lambda_1 = \lambda_i - \lambda_j \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}}), \quad \lambda_2 = \lambda - \lambda_j \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}}).$$

The conditions of the theorem imply that $\lambda_i - \lambda = \lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$, a contradiction. Then the assertion follows.

Set

$$\Lambda_i = \left\{\lambda \in \Lambda :\ \lambda - \lambda_i \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})\right\} \cup \{\lambda_i\}, \quad 1 \le i \le t.$$

Then

$$\Lambda = \bigcup_{i=1}^{t} \Lambda_i,$$

where $\Lambda_i \cap \Lambda_j = \emptyset$ for any $i \neq j$. Now we need the following two claims to complete the proof.

**Claim 3.2.** $(\Lambda_i - \Lambda_i) \setminus \{0\} \subseteq \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$ *for any* $1 \le i \le t$.

*Proof.* Fix $1 \le i \le t$. For any $\lambda \neq \lambda' \in \Lambda_i$, we have

$$\lambda - \lambda_i, \quad \lambda' - \lambda_i \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$$

or

$$\lambda - \lambda_i \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}}), \quad \lambda' = \lambda_i.$$

The first case follows directly from Lemma 3.1. And it is obvious that $\lambda - \lambda' \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$ in the second case above. Hence the claim is proved. ☐

**Claim 3.3.** $\Lambda_i - \Lambda_j \subseteq \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$ *for any* $i \neq j$.

*Proof.* For any $\lambda \in \Lambda_i$, $\lambda' \in \Lambda_j$, we have

$$\lambda - \lambda_i \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}}) \quad \text{or} \quad \lambda = \lambda_i$$

and

$$\lambda' - \lambda_j \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}}) \quad \text{or} \quad \lambda' = \lambda_j.$$

For the case in which $\lambda - \lambda_i,\ \lambda' - \lambda_j \in \mathcal{Z}(\hat{\nu}) \setminus \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$, it follows from $\lambda \notin \Lambda_j$ that $\lambda - \lambda_j \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$. Then

$$\lambda - \lambda' = (\lambda - \lambda_j) - (\lambda' - \lambda_j) \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}}).$$

For the remaining three cases, it is easy to verify that $\lambda - \lambda' \in \mathcal{Z}(\hat{\delta}_{\mathcal{B}})$. Then the claim follows. ☐

Due to $\Lambda_i \setminus \{0\} \subset \Lambda \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}) \subset A^{-1}\mathbb{Z}^2$ for some integral invertible matrix $A$, we know that $\Lambda_i/\sim$ is a finite set and $\Lambda_i/\sim$ is a partition of $\Lambda_i$. And thus we write $\Lambda_i/\sim\ = \{[\lambda_{i,1}], \ldots, [\lambda_{i,n_i}]\}$. Since $\mathcal{B} \subset \mathbb{Z}^2$, for any $\xi \in (0,1)^2$ and $1 \le i \le t$ we have

$$\left\{|\hat{\delta}_{\mathcal{B}}\left(\xi + \lambda\right)|^2 :\ \lambda \in \Lambda_i\right\} = \left\{|\hat{\delta}_{\mathcal{B}}\left(\xi + \lambda_{i,k}\right)|^2 : 1 \le k \le n_i\right\}.$$

It follows that for any $\xi \in (0,1)^2$ and $1 \le i \le t$, there exists $\lambda_{i,\xi(i)}$ with $\xi(i) \in \{1,2,\ldots,n_1\}$ such that

$$|\hat{\delta}_{\mathcal{B}}(\xi+\lambda_{i,\xi(i)})|^2 = \max\left\{|\hat{\delta}_{\mathcal{B}}\left(\xi+\lambda\right)|^2 : \lambda \in \Lambda_i\right\}.$$

That is, for any $\xi \in (0,1)^2$, there exist $\{\lambda_{i,\xi(i)}\}_{i=1}^t$ corresponding to it. As $\Lambda_i/{\sim}$ is a finite set for each $1 \le i \le t$ but there are infinitely many points in $(0,1)^2$, we can find a finite set $\{\widetilde{\lambda}_i\}_{i=1}^t$ in which $\widetilde{\lambda}_i = \lambda_{i,\xi(i)}$ for infinitely many $\xi \in I \subset (0,1)^2$. Then, for any $\xi \in I$, we have

$$\begin{aligned} 1 \equiv Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi+\lambda)|^2 &= \sum_{i=1}^t \sum_{\lambda\in\Lambda_i} |\hat{\delta}_{\mathcal{B}}(\xi+\lambda)|^2 |\hat{\nu}(\xi+\lambda)|^2 \\ &\le \sum_{i=1}^t |\hat{\delta}_{\mathcal{B}}(\xi+\widetilde{\lambda}_i)|^2 \sum_{\lambda\in\Lambda_i} |\hat{\nu}(\xi+\lambda)|^2 \qquad (3.1) \\ &\le \sum_{i=1}^t |\hat{\delta}_{\mathcal{B}}(\xi+\widetilde{\lambda}_i)|^2 \\ &\le 1. \end{aligned}$$

We know from Claim 3.2 that $\Lambda_i$ is an orthogonal set of $\nu$ for each $1 \le i \le t$; then the second to last inequality in (3.1) follows from Theorem 2.1. Similarly, the last inequality in (3.1) follows from Claim 3.3 and Theorem 2.1. And (3.1) implies that, for any $\xi \in I$,

$$\sum_{\lambda\in\Lambda_i} |\hat{\nu}(\xi+\lambda)|^2 \equiv 1 \quad \text{and} \quad \sum_{i=1}^t |\hat{\delta}_{\mathcal{B}}(\xi+\widetilde{\lambda}_i)|^2 \equiv 1.$$

