

DUALITY FOR INFINITE-DIMENSIONAL BRAIDED BIALGEBRAS AND THEIR (CO)MODULES

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ABSTRACT. The paper presents a detailed description of duality for braided algebras, coalgebras, bialgebras, Hopf algebras, and their modules and comodules in the infinite setting. Assuming that the dual objects exist, it is shown how a given braiding induces compatible braidings for the dual objects, and how actions (resp., coactions) can be turned into coactions (resp., actions) of the dual coalgebra (resp., algebra), with an emphasis on braided bialgebras and their braided (co)module algebras. Examples are provided by considering these structures in a graded (or filtered) setting, where each degree is finite-dimensional.

1. INTRODUCTION

The objective of this paper is to give a detailed description of duality for braided algebras, coalgebras, bi- and Hopf algebras, and their modules and comodules in the infinite setting. The interest in duality for these structures goes back to Majid [21, 24], who did much of the pioneering work on braided Hopf algebras [19, 20, 22, 23, 25], and to Takeuchi [29, 30]. Focusing on specific applications, duality for braided Hopf algebras has been studied (with different definitions) by Heckenberger and Schneider [14] in the setting of Yetter–Drinfeld modules, by Da Rocha, J. A. Guccione and J. J. Guccione [10] in connection with crossed products, and by Lyuvashenko [18] and by Guo and Zhang [13] in the context of integrals. The theory of braided Hopf crossed products reveals in particular the relevance of braided (co)module algebras [11, 12] which serve as a guiding principle for this paper. Duality for infinite braided Hopf algebras, under the assumption that the braiding is symmetric, has been considered in the Yetter–Drinfeld module category in the light of the Blattner–Montgomery duality theorem by Han and Zhang [32], and by Cheng, Xu and Zhang [31].

In the finite setting, duality is most conveniently studied in a rigid monoidal category [24, 29]. The problem in the infinite setting is the lack of a so-called coevaluation map. To avoid this problem, we do not follow a categorical approach but define the dual objects by a set of conditions, similar to the path taken by

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Takeuchi in [30]. In particular, we will neither prove the existence nor the uniqueness of a dual space with the desired properties, so we will not define a functor into a dual category. This may be especially useful in a topological setting, where it is not always practical to work with the full dual space, for instance, if there are unbounded (braiding) operators involved. However, the entire paper is kept completely algebraic even though the more interesting examples arise in a topological framework. In this sense, one may consider all tensor products as algebraic tensor products of linear spaces over a field \mathbb{K} .

Our first aim is to establish a duality theory for infinite-dimensional braided bialgebras and Hopf algebras. This will be done in Section 4. The guiding principle emanates from the definition of a dual pairing between braided bialgebras in Definition 4.2. To take into account the braided setting, we include in this definition a braiding between dual spaces so that the dual pairing of two-fold tensor products can be realized by evaluating simultaneously adjacent tensor factors. The fundamental idea of our approach is that all structures on dual spaces should be induced from the given ones, including the braiding appearing in the dual pairing. In this sense, our method is constructive, only the existence of a dual space with the required properties will be assumed, whereas all algebraic properties will be deduced from the original source.

To develop the theory step by step, we start by elaborating a duality theory for infinite-dimensional algebras and coalgebras in Propositions 4.3 and 4.4. Before doing so, we show in Lemma 4.1 how a given braiding induces braidings on a dual space and between the space and its dual which are compatible with additional algebraic structures like multiplication or comultiplication. Our definition of a product or coproduct on the dual space is intimately related to the compatibility conditions of a dual pairing. The construction of a dual braided bialgebra will be achieved in Theorem 4.5 under certain assumptions on the chosen dual space which guarantee that the induced braidings define bijective maps into the correct tensor products and that the product and coproduct are well-defined. Proposition 4.7 shows that the construction is reflexive in the sense that taking twice the dual gives back the same braided bialgebra. The extension of these results to braided Hopf algebras requires only a minor condition regarding the antipode.

In Section 5, we address duality for braided modules and comodules. The starting point is again to induce new braidings for dual spaces from a given braiding between a (co)algebra and a (co)module in such a way that the compatibility properties are maintained. This will be done step by step in Lemmas 5.1-5.3, each time replacing one of the two involved spaces by a dual space. Since a left braided vector space induces the structure of a right braided vector space on duals, and vice versa, we will frequently use the inverse of a braiding to recover a left-handed or right-handed version. In fact, one of the purposes of this paper is to single out the correct braidings and formulas so that it may serve as a reference for others.

Theorems 5.4 and 5.6 are the main results of Section 5. There it is shown how to transform a braided comodule into a braided module of a dual (bi)algebra, and a braided module into a braided comodule of a dual co- or bialgebra. Interestingly,

in the latter case, it will not exactly yield a comodule of a dual bialgebra but a version of it that corresponds to taking twice the braided opposite and co-opposite bialgebra. These structures, where the product or coproduct is flipped by a power of the braiding, are discussed at greater length in Section 3 since such considerations do play a role in subsequent results. For instance, they justify to present only one version of a braided dual (co)algebra in Propositions 4.3 and 4.4, other versions can be obtained by combining the constructed (co)product with powers of the braiding. Moreover, twisting products, coproducts, actions or coactions with a braiding may give rise to whole families of new structures as illustrated in Proposition 3.1 and Corollary 5.7. In Section 3, we also review the significance of the antipode in the braided setting for turning left (co)actions into right (co)actions and vice versa.

In Proposition 5.8, we dualize a coaction on a comodule to an action of a dual algebra on a dual of the comodule, and in Proposition 5.9, we dualize an action on a module to a coaction of a dual coalgebra on a dual of the module. Despite the fact that our main interest lies in module and comodule algebras of braided bialgebras, Propositions 5.8 and 5.9 do not consider these topics since we would then have to introduce the dual objects, namely module and comodule coalgebras. To keep the length of the paper reasonable, we refrain from introducing (co)module coalgebras. With a detailed description of braided (co)module algebras at hand, it should be clear how to dualize these notions to braided (co)module coalgebras.

Of course, finite-dimensional examples fit into our framework. It will be immediately clear from the constructions that the results can be extended to graded braided (co)algebras, bi- and Hopf algebras and their graded (co)modules whenever the spaces of homogeneous elements are finite-dimensional for all degrees. The obvious strategy is to deal with each degree individually. This observation leads already to a vast family of infinite-dimensional examples (see Section 6 for more details).

For brevity, we completely avoid braid diagrams since the proofs presented by braid diagrams would occupy considerably more space. Instead of braid diagrams, we introduce a Sweedler-type notation and annotate the employed relations over the equality signs. Once the reader gets used to this notation, it shouldn't be a problem to draw the corresponding braid diagrams, one only has to be careful with the chosen crossings. For instance, the crossings of the induced braidings have to be compatible with the given braidings, and the braidings obtained from an inverse braiding should be denoted differently than those induced from a given one. Although the proofs are rather straightforward, we present most of them in order to show where the involved braidings and compatibility relations are used.

2. PRELIMINARIES ON BRAIDED BIALGEBRAS AND THEIR (CO)MODULES

In this section, we give a working definition for braided bi- and Hopf algebras without using braided tensor categories. The reason is that we want to establish a duality theory for infinite-dimensional braided bialgebras. The categorical approach works well in rigid monoidal categories, but for infinite-dimensional examples, there is a problem with the rigorous definition of a co-evaluation map (see

e.g. [10, 30]). Moreover, a dual braided bialgebra may not exist, and if it exists, it may not be unique, therefore we do not aim at defining a functor into a dual category.

Throughout this paper, the letter \mathbb{K} stands for an arbitrary field. A braiding for a vector space V is a bijective linear map $\Psi_{VV} : V \otimes V \rightarrow V \otimes V$ fulfilling the Yang–Baxter equation

$$(\Psi_{VV} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{VV}) \circ (\Psi_{VV} \otimes \text{id}) = (\text{id} \otimes \Psi_{VV}) \circ (\Psi_{VV} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{VV}). \tag{1}$$

Let V be a braided vector space. A *left V -braided vector space* is a vector space W together with a bijective linear map $\Psi_{VW} : V \otimes W \rightarrow W \otimes V$ such that

$$(\Psi_{VW} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{VW}) \circ (\Psi_{VW} \otimes \text{id}) = (\text{id} \otimes \Psi_{VW}) \circ (\Psi_{VW} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{VW}). \tag{2}$$

Similarly, V is called a *right W -braided vector space* if W is a braided vector space and the bijective linear map $\Psi_{VW} : V \otimes W \rightarrow W \otimes V$ satisfies

$$(\text{id} \otimes \Psi_{VW}) \circ (\Psi_{VW} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{WW}) = (\Psi_{WW} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{VW}) \circ (\Psi_{VW} \otimes \text{id}). \tag{3}$$

The archetypal example, also in the case $V = W$, is given by the flip:

$$\tau : V \otimes W \longrightarrow W \otimes V, \quad \tau(v \otimes w) := w \otimes v. \tag{4}$$

Let V be an algebra with multiplication $m_V : V \otimes V \rightarrow V$. If W is a left V -braided vector space, or if V is a right W -braided vector space, then we say that the braiding Ψ_{VW} is compatible with the multiplication if

$$\Psi_{VW} \circ (m_V \otimes \text{id}) = (\text{id} \otimes m_V) \circ (\Psi_{VW} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{VW}) \tag{5}$$

and, if $1 \in V$,

$$\Psi_{VW}(1 \otimes w) = w \otimes 1, \quad w \in W. \tag{6}$$

Assume now that W is an algebra with multiplication $m_W : W \otimes W \rightarrow W$ and that W is a left V -braided vector space or that V is a right W -braided vector space. Then we say that the braiding Ψ_{VW} is compatible with the multiplication if

$$\Psi_{VW} \circ (\text{id} \otimes m_W) = (m_W \otimes \text{id}) \circ (\text{id} \otimes \Psi_{VW}) \circ (\Psi_{VW} \otimes \text{id}) \tag{7}$$

and

$$\Psi_{VW}(v \otimes 1) = 1 \otimes v, \quad v \in V. \tag{8}$$

A *braided algebra* is an algebra (A, m) which is a braided vector space such that the braiding is compatible with the multiplication. In this case, it follows from (5) and (7) that

$$\Psi_{AA} \circ (m \otimes m) = (m \otimes m) \circ (\text{id} \otimes \Psi_{AA} \otimes \text{id}) \circ (\Psi_{AA} \otimes \Psi_{AA}) \circ (\text{id} \otimes \Psi_{AA} \otimes \text{id}). \tag{9}$$

Clearly, each algebra A becomes a braided algebra with the usual flip defined in (4) as braiding isomorphism.

Suppose now that (V, Δ, ε) is a coalgebra and W is a left V -braided vector space or V is a right W -braided vector space. We say that the braiding $\Psi_{VW} : V \otimes W \rightarrow W \otimes V$ is compatible with the comultiplication of V if

$$(\text{id} \otimes \Delta) \circ \Psi_{VW} = (\Psi_{VW} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{VW}) \circ (\Delta \otimes \text{id}), \quad (\text{id} \otimes \varepsilon) \circ \Psi_{VW} = \varepsilon \otimes \text{id}. \tag{10}$$

If (W, Δ, ε) is a coalgebra and V is a right W -braided vector space or W is a left V -braided vector space, then the analogous definitions read

$$(\Delta \otimes \text{id}) \circ \Psi_{VW} = (\text{id} \otimes \Psi_{VW}) \circ (\Psi_{VW} \otimes \text{id}) \circ (\text{id} \otimes \Delta), \quad (\varepsilon \otimes \text{id}) \circ \Psi_{VW} = \text{id} \otimes \varepsilon. \tag{11}$$

A *braided coalgebra* is a coalgebra (H, Δ, ε) which is a braided vector space such that the braiding is compatible with the comultiplication. In this case, combining (10) and (11) yields

$$(\Delta \otimes \Delta) \circ \Psi_{HH} = (\text{id} \otimes \Psi_{HH} \otimes \text{id}) \circ (\Psi_{HH} \otimes \Psi_{HH}) \circ (\text{id} \otimes \Psi_{HH} \otimes \text{id}) \circ (\Delta \otimes \Delta). \tag{12}$$

As for algebras, each coalgebra becomes a braided coalgebra with the braiding defined by the flip τ given in (4).

The compatibility conditions permit us to extend the (co)algebra structures to tensor products. If (A, m_A) and (B, m_B) are algebras such that A is a right B -braided vector space, B is a left A -braided vector space, and the braiding $\Psi_{BA} : B \otimes A \rightarrow A \otimes B$ is compatible with the multiplications of A and B , then

$$m : (A \otimes B) \otimes (A \otimes B) \longrightarrow A \otimes B, \quad m := (m_A \otimes m_B) \circ (\text{id} \otimes \Psi_{BA} \otimes \text{id}), \tag{13}$$

defines a product on $A \otimes B$ turning it into an associative algebra denoted by $A \underline{\otimes} B$. If A and B are unital, then $1 \otimes 1$ yields the unit of $A \underline{\otimes} B$.

If H and G are coalgebras such that H is a right G -braided vector space, G is a left H -braided vector space, and the braiding $\Psi_{HG} : H \otimes G \rightarrow G \otimes H$ is compatible with the comultiplications of H and G , then the coproduct

$$\Delta : H \otimes G \longrightarrow (H \otimes G) \otimes (H \otimes G), \quad \Delta := (\text{id} \otimes \Psi_{HG} \otimes \text{id}) \circ (\Delta \otimes \Delta), \tag{14}$$

turns $H \otimes G$ into a coalgebra with counit $\varepsilon := \varepsilon \otimes \varepsilon$.

Recall that, for a coalgebra (H, Δ, ε) and a unital algebra (A, m) , the space $L(H, A)$ of all linear mappings from H to A becomes an associative unital algebra under the convolution product

$$(\phi * \psi)(h) := m \circ (\phi \otimes \psi) \circ \Delta(h), \quad h \in H, \phi, \psi \in L(H, A), \tag{15}$$

and with unit $h \mapsto \varepsilon(h)1$. In this picture, the antipode of a Hopf algebra can be viewed as the convolution inverse of the identity map $\text{id} : H \rightarrow H$.

The central objects of this paper are braided bialgebras which will be defined below. We include there the definition of a braided Hopf algebra although the existence of an antipode will play rather a minor role in our presentation.

Definition 2.1 ([19, 30]). A *braided bialgebra* is a unital algebra (H, m) together with a coalgebra structure (H, Δ, ε) and a braiding $\Psi_{HH} : H \otimes H \rightarrow H$ such that the following compatibility conditions hold: H is a braided vector space, the braiding Ψ_{HH} is compatible with the multiplication and the comultiplication of H , and the coproduct is an algebra homomorphism $\Delta : H \rightarrow H \underline{\otimes} H$, i.e.,

$$\Delta \circ m = (m \otimes m) \circ (\text{id} \otimes \Psi_{HH} \otimes \text{id}) \circ (\Delta \otimes \Delta). \tag{16}$$

A *braided Hopf algebra* is a braided bialgebra H such that the identity map has a convolution inverse $S : H \rightarrow H$ called the *antipode*.

For a braided Hopf algebra H , it can be shown that

$$\begin{aligned} \Psi_{HH} \circ (S \otimes \text{id}) &= (\text{id} \otimes S) \circ \Psi_{HH}, & \Psi_{HH} \circ (\text{id} \otimes S) &= (S \otimes \text{id}) \circ \Psi_{HH}, \\ S \circ m &= m \circ \Psi_{HH} \circ (S \otimes S), & \Delta \circ S &= \Psi_{HH} \circ (S \otimes S) \circ \Delta, & \varepsilon \circ S &= \varepsilon, \end{aligned} \tag{17}$$

(see e.g. [24, 30]).

If A and B are braided bialgebras with a braiding Ψ_{AB} satisfying the compatibility conditions on the multiplication and comultiplication, then $A \otimes B$ becomes a braided bialgebra with braiding

$$\Psi_{A \otimes B, A \otimes B} := (\text{id} \otimes \Psi_{AB} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \Psi_{BB}) \circ (\Psi_{AA} \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{AB}^{-1} \otimes \text{id}), \tag{18}$$

and multiplication and comultiplication defined in (13) and (14), respectively.

The following presentation of actions and coactions in the braided setting is taken from [10]. Let (A, m) be a braided (unital) algebra, and let W be a left A -braided vector space such that the braiding is compatible with the multiplication. We say that W is a braided left A -module if there is a map $\nu_L : A \otimes W \rightarrow W$ satisfying

$$\nu_L \circ (\text{id} \otimes \nu_L) = \nu_L \circ (m \otimes \text{id}), \quad \nu_L(1 \otimes w) = w, \quad w \in W, \tag{19}$$

$$\Psi_{AW} \circ (\text{id} \otimes \nu_L) = (\nu_L \otimes \text{id}) \circ (\text{id} \otimes \Psi_{AW}) \circ (\Psi_{AA} \otimes \text{id}). \tag{20}$$

Equation (19) says that ν_L is an algebra action in the usual sense, and (20) means that ν_L is compatible with the braiding. A braided right A -module V is defined analogously, i.e., V is a right A -braided vector space, the braiding is compatible with the multiplication, and the right action $\nu_R : V \otimes A \rightarrow V$ satisfies

$$\nu_R \circ (\nu_R \otimes \text{id}) = \nu_R \circ (\text{id} \otimes m), \quad \nu_R(v \otimes 1) = v, \quad v \in V, \tag{21}$$

$$\Psi_{VA} \circ (\nu_R \otimes \text{id}) = (\text{id} \otimes \nu_R) \circ (\Psi_{VA} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{AA}). \tag{22}$$

Applying the inverse braidings to the compatibility relations yields

$$(\text{id} \otimes \nu_L) \circ (\Psi_{AA}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{AW}^{-1}) = \Psi_{AW}^{-1} \circ (\nu_L \otimes \text{id}), \tag{23}$$

$$(\nu_R \otimes \text{id}) \circ (\text{id} \otimes \Psi_{AA}^{-1}) \circ (\Psi_{VA}^{-1} \otimes \text{id}) = \Psi_{VA}^{-1} \circ (\text{id} \otimes \nu_R). \tag{24}$$

To shorten notation, we often write

$$\nu_L(a \otimes w) := a \triangleright w, \quad \nu_R(v \otimes a) := v \triangleleft a, \quad w \in W, v \in V, a \in A. \tag{25}$$

Let (H, Δ, ε) be a braided coalgebra, and let V be a right H braided vector space such that the braiding is compatible with the comultiplication. Recall that a left coaction on a vector space V is a linear map $\rho_L : V \rightarrow H \otimes V$ satisfying

$$(\text{id} \otimes \rho_L) \circ \rho_L = (\Delta \otimes \text{id}) \circ \rho_L, \quad (\varepsilon \otimes \text{id}) \circ \rho_L = \text{id}. \tag{26}$$

We say that the coaction is compatible with the braiding if

$$(\text{id} \otimes \rho_L) \circ \Psi_{VH} = (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{VH}) \circ (\rho_L \otimes \text{id}). \tag{27}$$

In this case, V is called a *braided left H -comodule*.

Likewise, for a braided right H -comodule W with a right coaction $\rho_R : W \rightarrow W \otimes H$, we require that W be a left H braided vector space and that

$$(\rho_R \otimes \text{id}) \circ \rho_R = (\text{id} \otimes \Delta) \circ \rho_R, \quad (\text{id} \otimes \varepsilon) \circ \rho_R = \text{id}, \tag{28}$$

$$(\rho_R \otimes \text{id}) \circ \Psi_{HW} = (\text{id} \otimes \Psi_{HH}) \circ (\Psi_{HW} \otimes \text{id}) \circ (\text{id} \otimes \rho_R). \tag{29}$$

Note that we defined a braided *left* H -comodule for a *right* H braided vector space and vice versa. By Lemma 2.4 below, the inverse braidings turn a right (resp., left) H braided vector space into a left (resp., right) H braided vector space. For the inverse braidings, the compatibility conditions read

$$(\text{id} \otimes \Psi_{VH}^{-1}) \circ (\Psi_{HH}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \rho_L) = (\rho_L \otimes \text{id}) \circ \Psi_{VH}^{-1}, \tag{30}$$

$$(\Psi_{HW}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}^{-1}) \circ (\rho_R \otimes \text{id}) = (\text{id} \otimes \rho_R) \circ \Psi_{HW}^{-1}. \tag{31}$$

The objects of our interest are H -module algebras and H -comodule algebras for a braided bialgebra H , so we will highlight them in a separate definition.

Definition 2.2. Let H be a braided bialgebra and let B be a braided algebra such that B is a left H -braided vector space and the braiding Ψ_{HB} is compatible with the multiplications of H and B and with the comultiplication of H .

Assume that B is a braided left H -module with left action $\nu_L : H \otimes B \rightarrow B$. We say that B is a *braided left H -module algebra* if the left action ν_L and the multiplication m_B of B satisfy the compatibility condition

$$\nu_L \circ (\text{id} \otimes m_B) = m_B \circ (\nu_L \otimes \nu_L) \circ (\text{id} \otimes \Psi_{HB} \otimes \text{id}) \circ (\Delta \otimes \text{id} \otimes \text{id}). \tag{32}$$

If $1 \in B$, then it is additionally required that

$$\nu_L(f \otimes 1) = \varepsilon(f)1, \quad f \in H. \tag{33}$$

Assume that B is a braided right H -comodule with right coaction $\rho_R : B \rightarrow B \otimes H$. Then B is called a *braided right H -comodule algebra* if

$$\rho_R \circ m_B = (m_B \otimes m_H) \circ (\text{id} \otimes \Psi_{HB} \otimes \text{id}) \circ (\rho_R \otimes \rho_R). \tag{34}$$

If $1 \in B$, then it is additionally required that

$$\rho_R(1) = 1 \otimes 1. \tag{35}$$

A braided right H -module algebra and braided left H -comodule algebra are defined analogously by flipping the tensor products and replacing Ψ_{HB} by Ψ_{BH} .