The property of entire function implies that, for any $\xi \in \mathbb{R}^2$,

$$\sum_{\lambda\in\Lambda_i} |\hat{\nu}(\xi+\lambda)|^2 \equiv 1 \quad \text{and} \quad \sum_{i=1}^t |\hat{\delta}_{\mathcal{B}}(\xi+\widetilde{\lambda}_i)|^2 \equiv 1.$$

Hence $\{\widetilde{\lambda}_i\}_{i=1}^t$ is a spectrum of $\delta_{\mathcal{B}}$ and each $\Lambda_i$ is a spectrum of $\nu$. □

## 4. Sufficiency of Theorem 1.1

We will prove the sufficiency of Theorem 1.1 in this section. Wang and Dong [31] proved the sufficient case for more general 3-digit sets. In this section, we give another simple proof for it. This proof depends closely on the zero set of the Fourier transform $\hat{\mu}_{\{M_n\},D}$. By the definition of Fourier transform of $\hat{\mu}_{\{M_n\},D}$ and (1.1), for any $\xi \in \mathbb{R}^2$ we have

$$\hat{\mu}_{\{M_n\},D}(\xi) = \prod_{n=1}^{\infty} \hat{\delta}_{(M_n\cdots M_2M_1)^{-1}D}(\xi).$$

Then we have

$$\mathcal{Z}(\hat{\mu}_{\{M_n\},D}) = \bigcup_{n=1}^{\infty} \mathcal{Z}(\hat{\delta}_{(M_n\cdots M_2M_1)^{-1}D}) = \bigcup_{n=1}^{\infty} M_1\cdots M_n\mathcal{Z}(\hat{\delta}_D).$$

By calculation, we obtain

$$\mathcal{Z}(\hat{\delta}_D) = \left(\frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix} + \mathbb{Z}^2\right) \cup \left(\frac{1}{3}\begin{pmatrix}2\\1\end{pmatrix} + \mathbb{Z}^2\right) := \frac{1}{3}A_1 \cup \frac{1}{3}A_2, \tag{4.1}$$

where

$$A_1 = \begin{pmatrix}1\\2\end{pmatrix} + 3\mathbb{Z}^2, \quad A_2 = \begin{pmatrix}2\\1\end{pmatrix} + 3\mathbb{Z}^2.$$

Then

$$\mathcal{Z}\left(\hat{\mu}_{\{M_n\},D}\right) = \bigcup_{n=1}^{\infty} M_1\cdots M_n\mathcal{Z}(\hat{\delta}_D) = \bigcup_{n=1}^{\infty} \frac{M_1\cdots M_n}{3}(A_1\cup A_2). \tag{4.2}$$

For any $k \geq 1$, we define

$$\mu_k = \delta_{M_1^{-1}D} * \delta_{(M_2M_1)^{-1}D} * \cdots * \delta_{(M_k\cdots M_1)^{-1}D},$$
$$\mu_{>k} = \delta_{(M_{k+1}\cdots M_1)^{-1}D} * \delta_{(M_{k+2}\cdots M_1)^{-1}D} * \cdots .$$

Then

$$\mu_{\{M_n\},D} = \mu_k * \mu_{>k}.$$

Write

$$C = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{3}\\-\frac{1}{3}\end{pmatrix}, \begin{pmatrix}-\frac{1}{3}\\\frac{1}{3}\end{pmatrix}\right\}. \tag{4.3}$$

Then we have the following result. The sufficiency of Theorem 1.1 follows immediately from it.

**Theorem 4.1.** *Let $\mu_{\{M_n\},D}$ be the Moran–Sierpinski-type measure defined as in* (1.1)*. If $3 \mid p_n$ and $3 \mid q_n$ for all $n \geq 2$, then $\mu_{\{M_n\},D}$ is a spectral measure with a spectrum*

$$\Lambda = \left\{\sum_{k=1}^{m} M_1\cdots M_k c_k : c_k \in C \text{ and } m \geq 1\right\},$$

*where $C$ is defined as in* (4.3)*.*

*Proof.* Firstly, we will show that $\Lambda$ is a bi-zero set of $\mu_{\{M_n\},D}$. For any two distinct elements $\lambda, \lambda' \in \Lambda$, we can write

$$\lambda = \frac{1}{3}\begin{pmatrix}\sum_{i=1}^{m}(p_1p_2\cdots p_i)c_{i1}\\ \sum_{i=1}^{m}(q_1q_2\cdots q_i)c_{i2}\end{pmatrix}, \quad \lambda' = \frac{1}{3}\begin{pmatrix}\sum_{i=1}^{l}(p_1p_2\cdots p_i)c'_{i1}\\ \sum_{i=1}^{l}(q_1q_2\cdots q_i)c'_{i2}\end{pmatrix},$$

where $m, l \geq 1$ and $\begin{pmatrix} c_{i1} \\ c_{i2} \end{pmatrix}, \begin{pmatrix} c_{i1} \\ c_{i2} \end{pmatrix} \in 3C$ for each $i$. Let $s \geq 1$ be the first index such that $\begin{pmatrix} c_{s1} \\ c_{s2} \end{pmatrix} \neq \begin{pmatrix} c'_{s1} \\ c'_{s2} \end{pmatrix}$. It follows that $\begin{pmatrix} c_{s1} - c'_{s1} \\ c_{s2} - c'_{s2} \end{pmatrix} \in A_1 \cup A_2$, hence

$$\lambda - \lambda' = \frac{M_1 \cdots M_s}{3} \begin{pmatrix} (c_{s1} - c'_{s1}) + 3N_1 \\ (c_{s2} - c'_{s2}) + 3N_2 \end{pmatrix}$$

for some $N_1, N_2 \in \mathbb{Z}$. This together with (4.2) implies that

$$\lambda - \lambda' \in M_1 \cdots M_s \mathcal{Z}(\hat{\delta}_D) \subseteq \mathcal{Z}\left(\hat{\mu}_{\{M_n\},D}\right).$$

Therefore $\Lambda$ is a bi-zero set of $\mu_{\{M_n\},D}$.