Remark 2.3. The dual notions of H -module algebra and H -comodule algebra are H -module coalgebra and H -comodule coalgebra, respectively. The compatibility conditions are the dual versions of those in Definition 2.2. To keep the size of the paper in reasonable limits, we will not discuss these structures here.

An important source for braided vector spaces, or more precisely, for braided monoidal categories, are Yetter–Drinfeld modules over a Hopf algebra H . Here, H denotes a Hopf algebra with bijective antipode S in the unbraided setting. In other words, H is a Hopf algebra according to Definition 2.1 with braiding Ψ_{HH} given by the flip τ as defined in (4). A *(left-left) Yetter–Drinfeld module* is a vector space V with a left H -action $\nu_L : H \otimes V \rightarrow V$ satisfying (19) and a left H -coaction $\rho_L : V \rightarrow H \otimes V$ satisfying (26) such that

$$\rho_L(h \triangleright v) = \sum_{j,k} h_{(1)}^j v_{(-1)}^k S(h_{(3)}^j) \otimes h_{(2)}^j \triangleright v_{(0)}^k, \quad h \in H, v \in V,$$

where $(\Delta \otimes \text{id}) \circ \Delta(h) = \sum_j h_{(1)}^j \otimes h_{(2)}^j \otimes h_{(3)}^j$ and $\rho_L(v) = \sum_k v_{(-1)}^k \otimes v_{(0)}^k$ (cf. Sweedler-type notation in Section 4). Given two Yetter–Drinfeld modules V and W over H , the formula

$$\Psi_{VW}^{\mathcal{D}\mathcal{D}}(v \otimes w) := \sum_k v_{(-1)}^k \triangleright w \otimes v_{(0)}^k$$

defines a braiding which fulfills (1)–(3). It is also noteworthy that the elements $v \in V$ generate a unique braided Hopf algebra with braiding $\Psi_{VV}^{\mathcal{D}\mathcal{D}}$, called *Nichols algebra*, satisfying a certain universal condition and

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \quad \varepsilon(v) = 0, \quad S(v) = -v, \quad v \in V.$$

Moreover, Nichols algebras are graded algebras with $\text{gr}(v) = 1$ for all $v \in V$ and coradically graded with coradical $C_0 = \mathbb{K}$ and $C_1 = V$ (see, e.g., [1, 9]).

Throughout our presentation of duality, we will make frequent use of the inverse of a given braiding. For later reference, we finish this section with a lemma that summarizes some properties of inverse braidings. It is proved by applying repeatedly the corresponding inverse morphism on both sides of the defining relations.

Lemma 2.4. *Let H be a braided vector space with braiding Ψ_{HH} . Then Ψ_{HH}^{-1} defines a braiding on H . If V carries the structure of a left (resp., right) H -braided vector space with respect to Ψ_{HH} , then it does so with respect to Ψ_{HH}^{-1} , and it becomes a right (resp., left) H -braided vector space with respect to Ψ_{HV}^{-1} (resp., Ψ_{VH}^{-1}). If H or V is an algebra and Ψ_{HV} (resp., Ψ_{VH}) is compatible with the multiplication, then Ψ_{HV}^{-1} (resp., Ψ_{VH}^{-1}) is also compatible with the multiplication. If H or V is a coalgebra and Ψ_{HV} (resp., Ψ_{VH}) is compatible with the comultiplication, then Ψ_{HV}^{-1} (resp., Ψ_{VH}^{-1}) is also compatible with the comultiplication. In particular, if H is a braided (co)algebra with respect to Ψ_{HH} , then it is also one with respect to Ψ_{HH}^{-1} .*

3. BRAIDED PRODUCTS, COPRODUCTS AND (CO)ACTIONS

The purpose of this section is to discuss generalizations of opposite algebras and coalgebras in the braiding setting. It is also shown how to use the braiding, and possibly the inverse of the antipode, to turn right (co)actions into left (co)actions and vice versa.

Given a braided bialgebra H , there exists a different braided bialgebra structure on H with the multiplication m and the coproduct Δ replaced by

$$m_1 := m \circ \Psi_{HH} : H \otimes H \longrightarrow H, \quad \Delta_{-1} := \Psi_{HH}^{-1} \circ \Delta : H \longrightarrow H \otimes H. \quad (36)$$

This bialgebra will be denoted by $H^{(1,-1)}$, differing from the standard notation $H^{\text{op},\text{cop}}$ for reasons that will become clear below. If H is a braided Hopf algebra, then it follows from (17) that $H^{(1,-1)}$ is also one with the same antipode S .

The opposite algebra, say $H^{(-1,0)}$, with m replaced by $m_{-1} := m \circ \Psi_{HH}^{-1}$ and the opposite coalgebra, say $H^{(0,-1)}$, with Δ replaced by $\Delta_{-1} := \Psi_{HH}^{-1} \circ \Delta$ will also yield braided bialgebras, but with respect to the inverse braiding. These constructions may yield an infinite family of braided bialgebras provided that Ψ_{HH} is infinite cyclic. For later use, and since we did not find it in the literature, we will state

the result in the following proposition. In the subsequent corollary, it is shown that, for a braided Hopf algebra H with bijective antipode, H is isomorphic to its $H^{\text{op},\text{cop}}$ -version (see also [8]). For more information on braided Hopf algebras, we refer the reader to [7, 16, 25, 30].

Proposition 3.1. *Let (H, m) be a braided (unital) algebra and let $k \in \mathbb{Z}$. Then*

$$m_k := m \circ \Psi_{HH}^k : H \otimes H \longrightarrow H \tag{37}$$

defines a product on H turning it again into a braided (unital) algebra (with the same unit element). If V is a left or right H -braided vector space such that the braiding is compatible with the multiplication m on H , then the braiding is also compatible with the multiplication m_k .

Let (H, Δ, ε) be a braided coalgebra and let $n \in \mathbb{Z}$. Then

$$\Delta_n := \Psi_{HH}^n \circ \Delta : H \longrightarrow H \otimes H \tag{38}$$

defines a coproduct on H turning it again into a braided coalgebra with the same counit. If V is a left or right H -braided vector space such that the braiding is compatible with the comultiplication of H , then the braiding is also compatible with the coproduct Δ_n .

Assume that H is a braided bialgebra and let $H^{(k,n)}$ denote the linear space H equipped with the product m_k , the unmodified unit element, the coproduct Δ_n and the unmodified counit. Then, for all $n \in \mathbb{Z}$, $H^{(n,-n)}$ is a braided bialgebra with respect to the braiding Ψ_{HH} , and $H^{(n-1,-n)}$ is a braided bialgebra with respect to the braiding Ψ_{HH}^{-1} .

If H is a braided Hopf algebra with antipode S , then all $H^{(n,-n)}$ are braided Hopf algebras with the unmodified antipode S . If S is invertible, then all $H^{(n-1,-n)}$ are braided Hopf algebras with antipode S^{-1} .

Proof. We begin by showing that m_1 defines an associative product:

$$\begin{aligned} m_1 \circ (\text{id} \otimes m_1) &\stackrel{(37)}{=} m \circ \Psi_{HH} \circ (\text{id} \otimes m) \circ (\text{id} \otimes \Psi_{HH}) \\ &\stackrel{(7)}{=} m \circ (m \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \circ (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \\ &\stackrel{(1)}{=} m \circ (\text{id} \otimes m) \circ (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \circ (\Psi_{HH} \otimes \text{id}) \\ &\stackrel{(5)}{=} m \circ \Psi_{HH} \circ (m \otimes \text{id}) \circ (\Psi_{HH} \otimes \text{id}) = m_1 \circ (m_1 \otimes \text{id}). \end{aligned}$$

If $1 \in H$, then clearly $m_1(1 \otimes h) = h = m_1(h \otimes 1)$ for all $h \in H$ by (6) and (8).

To show the compatibility of Ψ_{HH} with m_1 , we compute that

$$\begin{aligned} \Psi_{HH} \circ (m_1 \otimes \text{id}) &\stackrel{(5),(37)}{=} (\text{id} \otimes m) \circ (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \circ (\Psi_{HH} \otimes \text{id}) \\ &\stackrel{(1)}{=} (\text{id} \otimes m) \circ (\text{id} \otimes \Psi_{HH}) \circ (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \\ &\stackrel{(37)}{=} (\text{id} \otimes m_1) \circ (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}), \end{aligned}$$

which proves (5). The proof of (7) is completely analogous. Moreover, if $1 \in H$, then (6) and (8) are trivially satisfied. Therefore H is again a braided algebra with respect to the multiplication m_1 and the braiding Ψ_{HH} .

Since $m_{k+1} = m_k \circ \Psi_{HH}$, we conclude by induction that the same holds for all $k \in \mathbb{N}$. From Lemma 2.4 and the previous computations, it follows that m_{-1} turns H also into a braided algebra, and again by induction, we obtain the result for all m_{-k} , $k \in \mathbb{N}$.

Next we prove that $(H, \Delta_1, \varepsilon)$ yields a braided coalgebra. The coassociativity follows from

$$\begin{aligned} ((\Psi_{HH} \circ \Delta) \otimes \text{id}) \circ (\Psi_{HH} \circ \Delta) &\stackrel{(11)}{=} (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \circ (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Delta) \circ \Delta \\ &\stackrel{(1)}{=} (\text{id} \otimes \Psi_{HH}) \circ (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \circ (\Delta \otimes \text{id}) \circ \Delta \\ &\stackrel{(10)}{=} (\text{id} \otimes (\Psi_{HH} \circ \Delta)) \circ (\Psi_{HH} \circ \Delta). \end{aligned}$$

Furthermore, $(\text{id} \otimes \varepsilon) \circ (\Psi_{HH} \circ \Delta) = \text{id} = (\varepsilon \otimes \text{id}) \circ (\Psi_{HH} \circ \Delta)$ by the second relations in (10) and (11). To verify the compatibility with the braiding, we compute that

$$\begin{aligned} (\text{id} \otimes \Delta_1) \circ \Psi_{HH} &\stackrel{(10),(38)}{=} (\text{id} \otimes \Psi_{HH}) \circ (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \circ (\Delta \otimes \text{id}) \\ &\stackrel{(1)}{=} (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \circ (\Psi_{HH} \otimes \text{id}) \circ (\Delta \otimes \text{id}) \\ &\stackrel{(38)}{=} (\Psi_{HH} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}) \circ (\Delta_1 \otimes \text{id}). \end{aligned}$$

This shows the first relation of (10). The first relation of (11) is proved analogously, and the second relations in (10) and (11) are trivially satisfied. Therefore $(H, \Delta_1, \varepsilon)$ is a braided coalgebra.

By Lemma 2.4, the same arguments show that $(H, \Delta_{-1}, \varepsilon)$ yields also a braided coalgebra. Similar to the above, since $\Delta_{k\pm 1} = \Delta_k \circ \Psi_{HH}^{\pm 1}$, we can now proceed by induction to conclude that $(H, \Delta_k, \varepsilon)$ is a braided coalgebra for all $k \in \mathbb{Z}$.

Let H be a braided bialgebra. We first show that $H^{(-1,0)}$ is a braided bialgebra with respect to Ψ_{HH}^{-1} . From the first part of the proof and Lemma 2.4, we know that Ψ_{HH}^{-1} is compatible with the multiplication m_{-1} and the comultiplication Δ , so it remains to verify (16). Again by Lemma 2.4, we conclude that Ψ_{HH}^{-1} satisfies (12). Therefore,

$$\begin{aligned} \Delta \circ m_{-1} &\stackrel{(16)}{=} (m \otimes m) \circ (\text{id} \otimes \Psi_{HH} \otimes \text{id}) \circ (\Delta \otimes \Delta) \circ \Psi_{HH}^{-1} \\ &\stackrel{(12)}{=} (m \otimes m) \circ (\Psi_{HH}^{-1} \otimes \Psi_{HH}^{-1}) \circ (\text{id} \otimes \Psi_{HH}^{-1} \otimes \text{id}) \circ (\Delta \otimes \Delta) \\ &\stackrel{(37)}{=} (m_{-1} \otimes m_{-1}) \circ (\text{id} \otimes \Psi_{HH}^{-1} \otimes \text{id}) \circ (\Delta \otimes \Delta). \end{aligned}$$

This finishes the proof that $H^{(-1,0)}$ is a braided bialgebra with respect to Ψ_{HH}^{-1} .

To conclude the same for $H^{(0,-1)}$, note that (9) remains valid if we replace Ψ_{HH} by Ψ_{HH}^{-1} . Thus

$$\begin{aligned} \Delta_{-1} \circ m &\stackrel{(16)}{=} \Psi_{HH}^{-1} \circ (m \otimes m) \circ (\text{id} \otimes \Psi_{HH} \otimes \text{id}) \circ (\Delta \otimes \Delta) \\ &\stackrel{(9)}{=} (m \otimes m) \circ (\text{id} \otimes \Psi_{HH}^{-1} \otimes \text{id}) \circ (\Psi_{HH}^{-1} \otimes \Psi_{HH}^{-1}) \circ (\Delta \otimes \Delta) \\ &\stackrel{(38)}{=} (m \otimes m) \circ (\text{id} \otimes \Psi_{HH}^{-1} \otimes \text{id}) \circ (\Delta_{-1} \otimes \Delta_{-1}), \end{aligned}$$

hence (16) is satisfied. Together with the previous results, it follows that $H^{(0,-1)}$ is a braided bialgebra with respect to Ψ_{HH}^{-1} .

Now we proceed by induction. Let $n \in \mathbb{N}$ and assume that $H^{(n-1,-n)}$ is a braided bialgebra with respect to Ψ_{HH}^{-1} . From what has already been shown and since $\Psi_{HH} = (\Psi_{HH}^{-1})^{-1}$, we conclude that $H^{(n,-n)}$ with $m_n = m_{n-1} \circ \Psi_{HH}$ and Δ_{-n} is a braided bialgebra with respect to Ψ_{HH} . Likewise, if $H^{(-n,n-1)}$ is a braided bialgebra with respect to Ψ_{HH}^{-1} , then $H^{(-n,n)}$ with m_{-n} and $\Delta_n = \Delta_{n-1} \circ \Psi_{HH}$ is a braided bialgebra with respect to Ψ_{HH} . Continuing in this way, if $H^{(-n,n)}$ is a braided bialgebra with respect to Ψ_{HH} , we can replace in the above calculations H by $H^{(-n,n)}$ and see that $H^{(-(n+1),n)}$ with $m_{-(n+1)} = m_{-n} \circ \Psi_{HH}^{-1}$ and Δ_{-n} is a braided bialgebra with respect to Ψ_{HH}^{-1} . Finally, if $H^{(n,-n)}$ is a braided bialgebra with respect to Ψ_{HH} , it follows that $H^{(n,-(n+1))}$ with m_n and $\Delta_{-(n+1)} = \Delta_{-n} \circ \Psi_{HH}^{-1}$ is a braided bialgebra with respect to Ψ_{HH}^{-1} . Thus the usual induction argument yields the result.

If H is a braided Hopf algebra, then, by (17),

$$m_n \circ (S \otimes \text{id}) \circ \Delta_{-n} = m \circ \Psi_{HH}^n \circ (S \otimes \text{id}) \circ \Psi_{HH}^{-n} \circ \Delta = \begin{cases} m \circ (S \otimes \text{id}) \circ \Delta, & n \in 2\mathbb{Z}, \\ m \circ (\text{id} \otimes S) \circ \Delta, & n \in 2\mathbb{Z} + 1. \end{cases}$$

Thus $m_n \circ (S \otimes \text{id}) \circ \Delta_{-n} = 1\varepsilon$, and similarly, $m_n \circ (\text{id} \otimes S) \circ \Delta_{-n} = 1\varepsilon$. Therefore $H^{(n,-n)}$ is a Hopf algebra with antipode S . If S^{-1} exists, we obtain from (17) for $n \in 2\mathbb{Z}$ that

$$\begin{aligned} m_{n-1} \circ (\text{id} \otimes S^{-1}) \circ \Delta_{-n} &= m \circ \Psi_{HH}^{n-1} \circ (\text{id} \otimes S^{-1}) \circ \Psi_{HH}^{-n} \circ \Delta \\ &= m \circ (S^{-1} \otimes \text{id}) \circ \Psi_{HH}^{-1} \circ \Delta \\ &= m \circ (\text{id} \otimes S) \circ (S^{-1} \otimes S^{-1}) \circ \Psi_{HH}^{-1} \circ \Delta \\ &= m \circ (\text{id} \otimes S) \circ \Delta \circ S^{-1} = 1\varepsilon \circ S^{-1} = 1\varepsilon. \end{aligned}$$

The remaining cases, which prove that S^{-1} yields an antipode for $H^{(n-1,-n)}$, are shown analogously.

Finally, let V be a left H -braided vector space. If Ψ_{HV} is compatible with the multiplication on H , then

$$\begin{aligned} \Psi_{HV} \circ (m_k \otimes \text{id}) &\stackrel{(5),(37)}{=} (\text{id} \otimes m) \circ (\Psi_{HV} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}) \circ (\Psi_{HH}^k \otimes \text{id}) \\ &\stackrel{(2)}{=} (\text{id} \otimes m) \circ (\text{id} \otimes \Psi_{HH}^k) \circ (\Psi_{HV} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}) \\ &\stackrel{(37)}{=} (\text{id} \otimes m_k) \circ (\Psi_{HV} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}), \end{aligned}$$

which shows the compatibility of Ψ_{HV} with the multiplication m_k . Likewise, if Ψ_{HV} is compatible with the comultiplication on H , then

$$\begin{aligned} (\text{id} \otimes \Delta_n) \circ \Psi_{HV} &\stackrel{(10),(38)}{=} (\text{id} \otimes \Psi_{HH}^n) \circ (\Psi_{HV} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}) \circ (\Delta \otimes \text{id}) \\ &\stackrel{(2)}{=} (\Psi_{HV} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}) \circ (\Psi_{HH}^n \otimes \text{id}) \circ (\Delta \otimes \text{id}) \\ &\stackrel{(38)}{=} (\Psi_{HV} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}) \circ (\Delta_n \otimes \text{id}) \end{aligned}$$

proves the compatibility of Ψ_{HV} with the comultiplication Δ_n . The relations regarding the unmodified unit or counit remain trivially true. The proof for a right H -braided vector space is completely analogous. \square

Since, by (17), $(S \otimes S) \circ \Psi_{HH} = \Psi_{HH} \circ (S \otimes S)$ and

$$\begin{aligned} S^k \circ m &= m \circ \Psi_{HH}^k \circ (S^k \otimes S^k) = m_k \circ (S^k \otimes S^k), \\ (S^k \otimes S^k) \circ \Delta &= \Psi_{HH}^{-k} \circ \Delta \circ S^k = \Delta_{-k} \circ S^k, \end{aligned}$$

we obtain immediately the following corollary.

Corollary 3.2. *Let H be a braided Hopf algebra and $n \in \mathbb{Z}$. Then the antipode S defines braided Hopf algebra homomorphisms $S^k : H^{(n,-n)} \rightarrow H^{(n+k,-(n+k))}$ and braided bialgebra homomorphisms $S^k : H^{(n-1,-n)} \rightarrow H^{(n+k-1,-(n+k))}$, where $k \in \mathbb{Z}$ if S is invertible and $k \in \mathbb{N}$ otherwise. For invertible S , all these homomorphisms are isomorphisms of braided Hopf algebras.*

In the unbraided case, a left action of an algebra yields a right action of the opposite algebra with flipped multiplication, and a left coaction of a coalgebra defines a right coaction of the opposite coalgebra with the flipped coproduct. Evidently, the same holds if left and right are interchanged. However, the usual flip is in general not compatible with the braiding. A proper version in the braided setting is given in the next proposition.

Proposition 3.3. *Let (H, m) be a braided algebra and V a braided left H -module. Then*

$$\nu_R^\circ : V \otimes H \longrightarrow H, \quad \nu_R^\circ := \nu_L \circ \Psi_{HV}^{-1} \tag{39}$$

turns V into a braided right H -module with respect to the multiplication $m_{-1} := m \circ \Psi_{HH}^{-1}$ and the braiding Ψ_{HH}^{-1} on H . If H is a braided bialgebra and V is a braided left H -module algebra, then ν_R° transforms V into a braided right $H^{(-1,0)}$ -module algebra.

Analogously, if V is a braided right H -module, then

$$\nu_L^\circ : H \otimes V \longrightarrow H, \quad \nu_L^\circ := \nu_R \circ \Psi_{VH}^{-1}$$

turns V into a braided left H -module with respect to the multiplication m_{-1} and the braiding Ψ_{HH}^{-1} on H . Furthermore, a braided right H -module algebra becomes a braided left $H^{(-1,0)}$ -module algebra.

Given a coalgebra H with coproduct Δ and a braided right H -comodule V , the left coaction

$$\rho_L^\circ : V \longrightarrow H \otimes V, \quad \rho_L^\circ := \Psi_{HV}^{-1} \circ \rho_R$$

turns V into a braided left H -comodule with respect to the coproduct $\Delta_{-1} := \Delta \circ \Psi_{HH}^{-1}$ and the braiding Ψ_{HH}^{-1} on H . If H is a braided bialgebra and V is a braided right H -comodule algebra, then ρ_L° transforms V into a braided left $H^{(0,-1)}$ -comodule algebra.

For a braided left H -comodule V , the right coaction

$$\rho_R^\circ : V \longrightarrow V \otimes H, \quad \rho_R^\circ := \Psi_{VH}^{-1} \circ \rho_L$$

turns V into a braided left H -comodule with respect to the coproduct Δ_{-1} and the braiding Ψ_{HH}^{-1} on H . Moreover, a braided left H -comodule algebra becomes a braided right $H^{(0,-1)}$ -comodule algebra.