We now show the completeness of $\Lambda$. For any $m \geq 1$, set

$$\Lambda_m := \sum_{k=1}^{m} M_1 \cdots M_k C = \left\{ \sum_{k=1}^{m} M_1 \cdots M_k c_k : c_k \in C \right\},$$

where $C$ is defined as in (4.3). Proceeding as in the proof above, we know that $\Lambda_m$ is a bi-zero set of $\mu_m$. Notice that $\#\Lambda_m = 3^m = \dim(L^2(\mu_m))$. Hence $\Lambda_m$ is a spectrum of $\mu_m$, and Theorem 2.1 implies that, for any $\xi \in \mathbb{R}^2$, we have

$$Q_m(\xi) := \sum_{\lambda \in \Lambda_m} |\hat{\mu}_m(\xi + \lambda)|^2 \equiv 1, \quad Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}_{\{M_n\},D}(\xi + \lambda)|^2 \leq 1.$$

Fix $\xi \in \mathbb{R}^2$. Write

$$f_m(\lambda) = \begin{cases} |\hat{\mu}_m(\xi + \lambda)|^2, & \lambda \in \Lambda_m; \\ 0, & \lambda \notin \Lambda_m; \end{cases}$$

$$f(\lambda) = \begin{cases} |\hat{\mu}_{\{M_n\},D}(\xi + \lambda)|^2, & \lambda \in \Lambda; \\ 0, & \lambda \notin \Lambda. \end{cases}$$

Then, for any $\lambda \in \Lambda$, we have $f(\lambda) = \lim_{m \to \infty} f_m(\lambda)$ and

$$\sum_{\lambda \in \Lambda} f(\lambda) = \sum_{\lambda \in \Lambda} |\hat{\mu}_{\{M_n\},D}(\xi + \lambda)|^2 \leq 1.$$

Moreover,

$$\begin{aligned} f(\lambda) = |\hat{\mu}_{\{M_n\},D}(\xi + \lambda)|^2 &= |\hat{\mu}_m(\xi + \lambda)|^2 |\hat{\mu}_{>m}(\xi + \lambda)|^2 \\ &= f_m(\lambda) |\hat{\mu}_{>m}(\xi + \lambda)|^2 \quad \text{for all } \lambda \in \Lambda_m. \end{aligned} \tag{4.4}$$

We now claim that there exists a constant $c > 0$ such that for any $m \geq 1$,

$$|\hat{\mu}_{>m}(\xi + \lambda)|^2 \geq c > 0,$$

where $|\xi| < \frac{1}{3}$ and $\lambda \in \Lambda_m$. Note that

$$|\hat{\mu}_{>m}(\xi + \lambda)|^2 = \prod_{k=m+1}^{\infty} \left| \hat{\delta}_{(M_k \cdots M_1)^{-1} D}(\xi + \lambda) \right|^2.$$

So we need to estimate the values of $\left|\hat{\delta}_{(M_k\cdots M_1)^{-1}D}(\xi+\lambda)\right|^2$ for $k \geq m+1$. For any $\lambda \in \Lambda_m$, we can write

$$\lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^m (p_1\cdots p_i)c_{i1}\\ \sum_{i=1}^m (q_1\cdots q_i)c_{i2}\end{pmatrix},$$

where $\begin{pmatrix}c_{i1}\\ c_{i2}\end{pmatrix} \in C$. Then

$$\begin{aligned}
&\left|\hat{\delta}_{(M_k\cdots M_1)^{-1}D}(\xi+\lambda)\right|^2\\
&\quad= \left|\frac{1}{3}\left(1+e^{-2\pi i(p_1\cdots p_k)^{-1}(\xi_1+\lambda_1)}+e^{-2\pi i(q_1\cdots q_k)^{-1}(\xi_2+\lambda_2)}\right)\right|^2\\
&\quad= \frac{1}{9}\Bigg|1+\cos\frac{2\pi(\xi_1+\lambda_1)}{p_1\cdots p_k}+\cos\frac{2\pi(\xi_2+\lambda_2)}{q_1\cdots q_k}\\
&\qquad\quad -i\left(\sin\frac{2\pi(\xi_1+\lambda_1)}{p_1\cdots p_k}+\sin\frac{2\pi(\xi_2+\lambda_2)}{q_1\cdots q_k}\right)\Bigg|^2\\
&\quad= \frac{1}{9}\Bigg|3+2\left(\cos\frac{2\pi(\xi_1+\lambda_1)}{p_1\cdots p_k}+\cos\frac{2\pi(\xi_2+\lambda_2)}{q_1\cdots q_k}\right)\\
&\qquad\quad +2\cos\left(\frac{2\pi(\xi_1+\lambda_1)}{p_1\cdots p_k}-\frac{2\pi(\xi_2+\lambda_2)}{q_1\cdots q_k}\right)\Bigg|.
\end{aligned} \tag{4.5}$$