Proof. The compatibility of the inverse braidings with algebraic structures can be deduced from Lemma 2.4. In particular, the braiding Ψ_{HV}^{-1} turns V into a right H -braided vector space and, by Proposition 3.1, is compatible with the multiplication m_{-1} . Since

$$\begin{aligned} \nu_R^\circ \circ (\nu_R^\circ \otimes \text{id}) &\stackrel{(39)}{=} \nu_L \circ \Psi_{HV}^{-1} \circ (\nu_L \otimes \text{id}) \circ (\Psi_{HV}^{-1} \otimes \text{id}) \\ &\stackrel{(23)}{=} \nu_L \circ (\text{id} \otimes \nu_L) \circ (\Psi_{HH}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}^{-1}) \circ (\Psi_{HV}^{-1} \otimes \text{id}) \\ &\stackrel{(2),(19)}{=} \nu_L \circ (m \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}^{-1}) \circ (\Psi_{HV}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}^{-1}) \\ &\stackrel{(5)}{=} \nu_L \circ \Psi_{HV}^{-1} \circ (\text{id} \otimes m) \circ (\text{id} \otimes \Psi_{HH}^{-1}) \stackrel{(39)}{=} \nu_R^\circ \circ (\text{id} \otimes m_{-1}), \end{aligned}$$

and $\nu_R^\circ(v \otimes 1) = v$ by (6) and (19), it follows that ν_R° defines a right H -action with respect to the multiplication m_{-1} . Next,

$$\begin{aligned} \Psi_{HV}^{-1} \circ (\nu_R^\circ \otimes \text{id}) &\stackrel{(39)}{=} \Psi_{HV}^{-1} \circ (\nu_L \otimes \text{id}) \circ (\Psi_{HV}^{-1} \otimes \text{id}) \\ &\stackrel{(23)}{=} (\text{id} \otimes \nu_L) \circ (\Psi_{HH}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}^{-1}) \circ (\Psi_{HV}^{-1} \otimes \text{id}) \\ &\stackrel{(2),(39)}{=} (\text{id} \otimes \nu_R^\circ) \circ (\Psi_{HV}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HH}^{-1}) \end{aligned}$$

proves (22). Therefore ν_R° equips V with the structure of a braided right H -module with respect to the multiplication m_{-1} and the braidings Ψ_{HH}^{-1} and Ψ_{HV}^{-1} .

If H is a braided bialgebra and V is a braided left H -module algebra, then

$$\begin{aligned} \nu_R^\circ \circ (m_V \otimes \text{id}) &\stackrel{(7),(39)}{=} \nu_L \circ (\text{id} \otimes m_V) \circ (\Psi_{HV}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}^{-1}) \\ &\stackrel{(32)}{=} m_V \circ (\nu_L \otimes \nu_L) \circ (\text{id} \otimes \Psi_{HV} \otimes \text{id}) \circ (\Delta \otimes \text{id} \otimes \text{id}) \circ (\Psi_{HV}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{HV}^{-1}) \\ &\stackrel{(10)}{=} m_V \circ (\nu_L \otimes \nu_L) \circ (\Psi_{HV}^{-1} \otimes \Psi_{HV}^{-1}) \circ (\text{id} \otimes \Psi_{HV}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \Delta) \\ &\stackrel{(39)}{=} m_V \circ (\nu_R^\circ \otimes \nu_R^\circ) \circ (\text{id} \otimes \Psi_{HV}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \Delta). \end{aligned}$$

This shows that ν_R° satisfies the compatibility condition of a braided right H -module algebra with respect to the unmodified coproduct Δ and the braiding Ψ_{HV}^{-1} , i.e., ν_R° equips V with the structure of a braided right $H^{(-1,0)}$ -module algebra.

The proof of the opposite version and the proofs for the coactions are similar and left to the reader. \square

In the last proposition, we had to replace the product of the braided algebra by the opposite one in order to interchange left and right actions. If H is a braided Hopf algebra with bijective antipode, we can use the antipode to turn a left action into a right action of the *same* algebra, but with a modified coproduct. To see this, it suffices to observe that, if V is a braided left H -module and $\varphi : H_0 \rightarrow H$ is a Hopf algebra homomorphism, then V becomes a braided left H_0 -module in the obvious way. Thus, setting $H_0 := H^{(1,-1)}$ and $\varphi := S^{-1} : H^{(1,-1)} \rightarrow H$, we obtain from Corollary 3.2 and Proposition 3.3 a right action of the Hopf algebra

$H^{(0,-1)}$ with the unmodified product, the coproduct Δ_{-1} and the braiding Ψ_{HH}^{-1} . Similar arguments can be applied to right actions and left or right coactions. We summarize these observations in the next corollary for (co)module algebras.

Corollary 3.4. *Let H be a braided Hopf algebra with invertible antipode S . If V is a left H -module algebra, then the right action*

$$\nu_{R,S} : V \otimes H \longrightarrow V, \quad \nu_{R,S} := \nu_L \circ \Psi_{HV}^{-1} \circ (\text{id} \otimes S^{-1}),$$

turns V into a right $H^{(0,-1)}$ -module algebra. Analogously, a right H -module algebra V becomes a left $H^{(0,-1)}$ -module algebra for the left action defined by

$$\nu_{L,S} : V \otimes H \longrightarrow V, \quad \nu_{L,S} := \nu_R \circ \Psi_{VH}^{-1} \circ (S^{-1} \otimes \text{id}).$$

Given a right H -comodule algebra V , the left coaction

$$\rho_{L,S} : V \longrightarrow H \otimes V, \quad \rho_{L,S} := (S^{-1} \otimes \text{id}) \circ \Psi_{HV}^{-1} \circ \rho_R,$$

turns V into a left $H^{(-1,0)}$ -comodule algebra, and a left H -comodule algebra V becomes a right $H^{(-1,0)}$ -comodule algebra for the right coaction defined by

$$\rho_{R,S} : V \longrightarrow V \otimes H, \quad \rho_{R,S} := (\text{id} \otimes S^{-1}) \circ \Psi_{VH}^{-1} \circ \rho_L.$$

4. DUALITY FOR INFINITE-DIMENSIONAL BRAIDED ALGEBRAS, COALGEBRAS, BIALGEBRAS AND HOPF ALGEBRAS

This section provides a detailed description of duality for braided algebras, braided coalgebras, and both structures together, i.e., braided bialgebras and Hopf algebras. As dual objects may not exist in a braided monoidal category, specifically in the infinite-dimensional setting (cf. [29]), we continue with our non-categorical approach. That is, we assume the existence of a dual space with certain properties without proving its existence or uniqueness, which means that we will not define a functor into a dual category. Furthermore, our definitions will be rather constructive in the sense that they are expressed by explicit formulas derived from the given structures.

A dual pairing between two vector spaces U and H is a linear map $\langle \cdot, \cdot \rangle : U \otimes H \rightarrow \mathbb{K}$. Let H' denote the dual space of H . Given a subspace $U \subset H'$, we define a dual pairing between U and H by

$$\langle \cdot, \cdot \rangle : U \otimes H \longrightarrow \mathbb{K}, \quad \langle f, a \rangle := f(a). \tag{40}$$

Identifying by a slight abuse of notation H with its image $\iota(H) \subset H''$ under the canonical embedding $\iota : H \rightarrow H''$, $\iota(a)(f) := f(a)$, the dual pairing (40) becomes symmetric in the sense that $\langle f, a \rangle = \langle a, f \rangle$. A subspace $U \subset H'$ is called *non-degenerate*, or synonymously the dual pairing is called *non-degenerate*, if the associated bilinear map $\langle \cdot, \cdot \rangle : U \times H \rightarrow \mathbb{K}$ is non-degenerate.

The dual pairing $\langle \cdot, \cdot \rangle$ defined in (40) is actually the restriction of the fundamental evaluation map $\text{ev} : H' \otimes H \rightarrow \mathbb{K}$, $\text{ev}(f \otimes a) := f(a)$. The problem in the infinite setting is that the so-called coevaluation map $\text{coev} : \mathbb{K} \rightarrow H \otimes H'$ may not exist (see e.g. [29]). In other words, the braided monoidal category may not be rigid. Nevertheless, the evaluation map, or rather its restriction $\langle \cdot, \cdot \rangle$, will play a fundamental role in the dual pairing between tensor spaces. In particular, the

dual pairing between n -fold tensor product spaces will entirely be traced back to the evaluation map on adjacent tensor factors. That is, given linear spaces H_j and subspaces $U_j \subset H'_j$, $j = 1, \dots, n$, we define

$$\begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle &: (U_n \otimes \dots \otimes U_1) \otimes (H_1 \otimes \dots \otimes H_n) \longrightarrow \mathbb{K}, \\ \langle\langle \cdot, \cdot \rangle\rangle &:= \langle \cdot, \cdot \rangle \circ (\text{id} \otimes \langle \cdot, \cdot \rangle \otimes \text{id}) \circ \dots \circ (\text{id} \otimes \dots \otimes \langle \cdot, \cdot \rangle \otimes \dots \otimes \text{id}). \end{aligned} \tag{41}$$

This definition is consistent with the representation of braided monoidal categories by braided strings. Once a convention for a dual pairing between tensor product spaces is agreed upon, it should be avoided to use isomorphisms between tensor spaces in the dual pairing that do not arise from braidings. For instance, to pair the second leg in $H' \otimes H'$ with the second leg in $H \otimes H$, it is more appropriate to apply first a braiding $\Psi_{H'H}$ and to consider

$$(\langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle) \circ (\text{id} \otimes \Psi_{H'H} \otimes \text{id}) : H' \otimes H' \otimes H \otimes H \longrightarrow \mathbb{K}.$$

In contrast, we will also make use of the embedding $U_1 \otimes \dots \otimes U_n \subset (H_1 \otimes \dots \otimes H_n)'$ (mind the order). In this case, we write

$$(f_1 \otimes \dots \otimes f_n)(a_1 \otimes \dots \otimes a_n) := f_1(a_1) \dots f_n(a_n).$$

Now let H be a braided vector space with braiding Ψ_{HH} . Our first aim is to show that, for appropriate subspaces $U \subset H'$, Ψ_{HH} induces braidings on $U \otimes U$ and between U and H . Moreover, the braidings between U and H will be compatible with the multiplication and comultiplication on H if these structures are compatible with Ψ_{HH} . According to our non-categorical approach, we will not assume that U is unique nor prove that it always exists.

To begin, consider the linear map

$$\Psi_{H'H'} : H' \otimes H' \longrightarrow (H \otimes H)', \quad \Psi_{H'H'}(f \otimes g)(b \otimes a) := \langle\langle f \otimes g, \Psi_{HH}(a \otimes b) \rangle\rangle, \tag{42}$$

where $a, b \in H$ and $f, g \in H'$. Note that we do not assume that $\Psi_{H'H'}(H' \otimes H') \subset H' \otimes H'$. Similarly, using the fact that the canonical pairing $\langle \cdot, \cdot \rangle : H' \otimes H \rightarrow \mathbb{K}$ is non-degenerate, we define

$$\Psi_{H'H} : H' \otimes H \longrightarrow (H' \otimes H)', \quad \Psi_{H'H}(g \otimes a)(f \otimes b) := \langle\langle f \otimes g, \Psi_{HH}(a \otimes b) \rangle\rangle, \tag{43}$$

$$\Psi_{H'H}^\circ : H' \otimes H \longrightarrow (H' \otimes H)', \quad \Psi_{H'H}^\circ(g \otimes a)(f \otimes b) := \langle\langle f \otimes g, \Psi_{HH}^{-1}(a \otimes b) \rangle\rangle, \tag{44}$$

$$\Psi_{HH'} : H \otimes H' \longrightarrow (H \otimes H')', \quad \Psi_{HH'}(b \otimes f)(a \otimes g) := \langle\langle f \otimes g, \Psi_{HH}(a \otimes b) \rangle\rangle, \tag{45}$$

$$\Psi_{HH'}^\circ : H \otimes H' \longrightarrow (H \otimes H')', \quad \Psi_{HH'}^\circ(b \otimes f)(a \otimes g) := \langle\langle f \otimes g, \Psi_{HH}^{-1}(a \otimes b) \rangle\rangle. \tag{46}$$

It will be convenient to introduce some Sweedler-type notation. As usual, a coproduct is written $\Delta(a) = a_{(1)} \otimes a_{(2)}$ with increasing numbers for multiple coproducts. Analogously, left and right coactions are written $\rho_L(v) = v_{(-1)} \otimes v_{(0)}$ and $\rho_R(v) = v_{(0)} \otimes v_{(1)}$, respectively. For a given braiding on a vector space H , we employ the notation

$$\Psi_{HH}(a \otimes b) = b^{(1)} \otimes a^{(2)}, \quad a, b \in H.$$

For multiple braidings, an index will be used to indicate the chronological order. As an example, a combination of (1) and (10) gives for a and b from a braided

coalgebra

$$b^{(1)1} \otimes a^{(2)1(2)}^{(1)2} \otimes a^{(2)1(1)}^{(2)2} = b^{(1)2(1)3} \otimes a_{(2)}^{(1)1(2)3} \otimes a_{(1)}^{(2)1(2)2}. \tag{47}$$

We use a back-prime to denote the inverse of Ψ_{HH} , i.e.,

$$\Psi_{HH}^{-1}(a \otimes b) = b^{(1)\prime} \otimes a^{(2)\prime}, \quad a, b \in H. \tag{48}$$

Then clearly

$$a^{(2)\langle 1 \rangle\prime} \otimes b^{(1)\langle 2 \rangle\prime} = a \otimes b = a^{(2)\prime(1)} \otimes b^{(1)\prime(2)}, \quad a, b \in H. \tag{49}$$

In a similar vein, (1) and (49) yield the identity

$$c^{(1)\langle 1 \rangle\prime} \otimes b^{(2)\langle 1 \rangle\prime} \otimes a^{(2)\prime(1)\langle 2 \rangle\prime} = c^{(1)\prime(2)\langle 1 \rangle} \otimes b^{(1)\prime(1)\langle 2 \rangle} \otimes a^{(2)\prime(1)\langle 2 \rangle\prime}, \quad a, b, c \in H. \tag{50}$$

The same notations will be used for subspaces $U \subset H'$ such that $\Psi_{H'H'}(U \otimes U) \subset U \otimes U$. Then (42) reads for instance

$$f^{(2)}(a)g^{(1)}(b) = g(b^{(1)})f(a^{(2)}), \quad f, g \in U, \quad a, b \in H. \tag{51}$$

If $\Psi_{H'H}(g \otimes a), \Psi_{H'H}^\circ(g \otimes a) \in H \otimes U \subset (H' \otimes H)'$ for $g \otimes a \in U \otimes H$, we write

$$\Psi_{H'H}(g \otimes a) := a^{\{1\}} \otimes g^{\{2\}}, \quad \Psi_{H'H}^\circ(g \otimes a) := a^{\{1\}^\circ} \otimes g^{\{2\}^\circ}, \tag{52}$$

and a similar notation will be employed for $\Psi_{HH'}$ and $\Psi_{HH'}^\circ$. Under the assumption that all maps belong to the tensor products of the corresponding spaces, (42)–(46) yield in Sweedler-type notation

$$g(b^{(1)})f(a^{(2)}) = g^{(1)}(b)f^{(2)}(a) = f(a^{\{1\}})g^{\{2\}}(b) = f^{\{1\}}(a)g(b^{\{2\}}), \tag{53}$$

$$g(b^{(1)\prime})f(a^{(2)\prime}) = g^{(1)\prime}(b)f^{(2)\prime}(a) = f(a^{\{1\}^\circ})g^{\{2\}^\circ}(b) = f^{\{1\}^\circ}(a)g(b^{\{2\}^\circ}). \tag{54}$$

Furthermore, if $U \subset H'$ is non-degenerate, we conclude from (53) and (54) that

$$f(a^{\{1\}})g^{\{2\}} = f^{(2)}(a)g^{(1)}, \quad a^{\{1\}}g^{\{2\}}(b) = g(b^{(1)})a^{(2)}, \tag{55}$$

$$f(a^{\{1\}^\circ})g^{\{2\}^\circ} = f^{(2)\prime}(a)g^{(1)\prime}, \quad a^{\{1\}^\circ}g^{\{2\}^\circ}(b) = g(b^{(1)\prime})a^{(2)\prime}, \tag{56}$$

for all $f, g \in U$ and $a, b \in H$.

The next lemma shows that, under suitable conditions on $U \subset H'$, the braiding Ψ_{HH} induces braidings on $U \otimes U$ and between U and H which are compatible with possibly additional structures on H .

Lemma 4.1. *Let H be a braided vector space and $U \subset H'$ a non-degenerate subspace. Assume that*

$$\Psi_{UU} = \Psi_{H'H'} \upharpoonright_{U \otimes U} : U \otimes U \longrightarrow U \otimes U \tag{57}$$

is bijective. Then Ψ_{UU} defines a braiding on U . If

$$\Psi_{UH} := \Psi_{H'H} \upharpoonright_{U \otimes H} : U \otimes H \longrightarrow H \otimes U \tag{58}$$

is bijective, then Ψ_{UH} turns H into a left U -braided vector space and U into a right H -braided vector space with respect to the braidings Ψ_{HH} and Ψ_{HH}^{-1} on H , and Ψ_{UU} and Ψ_{UU}^{-1} on U .

In case

$$\Psi_{UH}^\circ := \Psi_{H'H}^\circ \upharpoonright_{U \otimes H} : U \otimes H \longrightarrow H \otimes U \tag{59}$$

is bijective, it also turns H into a left U -braided vector space and U into a right H -braided vector space with respect to the braidings Ψ_{HH} and Ψ_{HH}^{-1} on H , and Ψ_{UU} and Ψ_{UU}^{-1} on U .

If H is a braided (unital) algebra, then the braidings Ψ_{UH} and Ψ_{UH}° are compatible with the multiplication on H . If H is a braided coalgebra, then the braidings Ψ_{UH} and Ψ_{UH}° are compatible with the comultiplication on H .

The analogous statements hold for the opposite versions with respect to the braidings

$$\Psi_{HU} := \Psi_{HH'} \upharpoonright_{H \otimes U} : H \otimes U \longrightarrow U \otimes H,$$

and

$$\Psi_{HU}^\circ := \Psi_{HH'}^\circ \upharpoonright_{H \otimes U} : H \otimes U \longrightarrow U \otimes H.$$

Proof. Let $f \otimes g \otimes h \in U \otimes U \otimes U$. Since $H \otimes H \otimes H$ separates the points of $U \otimes U \otimes U \subset H' \otimes H' \otimes H'$, we can prove (1) by evaluating both sides on all $x \otimes y \otimes z \in H \otimes H \otimes H$. In Sweedler-type notation, we get

$$\begin{aligned} & h^{(1)_2 \langle 1 \rangle_3} (x) g^{(1)_1 \langle 2 \rangle_3} (y) f^{(2)_1 \langle 2 \rangle_2} (z) \stackrel{(51)}{=} h(x^{(1)_1 \langle 1 \rangle_2}) g(y^{(2)_1 \langle 1 \rangle_3}) f(z^{(2)_2 \langle 2 \rangle_3}) \\ & \stackrel{(1)}{=} h(x^{(1)_2 \langle 1 \rangle_3}) g(y^{(1)_1 \langle 2 \rangle_3}) f(z^{(2)_1 \langle 2 \rangle_2}) \stackrel{(51)}{=} h^{(1)_1 \langle 1 \rangle_2} (x) g^{(2)_1 \langle 1 \rangle_3} (y) f^{(2)_2 \langle 2 \rangle_3} (z). \end{aligned} \tag{60}$$

As $\langle \cdot, \cdot \rangle : U \otimes H \rightarrow \mathbb{K}$ is non-degenerate, we conclude that Ψ_{UU} satisfies (1), and so does Ψ_{UU}^{-1} by Lemma 2.4.

To prove (2), we use again the non-degeneracy of the pairing $\langle \cdot, \cdot \rangle : U \otimes H \rightarrow \mathbb{K}$ and evaluate $a^{\{1\}_1 \{1\}_2} \otimes g^{\langle 1 \rangle \{2\}_2} \otimes f^{\langle 2 \rangle \{2\}_1} \in H \otimes U \otimes U$ on all $h \otimes y \otimes z \in U \otimes H \otimes H$, where $a \in H$ and $f, g \in U$. This gives

$$\begin{aligned} & h(a^{\{1\}_1 \{1\}_2}) g^{\langle 1 \rangle \{2\}_2} (y) f^{\langle 2 \rangle \{2\}_1} (z) \stackrel{(53)}{=} h^{(2)_2} (a^{(2)}) g^{(1)_1 \langle 1 \rangle_2} (y) f^{(2)_1} (z^{(1)}) \\ & \stackrel{(51)}{=} h(a^{(2)_1 \langle 2 \rangle_2}) g(y^{(1)_2 \langle 1 \rangle_3}) f(z^{(1)_1 \langle 2 \rangle_3}) \stackrel{(1)}{=} h(a^{(2)_2 \langle 2 \rangle_3}) g(y^{(1)_1 \langle 1 \rangle_2}) f(z^{(2)_1 \langle 1 \rangle_3}) \\ & \stackrel{(51)}{=} h^{(2)} (a^{(2)_2}) g(y^{(1)_1 \langle 1 \rangle_2}) f^{(1)} (z^{(2)_1}) \stackrel{(53)}{=} h(a^{\{1\}_1 \{1\}_2}) g^{\{2\}_1} (y^{(1)}) f^{\{2\}_2} (z^{(2)}) \\ & \stackrel{(51)}{=} h(a^{\{1\}_1 \{1\}_2}) g^{\{2\}_1 \langle 1 \rangle} (y) f^{\{2\}_2 \langle 2 \rangle} (z). \end{aligned} \tag{61}$$

Since $U \otimes H \otimes H$ separates the points of $H \otimes U \otimes U$, these calculations show that (2) is satisfied.