Note that $c_{ij} \in \left\{0, \pm\frac{1}{3}\right\}$. Then, for any $k \geq m+1$ and $\xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ with $|\xi| < \frac{1}{3}$, we have

$$\begin{aligned}
\left|\frac{\xi_1+\lambda_1}{p_1\cdots p_k}\right| &= \left|\frac{\xi_1+\sum_{i=1}^m (p_1\cdots p_i)c_{i1}}{p_1\cdots p_k}\right|\\
&\leq \frac{1}{3}\left|\frac{1}{p_1\cdots p_k}+\frac{1}{p_2\cdots p_k}\cdots+\frac{1}{p_{m+1}\cdots p_k}\right|\\
&\leq \frac{1}{6}\frac{1}{3^{k-m-1}}.
\end{aligned} \tag{4.6}$$

Similarly,

$$\left|\frac{\xi_2+\lambda_2}{q_1\cdots q_k}\right| \leq \frac{1}{6}\frac{1}{3^{k-m-1}}. \tag{4.7}$$

Hence

$$\left|\frac{\xi_1+\lambda_1}{p_1\cdots p_k}-\frac{\xi_2+\lambda_2}{q_1\cdots q_k}\right| \leq \frac{1}{6}\left(\frac{1}{3^{k-m-1}}+\frac{1}{3^{k-m-1}}\right) \leq \frac{1}{3}\frac{1}{3^{k-m-1}}. \tag{4.8}$$

If $k = m+1$, it follows from (4.6), (4.7), and (4.8) that

$$\cos\frac{2\pi(\xi_1+\lambda_1)}{p_1\cdots p_{m+1}} \geq \cos\frac{\pi}{3} = \frac{1}{2}, \quad \cos\frac{2\pi(\xi_2+\lambda_2)}{q_1\cdots q_{m+1}} \geq \cos\frac{\pi}{3} = \frac{1}{2}.$$

Combining with (4.5), we have

$$\left|\hat{\delta}_{(M_{m+1}\cdots M_1)^{-1}D}(\xi+\lambda)\right|^2 \geq \frac{1}{9}\left|5+2\cos\left(\frac{2\pi(\xi_1+\lambda_1)}{p_1\cdots p_{m+1}}-\frac{2\pi(\xi_2+\lambda_2)}{q_1\cdots q_{m+1}}\right)\right| \geq \frac{1}{3}.$$

If $k \geq m+2$, then we know from (4.6), (4.7), and (4.8) that

$$\cos\frac{2\pi(\xi_1+\lambda_1)}{p_1\cdots p_k} \geq \cos\frac{\pi}{3}\frac{1}{3^{k-m-1}} \geq 1-\frac{\pi^2}{18}\frac{1}{9^{k-m-1}} > 0,$$
$$\cos\frac{2\pi(\xi_2+\lambda_2)}{q_1\cdots q_k} \geq \cos\frac{\pi}{3}\frac{1}{3^{k-m-1}} \geq 1-\frac{\pi^2}{18}\frac{1}{9^{k-m-1}} > 0,$$

and

$$\cos\left(\frac{2\pi(\xi_1+\lambda_1)}{p_1\cdots p_k}-\frac{2\pi(\xi_2+\lambda_2)}{q_1\cdots q_k}\right) \geq \cos\frac{2\pi}{3}\frac{1}{3^{k-m-1}} \geq 1-\frac{4\pi^2}{18}\frac{1}{9^{k-m-1}} > 0.$$

These together with (4.5) imply that

$$\left|\hat{\delta}_{(M_k\cdots M_1)^{-1}D}(\xi+\lambda)\right|^2 \geq 1-\frac{2\pi^2}{27}\frac{1}{9^{k-m-1}}.$$

Hence

$$\begin{aligned}|\hat{\mu}_{>m}(\xi+\lambda)|^2 &\geq \frac{1}{3}\prod_{k=m+2}^{\infty}\left(1-\frac{2\pi^2}{27}\frac{1}{9^{k-m-1}}\right)\\ &= \frac{1}{3}\prod_{k=1}^{\infty}\left(1-\frac{2\pi^2}{27}\frac{1}{9^k}\right) := c > 0.\end{aligned}$$

Thus the claim holds. Combining this claim with (4.4), we obtain

$$f_m(\lambda) \leq \frac{1}{c}f(\lambda) \quad \text{for all } \lambda \in \Lambda.$$

By the dominated convergence theorem, we conclude that

$$Q_\Lambda(\xi) = \lim_{m\to\infty} Q_m(\xi) = 1$$

for any $\xi \in \mathbb{R}^2$ with $|\xi| < \frac{1}{3}$. As $Q_\Lambda(\xi)$ is an entire function, we obtain that $Q_\Lambda(\xi) \equiv 1$ for any $\xi \in \mathbb{R}^2$. By Theorem 2.1, we know that $\mu_{\{M_n\},D}$ is a spectral measure. Now the proof is complete. □

## 5. Necessity of Theorem 1.1

In this section, we will give the proof of the necessity of Theorem 1.1. For that purpose, we need the following technical theorem, which plays a crucial role in the proof. Moreover, the following theorem shows that if $\mu_{\{M_n\},\mathcal{D}}$ is a spectral measure, then any "truncation" of it is still a spectral measure.

**Theorem 5.1.** *Let $\mu_{\{M_n\},D}$ be the Moran–Sierpinski-type measure defined by* (1.1) *and $p_n \equiv \pm q_n \pmod 3$ for all $n \geq 2$. If $\mu_{\{M_n\},D}$ is a spectral measure, then both $\mu_k$ and $\mu_{>k}$ are spectral measures for any $k \geq 1$.*

To prove Theorem 5.1, we need the following lemmas.

**Lemma 5.2.** *Let* $p_n \equiv \pm q_n \pmod 3$ *for all* $n \geq 2$. *Suppose that* $\mu_{\{M_n\},D}$ *is a spectral measure. If* $\{\lambda_1, \lambda_2\}$ *is a bi-zero set of* $\mu_{\{M_n\},D}$ *with* $\lambda_1 \in \mathcal{Z}(\hat{\mu}_k)$ *and* $\lambda_2 \in \mathcal{Z}(\hat{\mu}_{>k}) \setminus \mathcal{Z}(\hat{\mu}_k)$ *for any* $k \geq 1$, *then*

$$\lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\mu}_k).$$

*Proof.* Fix $k \geq 1$. Since $\lambda_1 \in \mathcal{Z}(\hat{\mu}_k)$ and $\lambda_2 \in \mathcal{Z}(\hat{\mu}_{>k}) \setminus \mathcal{Z}(\hat{\mu}_k)$, we can write

$$\lambda_1 = \frac{M_1 M_2 \cdots M_{j_1}}{3} a_1, \quad \lambda_2 = \frac{M_1 M_2 \cdots M_{j_2}}{3} a_2,$$

where $j_1 \leq k < j_2$ and $a_1, a_2 \in A_1 \cup A_2$. Suppose, on the contrary, that there exists $j > k$ such that $\lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\delta}_{(M_j \cdots M_2 M_1)^{-1} D}) \setminus \mathcal{Z}(\hat{\mu}_k)$. Then there exists $a_3 \in A_1 \cup A_2$ such that

$$\frac{M_1 M_2 \cdots M_{j_1}}{3} a_1 - \frac{M_1 M_2 \cdots M_{j_2}}{3} a_2 = \frac{M_1 M_2 \cdots M_j}{3} a_3,$$

i.e.,

$$a_1 = M_{j_1+1} \cdots M_{j_2} a_2 + M_{j_1+1} \cdots M_j a_3. \tag{5.1}$$

Write

$$a_1 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}, \quad a_2 = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}, \quad a_3 = \begin{pmatrix} a_{31} \\ a_{32} \end{pmatrix}.$$

Note that $a_i \in A_1 \cup A_2$ for all $i = 1, 2, 3$. This means that

$$a_{i1} \not\equiv a_{i2} \pmod 3 \qquad \text{and} \qquad a_{i1}, a_{i2} \in \mathbb{Z} \setminus 3\mathbb{Z}, \quad i = 1, 2, 3. \tag{5.2}$$

Without loss of generality, we assume that $j_2 \leq j$. If $j_2 = j$, then (5.1) implies

$$\begin{cases} a_{11} = p_{j_1+1} \cdots p_{j_2} (a_{21} + a_{31}) \\ a_{12} = q_{j_1+1} \cdots q_{j_2} (a_{22} + a_{32}). \end{cases}$$

Combining with (5.2), we obtain

$$p_{j_1+1} \cdots p_{j_2} \in \mathbb{Z} \setminus 3\mathbb{Z} \qquad \text{and} \qquad q_{j_1+1} \cdots q_{j_2} \in \mathbb{Z} \setminus 3\mathbb{Z}.$$

Furthermore, we have

$$p_{j_1+1} \cdots p_{j_2} \equiv q_{j_1+1} \cdots q_{j_2} \pmod 3.$$

Hence

$$\begin{aligned} \mathcal{Z}(\hat{\delta}_{(M_{j_2} \cdots M_1)^{-1} D}) &= \frac{M_1 \cdots M_{j_1} M_{j_1+1} \cdots M_{j_2}}{3} (A_1 \cup A_2) \\ &\subset \frac{M_1 \cdots M_{j_1}}{3} (A_1 \cup A_2) = \mathcal{Z}(\hat{\delta}_{(M_{j_1} \cdots M_1)^{-1} D}). \end{aligned}$$

Write

$$\nu = \delta_{M_1^{-1} D} * \cdots * \delta_{(M_{j_2-1} \cdots M_1)^{-1} D} * \delta_{(M_{j_2+1} \cdots M_1)^{-1} D} * \cdots .$$

Then

$$\mu_{\{M_n\},D} = \delta_{(M_{j_2} \cdots M_1)^{-1} D} * \nu \qquad \text{and} \qquad \mathcal{Z}(\hat{\mu}_{\{M_n\},D}) \subseteq \mathcal{Z}(\hat{\nu}).$$

Let $\Lambda$ denote a bi-zero set of $\mu_{\{M_n\},D}$. Then $\Lambda$ is also a bi-zero set of $\nu$. Hence it follows from Theorem 2.2 that $\Lambda$ is not a spectrum of $\mu_{\{M_n\},D}$. Therefore, $\mu_{\{M_n\},D}$ is not a spectral measure, a contradiction.

Now we consider the case $j_2 < j$. Then (5.1) implies

$$\begin{cases} a_{11} = p_{j_1+1}\cdots p_{j_2}(a_{21} + p_{j_2+1}\cdots p_j a_{31}) \\ a_{12} = q_{j_1+1}\cdots q_{j_2}(a_{22} + q_{j_2+1}\cdots q_j a_{32}). \end{cases} \tag{5.3}$$

This together with (5.2) implies that

$$p_{j_1+1}\cdots p_{j_2} \in \mathbb{Z}\setminus 3\mathbb{Z}, \qquad q_{j_1+1}\cdots q_{j_2} \in \mathbb{Z}\setminus 3\mathbb{Z}$$

and

$$a_{21} + p_{j_2+1}\cdots p_j a_{31} \in \mathbb{Z}\setminus 3\mathbb{Z}, \qquad a_{22} + q_{j_2+1}\cdots q_j a_{32} \in \mathbb{Z}\setminus 3\mathbb{Z}.$$

Moreover, applying the condition of $p_n \equiv \pm q_n \pmod 3$ for $n \geq 2$, it follows that

$$a_{21} + p_{j_2+1}\cdots p_j a_{31} \not\equiv a_{22} + q_{j_2+1}\cdots q_j a_{32} \pmod 3.$$