Much in the same way, for all $f \otimes a \otimes b \in U \otimes H \otimes H$ and $g \otimes h \otimes z \in U \otimes U \otimes H$, we compute

$$\begin{aligned} & g(b^{(1)\{1\}_1}) h(a^{(2)\{1\}_2}) f^{\{2\}_1 \{2\}_2} (z) \stackrel{(53)}{=} g(b^{(1)_1 \langle 2 \rangle_3}) h(a^{(2)_1 \langle 2 \rangle_2}) f(z^{(2)_2 \langle 1 \rangle_3}) \\ & \stackrel{(1)}{=} g(b^{(2)_1 \langle 1 \rangle_3}) h(a^{(2)_2 \langle 2 \rangle_3}) f(z^{(1)_1 \langle 1 \rangle_2}) \stackrel{(53)}{=} g^{(1)} (b^{\{1\}_2}) h^{(2)} (a^{\{1\}_1}) f^{\{2\}_1 \{2\}_2} (z) \\ & \stackrel{(51)}{=} g(b^{\{1\}_2 \langle 1 \rangle}) h(a^{\{1\}_1 \langle 2 \rangle}) f^{\{2\}_1 \{2\}_2} (z), \end{aligned}$$

which proves (3). This finishes the proof of first part of the lemma regarding the braidings Ψ_{HH} , Ψ_{UU} and Ψ_{UH} . By Lemma 2.4, the same holds with respect to the braidings Ψ_{HH}^{-1} , Ψ_{UU}^{-1} and Ψ_{UH} . Replacing in the above calculations Ψ_{HH} by Ψ_{HH}^{-1} and Ψ_{UU} by Ψ_{UU}^{-1} shows the analogous results for Ψ_{UH}° .

Let H be a braided algebra. Using the fact that Ψ_{HH} satisfies (5) and (6), we get for all $f, g \in U$ and $a, b \in H$,

$$f((ab)^{\{1\}})g^{\{2\}}(c) \stackrel{(53)}{=} f((ab)^{\langle 2 \rangle})g(c^{\langle 1 \rangle}) \stackrel{(5)}{=} f(a^{\langle 2 \rangle_2}b^{\langle 2 \rangle_1})g(c^{\langle 1 \rangle_1}\langle 1 \rangle_2) \stackrel{(53)}{=} f(a^{\{1\}_1}b^{\{1\}_2})g^{\{2\}_1\{2\}_2}(c) \tag{62}$$

and

$$f(1^{\{1\}})g^{\{2\}}(a) \stackrel{(53)}{=} f(1^{\langle 2 \rangle})g(a^{\langle 1 \rangle}) \stackrel{(6)}{=} f(1)g(a). \tag{63}$$

This implies the compatibility of Ψ_{UH} with the multiplication on H .

By Lemma 2.4, Ψ_{HH}^{-1} is also compatible with the multiplication on H . Replacing $\{k\}$ by $\{k\}^\circ$ and $\langle k \rangle$ by $\langle k \rangle'$ in (62) and (63) shows the compatibility of Ψ_{UH}° with the multiplication on H . Similarly, if H is a braided coalgebra, we obtain for $f, g, h \in U$ and $a, b \in H$ that

$$f(a^{\{1\}_{(1)}})g(a^{\{1\}_{(2)}})h^{\{2\}}(b) \stackrel{(55)}{=} f(a^{\langle 2 \rangle_{(1)}})g(a^{\langle 2 \rangle_{(2)}})h(b^{\langle 1 \rangle}) \stackrel{(10)}{=} f(a_{(1)}^{\langle 2 \rangle_2})g(a_{(2)}^{\langle 2 \rangle_1})h(b^{\langle 1 \rangle_1}\langle 1 \rangle_2) \stackrel{(53)}{=} f(a_{(1)}^{\{1\}_1})g(a_{(2)}^{\{1\}_2})h^{\{2\}_1\{2\}_2}(b)$$

and

$$\varepsilon(a^{\{1\}})h^{\{2\}}(b) \stackrel{(55)}{=} \varepsilon(a^{\langle 2 \rangle})h(b^{\langle 1 \rangle}) \stackrel{(10)}{=} \varepsilon(a)h(b),$$

which proves (11) for Ψ_{UH} .

The same proof with the notational changes mentioned above shows the compatibility of Ψ_{UH}° with the multiplication or the comultiplication (as applicable) on H . The statements of the opposite versions are proved analogously. \square

The following definition of a dual pairing between braided bialgebras is the central definition of this section because it will also serve as a guiding principle for duality between braided algebras and braided coalgebras. Similar definitions can be found in [10, 13, 32, 26].

Definition 4.2. Let U and H be braided bialgebras and let $\Upsilon_{UH} : U \otimes H \rightarrow H \otimes U$ be a braiding such that H is a left U -braided vector space, U is a right H -braided vector space, and the braiding is compatible with the multiplications and comultiplications of U and H . A dual pairing between U and H is a linear map $\langle \cdot, \cdot \rangle : U \otimes H \rightarrow \mathbb{K}$ such that

$$\langle \cdot, \cdot \rangle \circ (m \otimes \text{id}) = (\langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle) \circ (\text{id} \otimes \Upsilon_{UH} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \Delta), \tag{64}$$

$$\langle \cdot, \cdot \rangle \circ (\text{id} \otimes m) = (\langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle) \circ (\text{id} \otimes \Upsilon_{UH} \otimes \text{id}) \circ (\Delta \otimes \text{id} \otimes \text{id}), \tag{65}$$

$$\langle 1, a \rangle = \varepsilon(a), \quad \langle h, 1 \rangle = \varepsilon(h), \quad a \in H, \quad h \in U. \tag{66}$$

For a dual pairing between braided Hopf algebras, it is additionally required that

$$\langle \cdot, \cdot \rangle \circ (S \otimes \text{id}) = \langle \cdot, \cdot \rangle \circ (\text{id} \otimes S). \tag{67}$$

If the dual pairing is non-degenerate, U is called a *left dual* of H , and H is called a *right dual* of U .

Given a braided coalgebra H , the convolution product (15) turns H' into an associative algebra. In general, this product will not be compatible with the dual pairing of Definition 4.2. On the other hand, under the assumptions of Definition 4.2, the product on $U \subset H'$ is uniquely determined by (64) since H separates the points of H' . For this reason, we will consider an alternative convolution product on H' such that the equality in (64) is automatically met for subalgebras $U \subset H'$ satisfying the assumptions of Lemma 4.1.

Proposition 4.3. *Let H be a braided coalgebra and set*

$$\underline{*} : H' \otimes H' \longrightarrow H', \quad f \underline{*} g(a) := \langle\langle f \otimes g, \Psi_{HH} \circ \Delta(a) \rangle\rangle = f(a_{(1)}^{(2)})g(a_{(2)}^{(1)}), \quad (68)$$

where $f, g \in H'$ and $a \in H$. Then (68) turns H' into an associative unital algebra with the unit element given by the counit of H .

Suppose that $U \subset H'$ is a (unital) subalgebra separating the points of H . If Ψ_{UU} defined in (57) is bijective, then it turns U into a braided algebra. In case Ψ_{UH} or Ψ_{UH}° satisfies the assumptions of Lemma 4.1, it defines a braiding that is compatible with the multiplication on U . The same remains true for the opposite version with Ψ_{UH} and Ψ_{UH}° replaced by Ψ_{HU} and Ψ_{HU}° , respectively.

Proof. The associativity of the product $\underline{*}$ is equivalent to coassociativity of $\Delta_1 := \Psi_{HH} \circ \Delta$ which was proved in Proposition 3.1. Moreover, by (68) and the second identity in (11),

$$f \underline{*} \varepsilon(a) = f \circ (\varepsilon \otimes \text{id}) \circ \Psi_{HH} \circ \Delta(a) = f \circ (\text{id} \otimes \varepsilon) \circ \Delta(a) = f(a),$$

and similarly $\varepsilon \underline{*} f(a) = f(a)$ for all $f \in H'$ and $a \in H$. Therefore ε yields the unit element in H' with respect to the product $\underline{*}$.

To show the compatibility of the multiplication with the braiding, we have to verify Equations (5)–(8) for Ψ_{UU} but only (5) and (6) for Ψ_{UH} . Since U separates the points of H and vice versa, we may again prove the required relations by evaluating both sides on elements from U and H . Let $a, b, c \in H$ and $f, g, h \in U$. In Sweedler-type notation, the proof of (5) reads as follows:

$$\begin{aligned} & h^{(1)}(b)(f \underline{*} g)^{(2)}(a) \stackrel{(51),(68)}{=} h(b^{(1)}_1)g(a^{(2)}_1)^{(1)}_2 f(a^{(2)}_1)^{(2)}_2 \\ & \stackrel{(47)}{=} h(b^{(1)}_2)^{(1)}_3 g(a_{(2)}^{(1)}_1)^{(2)}_3 f(a_{(1)}^{(2)}_1)^{(2)}_2 \stackrel{(51),(68)}{=} h^{(1)}_1(b)(f^{(2)}_2 \underline{*} g^{(2)}_1)(a). \end{aligned} \quad (69)$$

Similarly,

$$\begin{aligned} & (f \underline{*} g)^{(1)}(b)h^{(2)}(a) \stackrel{(51),(68)}{=} g(b^{(1)}_1)^{(1)}_2 f(b^{(1)}_1)^{(2)}_2 h(a^{(2)}_1) \\ & \stackrel{(1),(11)}{=} g(b_{(2)}^{(1)}_1)^{(1)}_2 f(b_{(1)}^{(2)}_1)^{(1)}_3 h(a^{(2)}_2)^{(2)}_3 \stackrel{(51),(68)}{=} (f^{(1)}_1 \underline{*} g^{(1)}_2)(b)h^{(2)}_1(a), \end{aligned} \quad (70)$$

which yields (7) for Ψ_{UU} . Moreover,

$$f^{(1)}(b)\varepsilon^{(2)}(a) \stackrel{(51)}{=} f(b^{(1)})\varepsilon(a^{(2)}) \stackrel{(10)}{=} \varepsilon(a)f(b) \stackrel{(11)}{=} \varepsilon(a^{(1)})f(b^{(2)}) \stackrel{(51)}{=} \varepsilon^{(1)}(a)f^{(2)}(b) \quad (71)$$

shows that Ψ_{UU} fulfills (6) and (8). This finishes the proof that Ψ_{UU} is compatible with the multiplication of U .

Applying (53) to both sides of (70) and the right side of (71) gives

$$h(a^{\{1\}})(f * g)^{\{2\}}(b) = h(a^{\{1\}_1\{1\}_2})(f^{\{2\}_2} * g^{\{2\}_1})(b), \quad f(b^{\{1\}})\varepsilon^{\{2\}}(a) = f(b)\varepsilon(a),$$

so that Ψ_{UH} is also compatible with the multiplication on U .

By Lemma 2.4, Ψ_{HH}^{-1} is compatible with the comultiplication on H . Similar as above,

$$\begin{aligned} & h(a^{\{1\}^\circ})(f * g)^{\{2\}^\circ}(b) \stackrel{(54),(68)}{=} h(a^{\langle 2 \rangle^\backslash})g(b^{\langle 1 \rangle^\backslash}{}_{(2)}^{\langle 1 \rangle})f(b^{\langle 1 \rangle^\backslash}{}_{(1)}^{\langle 2 \rangle}) \\ & \stackrel{(11)}{=} h(a^{\langle 2 \rangle^\backslash}_1 \langle 2 \rangle^\backslash_2)g(b_{(2)}^{\langle 1 \rangle^\backslash}_1 \langle 1 \rangle) f(b_{(1)}^{\langle 1 \rangle^\backslash}_1 \langle 2 \rangle) \stackrel{(50)}{=} h(a^{\langle 2 \rangle^\backslash}_1 \langle 2 \rangle^\backslash_2)g(b_{(2)}^{\langle 1 \rangle^\backslash}_1 \langle 1 \rangle_1) f(b_{(1)}^{\langle 2 \rangle^\backslash}_1 \langle 1 \rangle_2) \\ & \stackrel{(54)}{=} h^{\langle 2 \rangle^\backslash}(a^{\{1\}^\circ})g^{\{2\}^\circ}(b_{(2)}^{\langle 1 \rangle})f^{\langle 1 \rangle^\backslash}(b_{(1)}^{\langle 2 \rangle}) \stackrel{(54)}{=} h(a^{\{1\}_1^\circ \{1\}_2^\circ})g^{\{2\}_1^\circ}(b_{(2)}^{\langle 1 \rangle})f^{\{2\}_2^\circ}(b_{(1)}^{\langle 2 \rangle}) \\ & \stackrel{(68)}{=} h(a^{\{1\}_1^\circ \{1\}_2^\circ})(f^{\{2\}_2^\circ} * g^{\{2\}_1^\circ})(b), \end{aligned}$$

hence Ψ_{UH}° satisfies (5). By Lemma 2.4 and the second relation of (11),

$$f(a^{\{1\}^\circ})\varepsilon^{\{2\}^\circ}(b) = f(a^{\langle 2 \rangle^\backslash})\varepsilon(b^{\langle 1 \rangle^\backslash}) = f(a)\varepsilon(b),$$

which shows that Ψ_{UH}° satisfies also (6). This proves the compatibility of Ψ_{UH}° with the multiplication on U .

The opposite version is shown in the same way. □

Recall that, for any unital algebra H , the dual space H' contains a largest coalgebra H° such that $\Delta : H^\circ \rightarrow H^\circ \otimes H^\circ$, $\Delta(f)(a \otimes b) := f(ab)$ for $a, b \in H$, and $\varepsilon(f) := f(1)$ (see e.g. [17]). However, the compatibility condition (65) of the dual pairing requires to consider a modified coproduct, say Δ' , on suitable subspaces of H' . To state an explicit formula, assume that H is a braided algebra and suppose that U is a linear subspace of H' satisfying the assumptions of Lemma 4.1. Then, for Δ' on U and $\Upsilon_{UH} = \Psi_{UH}$, (65) is equivalent to

$$\langle\langle \Delta'(f), \Psi_{HH}(a \otimes b) \rangle\rangle = \langle f, m(a \otimes b) \rangle, \quad f \in U, a, b \in H,$$

which leads to

$$\langle\langle \Delta'(f), a \otimes b \rangle\rangle = \langle f, m \circ \Psi_{HH}^{-1}(a \otimes b) \rangle \quad f \in U, a, b \in H. \tag{72}$$

If we replace Ψ_{UH} by Ψ_{UH}° in (65), then Ψ_{HH}^{-1} needs to be replaced by Ψ_{HH} in (72).

Note that (72) determines uniquely $\Delta'(f) \in (H \otimes H)'$ but $\Delta'(f)$ defined by the right hand side of (72) may not belong to $H' \otimes H'$. The next proposition shows that, similar to the unbraided case, H' contains a largest coalgebra such that the coproduct is given as in (72) and any subcoalgebra U satisfying the assumptions of Lemma 4.1 becomes a braided coalgebra such that the comultiplication is compatible with the braidings Ψ_{UH} and Ψ_{UH}° .

Proposition 4.4. *Let H be a braided unital algebra and consider*

$$\underline{\Delta} : H' \longrightarrow (H \otimes H)', \quad \langle \underline{\Delta}(f), a \otimes b \rangle := \langle f, m \circ \Psi_{HH}(a \otimes b) \rangle = f(b^{(1)}a^{(2)}), \quad (73)$$

where $f \in H'$, $a, b \in H$. Then there exists a largest coalgebra H° in H' such that the coproduct is given by (73) and $\varepsilon(f) := f(1)$.

Suppose that $U \subset H^\circ$ is a subcoalgebra separating the points of H . If Ψ_{UU} defined in (57) is bijective, then it turns U into a braided coalgebra. In case Ψ_{UH} or Ψ_{UH}° satisfies the assumptions of Lemma 4.1, it defines a braiding that is compatible with the comultiplication on U . The same remains true for the opposite version with Ψ_{UH} and Ψ_{UH}° replaced by Ψ_{HU} and Ψ_{HU}° , respectively.

Proof. By Proposition 3.1, $m_1 := m \circ \Psi_{HH}$ defines an associative multiplication on H with unit element $1 \in H$. Therefore the existence of a largest coalgebra follows from the known result of the unbraided case, i.e., there exists a largest coalgebra, say H° , in H' with the coproduct given by (73) and $\varepsilon(f) := f(1)$, see e.g. [17, Section 1.2.8].

Assume that the subcoalgebra $U \subset H^\circ$ satisfies the assumptions of the proposition which guarantee that Ψ_{UU} and Ψ_{UH} are well-defined. Then, for all $f, g \in U$ and $a, b, c \in H$, we have

$$\begin{aligned} g^{(1)}_{(1)}(c)g^{(1)}_{(2)}(b)f^{(2)}(a) &\stackrel{(73)}{=} g^{(1)}_2(c^{(1)}_1b^{(2)}_1)f^{(2)}_2(a) \stackrel{(53)}{=} g((c^{(1)}_1b^{(2)}_1)^{(1)}_2)f(a^{(2)}_2) \\ &\stackrel{(7)}{=} g(c^{(1)}_1{}^{(1)}_2b^{(2)}_1{}^{(1)}_3)f(a^{(2)}_2{}^{(2)}_3) \stackrel{(1)}{=} g(c^{(1)}_2{}^{(1)}_3b^{(1)}_1{}^{(2)}_3)f(a^{(2)}_1{}^{(2)}_2) \quad (74) \\ &\stackrel{(73)}{=} g_{(1)}(c^{(1)}_2)g_{(2)}(b^{(1)}_1)f(a^{(2)}_1{}^{(2)}_2) \stackrel{(53)}{=} g_{(1)}{}^{(1)}_1(c)g_{(2)}{}^{(1)}_2(b)f^{(2)}_1{}^{(2)}_2(a) \end{aligned}$$

which proves that Ψ_{UU} satisfies the first relation of (11). The second relation of (11) follows from

$$\varepsilon(g^{(1)})f^{(2)}(a) = g^{(1)}(1)f^{(2)}(a) \stackrel{(53)}{=} g(1^{(1)})f(a^{(2)}) \stackrel{(8)}{=} g(1)f(a) = \varepsilon(g)f(a).$$

In exactly the same way, one shows that Ψ_{UU} satisfies (10), hence U is a braided coalgebra with respect to braiding Ψ_{UU} .

Applying $\underline{\Delta}$ to (55), evaluating on $b \otimes c \in H \otimes H$ and using (74) gives

$$\begin{aligned} f(a^{\{1\}})g^{\{2\}}_{(1)}(c)g^{\{2\}}_{(2)}(b) &\stackrel{(55)}{=} f^{(2)}(a)g^{(1)}_{(1)}(c)g^{(1)}_{(2)}(b) \\ &\stackrel{(74)}{=} f^{(2)}_1{}^{(2)}_2(a)g_{(1)}{}^{(1)}_1(c)g_{(2)}{}^{(1)}_2(b) \stackrel{(53)}{=} f(a^{\{1\}}_1{}^{\{1\}}_2)g_{(1)}^{\{2\}}_2(c)g_{(2)}^{\{2\}}_1(b). \end{aligned}$$

Moreover,

$$f(a^{\{1\}}\varepsilon(g^{\{2\}})) = g^{\{2\}}(1)f(a^{\{1\}}) \stackrel{(53)}{=} g(1^{(1)})f(a^{(2)}) \stackrel{(8)}{=} g(1)f(a) = f(\varepsilon(g)a).$$

From the last two computations, we conclude that Ψ_{UH} satisfies (10), i.e., the braiding Ψ_{UH} is compatible with the comultiplication on U .

Much in the same way, by applying Lemma 2.4 to Ψ_{HH} ,

$$\begin{aligned} f(a^{\{1\}^\circ})g^{\{2\}^\circ}_{(1)}(c)g^{\{2\}^\circ}_{(2)}(b) & \stackrel{(56)}{=} f^{(2)\setminus}(a)g^{(1)\setminus}_{(1)}(c)g^{(1)\setminus}_{(2)}(b) \stackrel{(73)}{=} f^{(2)\setminus}(a)g^{(1)\setminus}(c^{(1)}b^{(2)}) \\ & \stackrel{(54)}{=} f(a^{(2)\setminus}_2)g((c^{(1)\setminus}_1b^{(2)\setminus}_1)^{(1)\setminus}_2) \stackrel{(7)}{=} f(a^{(2)\setminus}_1\langle 2 \rangle_2)g(c^{(1)\setminus}_1\langle 1 \rangle_1b^{(2)\setminus}_1\langle 1 \rangle_2) \\ & \stackrel{(50)}{=} f(a^{(2)\setminus}_1\langle 2 \rangle_2)g(c^{(1)\setminus}_2\langle 1 \rangle_1b^{(1)\setminus}_1\langle 2 \rangle_2) \stackrel{(73)}{=} f(a^{(2)\setminus}_1\langle 2 \rangle_2)g_{(1)}(c^{(1)\setminus}_2)g_{(2)}(b^{(1)\setminus}_1) \\ & \stackrel{(54)}{=} f^{(2)\setminus}_2(a^{(2)\setminus}_1)g_{(1)}\langle 1 \rangle_2(c)g_{(2)}(b^{(1)\setminus}_1) \stackrel{(54)}{=} f(a^{\{1\}^\circ\{1\}^\circ_2})g_{(1)}\{2\}^\circ_2(c)g_{(2)}\{2\}^\circ_1(b), \end{aligned}$$

and

$$f(a^{\{1\}}\varepsilon(g^{\{2\}^\circ})) = g^{\{2\}^\circ}(1)f(a^{\{1\}^\circ}) \stackrel{(54)}{=} g(1^{(1)\setminus})f(a^{(2)\setminus}) \stackrel{(6)}{=} g(1)f(a) = f(\varepsilon(g)a).$$

Therefore, if the braiding Ψ_{UH}° exists, it is compatible with the comultiplication on U .

The opposite version is proved analogously. □

Given a braided bialgebra H , Lemma 4.1, Proposition 4.3 and Proposition 4.4 tell us how to define braidings, a product and a coproduct, respectively, on appropriate subspaces of H' . The next theorem shows that these structures fit well together, i.e., they can be used to obtain a braided bialgebra $U \subset H'$ such that the canonical pairing yields a dual pairing between braided bialgebras. It is noteworthy to mention that the braiding Υ_{UH} in Definition 4.2 will not be implemented by Ψ_{UH} , but by Ψ_{UH}° .

Theorem 4.5. *Let H be a braided bialgebra and consider the product $\underline{m} : H' \otimes H' \rightarrow H'$ defined by*

$$\langle \underline{m}(f \otimes g), a \rangle := \langle \langle f \otimes g, \Psi_{HH}^{-1} \circ \Delta(a) \rangle \rangle = f(a_{(1)}^{\langle 2 \rangle})g(a_{(2)}^{\langle 1 \rangle}), \quad a \in H, f, g \in H'. \tag{75}$$

Assume that $U \subset H'$ is a unital subalgebra which is also a subcoalgebra of H° for the coproduct $\underline{\Delta}$ defined in Proposition 4.4.

Suppose that U satisfies the left-handed version of Lemma 4.1 such that Ψ_{UU} and Ψ_{UH}° define braidings. Then U is a braided bialgebra with respect to the braiding Ψ_{UU} , and the canonical pairing $\langle \cdot, \cdot \rangle : U \otimes H \rightarrow \mathbb{K}$ defines a pairing between braided bialgebras with respect to the braiding Ψ_{UH}° such that U becomes a left dual of H .

In case U satisfies the right-handed version of Lemma 4.1 such that Ψ_{UU} and Ψ_{HU}° yield braidings, the canonical pairing $\langle \cdot, \cdot \rangle : H \otimes U \rightarrow \mathbb{K}$ defines a pairing between braided bialgebras with respect to the braiding Ψ_{HU}° and U becomes a right dual of H .

If H is a braided Hopf algebra and $f \circ S \in U$ for all $f \in U$, then U is a braided Hopf algebra with antipode $S(f) := f \circ S$.

Proof. By Lemma 2.4, Ψ_{HH}^{-1} defines a braiding on H which is compatible with the comultiplication on H . Therefore, by Proposition 4.3, the product (75) is well-defined. Note that the braiding in (75) is the inverse of that in (68). Combining Proposition 4.3 with Lemma 2.4 shows that the multiplication on U is compatible

with Ψ_{UU} so that U becomes a braided algebra. From Proposition 4.4, it follows directly that U is a braided coalgebra with respect to Ψ_{UU} . Therefore, to complete the proof that U yields a braided bialgebra, it suffices to prove (16).

As in the previous proofs, we will verify (16) by evaluating both sides on $a \otimes b \in H \otimes H$. It was shown in Proposition 3.1 that $H^{(1,-1)}$ is a braided bialgebra with respect to Ψ_{HH} . Therefore, for all $f \otimes g \in U \otimes U$,

$$\begin{aligned} & \langle \underline{\Delta} \circ \underline{m}(f \otimes g), a \otimes b \rangle \stackrel{(73),(75)}{=} \langle f \otimes g, \Delta_{-1} \circ m_1(a \otimes b) \rangle \\ & \stackrel{(16)}{=} \langle f \otimes g, (m_1 \otimes m_1) \circ (\text{id} \otimes \Psi_{HH} \otimes \text{id}) \circ (\Delta_{-1} \otimes \Delta_{-1})(a \otimes b) \rangle \\ & \stackrel{(73)}{=} \langle (\underline{\Delta} \otimes \underline{\Delta})(f \otimes g), (\text{id} \otimes \Psi_{HH} \otimes \text{id}) \circ (\Delta_{-1} \otimes \Delta_{-1})(a \otimes b) \rangle \\ & \stackrel{(42),(57)}{=} \langle (\text{id} \otimes \Psi_{UU} \otimes \text{id}) \circ (\underline{\Delta} \otimes \underline{\Delta})(f \otimes g), \Delta_{-1} \otimes \Delta_{-1}(a \otimes b) \rangle \\ & \stackrel{(75)}{=} \langle (\underline{m} \otimes \underline{m}) \circ (\text{id} \otimes \Psi_{UU} \otimes \text{id}) \circ (\underline{\Delta} \otimes \underline{\Delta})(f \otimes g), a \otimes b \rangle, \end{aligned}$$

which implies that the product and the coproduct on U satisfy the compatibility condition (16) with respect to the braiding Ψ_{UU} . Thus we have proved that U is a braided bialgebra.

Our next aim is to show that the canonical pairing defines a pairing between braided bialgebras with respect to the braiding Ψ_{UH}° . From Lemma 4.1, Proposition 4.3 and Proposition 4.4, we conclude that Ψ_{UH}° is compatible with the multiplications and the comultiplications on U and H . By Definition 4.2, it remains to prove Equations (64)–(66).

Equation (66) is trivially satisfied by the stated unit element in Proposition 4.3 and the definition of the counit in Proposition 4.4. Let $f, g \in U$ and $a, b \in H$. Equation (64) follows from

$$\underline{m}(f \otimes g)(a) \stackrel{(75)}{=} g(a_{(2)} \langle 1 \rangle) f(a_{(1)} \langle 2 \rangle) \stackrel{(54)}{=} f(a_{(1)} \{1\}^{\circ}) g \{2\}^{\circ}(a_{(2)}),$$

and

$$f(ab) \stackrel{(49)}{=} f(a \langle 2 \rangle \langle 1 \rangle b \langle 1 \rangle \langle 2 \rangle) \stackrel{(73)}{=} f_{(1)}(a \langle 2 \rangle) f_{(2)}(b \langle 1 \rangle) \stackrel{(54)}{=} f_{(1)}(a \{1\}^{\circ}) f_{(2)} \{2\}^{\circ}(b)$$

implies (65).

Now assume that H is a braided Hopf algebra with antipode S and that $f \circ S \in U$ for all $f \in U$. With the definition $S(f) := f \circ S$, we compute for all $a \in H$

$$\begin{aligned} \langle \underline{m} \circ (\text{id} \otimes S) \circ \underline{\Delta}(f), a \rangle & \stackrel{(75)}{=} \langle (\text{id} \otimes S) \circ \underline{\Delta}(f), \Delta_{-1}(a) \rangle \stackrel{(41)}{=} \langle \underline{\Delta}(f), (S \otimes \text{id}) \circ \Delta_{-1}(a) \rangle \\ & \stackrel{(73)}{=} \langle f, m_1 \circ (S \otimes \text{id}) \circ \Delta_{-1}(a) \rangle = \varepsilon(a) \langle f, 1 \rangle = \langle \varepsilon(f) 1, a \rangle, \end{aligned}$$

where we used the fact from Proposition 3.1 that $H^{(1,-1)}$ is a braided Hopf algebra with antipode S . This yields $\underline{m} \circ (\text{id} \otimes S) \circ \underline{\Delta}(f) = \varepsilon(f)1$. Interchanging the positions of S and id in the above calculations shows that also $\underline{m} \circ (S \otimes \text{id}) \circ \underline{\Delta}(f) = \varepsilon(f)1$, therefore the linear mapping $S : U \rightarrow U$ defined above turns U into a braided Hopf algebra.

Since $f^{\{1\}^\circ}(a)g(b^{\{2\}^\circ}) = f(a^{\{2\}^\circ})g(b^{\{1\}^\circ}) = f(a^{\{1\}^\circ})g^{\{2\}^\circ}(b)$ by (54), and $\langle \cdot, \cdot \rangle$ is symmetric in the sense that $\langle a, f \rangle = f(a) = \langle f, a \rangle$, the proof of the opposite version is essentially the same. \square

Note that the same bialgebra U serves as a left and as a right dual of H by considering either the braiding Ψ_{UH}° or the braiding Ψ_{HU}° . From now on, we restrict ourselves mainly to the left version, the right version differs essentially only in notation.

Our definitions of the product and the coproduct on $U \subset H'$ deviate from those given in [24] for rigid (= “finite”) braided monoidal categories. By evaluating elements of B^* ($\sim U$) on elements of B ($\sim H$) in [24], it can be seen that the main difference boils down to omitting the braiding Υ_{UH} in Definition 4.2 and using the pairing $\langle\langle \cdot, \cdot \rangle\rangle$ instead. For a finite dimensional braided bialgebra, the dual braided bialgebra in the sense of [24, Proposition 2.4] is a special case of our construction and corresponds to $U^{(1,-1)}$. We present this result in the following corollary, whose proof is straightforward by applying the corresponding definitions.

Corollary 4.6. *Let H be a braided bialgebra and assume that $U \subset H'$ fulfills the assumptions of Theorem 4.5. Then the product m_1 and the coproduct Δ_{-1} of $U^{(1,-1)}$ satisfy*

$$\langle m_1(f \otimes g), a \rangle = \langle\langle f \otimes g, \Delta(a) \rangle\rangle, \quad \langle\langle \Delta_{-1}(f), a \otimes b \rangle\rangle = \langle f, m(a \otimes b) \rangle$$

for all $f, g \in U$ and $a, b \in H$.

Note that, for a finite-dimensional braided bialgebra H , the non-degeneracy condition implies $U = H'$, and H' satisfies automatically the assumptions on U in the above theorem. In this case, the left dual $U = H'$ is unique. For an infinite-dimensional braided bialgebra, this need not be the case (see e.g. [17, Section 11.2.3]) with all braidings given by the usual flip.

Now let H be a possibly infinite-dimensional braided bialgebra and $U \subset H'$ a left dual of H . If $H \subset U'$ satisfies the conditions in Lemma 4.1, then we can apply Theorem 4.5 to construct a multiplication \underline{m} and a coproduct $\underline{\Delta}$ on H such that H becomes a left dual of U . The next proposition shows that this left dual is isomorphic to the braided bialgebra H . In this sense the construction is reflexive, i.e., taking twice the left dual yields the same braided bialgebra.

Proposition 4.7. *Let H be a braided bialgebra and let $U \subset H'$ be a left dual of H with braiding Ψ_{UU} satisfying (57). Assume that the map Ψ_{HU}° defined in Lemma 4.1 is a bijection. Consider the canonical embedding $\iota : H \rightarrow \iota(H) \subset U'$ given by $\iota(a)(f) := f(a)$. Then $\iota(H)$ with the multiplication \underline{m} and the comultiplication $\underline{\Delta}$ from Theorem 4.5 becomes a left dual of U , and $\iota : H \rightarrow \iota(H)$ yields an isomorphism of braided bialgebras.*

Analogously, if $U \subset H'$ is a right dual of H and Ψ_{UH}° in (59) is a bijection, then $\iota(H) \cong H$ becomes a right dual of U .

If H is a braided Hopf algebra, then ι yields an isomorphism of braided Hopf algebras.

Proof. Let $a, b \in H$ and $f, g \in U$. Since

$$\Psi_{\iota(H)\iota(H)}(\iota(a) \otimes \iota(b))(g \otimes f) \stackrel{(42)}{=} \langle\langle a \otimes b, \Psi_{UU}(f \otimes g) \rangle\rangle \stackrel{(57)}{=} \langle\langle (\iota \otimes \iota) \circ \Psi_{HH}(a \otimes b), f \otimes g \rangle\rangle,$$

we conclude that $\Psi_{\iota(H)\iota(H)} : \iota(H) \otimes \iota(H) \rightarrow \iota(H) \otimes \iota(H)$ yields a bijection and that ι intertwines the braidings on H and $\iota(H)$. In the same manner, it follows from (42) and (46) that $\Psi_{\iota(H)U}^\circ \circ (\iota \otimes \text{id}) = (\text{id} \otimes \iota) \circ \Psi_{HU}^\circ$ which shows that $\Psi_{\iota(H)U}^\circ$ is bijective. Thus, by Lemma 4.1, it defines a braiding.

By (42), (73) and (75), we have

$$\langle \underline{m}(\iota(a) \otimes \iota(b)), f \rangle = \langle\langle a \otimes b, \Psi_{UU}^{-1} \circ \underline{\Delta}(f) \rangle\rangle = \langle m(a \otimes b), f \rangle = \langle \iota \circ m(a \otimes b), f \rangle,$$

hence $\iota \circ m = \underline{m} \circ (\iota \otimes \iota)$. This implies that $\iota(H) \subset U'$ is a unital subalgebra and that ι yields an algebra isomorphism.

Next we prove that ι determines a coalgebra isomorphism. From

$$\begin{aligned} \langle \underline{\Delta}(\iota(a)), f \otimes g \rangle &\stackrel{(73)}{=} \langle a, \underline{m} \circ \Psi_{UU}(f \otimes g) \rangle \\ &\stackrel{(42),(75)}{=} \langle\langle \Delta(a), f \otimes g \rangle\rangle = \langle\langle (\iota \otimes \iota) \circ \Delta(a), f \otimes g \rangle\rangle \end{aligned} \tag{76}$$

we conclude that $(\iota \otimes \iota) \circ \Delta = \underline{\Delta} \circ \iota$. As a consequence, $\underline{\Delta}(\iota(H)) \subset \iota(H) \otimes \iota(H)$. Moreover, $\varepsilon(\iota(a)) = \langle a, 1 \rangle = \varepsilon(a)$. Therefore $\iota(H) \subset U^\circ$ is a subcoalgebra isomorphic to H .

Summarizing, we have shown that $\iota(H)$ satisfies the assumption of Theorem 4.5 so that it becomes a left dual of U with multiplication and comultiplication given in (75) and (73), and that $\iota : H \rightarrow \iota(H)$ yields an isomorphism of braided bialgebras. If H is a braided Hopf algebra, then ι lifts to a braided Hopf algebra isomorphism with $S \circ \iota = \iota \circ S$. The opposite version is proved analogously. \square

Given a braided bialgebra H , Proposition 3.1 allows us to construct a countable family of braided bialgebras. The next proposition gives a description of the corresponding dual bialgebras obtained from a left (or right) dual of H .

Proposition 4.8. *Let H be a braided bialgebra and let U be a left dual of H with respect to the braidings Ψ_{UU} and Ψ_{UH}° given in (57) and (59), respectively. For $n \in \mathbb{Z}$, $U^{(-n,n)}$ is a left dual of $H^{(n,-n)}$ with respect to the braiding Ψ_{UH}° , and if Ψ_{UH} is bijective, then $U^{(-n,n-1)}$ is a left dual of $H^{(n-1,n)}$ with respect to the braiding Ψ_{UH} . The analogous statements are true for Hopf algebras, and for the opposite versions with Ψ_{UH}° and Ψ_{UH} replaced by Ψ_{HU}° and Ψ_{HU} , respectively.*

Proof. Let $n \in \mathbb{Z}$. As

$$\begin{aligned} \langle \underline{m}_{-n}(f \otimes g), a \rangle &\stackrel{(37)}{=} \langle \underline{m} \circ \Psi_{UU}^{-n}(f \otimes g), a \rangle \stackrel{(42),(75)}{=} \langle\langle f \otimes g, \Psi_{HH}^{-n} \circ \Psi_{HH}^{-1} \circ \Delta(a) \rangle\rangle \\ &\stackrel{(38)}{=} \langle\langle f \otimes g, \Psi_{HH}^{-1} \circ \Delta_{-n}(a) \rangle\rangle \end{aligned}$$

and

$$\begin{aligned} \langle \underline{\Delta}_n(f), a \otimes b \rangle &\stackrel{(38)}{=} \langle\langle \Psi_{UU}^n \circ \underline{\Delta}(f), a \otimes b \rangle\rangle \stackrel{(42),(73)}{=} \langle f, m \circ \Psi_{HH}^n \circ \Psi_{HH}^n(a \otimes b) \rangle \\ &\stackrel{(37)}{=} \langle f, m_n \circ \Psi_{HH}(a \otimes b) \rangle \end{aligned}$$

for $f, g \in U$ and $a, b \in H$, it follows from Proposition 3.1 and Theorem 4.5 that $U^{(-n,n)}$ is a left dual of $H^{(n,-n)}$.

In the case of $H^{(n-1,n)}$, we have to replace the braiding Ψ_{HH} on H by Ψ_{HH}^{-1} and therefore Ψ_{UH}° by Ψ_{UH} . A careful look at Proposition 4.8, Lemma 4.1 and Theorem 4.5 reveals that the assumptions of Theorem 4.5 are still satisfied after these substitutions. Now the same calculations, but with $\underline{\Delta}_n$ replaced by $\underline{\Delta}_{n-1}$, show the result.

The opposite versions are proved similarly. The statement about Hopf algebras follows from Proposition 3.1 and the definition of the antipode on the dual Hopf algebras. \square

5. MODULES AND COMODULES IN THE BRAIDED SETTING

A typical application of duality is to turn a comodule of a coalgebra into a module of a dual algebra. Adding more structure, a braided comodule algebra of a braided bialgebra should become a braided module algebra of a dual braided bialgebra. This will be discussed in Theorem 5.4. The dual version, i.e., turning a braided module into a braided comodule of a dual coalgebra, will be presented in Theorem 5.6. Propositions 5.8 and 5.9 elaborate the same idea on duals of (co)modules. Similarly to Lemma 4.1, we start by lifting braidings on vector spaces to dual spaces.

As in the previous section, we will make frequent use of the Sweedler-type notation:

$$\begin{aligned} \Psi_{VW}(v \otimes w) &:= w^{\{1\}} \otimes v^{\{2\}}, & \Psi_{VW}^{-1}(w \otimes v) &:= v^{\{1\}'} \otimes w^{\{2\}'}, \\ \Psi_{VW}^\circ(v \otimes w) &:= w^{\{1\}^\circ} \otimes v^{\{2\}^\circ}, & \Psi_{VW}^{\circ-1}(w \otimes v) &:= v^{\{1\}^{\circ\prime}} \otimes w^{\{2\}^{\circ\prime}}, \end{aligned}$$

where $v \in V, w \in W$, and Ψ_{VW}° denotes a braiding that is constructed from the inverse of a given one. In this notation, we have

$$w^{\{2\}'\{1\}} \otimes v^{\{1\}'\{2\}} = w \otimes v, \quad v^{\{2\}\{1\}'} \otimes w^{\{1\}\{2\}'} = v \otimes w, \tag{77}$$

and the same holds for Ψ_{VW}° . It turns out that the braidings of the type Ψ_{VW}° will be the correct ones for obtaining our desired result as it happened in Theorem 4.5. For this reason, we will give four versions of the auxiliary results on braidings, a left version, a right version and the corresponding versions arising from the inverse braidings.

Lemma 5.1. *Let V be a right H -braided vector space. Consider the linear map*

$$\Psi_{HV'} : H \otimes V' \longrightarrow (V \otimes H) ', \quad \Psi_{HV'}(a \otimes e)(v \otimes f) := \langle\langle e \otimes f, \Psi_{VH}(v \otimes a) \rangle\rangle, \tag{78}$$

for $a \in H, v \in V, f \in H'$ and $e \in V'$. Let $W \subset V'$ be a non-degenerate subspace such that

$$\Psi_{HW} := \Psi_{HV'} \upharpoonright_{H \otimes W} : H \otimes W \longrightarrow W \otimes H \subset (V \otimes H) ' \tag{79}$$

is bijective. Then Ψ_{HW} turns W into a left H -braided vector space.

If H is a braided coalgebra and Ψ_{VH} is compatible with the comultiplication, then Ψ_{HW} is also compatible with the comultiplication of H . If H is a braided (unital)

algebra and Ψ_{VH} is compatible with the multiplication, then Ψ_{HW} is compatible with the multiplication of H .

For a left H -braided vector space V , it is required that

$$\Psi_{WH} := \Psi_{V'H} \upharpoonright_{W \otimes H} : W \otimes H \longrightarrow H \otimes W \subset (H' \otimes V)', \tag{80}$$

is bijective, where

$$\Psi_{V'H} : V' \otimes H \longrightarrow (H' \otimes V)', \quad \Psi_{V'H}(e \otimes a)(f \otimes v) := \langle\langle f \otimes e, \Psi_{HV}(a \otimes v) \rangle\rangle. \tag{81}$$

In this case, Ψ_{WH} turns W into a right H -braided vector space and the other implications remain the same under identical assumptions.

The analogous statements hold for Ψ_{WH}° and Ψ_{HW}° if

$$\Psi_{V'H}^\circ : V' \otimes H \longrightarrow (H' \otimes V)', \quad \Psi_{V'H}^\circ(e \otimes a)(f \otimes v) := \langle\langle f \otimes e, \Psi_{VH}^{-1}(a \otimes v) \rangle\rangle \tag{82}$$

yields a bijective map

$$\Psi_{WH}^\circ := \Psi_{V'H}^\circ \upharpoonright_{W \otimes H} : W \otimes H \longrightarrow H \otimes W \subset (H' \otimes V)', \tag{83}$$

and if

$$\Psi_{HV'}^\circ : H \otimes V' \longrightarrow (V \otimes H)', \quad \Psi_{HV'}^\circ(a \otimes e)(v \otimes f) := \langle\langle e \otimes f, \Psi_{HV}^{-1}(v \otimes a) \rangle\rangle \tag{84}$$

yields a bijective map

$$\Psi_{HW}^\circ := \Psi_{HV'}^\circ \upharpoonright_{H \otimes W} : H \otimes W \longrightarrow W \otimes H \subset (V \otimes H)'. \tag{85}$$

Proof. Let $a, b \in H, f \in H', v \in V$ and $e \in W$. In Sweedler-type notation, we can write (78) in the form $e^{\{1\}}(v) f(a^{\{2\}}) = e(v^{\{2\}}) f(a^{\{1\}})$, which is equivalent to

$$e^{\{1\}}(v) a^{\{2\}} = e(v^{\{2\}}) a^{\{1\}}. \tag{86}$$

The proof is now straightforward, nevertheless we give parts of the proof in order to show how (86) enables us to move the action of the new braiding to the spaces where the given braiding is defined. For instance,

$$\begin{aligned} e^{\{1\}_1\{1\}_2}(v) b^{\langle 1 \rangle\{2\}_2} \otimes a^{\langle 2 \rangle\{2\}_1} &\stackrel{(86)}{=} e(v^{\{2\}_1\{2\}_2}) b^{\langle 1 \rangle\{1\}_1} \otimes a^{\langle 2 \rangle\{1\}_2} \\ &\stackrel{(3)}{=} e(v^{\{2\}_1\{2\}_2}) b^{\{1\}_2\langle 1 \rangle} \otimes a^{\{1\}_1\langle 2 \rangle} \stackrel{(86)}{=} e^{\{1\}_1\{1\}_2}(v) b^{\{2\}_1\langle 1 \rangle} \otimes a^{\{2\}_2\langle 2 \rangle} \end{aligned}$$

proves (2). Furthermore,

$$\begin{aligned} e^{\{1\}_1\{1\}_2}(v) a_{(1)}^{\{2\}_2} \otimes a_{(2)}^{\{2\}_1} &\stackrel{(86)}{=} e(v^{\{2\}_1\{2\}_2}) a_{(1)}^{\{1\}_1} \otimes a_{(2)}^{\{1\}_2} \\ &\stackrel{(11)}{=} e(v^{\{2\}}) a^{\{1\}}_{(1)} \otimes a^{\{1\}}_{(2)} \stackrel{(86)}{=} e^{\{1\}}(v) a^{\{2\}}_{(1)} \otimes a^{\{2\}}_{(2)} \end{aligned}$$

implies the first relation of (10). The second relation follows from

$$\varepsilon(a^{\{2\}}) e^{\{1\}}(v) \stackrel{(86)}{=} \varepsilon(a^{\{1\}}) e(v^{\{2\}}) \stackrel{(11)}{=} \varepsilon(a) e(v).$$

Likewise,

$$\begin{aligned} e^{\{1\}}(v) (ab)^{\{2\}} &\stackrel{(86)}{=} e(v^{\{2\}}) (ab)^{\{1\}} \stackrel{(7)}{=} e(v^{\{2\}_1\{2\}_2}) a^{\{1\}_1} b^{\{1\}_2} \\ &\stackrel{(86)}{=} e^{\{1\}_1\{1\}_2}(v) a^{\{2\}_2} b^{\{2\}_1}, \end{aligned}$$

which yields (5). If $1 \in H$, then (6) follows from (8) and (86) with $a = 1$.