Applying (5.2) and (5.3) again, we get

$$p_{j_1+1}\cdots p_{j_2} \equiv q_{j_1+1}\cdots q_{j_2} \pmod 3.$$

Proceeding as in the proof of the above case, we get a contradiction. Then we complete the proof of the lemma. □

**Lemma 5.3.** *Let $R$ be an invertible diagonal matrix. Then $\Lambda$ is a spectrum of $\mu_{\{M_n\},D}$ if and only if $R^{-1}\Lambda$ is a spectrum of $\mu_{\{M_n\},RD}$.*

*Proof.* Note that

$$\begin{aligned} \hat{\delta}_{(M_n\cdots M_1)^{-1}RD}(\xi) &= \frac{1}{\#D}\sum_{d\in D} e^{-2\pi i\langle (M_n\cdots M_1)^{-1}Rd,\xi\rangle} \\ &= \frac{1}{\#D}\sum_{d\in D} e^{-2\pi i\langle (M_n\cdots M_1)^{-1}d,R\xi\rangle} = \hat{\delta}_{(M_n\cdots M_1)^{-1}D}(R\xi) \end{aligned}$$

for any $\xi \in \mathbb{R}^2$ and $n \geq 1$. Then we have

$$\hat{\mu}_{\{M_n\},RD}(\xi) = \prod_{n=1}^{\infty} \hat{\delta}_{(M_n\cdots M_1)^{-1}RD}(\xi) = \prod_{n=1}^{\infty} \hat{\delta}_{(M_n\cdots M_1)^{-1}D}(R\xi) = \hat{\mu}_{\{M_n\},D}(R\xi).$$

Hence

$$\sum_{\lambda\in\Lambda} |\hat{\mu}_{\{M_n\},RD}(\xi + R^{-1}\lambda)|^2 = \sum_{\lambda\in\Lambda} |\hat{\mu}_{\{M_n\},D}(R\xi + \lambda)|^2.$$

The conclusion follows from Theorem 2.1. □

*Proof of Theorem 5.1.* For any $k \geq 1$, we write

$$\mathcal{B} = (M_k M_{k-1}\cdots M_2)D + (M_k M_{k-1}\cdots M_3)D + \cdots + D.$$

Then we have

$$\mu_k = \delta_{(M_k M_{k-1}\cdots M_1)^{-1}\mathcal{B}}.$$

Set

$$\nu = \delta_{M_{k+1}^{-1}D} * \delta_{(M_{k+2}M_{k+1})^{-1}D} * \cdots .$$

Then

$$\mu_{\{M_n\},(M_k\cdots M_2M_1)D} = \delta_{\mathcal{B}} * \nu, \quad \mathcal{Z}(\hat{\mu}_{\{M_n\},(M_k\cdots M_2M_1)D}) = \mathcal{Z}(\hat{\delta}_{\mathcal{B}}) \cup \mathcal{Z}(\hat{\nu}).$$

Notice that

$$\mathcal{Z}(\hat{\delta}_{\mathcal{B}}) = (M_1M_2\cdots M_k)^{-1}\mathcal{Z}(\hat{\mu}_k), \quad \mathcal{Z}(\hat{\nu}) = (M_1M_2\cdots M_k)^{-1}\mathcal{Z}(\hat{\mu}_{>k}). \tag{5.4}$$

We know from (4.2) that $\mathcal{Z}\left(\hat{\mu}_{\{M_n\},D}\right) \subset \frac{1}{3}\mathbb{Z}^2$. This together with (5.4) implies that

$$\mathcal{Z}(\hat{\mu}_{\{M_n\},(M_k\cdots M_2M_1)D}) \subset (3M_1M_2\cdots M_k)^{-1}\mathbb{Z}^2 := A^{-1}\mathbb{Z}^2,$$

where $A = 3M_1M_2\cdots M_k$ is an integral invertible matrix. Then we know from Lemma 5.2 and Theorem 1.2 that both $\delta_{\mathcal{B}}$ and $\nu$ are spectral measures. Applying Lemma 5.3, we obtain that $\mu_k$ and $\mu_{>k}$ are all spectral measures. The proof is completed. □

Recall that

$$\mu_2 = \delta_{M_1^{-1}D} * \delta_{(M_2M_1)^{-1}D}.$$

Then

$$\mathcal{Z}(\hat{\mu}_2) = \mathcal{Z}(\hat{\delta}_{M_1^{-1}D}) \cup \mathcal{Z}(\hat{\delta}_{(M_2M_1)^{-1}D}) = \frac{M_1}{3}(A_1 \cup A_2) \cup \frac{M_1M_2}{3}(A_1 \cup A_2).$$

**Theorem 5.4.** *Let $p_2 \equiv \pm q_2 \pmod 3$. If $\mu_2$ is a spectral measure, then $3 \mid p_2$ and $3 \mid q_2$.*

*Proof.* Suppose, on the contrary, that $3 \nmid p_2$ or $3 \nmid q_2$. We just prove the case in which $3 \nmid p_2$. The proof of the remaining case is similar and we omit it here. Let $\Lambda$ denote a bi-zero set of $\mu_2$. Then

$$\Lambda \subseteq \mathcal{Z}(\hat{\delta}_{M_1^{-1}D}) \cup \mathcal{Z}(\hat{\delta}_{(M_2M_1)^{-1}D}) \cup \{0\}.$$

As $p_2 \equiv \pm q_2 \pmod 3$, we know from Lemma 3.1 and Lemma 5.2 that

$$\Big(\Lambda \cap \mathcal{Z}(\hat{\delta}_{(M_2M_1)^{-1}D}) - \Lambda \cap \mathcal{Z}(\hat{\delta}_{(M_2M_1)^{-1}D})\Big) \setminus \{0\} \subset \mathcal{Z}(\hat{\delta}_{(M_2M_1)^{-1}D}).$$

Then $\Lambda \cap \mathcal{Z}(\hat{\delta}_{(M_2M_1)^{-1}D})$ is an orthogonal set of $\delta_{(M_2M_1)^{-1}D}$, and thus

$$\#(\Lambda \cap \mathcal{Z}(\hat{\delta}_{(M_2M_1)^{-1}D})) \leq 3. \tag{5.5}$$

For the set $\Lambda \cap \mathcal{Z}(\hat{\delta}_{M_1^{-1}D})$, we make the following claim.