The proof of the opposite version with the braiding Ψ_{WH} uses similar arguments. Combining the obtained results with Lemmas 2.4 proves the statements for Ψ_{WH}° and Ψ_{HW}° . \square

Given an H -braided vector space V , the previous lemma showed how to define braidings between H and appropriate subspaces of V' . The next lemma fixes V and shows how to induce braidings between V and appropriate subspaces $U \subset H'$. Moreover, if U inherits a (co)multiplication from H , then the braidings between U and V inherit the compatibility properties from the corresponding braiding between H and V .

Lemma 5.2. *Let V be a right H -braided vector space and consider the linear map*

$$\Psi_{H'V} : H' \otimes V \longrightarrow (V' \otimes H)', \quad \Psi_{H'V}(f \otimes v)(e \otimes a) := \langle\langle e \otimes f, \Psi_{VH}(v \otimes a) \rangle\rangle, \tag{87}$$

for $a \in H$, $v \in V$, $f \in H'$ and $e \in V'$. Assume that $U \subset H'$ is a non-degenerate subspace such that Ψ_{UU} given in (57) defines a braiding. If

$$\Psi_{UV} := \Psi_{H'V}|_{U \otimes V} : U \otimes V \longrightarrow V \otimes U \subset (V' \otimes H)' \tag{88}$$

is bijective, then Ψ_{UV} turns V into a left U -braided vector space.

If H is a braided coalgebra, Ψ_{VH} is compatible with the multiplication on H , and U satisfies the assumptions of Proposition 4.3, then Ψ_{UV} is compatible with the multiplication $\underline{*}$ from (68) on U . If H is a braided unital algebra, Ψ_{VH} is compatible with the multiplication on H , and U satisfies the assumptions of Proposition 4.4, then Ψ_{UV} is compatible with the comultiplication $\underline{\Delta}$ from (73) on U .

Assume that V is an algebra and Ψ_{VH} is compatible with the multiplication on V . Then Ψ_{UV} is compatible with the multiplication on V . If V is a coalgebra and Ψ_{VH} is compatible with the comultiplication on V , then Ψ_{UV} is also compatible with the multiplication on V .

Given a left H -braided vector space V such that

$$\Psi_{VU} := \Psi_{VH'}|_{V \otimes U} : V \otimes U \longrightarrow U \otimes V \subset (H \otimes V)'$$

is bijective, where

$$\Psi_{VH'} : V \otimes H' \longrightarrow (H \otimes V)', \quad \Psi_{VH'}(v \otimes f)(a \otimes e) := \langle\langle f \otimes e, \Psi_{HV}(a \otimes v) \rangle\rangle, \tag{89}$$

the map Ψ_{VU} defines a braiding that turns V into a right U -braided vector space and the opposite versions of the above compatibility statements remain true.

The analogous assertions hold if

$$\Psi_{VH'}^\circ : V \otimes H' \longrightarrow (H \otimes V)', \quad \Psi_{VH'}^\circ(v \otimes f)(a \otimes e) := \langle\langle f \otimes e, \Psi_{VH}^{-1}(a \otimes v) \rangle\rangle, \tag{90}$$

yields a bijective map

$$\Psi_{VU}^\circ := \Psi_{VH'}^\circ|_{V \otimes U} : V \otimes U \longrightarrow U \otimes V \subset (H \otimes V)', \tag{91}$$

and if

$$\Psi_{H'V}^\circ : H' \otimes V \longrightarrow (V' \otimes H)', \quad \Psi_{H'V}^\circ(f \otimes v)(e \otimes a) := \langle\langle e \otimes f, \Psi_{HV}^{-1}(v \otimes a) \rangle\rangle, \tag{92}$$

yields a bijective map

$$\Psi_{UV}^\circ := \Psi_{H'V}^\circ|_{U \otimes V} : U \otimes V \longrightarrow V \otimes U \subset (V' \otimes H)'. \tag{93}$$

Proof. Although the lemma is proved along the lines of the previous ones, we will state the proof in order to demonstrate where the duality between H and U is used. Let $f, g \in U$, $u, v \in V$, $a, b \in H$ and $e \in W$. First note that $e(v^{\{1\}})f^{\{2\}}(a) \stackrel{(87)}{=} e(v^{\{2\}})f(a^{\{1\}})$ implies

$$f^{\{2\}}(a)v^{\{1\}} = f(a^{\{1\}})v^{\{2\}} \tag{94}$$

and

$$\begin{aligned} g^{\{2\}_2}(b)f^{\{2\}_1}(a)v^{\{1\}_1\{1\}_2} &\stackrel{(94)}{=} g^{\{2\}_2}(b)f(a^{\{1\}_1})v^{\{2\}_1\{1\}_2} \\ &\stackrel{(94)}{=} g(b^{\{1\}_2})f(a^{\{1\}_1})v^{\{2\}_1\{2\}_2}. \end{aligned} \tag{95}$$

Thus

$$\begin{aligned} f^{\{2\}\{2\}_1}(a)g^{\langle 1 \rangle\{2\}_2}(b)v^{\{1\}_1\{1\}_2} &\stackrel{(53),(95)}{=} f(a^{\{1\}_1\langle 2 \rangle})g(b^{\{1\}_2\langle 1 \rangle})v^{\{2\}_1\{2\}_2} \\ &\stackrel{(3)}{=} f(a^{\langle 2 \rangle\{1\}_2})g(b^{\langle 1 \rangle\{1\}_1})v^{\{2\}_1\{2\}_2} \stackrel{(53),(95)}{=} f^{\{2\}_2\langle 2 \rangle}(a)g^{\{2\}_1\langle 1 \rangle}(b)v^{\{1\}_1\{1\}_2}, \end{aligned}$$

which proves (2), so V becomes a left U -braided vector space with respect to the braidings Ψ_{UU} and Ψ_{UV} .

To prove compatibility with the multiplication \ast from (68) on U , we compute

$$\begin{aligned} (f\ast g)^{\{2\}}(a)v^{\{1\}} &\stackrel{(94)}{=} (f\ast g)(a^{\{1\}})v^{\{2\}} \stackrel{(68)}{=} f(a^{\{1\}}_{(1)\langle 2 \rangle})g(a^{\{1\}}_{(2)\langle 1 \rangle})v^{\{2\}} \\ &\stackrel{(3),(11)}{=} f(a_{(1)\langle 2 \rangle\{1\}_2})g(a_{(2)\langle 1 \rangle\{1\}_1})v^{\{2\}_1\{2\}_2} \stackrel{(95)}{=} f^{\{2\}_2}(a_{(1)\langle 2 \rangle})g^{\{2\}_1}(a_{(2)\langle 1 \rangle})v^{\{1\}_1\{1\}_2} \\ &\stackrel{(68)}{=} (f^{\{2\}_2}\ast g^{\{2\}_1})(a)v^{\{1\}_1\{1\}_2}. \end{aligned}$$

This implies (5), and (6) follows from $\varepsilon^{\{2\}}(a)v^{\{1\}} \stackrel{(94)}{=} \varepsilon(a^{\{1\}})v^{\{2\}} \stackrel{(11)}{=} \varepsilon(a)v$ since ε yields the unit element in dual algebra $U \subset H'$.

To prove the compatibility with the comultiplication of U , we proceed in the same manner:

$$\begin{aligned} f^{\{2\}}_{(1)}(b)f^{\{2\}}_{(2)}(a)v^{\{1\}} &\stackrel{(73)}{=} f^{\{2\}}(b^{\langle 1 \rangle}a^{\langle 2 \rangle})v^{\{1\}} \stackrel{(94)}{=} f((b^{\langle 1 \rangle}a^{\langle 2 \rangle})^{\{1\}})v^{\{2\}} \\ &\stackrel{(7)}{=} f(b^{\langle 1 \rangle\{1\}_1}a^{\langle 2 \rangle\{1\}_2})v^{\{2\}_1\{2\}_2} \stackrel{(3)}{=} f(b^{\{1\}_2\langle 1 \rangle}a^{\{1\}_1\langle 2 \rangle})v^{\{2\}_1\{2\}_2} \\ &\stackrel{(73)}{=} f_{(1)}(b^{\{1\}_2})f_{(2)}(a^{\{1\}_1})v^{\{2\}_1\{2\}_2} \stackrel{(95)}{=} f_{(1)}^{\{2\}_2}(b)f_{(2)}^{\{2\}_1}(a)v^{\{1\}_1\{1\}_2} \end{aligned}$$

shows the first relation of (10). The second relation of (10) follows from

$$\varepsilon(f^{\{2\}})v^{\{1\}} = f^{\{2\}}(1)v^{\{1\}} \stackrel{(94)}{=} f(1^{\{1\}})v^{\{2\}} \stackrel{(8)}{=} f(1)v = \varepsilon(f)v.$$

If V is an algebra and Ψ_{VH} is compatible with the multiplication, then

$$\begin{aligned} f^{\{2\}}(a)(vu)^{\{1\}} &\stackrel{(94)}{=} f(a^{\{1\}})(vu)^{\{2\}} \stackrel{(5)}{=} f(a^{\{1\}_1\{1\}_2})v^{\{2\}_2}u^{\{2\}_1} \\ &\stackrel{(94)}{=} f^{\{2\}_1\{2\}_2}(a)v^{\{1\}_1}u^{\{1\}_2}. \end{aligned}$$

Furthermore, if $1 \in V$, we have $f^{\{2\}}(a)1^{\{1\}} \stackrel{(94)}{=} f(a^{\{1\}})1^{\{2\}} \stackrel{(6)}{=} f(a)1$. This and the previous computation show the compatibility of Ψ_{UV} with the multiplication of V .

Assume now that V is a coalgebra and that Ψ_{VH} is compatible with the comultiplication. Then

$$\begin{aligned} f^{\{2\}_1\{2\}_2}(a)v_{(1)}^{\{1\}_1} \otimes v_{(2)}^{\{1\}_2} &\stackrel{(94)}{=} f(a^{\{1\}_1\{1\}_2})v_{(1)}^{\{2\}_2} \otimes v_{(2)}^{\{2\}_1} \\ &\stackrel{(10)}{=} f(a^{\{1\}})v_{(1)}^{\{2\}} \otimes v_{(2)}^{\{2\}} \\ &\stackrel{(94)}{=} f^{\{2\}}(a)v_{(1)}^{\{1\}} \otimes v_{(2)}^{\{1\}}, \end{aligned}$$

and $f^{\{2\}}(a)\varepsilon(v^{\{1\}}) \stackrel{(94)}{=} f(a^{\{1\}})\varepsilon(v^{\{2\}}) \stackrel{(10)}{=} f(a)\varepsilon(v)$. Hence Ψ_{UV} is also compatible with the comultiplication of V .

The opposite versions are proved analogously, and the last part of the lemma follows from the first part by applying Lemma 2.4. □

Given a left H -braided vector space V and subspaces $U \subset H'$ and $W \subset V'$ satisfying the assumptions of Lemmas 5.1 and 5.2, there are two ways of constructing a braiding Ψ_{UW} on $U \otimes W$, either by the restriction of $\Psi_{UV'}$ from (78) or by the restriction of $\Psi_{H'W}$ from (87). The next lemma shows that both constructions coincide whenever one of them can be realized. Equally, we can use either Ψ_{VU}° or Ψ_{WH}° to construct a braiding on $U \otimes W$. In this case, the resulting braiding will be denoted by Ψ_{UW}^\bullet . Analogous results hold for right H -braided vector spaces.

Lemma 5.3. *Let V be a left H -braided vector space. Assume that $U \subset H'$ and $W \subset V'$ satisfy the conditions of Lemmas 5.1 and 5.2 such that the braidings Ψ_{WH} and Ψ_{VU} are well-defined. If either $\Psi_{H'W}|_{U \otimes W}: U \otimes W \rightarrow W \otimes U$ or $\Psi_{UV'}|_{U \otimes W}: U \otimes W \rightarrow W \otimes U$ is bijective, then so is the other and $\Psi_{H'W}|_{U \otimes W} = \Psi_{UV'}|_{U \otimes W} =: \Psi_{UW}$.*

Similarly, if Ψ_{HW}° and Ψ_{UV}° are well-defined and if either $\Psi_{V'U}^\bullet|_{W \otimes U}$ or $\Psi_{WH'}^\bullet|_{W \otimes U}$ yields a bijective map between $W \otimes U$ and $U \otimes W$, then $\Psi_{V'U}^\bullet|_{W \otimes U} = \Psi_{WH'}^\bullet|_{W \otimes U} =: \Psi_{WU}^\bullet$, where

$$\Psi_{V'U}^\bullet: V' \otimes U \longrightarrow (H \otimes V)', \quad \Psi_{V'U}^\bullet(e \otimes f)(a \otimes v) := \langle\langle a \otimes e, \Psi_{UV}^\circ(f \otimes v) \rangle\rangle, \quad (96)$$

$$\Psi_{WH'}^\bullet: W \otimes H' \longrightarrow (H \otimes V)', \quad \Psi_{WH'}^\bullet(e \otimes f)(a \otimes v) := \langle\langle f \otimes v, \Psi_{HW}^\circ(a \otimes e) \rangle\rangle. \quad (97)$$

If V is a right H -braided vector space, then the analogous statements hold for Ψ_{WU} and Ψ_{UW}^\bullet under homologous assumptions, where

$$\Psi_{UW}^\bullet(f \otimes e)(a \otimes v) = \langle\langle e \otimes a, \Psi_{VU}^\circ(v \otimes f) \rangle\rangle = \langle\langle v \otimes f, \Psi_{WH}^\circ(e \otimes a) \rangle\rangle.$$

Proof. The claim $\Psi_{H'W}|_{U \otimes W} = \Psi_{UV'}|_{U \otimes W}$ follows from the equation

$$\begin{aligned} \Psi_{UV'}(f \otimes e)(v \otimes a) &\stackrel{(78)}{=} e(v^{\{2\}})f^{\{1\}}(a) \stackrel{(89)}{=} e(v^{\{1\}})f(a^{\{2\}}) \stackrel{(81)}{=} e^{\{2\}}(v)f(a^{\{1\}}) \\ &\stackrel{(87)}{=} \Psi_{H'W}(f \otimes e)(v \otimes a) \end{aligned}$$

for $a \in H, v \in V, f \in U$ and $e \in W$. Similarly,

$$\begin{aligned} \Psi_{V'U}^\bullet(e \otimes f)(a \otimes v) &\stackrel{(96)}{=} e(v^{\{1\}^\circ})f^{\{2\}^\circ}(a) \stackrel{(92)}{=} e(v^{\{2\}'})f(a^{\{1\}'}) \stackrel{(84)}{=} e^{\{1\}^\circ}(v)f(a^{\{2\}^\circ}) \\ &\stackrel{(97)}{=} \Psi_{WH'}^\bullet(e \otimes f)(a \otimes v) \end{aligned}$$

implies that $\Psi_{V,U}^\bullet \lrcorner_{W \otimes U} = \Psi_{W,H'}^\bullet \lrcorner_{W \otimes U}$. The opposite versions are proved analogously. \square

The next theorem shows how to transform a comodule V of a coalgebra H into a module of a dual algebra of H . Note that we will use again a braiding that is constructed from the inverse of the given one as it happened in Theorem 4.5. The same observation can be made for subsequent results.

Theorem 5.4. *Let H be a braided coalgebra and V a braided right H -comodule with coaction $\rho_R : V \rightarrow V \otimes H$. Let $U \subset H'$ be a non-degenerate (unital) subalgebra with product given by (75) such that Ψ_{UU} and Ψ_{UV}° introduced in (57) and (93), respectively, are bijective. Then V becomes a braided left U -module with respect to the braiding Ψ_{UV}° and the left action $\nu_L : U \otimes V \rightarrow V$ given by*

$$\nu_L(f \otimes v) := \langle f, v_{(1)}^{\{1\}'} \rangle v_{(0)}^{\{2\}'} \tag{98}$$

If H is a braided bialgebra, V is a braided right H -comodule algebra, and U is a left dual of H as in Theorem 4.5, then the action ν_L turns V into a braided left U -module algebra.

In case V is a braided left H -comodule, the analogous statements for the opposite versions hold under homologous assumptions for the right action $\nu_R : V \otimes U \rightarrow V$ given by

$$\nu_R(v \otimes f) := \langle f, v_{(-1)}^{\{2\}'} \rangle v_{(0)}^{\{1\}'} \tag{99}$$

and with respect to the braiding Ψ_{VU}° .

Proof. From Proposition 4.3 and Lemma 5.2, we conclude that U is a braided algebra with respect to braiding Ψ_{UU} and V is a left U -braided vector space such that the braiding Ψ_{UV}° is compatible with the multiplication of U . To show that the action ν_L equips V with the structure of a left braided U -module, we need to prove (19) and (20).

Let $f, g \in U$ and $v \in V$. Using the notation (25), we compute that

$$\begin{aligned} f \triangleright (g \triangleright v) &\stackrel{(98)}{=} g(v_{(1)}^{\{1\}'} f(v_{(0)}^{\{2\}'_{(1)} \{1\}'_2} v_{(0)}^{\{2\}'_{(1)} \{2\}'_2} \\ &\stackrel{(28),(31)}{=} g(v_{(2)}^{\langle 1 \rangle' \{1\}'_1} f(v_{(1)}^{\langle 2 \rangle' \{1\}'_2} v_{(0)}^{\{2\}'_1 \{2\}'_2} \\ &\stackrel{(3)}{=} g(v_{(2)}^{\{1\}'_2 \langle 1 \rangle'} f(v_{(1)}^{\{1\}'_1 \langle 2 \rangle'} v_{(0)}^{\{2\}'_1 \{2\}'_2} \\ &\stackrel{(11)}{=} g(v_{(1)}^{\{1\}'_{(2)} \langle 1 \rangle'} f(v_{(1)}^{\{1\}'_{(1)} \langle 2 \rangle'} v_{(0)}^{\{2\}'}) \stackrel{(75)}{=} f \star g(v_{(1)}^{\{1\}'}) v_{(0)}^{\{2\}'} \stackrel{(98)}{=} (f \star g) \triangleright v. \end{aligned}$$

This yields (19).

Note that, since V' separates the points of V , Equations (92) and (93) give

$$f^{\{2\}'^\circ}(a) v^{\{1\}'^\circ} = f(a^{\{1\}'}) v^{\{2\}'}, \quad f \in U, v \in V, a \in H. \tag{100}$$

Therefore, for $f, g \in U$, $v \in V$ and $a \in H$,

$$\begin{aligned}
 f^{(2)\{2\}^\circ}(a)(g^{(1)} \triangleright v^{\{1\}^\circ}) &\stackrel{(98)}{=} f^{(2)\{2\}^\circ}(a)g^{(1)}(v^{\{1\}^\circ}_{(1)}\{1\}^\circ)v^{\{1\}^\circ}_{(0)}\{2\}^\circ \\
 &\stackrel{(100)}{=} f^{(2)}(a^{\{1\}^\circ}_1)g^{(1)}(v^{\{2\}^\circ}_1\{1\}^\circ)v^{\{2\}^\circ}_1\{2\}^\circ \\
 &\stackrel{(31)}{=} f^{(2)}(a^{\langle 1 \rangle\{1\}^\circ}_1)g^{(1)}(v_{(1)}^{\langle 2 \rangle\{1\}^\circ})v_{(0)}^{\{2\}^\circ}\{2\}^\circ \\
 &\stackrel{(3)}{=} f^{(2)}(a^{\{1\}^\circ\langle 1 \rangle^\circ})g^{(1)}(v_{(1)}^{\{1\}^\circ\langle 2 \rangle^\circ})v_{(0)}^{\{2\}^\circ}\{2\}^\circ \\
 &\stackrel{(49),(51)}{=} f(a^{\{1\}^\circ}_2)g(v_{(1)}^{\{1\}^\circ})v_{(0)}^{\{2\}^\circ}\{2\}^\circ \\
 &\stackrel{(98)}{=} f(a^{\{1\}^\circ})(g \triangleright v)^{\{2\}^\circ} \stackrel{(100)}{=} f^{(2)\circ}(a)(g \triangleright v)^{\{1\}^\circ},
 \end{aligned}$$

which implies (20).

Now assume that H is a braided bialgebra, V is a braided right H -comodule algebra, and U is a left dual of H . Then, for $u, v \in V$ and $f \in U$,

$$\begin{aligned}
 (f_{(1)} \triangleright u^{\{1\}^\circ})(f_{(2)}^{\{2\}^\circ} \triangleright v) &\stackrel{(98)}{=} f_{(1)}(u^{\{1\}^\circ}_{(1)}\{1\}^\circ)f_{(2)}^{\{2\}^\circ}(v_{(1)}^{\{1\}^\circ})u^{\{1\}^\circ}_{(0)}\{2\}^\circ v_{(0)}^{\{2\}^\circ} \\
 &\stackrel{(100)}{=} f_{(1)}(u^{\{2\}^\circ}_2\{1\}^\circ)f_{(2)}(v_{(1)}^{\{1\}^\circ}\{1\}^\circ)u^{\{2\}^\circ}_2\{2\}^\circ v_{(0)}^{\{2\}^\circ} \\
 &\stackrel{(31)}{=} f_{(1)}(u_{(1)}^{\langle 2 \rangle\{1\}^\circ})f_{(2)}(v_{(1)}^{\{1\}^\circ\langle 1 \rangle^\circ})u_{(0)}^{\{2\}^\circ}\{2\}^\circ v_{(0)}^{\{2\}^\circ} \\
 &\stackrel{(3)}{=} f_{(1)}(u_{(1)}^{\{1\}^\circ\langle 2 \rangle^\circ})f_{(2)}(v_{(1)}^{\{1\}^\circ\{1\}^\circ\langle 1 \rangle^\circ})u_{(0)}^{\{2\}^\circ}\{2\}^\circ v_{(0)}^{\{2\}^\circ} \\
 &\stackrel{(49),(73)}{=} f(u_{(1)}^{\{1\}^\circ}v_{(1)}^{\{1\}^\circ\{1\}^\circ})u_{(0)}^{\{2\}^\circ}\{2\}^\circ v_{(0)}^{\{2\}^\circ} \\
 &\stackrel{(7)}{=} f((u_{(1)}v_{(1)}^{\{1\}^\circ})^{\{1\}^\circ})u_{(0)}^{\{2\}^\circ}v_{(0)}^{\{2\}^\circ} \\
 &\stackrel{(77)}{=} f((u_{(1)}^{\{2\}\{1\}^\circ}v_{(1)}^{\{1\}^\circ})^{\{1\}^\circ})u_{(0)}^{\{2\}^\circ}v_{(0)}^{\{1\}\{2\}^\circ} \\
 &\stackrel{(7)}{=} f((u_{(1)}^{\{2\}}v_{(1)}^{\{1\}^\circ})^{\{1\}^\circ})u_{(0)}^{\{2\}^\circ}v_{(0)}^{\{1\}\{2\}^\circ} \\
 &\stackrel{(5)}{=} f((u_{(1)}^{\{2\}}v_{(1)}^{\{1\}^\circ})^{\{1\}^\circ})(u_{(0)}v_{(0)}^{\{1\}^\circ})^{\{2\}^\circ} \\
 &\stackrel{(34)}{=} f((uv)_{(1)}^{\{1\}^\circ})(uv)_{(0)}^{\{2\}^\circ} \stackrel{(98)}{=} f \triangleright (uv).
 \end{aligned}$$

This proves (32). Furthermore, if $1 \in V$, then $f \triangleright 1 \stackrel{(35)}{=} f(1^{\{1\}^\circ})1^{\{2\}^\circ} \stackrel{(6)}{=} f(1)1 = \varepsilon(f)1$. Since, by Lemma 5.2, Ψ_{UV}° is compatible with the comultiplication of U and with the multiplications of U and V , we conclude that V is a braided left U -module algebra.

The opposite versions are shown analogously. □

Let H be a braided bialgebra and U a left dual of H . Since $U \subset H'$ is an algebra, there is a natural left (resp., right) U -action on H given by right (resp., left) multiplication on U . On the other hand, the coproduct on H equips H trivially with the structure of a right (resp., left) H -comodule so that we may consider the left (resp., right) U -action on H described in Theorem 5.4. The next corollary

shows that these actions coincide and turn H into a left (resp., right) U -module algebra. Similar results hold for U and H interchanged.

Corollary 5.5. *Let H be a braided bialgebra and let U be a left dual of H as in Theorem 4.5. Then the natural left U -action $\triangleright : U \otimes H \rightarrow H$ defined by $f(g \triangleright a) := (fg)(a)$, $a \in H$, $f, g \in U$, satisfies*

$$g \triangleright a = \langle g, a_{(2)}^{(1)\prime} \rangle a_{(1)}^{(2)\prime} \tag{101}$$

and turns H into a left U -module algebra. The natural right U -action $\triangleleft : H \otimes U \rightarrow H$ defined by $g(a \triangleleft f) := (fg)(a)$ satisfies

$$a \triangleleft f = \langle f, a_{(1)}^{(2)\prime} \rangle a_{(2)}^{(1)\prime} \tag{102}$$

and turns H into a right U -module algebra.

The same formulas with $a \in U$, $f, g \in H$ hold for the natural left (resp., right) H -action on U and turn U into a left (resp., right) H -module algebra.

Proof. Setting $V := H$ in Theorem 5.4 shows that the right-hand side in (101) (resp., (102)) defines a left (resp., right) U -action on H such that H becomes a left (resp., right) U -module algebra. The equalities in (101) and (102) follow from (64) for the braiding Ψ_{UH}° by applying (54). The proof for the H -actions on U is identical but uses (65) instead of (64). \square

Theorem 5.4 shows how to turn a comodule of a braided coalgebra into a module of an appropriate dual algebra. The dual construction corresponds to turning a module of a braided algebra into a comodule of a dual coalgebra. This will be done in the next theorem. Similar to Proposition 4.4, we will need an additional condition to ensure that the coaction belongs to the correct (algebraic) tensor product.

Theorem 5.6. *Let H be a braided unital algebra and V a braided left H -module with action $\nu_L : H \otimes V \rightarrow V$ denoted by $\nu_L(a \otimes v) := a \triangleright v$. Let U be a non-degenerate subcoalgebra of H^2 as defined in Proposition 4.4 such that Ψ_{UU} and Ψ_{UV}° given in (57) and (93), respectively, are bijective. Consider the linear map*

$$\rho_R : V \longrightarrow (V' \otimes H)', \quad \rho_R(v)(e \otimes a) := e(a^{\{1\}' \triangleright v^{\{2\}'}}), \tag{103}$$

where $v \in V$, $a \in H$ and $e \in V'$. If

$$\rho_R : V \longrightarrow V \otimes U \subset (V' \otimes H)', \tag{104}$$

then ρ_R yields a right U -coaction on V such that V becomes a braided right U -comodule with respect to the braiding Ψ_{UV}° and the coproduct $\underline{\Delta}_{-2} := \Psi_{UV}^{-2} \circ \underline{\Delta}$ on U , where $\underline{\Delta}$ denotes the coproduct introduced in (73).

If H is a braided bialgebra, V is a braided left H -module algebra, and U is a left dual of H as in Theorem 4.5, then the coaction ρ_R turns V into a braided right $U^{(2,-2)}$ -comodule algebra.

In case V is a braided right H -module, the analogous statements hold under homologous assumptions for the opposite versions with respect to the left coaction

determined by

$$\rho_L : V \longrightarrow U \otimes V \subset (H \otimes V)’, \quad \rho_L(v)(a \otimes e) := e(v^{\{1\} \triangleleft a^{\{2\}}), \tag{105}$$

and again for the braided coalgebra $(U, \underline{\Delta}_{-2}, \varepsilon)$ and the braided bialgebra $U^{(2, -2)}$.

Proof. As shown in Proposition 4.4, the braiding Ψ_{UU} equips $(U, \underline{\Delta}, \varepsilon)$ with the structure of a braided coalgebra. From Lemma 5.2, we know that Ψ_{UV}° turns V into a left U -braided vector space such that the braiding is compatible with the comultiplication of U and, if defined, with the multiplications of U and V . Furthermore, by Proposition 3.1, the compatibility conditions are also fulfilled with respect to the modified coproduct $\underline{\Delta}_{-2} = \Psi_{UU}^{-2} \circ \underline{\Delta}$ and the modified product $m_2 = m \circ \Psi_{UU}^2$.

Throughout this proof, let $u, v \in V, a, b \in H$ and $f, g \in U$. To show that ρ_R turns V into a braided right U -comodule, it remains to prove (28) and (29). First note that, since V' separates the points of V , (103) is equivalent to

$$v_{(0)} v_{(1)}(a) = a^{\{1\} \triangleright v^{\{2\}}. \tag{106}$$

Moreover, to distinguish between $\underline{\Delta}_{-2}$ and the coproduct $\underline{\Delta}$ on U defined in Proposition 4.4, we will employ the Sweedler notation $\underline{\Delta}_{-2}(f) := f_{(1)} \otimes f_{(2)}$. Then it follows from (42), (57) and (73) that

$$f_{(1)}(b) f_{(2)}(a) = f(b^{\langle 1 \rangle} a^{\langle 2 \rangle}). \tag{107}$$

Using Lemma 2.4 for the relations concerning the inverse braiding, we obtain

$$\begin{aligned} (v_{(0)})_{(0)} v_{(1)}(a) (v_{(0)})_{(1)}(b) &\stackrel{(106)}{=} b^{\{1\}_2 \triangleright (a^{\{1\}_1 \triangleright v^{\{2\}_1})\{2\}_2} \\ &\stackrel{(23)}{=} b^{\{1\}_2 \langle 1 \rangle \triangleright (a^{\{1\}_1 \langle 2 \rangle \triangleright v^{\{2\}_1 \{2\}_2}) \stackrel{(2)}{=} b^{\langle 1 \rangle \{1\}_1 \triangleright (a^{\langle 2 \rangle \{1\}_2 \triangleright v^{\{2\}_1 \{2\}_2}) \\ &\stackrel{(19)}{=} (b^{\langle 1 \rangle \{1\}_1} a^{\langle 2 \rangle \{1\}_2}) \triangleright v^{\{2\}_1 \{2\}_2} \stackrel{(5)}{=} (b^{\langle 1 \rangle} a^{\langle 2 \rangle}) \{1\} \triangleright v^{\{2\}} \\ &\stackrel{(106)}{=} v_{(0)} v_{(1)}(b^{\langle 1 \rangle} a^{\langle 2 \rangle}) \stackrel{(107)}{=} v_{(0)} (v_{(1)})_{(1)}(b) (v_{(1)})_{(2)}(a). \end{aligned}$$

Therefore

$$v_{(0)} \otimes (v_{(1)})_{(1)} \otimes (v_{(1)})_{(2)} = (v_{(0)})_{(0)} \otimes (v_{(0)})_{(1)} \otimes v_{(1)} =: v_{(0)} \otimes v_{(1)} \otimes v_{(2)},$$

which shows the first relation of (28). The second relation of (28) follows from the definition of ε in Proposition 4.4 and

$$v_{(0)} \varepsilon(v_{(1)}) = v_{(0)} v_{(1)}(1) \stackrel{(106)}{=} 1^{\{1\} \triangleright v^{\{2\}} \stackrel{(6)}{=} 1 \triangleright v \stackrel{(19)}{=} v.$$

Furthermore,

$$\begin{aligned} v_{(0)}^{\{1\}^\circ} v_{(1)}^{\langle 1 \rangle}(b) f^{\{2\}^\circ \langle 2 \rangle}(a) &\stackrel{(53)}{=} v_{(0)}^{\{1\}^\circ} v_{(1)}(b^{\langle 1 \rangle}) f^{\{2\}^\circ}(a^{\langle 2 \rangle}) \\ &\stackrel{(100)}{=} v_{(0)}^{\{2\}} v_{(1)}(b^{\langle 1 \rangle}) f(a^{\langle 2 \rangle \{1\}}) \stackrel{(106)}{=} (b^{\langle 1 \rangle \{1\}_1 \triangleright v^{\{2\}_1})\{2\}_2 f(a^{\langle 2 \rangle \{1\}_2}) \\ &\stackrel{(23)}{=} (b^{\langle 1 \rangle \{1\}_1 \langle 2 \rangle \triangleright v^{\{2\}_1 \{2\}_2}) f(a^{\langle 2 \rangle \{1\}_2 \langle 1 \rangle}) \stackrel{(2), (49)}{=} (b^{\{1\}_2 \triangleright v^{\{2\}_1 \{2\}_2}) f(a^{\{1\}_1}) \\ &\stackrel{(106)}{=} v^{\{2\}}_{(0)} v^{\{2\}}_{(1)}(b) f(a^{\{1\}}) \stackrel{(100)}{=} v^{\{1\}^\circ}_{(0)} v^{\{1\}^\circ}_{(1)}(b) f^{\{2\}^\circ}(a), \end{aligned}$$

which proves (29). Hence ρ_R turns V into a braided right U -module.

Now let H be a braided bialgebra, V a braided right H -module algebra, and U a left dual of H as in Theorem 4.5. The compatibility of the braiding Ψ_{UV}° with the (modified) comultiplication on U and the (modified) multiplications on U and V has been discussed in the beginning of the proof. To show that V becomes a braided right $U^{(2,-2)}$ -comodule algebra, we need to verify (34).

First note that

$$\begin{aligned} \underline{m}_2(f \otimes g)(a) &\stackrel{(37),(75)}{=} \langle\langle \Psi_{UU}^2(f \otimes g), \Psi_{HH}^{-1} \circ \Delta(a) \rangle\rangle \stackrel{(42),(57)}{=} \langle\langle f \otimes g, \Psi_{HH} \circ \Delta(a) \rangle\rangle \\ &\stackrel{(41)}{=} f(a_{(1)}\langle 2 \rangle) g(a_{(2)}\langle 1 \rangle) \stackrel{(68)}{=} f \star g(a). \end{aligned} \tag{108}$$

Starting with the right hand side of (34), we compute that

$$\begin{aligned} u_{(0)}v_{(0)}\langle 1 \rangle^\circ (u_{(1)}\langle 2 \rangle^\circ \star v_{(1)})(a) &\stackrel{(108)}{=} u_{(0)}v_{(0)}\langle 1 \rangle^\circ u_{(1)}\langle 2 \rangle^\circ (a_{(1)}\langle 2 \rangle) v_{(1)}(a_{(2)}\langle 1 \rangle) \\ &\stackrel{(100)}{=} u_{(0)}v_{(0)}\langle 2 \rangle^\circ u_{(1)}(a_{(1)}\langle 2 \rangle\langle 1 \rangle^\circ) v_{(1)}(a_{(2)}\langle 1 \rangle) \\ &\stackrel{(106)}{=} u_{(0)}(a_{(2)}\langle 1 \rangle\langle 1 \rangle_1^\circ \triangleright v\langle 2 \rangle_1^\circ)\langle 2 \rangle_2^\circ u_{(1)}(a_{(1)}\langle 2 \rangle\langle 1 \rangle_2^\circ) \\ &\stackrel{(23)}{=} u_{(0)}(a_{(2)}\langle 1 \rangle\langle 1 \rangle_1^\circ\langle 2 \rangle^\circ \triangleright v\langle 2 \rangle_1^\circ\langle 2 \rangle_2^\circ) u_{(1)}(a_{(1)}\langle 2 \rangle\langle 1 \rangle_2^\circ\langle 1 \rangle^\circ) \\ &\stackrel{(2),(49)}{=} u_{(0)}(a_{(2)}\langle 1 \rangle_2^\circ \triangleright v\langle 2 \rangle_1^\circ\langle 2 \rangle_2^\circ) u_{(1)}(a_{(1)}\langle 1 \rangle_1^\circ) \stackrel{(10)}{=} u_{(0)}(a\langle 1 \rangle_2^\circ \triangleright v\langle 2 \rangle^\circ) u_{(1)}(a\langle 1 \rangle_1^\circ) \\ &\stackrel{(106)}{=} (a\langle 1 \rangle_1^\circ\langle 1 \rangle_2^\circ \triangleright u\langle 2 \rangle_2^\circ)(a\langle 1 \rangle_1^\circ\langle 2 \rangle^\circ \triangleright v\langle 2 \rangle_1^\circ) \\ &\stackrel{(10)}{=} (a\langle 1 \rangle_1^\circ\langle 1 \rangle_2^\circ\langle 1 \rangle^\circ \triangleright u\langle 2 \rangle_2^\circ\langle 1 \rangle^\circ)(a\langle 1 \rangle_1^\circ\langle 1 \rangle_2^\circ\langle 2 \rangle^\circ \triangleright v\langle 2 \rangle_1^\circ) \stackrel{(32)}{=} a\langle 1 \rangle_1^\circ\langle 1 \rangle_2^\circ \triangleright (u\langle 2 \rangle_2^\circ v\langle 2 \rangle_1^\circ) \\ &\stackrel{(7)}{=} a\langle 1 \rangle^\circ \triangleright (uv)\langle 2 \rangle^\circ \stackrel{(106)}{=} (uv)_{(0)}(uv)_{(1)}(a), \end{aligned}$$

which implies (34). Finally, if $1 \in V$, then $1_{(0)}1_{(1)}(a) \stackrel{(106)}{=} a\langle 1 \rangle^\circ \triangleright 1\langle 2 \rangle^\circ \stackrel{(8)}{=} a \triangleright 1 \stackrel{(33)}{=} 1 \varepsilon(a)$, from which we conclude that $\rho_R(1) = 1 \otimes 1$. This finishes the proof that V is a braided right $U^{(2,-2)}$ -comodule algebra.

The opposite versions are proved analogously. □

Taking $V := H$ in the previous theorem and the multiplication of H as left action, the right coaction $\rho_R : H \rightarrow H \otimes U$ is determined by

$$\rho_R(a)(f \otimes b) \stackrel{(103)}{=} f(b\langle 1 \rangle^\circ a\langle 2 \rangle^\circ) \stackrel{(73)}{=} \langle\langle \underline{\Delta}_{-2}(f), a \otimes b \rangle\rangle,$$

which is equivalent to

$$a_{(0)}(f) a_{(1)} = f_{(2)}(a) f_{(1)}^\circ, \quad \underline{\Delta}_{-2}(f) := f_{(1)} \otimes f_{(2)}^\circ.$$

However, if we consider H as a braided algebra with respect to the inverse braiding Ψ_{HH}^{-1} , then $\rho_R(a)(f \otimes b) = f(b\langle 1 \rangle^\circ a\langle 2 \rangle^\circ)$ and thus

$$f(a_{(0)}) a_{(1)} = f_{(2)}(a) f_{(1)}, \quad \underline{\Delta}(f) := f_{(1)} \otimes f_{(2)}.$$

The corresponding left U -coaction $\rho_L(a)(b \otimes f) = f(a\langle 1 \rangle^\circ b\langle 2 \rangle^\circ)$ satisfies

$$f(a_{(0)}) a_{(-1)} = f_{(1)}(a) f_{(2)}.$$

These observations may be viewed as the dual version of Corollary 5.5.

Note that the coproduct of the dual coalgebra $(U, \underline{\Delta}, \varepsilon)$ had to be changed to $\underline{\Delta}_{-2}$ in Theorem 5.6. Therefore, combining repeatedly Theorems 5.4 and 5.6 may turn V into a (co)module for a whole family of (co)algebras, each time with respect to a potentially different (co)action. The starting point for this observation is the next corollary.

Corollary 5.7. (i) *Let (H, Δ, ε) be a braided coalgebra and V a braided right H -comodule with coaction $\rho_R : V \rightarrow V \otimes H$. Let $U \subset H'$ be a non-degenerate unital subalgebra with product given by (75) such that Ψ_{UU}, Ψ_{UH} and Ψ_{UV}° introduced in (57), (59) and (93), respectively, are bijective. Assume that*

$$\Psi_{HV}^{\circ\circ} : H \otimes V \longrightarrow (V' \otimes U)', \quad \Psi_{HV}^{\circ\circ}(a \otimes v)(e \otimes f) := \langle\langle e \otimes a, \Psi_{UV}^{\circ-1}(v \otimes f) \rangle\rangle, \quad (109)$$

yields a bijection $\Psi_{HV}^{\circ\circ} : H \otimes V \rightarrow V \otimes H \subset (V' \otimes U)'$. If the map

$$\rho_R^\flat : V \longrightarrow (V' \otimes U)', \quad \rho_R^\flat(v)(e \otimes f) := e(v_{(0)}) f^{\{1\}^{\circ\flat}}(v_{(1)}^{\{2\}^{\circ\flat}}) \quad (110)$$

satisfies $\rho_R^\flat : V \rightarrow V \otimes H \subset (V' \otimes U)'$, then it defines a right H -coaction that turns V into a braided right H -comodule with respect to the braiding $\Psi_{HV}^{\circ\circ}$ and the coproduct $\Delta_{-2} := \Psi_{HH}^{-2} \circ \Delta$ on H .

If H is a braided bialgebra, V is a braided right H -comodule algebra, and U is a left dual of H as in Theorem 4.5, then the right coaction defined in (110) turns V into a braided right $H^{(2,-2)}$ -comodule algebra.