**Claim 5.5.** $\#(\Lambda \cap \mathcal{Z}(\hat{\delta}_{M_1^{-1}D})) \leq 2.$

*Proof.* Otherwise, we have $\#(\Lambda\cap\mathcal{Z}(\hat{\delta}_{M_1^{-1}D})) \geq 3$. Let $\{\lambda_1, \lambda_2, \lambda_3\} \subseteq \Lambda\cap\mathcal{Z}(\hat{\delta}_{M_1^{-1}D})$ and write

$$\lambda_1 = \frac{1}{3}\begin{pmatrix} p_1a_{11} \\ q_1a_{12} \end{pmatrix}, \quad \lambda_2 = \frac{1}{3}\begin{pmatrix} p_1a_{21} \\ q_1a_{22} \end{pmatrix}, \quad \lambda_3 = \frac{1}{3}\begin{pmatrix} p_1a_{31} \\ q_1a_{32} \end{pmatrix},$$

where $\begin{pmatrix} a_{i1} \\ a_{i2} \end{pmatrix} \in A_1 \cup A_2$ for $i = 1, 2, 3$. Then by the pigeonhole principle, without loss of generality we assume that $\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}, \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} \in A_1$. This means that

$$\begin{pmatrix} a_{11} - a_{21} \\ a_{12} - a_{22} \end{pmatrix} \in 3\mathbb{Z}^2.$$

Notice that $3 \nmid p_2$. This together with (4.1) implies that

$$\lambda_1 - \lambda_2 = \frac{1}{3}\begin{pmatrix} p_1(a_{11}-a_{21}) \\ q_1(a_{12}-a_{22}) \end{pmatrix} \notin \frac{M_1}{3}(A_1 \cup A_2) \cup \frac{M_1 M_2}{3}(A_1 \cup A_2) = \mathcal{Z}(\hat{\mu}_2).$$

We get a contradiction, and thus $\#(\Lambda \cap \mathcal{Z}(\hat{\delta}_{M_1^{-1}D})) \leq 2$. ☐

Combining Claim 5.5 with (5.5), we obtain

$$\begin{aligned}\#\Lambda &= 1 + \#(\Lambda \cap \mathcal{Z}(\hat{\delta}_{M_1^{-1}D})) + \#(\Lambda \cap \mathcal{Z}(\hat{\delta}_{(M_2M_1)^{-1}D}) \setminus \mathcal{Z}(\hat{\delta}_{M_1^{-1}D})) \\ &\leq 1+2+3 = 6 < \dim L^2(\mu_2) = 9.\end{aligned}$$

Therefore, $\mu_2$ is not a spectral measure. This is a contradiction. ☐

Now we have all the ingredients for the proof of the necessity of Theorem 1.1.

**Theorem 5.6.** *Let $p_n \equiv \pm q_n \pmod 3$ for each $n \geq 2$. If $\mu_{\{M_n\},D}$ is a spectral measure, then $3 \mid p_n$ and $3 \mid q_n$ for all $n \geq 2$.*

*Proof.* Suppose, on the contrary, that there exists $n \geq 2$ such that $3 \nmid p_n$ or $3 \nmid q_n$. Set $n_0$ as the first index satisfying $3 \nmid p_{n_0}$ or $3 \nmid q_{n_0}$. From Theorem 5.4, we know that if $3 \nmid p_{n_0}$ or $3 \nmid q_{n_0}$, then $\delta_{M_{n_0-1}^{-1}D} * \delta_{(M_{n_0}M_{n_0-1})^{-1}D}$ is not a spectral measure. Set

$$\nu = \delta_{M_{n_0-1}^{-1}D} * \delta_{(M_{n_0}M_{n_0-1})^{-1}D} * \delta_{(M_{n_0+1}M_{n_0}M_{n_0-1})^{-1}D} * \cdots.$$

Then we know from Theorem 5.1 that $\nu$ is not a spectral measure. And thus $\mu_{>n_0-2}$ is not a spectral measure by Lemma 5.3. Applying Theorem 5.1 again, we have that $\mu_{\{M_n\},D}$ is not a spectral measure. This is a contradiction. ☐

## References


[1] L.-X. An and X.-G. He, A class of spectral Moran measures, *J. Funct. Anal.* **266** no. 1 (2014), 343–354. DOI MR Zbl

[2] L.-X. An, X.-G. He, and K.-S. Lau, Spectrality of a class of infinite convolutions, *Adv. Math.* **283** (2015), 362–376. DOI MR Zbl

[3] L.-X. An, X.-G. He, and H.-X. Li, Spectrality of infinite Bernoulli convolutions, *J. Funct. Anal.* **269** no. 5 (2015), 1571–1590. DOI MR Zbl

[4] L. An, X. Fu, and C.-K. Lai, On spectral Cantor-Moran measures and a variant of Bourgain's sum of sine problem, *Adv. Math.* **349** (2019), 84–124. DOI MR Zbl

[5] L. An and C. Wang, On self-similar spectral measures, *J. Funct. Anal.* **280** no. 3 (2021), Paper No. 108821, 31 pp. DOI MR Zbl

[6] X.-R. Dai, When does a Bernoulli convolution admit a spectrum?, *Adv. Math.* **231** no. 3-4 (2012), 1681–1693. DOI MR Zbl

[7] X.-R. Dai, Spectra of Cantor measures, *Math. Ann.* **366** no. 3-4 (2016), 1621–1647. DOI MR Zbl

[8] X.-R. Dai, X.-Y. Fu, and Z.-H. Yan, Spectrality of self-affine Sierpinski-type measures on $\mathbb{R}^2$, *Appl. Comput. Harmon. Anal.* **52** (2021), 63–81. DOI MR Zbl

[9] X.-R. Dai, X.-G. He, and K.-S. Lau, On spectral $N$-Bernoulli measures, *Adv. Math.* **259** (2014), 511–531. DOI MR Zbl

[10] Q.-R. Deng and X.-Y. Wang, On the spectra of self-affine measures with three digits, *Anal. Math.* **45** no. 2 (2019), 267–289. DOI MR Zbl

[11] Q.-R. Deng, On the spectra of Sierpinski-type self-affine measures, *J. Funct. Anal.* **270** no. 12 (2016), 4426–4442. DOI MR Zbl

[12] Q.-R. Deng and K.-S. Lau, Sierpinski-type spectral self-similar measures, *J. Funct. Anal.* **269** no. 5 (2015), 1310–1326. DOI MR Zbl

[13] D. E. Dutkay, D. Han, and Q. Sun, On the spectra of a Cantor measure, *Adv. Math.* **221** no. 1 (2009), 251–276. DOI MR Zbl

[14] D. E. Dutkay, J. Haussermann, and C.-K. Lai, Hadamard triples generate self-affine spectral measures, *Trans. Amer. Math. Soc.* **371** no. 2 (2019), 1439–1481. DOI MR Zbl

[15] K. Falconer, *Fractal geometry: mathematical foundations and applications*, John Wiley & Sons, Chichester, 1990. MR Zbl

[16] Y.-S. Fu, X.-G. He, and Z.-X. Wen, Spectra of Bernoulli convolutions and random convolutions, *J. Math. Pures Appl. (9)* **116** (2018), 105–131. DOI MR Zbl

[17] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, *J. Functional Analysis* **16** (1974), 101–121. DOI MR Zbl

[18] L. He and X.-G. He, On the Fourier orthonormal bases of Cantor-Moran measures, *J. Funct. Anal.* **272** no. 5 (2017), 1980–2004. DOI MR Zbl

[19] X.-G. He, M.-w. Tang, and Z.-Y. Wu, Spectral structure and spectral eigenvalue problems of a class of self-similar spectral measures, *J. Funct. Anal.* **277** no. 10 (2019), 3688–3722. DOI MR Zbl

[20] T.-Y. Hu and K.-S. Lau, Spectral property of the Bernoulli convolutions, *Adv. Math.* **219** no. 2 (2008), 554–567. DOI MR Zbl

[21] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** no. 5 (1981), 713–747. DOI MR Zbl

[22] P. E. T. Jorgensen and S. Pedersen, Dense analytic subspaces in fractal $L^2$-spaces, *J. Anal. Math.* **75** (1998), 185–228. DOI MR Zbl

[23] M. N. Kolountzakis and M. Matolcsi, Complex Hadamard matrices and the spectral set conjecture, *Collect. Math.* Vol. Extra (2006), 281–291. MR Zbl Available at https://eudml.org/doc/41780.

[24] M. N. Kolountzakis and M. Matolcsi, Tiles with no spectra, *Forum Math.* **18** no. 3 (2006), 519–528. DOI MR Zbl

[25] I. Łaba and Y. Wang, On spectral Cantor measures, *J. Funct. Anal.* **193** no. 2 (2002), 409–420. DOI MR Zbl

[26] J.-L. Li, Spectra of a class of self-affine measures, *J. Funct. Anal.* **260** no. 4 (2011), 1086–1095. DOI MR Zbl

[27] J. Li, Spectrality of a class of self-affine measures with decomposable digit sets, *Sci. China Math.* **55** no. 6 (2012), 1229–1242. DOI MR Zbl

[28] J. Li and Z. Wen, Spectrality of planar self-affine measures with two-element digit set, *Sci. China Math.* **55** no. 3 (2012), 593–605. DOI MR Zbl

[29] M. Matolcsi, Fuglede's conjecture fails in dimension 4, *Proc. Amer. Math. Soc.* **133** no. 10 (2005), 3021–3026. DOI MR Zbl

[30] T. Tao, Fuglede's conjecture is false in 5 and higher dimensions, *Math. Res. Lett.* **11** no. 2-3 (2004), 251–258. DOI MR Zbl

[31] Z.-Y. Wang and X.-H. Dong, Spectrality of Sierpinski-Moran measures, *Monatsh. Math.* **195** no. 4 (2021), 743–761. DOI MR Zbl

[32] Z.-H. Yan, Spectral properties of a class of Moran measures, *J. Math. Anal. Appl.* **470** no. 1 (2019), 375–387. DOI MR Zbl

*Qian Li*
School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, 430074, P. R. China
liqian303606@163.com

*Min-Min Zhang* ✉
Department of Applied Mathematics, Anhui University of Technology, Ma'anshan, 243002, P. R. China
zhangminmin0907@163.com

*Received: October 12, 2021*
*Accepted: July 5, 2022*