(ii) *Let (H, m) be a braided unital algebra and V a braided left H -module with left action $\nu_L : H \otimes V \rightarrow V$. Let U be a non-degenerate subcoalgebra of H° as defined in Proposition 4.4 such that Ψ_{UU} and Ψ_{UV}° given in (57) and (93), respectively, are bijective, and suppose that ρ_R defined in (103) satisfies (104). Assume that the map $\Psi_{HV}^{\circ\circ}$ introduced in (109) yields a bijection $\Psi_{HV}^{\circ\circ} : H \otimes V \rightarrow V \otimes H$, and that there exists a non-degenerate subspace $W \subset V'$ such that*

$$\Psi_{WH}^{\circ\circ} : W \otimes H \longrightarrow (U \otimes V)', \quad \Psi_{WH}^{\circ\circ}(e \otimes a)(f \otimes v) := \langle\langle f \otimes e, \Psi_{HV}^{\circ\circ}(a \otimes v) \rangle\rangle, \quad (111)$$

yields a bijection $\Psi_{WH}^{\circ\circ} : W \otimes H \rightarrow H \otimes W \subset (U \otimes V)'$. Then the map $\nu_L^\flat : H \otimes V \rightarrow W'$,

$$\nu_L^\flat(a \otimes v)(e) := \langle \cdot, \cdot \rangle \circ (\nu_L \otimes \text{id}) \circ (\Psi_{HV}^{-1} \otimes \text{id}) \circ (\text{id} \otimes \Psi_{WH}^{\circ\circ}) \circ (v \otimes e \otimes a), \quad (112)$$

turns V into a braided left H -module with respect to the braiding $\Psi_{HV}^{\circ\circ}$ and the multiplication $m_{-2} := m \circ \Psi_{HH}^{-2}$ on H .

If H is a braided bialgebra, V is a braided left H -module algebra, and U is a left dual of H as in Theorem 4.5, then the left action defined in (112) turns V into a braided left $H^{(-2,2)}$ -module algebra.

Analogous statements hold for the opposite versions.

Proof. (i) First note that the braided coalgebra (H, Δ, ε) and the subalgebra $U \subset H'$ in (i) satisfy the assumptions of Theorem 5.4. Hence the left action (98) turns V into a braided left U -module with respect to the braiding Ψ_{UV}° . From (76), it follows that $H \cong \iota(H) \subset U^\circ$ is a non-degenerate subcoalgebra. Furthermore, $\Psi_{HV}^{\circ\circ}$ corresponds to Ψ_{UV}° in Theorem 5.6, where H and U swap the roles in the present

situation. Applying Theorem 5.6, we conclude that the H -coaction defined in (103) turns V into a braided right H -comodule with respect to the braiding Ψ_{HV}° and the coproduct $\Delta_{-2} = \Psi_{HH}^{-2} \circ \Delta$. But, before we can apply Theorem 5.6, we need to prove that the coaction resulting from this construction coincides with (110) so that (104) is fulfilled by assumption.

Let $v \in V$. To distinguish the new coaction ρ_R^\setminus from the original one, we use the Sweedler notation $\rho_R^\setminus(v) := v_{(0)^\setminus} \otimes v_{(1)^\setminus}$. Evaluating $\rho_R^\setminus(v) \in V \otimes H$ on $e \otimes f \in W \otimes U$ yields

$$\begin{aligned} e(v_{(0)^\setminus}) f(v_{(1)^\setminus}) &\stackrel{(103)}{=} e(f^{\{1\}^\circ\setminus} \triangleright v^{\{1\}^\circ\setminus}) \stackrel{(98)}{=} f^{\{1\}^\circ\setminus}(v^{\{2\}^\circ\setminus}_{(1)} \{1\}^\setminus) e(v^{\{2\}^\circ\setminus}_{(0)} \{2\}^\setminus) \\ &\stackrel{(92)}{=} f^{\{1\}^\circ\setminus} \{2\}^\circ(v^{\{2\}^\circ\setminus}_{(1)}) e(v^{\{2\}^\circ\setminus}_{(0)} \{1\}^\circ), \end{aligned}$$

which is equivalent to

$$v_{(0)^\setminus} f(v_{(1)^\setminus}) = f^{\{1\}^\circ\setminus} \{2\}^\circ(v^{\{2\}^\circ\setminus}_{(1)}) v^{\{2\}^\circ\setminus}_{(0)} \{1\}^\circ. \tag{113}$$

Furthermore,

$$\begin{aligned} f^{\{2\}^\circ}(a) v^{\{1\}^\circ}_{(0)} \otimes v^{\{1\}^\circ}_{(1)} &\stackrel{(100)}{=} f(a^{\{1\}^\setminus}) v^{\{2\}^\setminus}_{(0)} \otimes v^{\{2\}^\setminus}_{(1)} \\ &\stackrel{(31)}{=} f(a^{\{1\}^\setminus} \{1\}^\setminus) v_{(0)}^{\{2\}^\setminus} \otimes v_{(1)}^{\{2\}^\setminus} \stackrel{(56), (100)}{=} f^{\{2\}^\circ_1 \{2\}^\circ_2}(a) v_{(0)}^{\{1\}^\circ_1} \otimes v_{(1)}^{\{1\}^\circ_2} \end{aligned}$$

gives

$$(\rho_R \otimes \text{id}) \circ \Psi_{UV}^\circ = (\text{id} \otimes \Psi_{UH}^\circ) \circ (\Psi_{UV}^\circ \otimes \text{id}) \circ (\text{id} \otimes \rho_R).$$

Applying on both sides the corresponding inverse braidings, we obtain

$$(\text{id} \otimes \Psi_{UH}^{\circ-1}) \circ (\rho_R \otimes \text{id}) = (\Psi_{UV}^\circ \otimes \text{id}) \circ (\text{id} \otimes \rho_R) \circ \Psi_{UV}^{\circ-1}. \tag{114}$$

Inserting (114) into (113) yields

$$v_{(0)^\setminus} f(v_{(1)^\setminus}) \stackrel{(114), (113)}{=} f^{\{1\}^\circ\setminus}(v_{(1)}^{\{2\}^\circ\setminus}) v_{(0)}.$$

This proves (110) for the given right H -coaction $\rho_R(v) = v_{(0)} \otimes v_{(1)}$ on V and with the braiding $\Psi_{UH}^{\circ-1}(a \otimes f) = f^{\{1\}^\circ\setminus} \otimes a^{\{2\}^\circ\setminus}$ on $H \otimes U$.

Assume now that H is a braided bialgebra and V is a braided right H -comodule algebra. Then, by Theorem 5.4, V becomes a left U -module algebra. In Proposition 4.7, it has been shown that $H \cong \iota(H)$ is a left dual of U . Applying Theorem 5.6 with the roles of U and H reversed shows that the new coaction ρ_R^\setminus turns V into a braided right $H^{(2,-2)}$ -comodule algebra.

(ii) Consider a braided unital algebra (H, m) , a braided left H -module V and a non-degenerate subcoalgebra $U \subset H^\circ$ as described in (ii). From Theorem 5.6, we conclude that V becomes a braided right U -comodule with respect to the coproduct $\underline{\Delta}_{-2} := \Psi_{UU}^{-2} \circ \underline{\Delta}$ on U , the right coaction defined in (103), and the braiding Ψ_{UV}° . As in the proof of (i), Ψ_{HV}° corresponds to the braiding Ψ_{UV}° in Theorem 5.4 with the roles of H and U reversed. Furthermore, the multiplication on $H \cong \iota(H) \subset U'$ corresponding to the coproduct $\underline{\Delta}_{-2}$ on U in (75) is given by $m_{-2} = m \circ \Psi_{HH}^{-2}$.

Indeed, for all $a, b \in H$ and $f \in U$, we have

$$\begin{aligned} \langle \underline{m}(\iota(a) \otimes \iota(b)), f \rangle &\stackrel{(75)}{=} \langle \langle a \otimes b, \Psi_{UU}^{-1} \circ \underline{\Delta}_{-2}(f) \rangle \rangle \stackrel{(38),(42)}{=} \langle \langle \Psi_{HH}^{-3}(a \otimes b), \underline{\Delta}(f) \rangle \rangle \\ &\stackrel{(73)}{=} \langle m \circ \Psi_{HH}^{-2}(a \otimes b), f \rangle \stackrel{(37)}{=} \langle m_{-2}(a \otimes b), f \rangle. \end{aligned}$$

Applying now Theorem 5.4 shows that V becomes a braided left H -module with respect to the left H -action defined in (98), the multiplication $m_{-2} := m \circ \Psi_{HH}^{-2}$ on H , and the braiding $\Psi_{HV}^{\circ\circ}$.

Let ν_L^λ denote the new left H -action on V . To show that ν_L^λ is given by (112), we use the Sweedler-type notation

$$\Psi_{HV}^{\circ\circ}(a \otimes v) := v^{\{1\}^{\circ\circ}} \otimes a^{\{2\}^{\circ\circ}}, \quad \Psi_{WH}^{\circ\circ}(e \otimes a) := a^{\{1\}^{\circ\circ}} \otimes e^{\{2\}^{\circ\circ}},$$

for $a \in H$, $v \in V$ and $e \in W$, and compute that

$$\begin{aligned} e(\nu_L^\lambda(a \otimes v)) &\stackrel{(98)}{=} e(v_{(0)}^{\{2\}^{\circ\circ}}) v_{(1)}^{\{1\}^{\circ\circ}}(a) \stackrel{(109)}{=} e(v_{(0)}^{\{1\}^{\circ\circ}}) v_{(1)}(a^{\{2\}^{\circ\circ}}) \\ &\stackrel{(111)}{=} e^{\{2\}^{\circ\circ}}(v_{(0)}) v_{(1)}(a^{\{1\}^{\circ\circ}}) \stackrel{(106)}{=} e^{\{2\}^{\circ\circ}}(a^{\{1\}^{\circ\circ}} \{1\} \triangleright v^{\{2\}^{\circ\circ}}). \end{aligned} \tag{115}$$

This shows that ν_L^λ is given by (112). In particular, we have $\nu_L^\lambda : H \otimes V \rightarrow V \subset W'$.

If H is a braided bialgebra, V is a braided left H -module algebra, and U is a left dual of H as in Theorem 4.5, then we conclude from Theorem 5.6 that V becomes a braided right $U^{(2,-2)}$ -comodule algebra. Since $H^{(-2,2)}$ is a left dual of $U^{(2,-2)}$ by Proposition 4.8, we deduce from Theorem 5.4 that the left action ν_L^λ turns V into a braided left $H^{(-2,2)}$ -module algebra.

The opposite versions are proved analogously. □

The results of the last corollary may be seen as an induction step. Starting with an H -(co)module V , Corollary 5.7 produces another H -(co)action on V but for the modified coproduct Δ_{-2} or product m_{-2} . If the assumptions of the corollary are again satisfied, we can repeat the process, and so on, obtaining thus for all $n \in \mathbb{N}$ an H -(co)action on V for the coproduct Δ_{-2n} or the product m_{-2n} . If H is a braided bialgebra and V a braided (co)module algebra, then we get a coaction of the braided bialgebra $H^{(2n,-2n)}$ or an action of $H^{(-2n,2n)}$.

Theorem 5.4 shows how to turn a braided comodule of a braided coalgebra into a braided module of the dual algebra. However, it is more natural to dualize a coaction in such a way that a *dual* space of the comodule becomes a module of the dual algebra. This will be done in the next proposition. Unlike Theorem 5.4, a braided comodule algebra will not dualize to a braided module algebra. The correct way would be to dualize it to a braided module coalgebra but, as mentioned in Remark 2.3, we do not discuss these structures here.

Proposition 5.8. *Let H be a braided coalgebra and V a braided right H -comodule with coaction $\rho_R : V \rightarrow V \otimes H$. Let $U \subset H'$ and $W \subset V'$ satisfy the assumptions of Lemmas 4.1, 5.1, 5.2 and 5.3 which guarantee that the braidings Ψ_{UU} , Ψ_{HW}° , Ψ_{UV}° and Ψ_{WU}^\bullet are well-defined. Assume that $U \subset H'$ is a (unital) subalgebra with respect to the product defined in (75). Consider*

$$\nu_R : W \otimes U \longrightarrow V', \quad \nu_R(e \otimes f)(v) := f(v_{(1)}^{\{1\}}) e(v_{(0)}^{\{2\}}). \tag{116}$$

If $\nu_R : W \otimes U \rightarrow W \subset V'$, then it defines a right U -action on W such that W becomes a braided right U -module with respect to the braidings Ψ_{UU}^{-1} and Ψ_{WU}^\bullet .

For a braided left H -comodule V , it is required that U and W satisfy the assumptions of Lemmas 4.1, 5.1, 5.2 and 5.3 which guarantee that the braidings $\Psi_{UU}, \Psi_{WH}^\circ, \Psi_{VU}^\circ$ and Ψ_{UW}^\bullet are well-defined. If

$$\nu_L : U \otimes W \longrightarrow V', \quad \nu_L(f \otimes e)(v) := f(v_{(-1)}^{\{2\}'}) e(v_{(0)}^{\{1\}'}) \tag{117}$$

yields a map $\nu_L : U \otimes W \rightarrow W \subset V'$, then it defines a left U -action on W such that W becomes a braided left U -module with respect to the braidings Ψ_{UU}^{-1} and Ψ_{UW}^\bullet .

Proof. As customary, we denote the map defined in (116) by $\nu_R(e \otimes f) := e \triangleleft f$ and the left action given in (98) by $\nu_L(f \otimes v) := f \triangleright v$. Since

$$(e \triangleleft f)(v) \stackrel{(116)}{=} f(v_{(1)}^{\{1\}'}) e(v_{(0)}^{\{2\}'}) \stackrel{(98)}{=} e(f \triangleright v), \quad f \in U, v \in V, e \in V',$$

it follows immediately from Theorem 5.4 that (116) defines a right U -action on V' . Assuming that $\nu_R : W \otimes U \rightarrow W$, we need to prove the compatibility with the braiding Ψ_{WU}^\bullet .

Although the proof of (22) goes along the lines of previous ones, we present the computations in order to show where all the listed braidings and Lemma 5.3 are needed. Let now $f, g \in U, a \in H, e \in W$ and $v \in V$. Employing the Sweedler-type notation $\Psi_{WU}^\bullet(e \otimes f) := f^{\{1\}\bullet} \otimes e^{\{2\}\bullet}$, we have

$$\begin{aligned} f^{\{1\}\bullet}(a) (e \triangleleft g)^{\{2\}\bullet}(v) &\stackrel{(96)}{=} f^{\{2\}\circ}(a) (e \triangleleft g)(v^{\{1\}\circ}) \stackrel{(92)}{=} f(a^{\{1\}'}) (e \triangleleft g)(v^{\{2\}'}) \\ &\stackrel{(116)}{=} f(a^{\{1\}'}_1) g(v^{\{2\}'}_1 \{1\}'_2) e(v^{\{2\}'}_1 \{2\}'_2) \\ &\stackrel{(2),(31)}{=} f(a^{\{1\}'_2 \{1\}'}) g(v_{(1)}^{\{1\}' \{2\}'}) e(v_{(0)}^{\{2\}'_1 \{2\}'_2}) \\ &\stackrel{(54),(84)}{=} f^{\{1\}'}(a^{\{2\}\circ}) g^{\{2\}'}(v_{(1)}^{\{1\}'}) e^{\{1\}\circ}(v_{(0)}^{\{2\}'}) \\ &\stackrel{(97)}{=} f^{\{1\}' \{1\}\bullet}(a) g^{\{2\}'}(v_{(1)}^{\{1\}'}) e^{\{2\}\bullet}(v_{(0)}^{\{2\}'}) \stackrel{(116)}{=} f^{\{1\}' \{1\}\bullet}(a) (e^{\{2\}\bullet} \triangleleft g^{\{2\}'})(v), \end{aligned}$$

which implies (22) for the braidings Ψ_{WU}^\bullet and Ψ_{UU}^{-1} . Furthermore, we know from Lemma 2.4 that U is a braided algebra with respect to the braiding Ψ_{UU}^{-1} , so this finishes the proof of the first part. The second part is proved similarly. \square

The analog of dualizing a coaction to an action as in the last proposition consists in dualizing an action to obtain a coaction of the dual coalgebra on the dual space. This is the purpose of our last proposition.

Proposition 5.9. *Let H be a braided unital algebra and V a braided left H -module with action $\nu_L : H \otimes V \rightarrow V$. Let $U \subset H'$ and $W \subset V'$ satisfy the assumptions of Lemmas 4.1, 5.1, 5.2 and 5.3 which guarantee that the braidings $\Psi_{UU}, \Psi_{HW}^\circ, \Psi_{UV}^\circ$ and Ψ_{WU}^\bullet are well-defined. Assume that $U \subset H^\circ$ is a subcoalgebra as in Proposition 4.4 but with respect to the braiding Ψ_{HH}^{-1} on H , i.e., $\underline{\Delta}^\circ(f) \in U \otimes U$ for all $f \in U$, where*

$$\langle\langle \underline{\Delta}^\circ(f), a \otimes b \rangle\rangle := \langle f, m \circ \Psi_{HH}^{-1}(a \otimes b) \rangle = f(b^{\{1\}'}) a^{\{2\}'}, \quad a, b \in H. \tag{118}$$

Consider the linear map

$$\rho_L : W \longrightarrow (H \otimes V)', \quad \rho_L(e)(a \otimes v) := e(a^{\{1\}' \triangleright} v^{\{2\}'}) \tag{119}$$

If $\rho_L : W \rightarrow U \otimes W \subset (H \otimes V)'$, then it defines a left U -coaction on W such that W becomes a braided left U -comodule with respect to the coalgebra $(U, \underline{\Delta}^\circ, \varepsilon)$, the braiding Ψ_{UU}^{-1} on U and the braiding Ψ_{WU}^\bullet between W and U .

For a braided right H -comodule V , it is required that U and W satisfy the assumptions of Lemmas 4.1, 5.1, 5.2 and 5.3 which guarantee that the braidings Ψ_{UU} , Ψ_{WH}° , Ψ_{VU}° and Ψ_{UW}^\bullet are well-defined. If the linear map

$$\rho_R : W \longrightarrow (V \otimes H)', \quad \rho_R(e)(v \otimes a) := e(v^{\{1\}' \triangleleft} a^{\{2\}'}) \tag{120}$$

fulfills $\rho_R : W \rightarrow W \otimes U \subset (V \otimes H)'$, then it defines a right U -coaction on W such that W becomes a braided right U -comodule with respect to the coalgebra $(U, \underline{\Delta}^\circ, \varepsilon)$, the braiding Ψ_{UU}^{-1} on U and the braiding Ψ_{UW}^\bullet between U and W .

Proof. As in the proof of the previous proposition, we prove (26) and (27) in order to show that the correct braidings and the correct coproduct are used. Let $f \in U$, $a, b \in H$, $e \in W$ and $v \in V$. Then

$$\begin{aligned} e_{(-1)}(a) (e_{(0)})_{(-1)}(b) (e_{(0)})_{(0)}(v) &\stackrel{(119)}{=} e(a^{\{1\}'_2 \triangleright} (b^{\{1\}'_1 \triangleright} v^{\{2\}'_1})^{\{2\}'_2}) \\ &\stackrel{(2),(23)}{=} e(a^{\langle 1 \rangle \{1\}'_1 \triangleright} (b^{\langle 2 \rangle \{1\}'_2 \triangleright} v^{\{2\}'_1 \{2\}'_2})) \stackrel{(19)}{=} e((a^{\langle 1 \rangle \{1\}'_1 \triangleright} b^{\langle 2 \rangle \{1\}'_2}) \triangleright v^{\{2\}'_1 \{2\}'_2}) \\ &\stackrel{(5)}{=} e((a^{\langle 1 \rangle \{1\}'_1 \triangleright} b^{\langle 2 \rangle \{1\}'_2}) \{1\}' \triangleright v^{\{2\}'}) \stackrel{(119)}{=} e_{(-1)}(a^{\langle 1 \rangle \{1\}'_1 \triangleright} b^{\langle 2 \rangle \{1\}'_2}) e_{(0)}(v) \\ &\stackrel{(118)}{=} (e_{(-1)})_{(1)}(a) (e_{(-1)})_{(2)}(b) e_{(0)}(v), \end{aligned}$$

so that the first relation of (26) holds for the coproduct given in (118). Moreover,

$$\varepsilon(e_{(-1)}) e_{(0)}(v) = e_{(-1)}(1) e_{(0)}(v) \stackrel{(119)}{=} e(1^{\{1\}' \triangleright} v^{\{2\}'}) \stackrel{(6),(19)}{=} e(v).$$

Hence ρ_L defines a left U -coaction for the coalgebra $(U, \underline{\Delta}^\circ, \varepsilon)$.

Recall from (54) that Ψ_{UU}^{-1} is the braiding on U which corresponds to Ψ_{HH}^{-1} on H according to (42). Now,

$$\begin{aligned} f^{\{1\}^\bullet \langle 1 \rangle \{1\}'_1} (a) e_{(-1)}^{\langle 2 \rangle \{1\}'_2} (b) e_{(0)}^{\{2\}^\bullet \{1\}'_1} (v) &\stackrel{(54)}{=} f^{\{1\}^\bullet \langle 1 \rangle \{1\}'_1} (a^{\langle 1 \rangle \{1\}'_1}) e_{(-1)}(b^{\langle 2 \rangle \{1\}'_2}) e_{(0)}^{\{2\}^\bullet \{1\}'_1} (v) \\ &\stackrel{(92),(96)}{=} f(a^{\langle 1 \rangle \{1\}'_1}) e_{(-1)}(b^{\langle 2 \rangle \{1\}'_2}) e_{(0)}(v^{\{2\}'_1}) \stackrel{(119)}{=} f(a^{\langle 1 \rangle \{1\}'_1}) e(b^{\langle 2 \rangle \{1\}'_2} \triangleright v^{\{2\}'_1 \{2\}'_2}) \\ &\stackrel{(2),(23)}{=} f(a^{\{1\}'_2}) e((b^{\{1\}'_1 \triangleright} v^{\{2\}'_1})^{\{2\}'_2}) \stackrel{(92),(96)}{=} f^{\{1\}^\bullet \langle 1 \rangle \{1\}'_1} (a) e^{\{2\}^\bullet \langle 1 \rangle \{1\}'_1} (b^{\{1\}'_1 \triangleright} v^{\{2\}'_1}) \\ &\stackrel{(119)}{=} f^{\{1\}^\bullet \langle 1 \rangle \{1\}'_1} (a) e^{\{2\}^\bullet \langle 1 \rangle \{1\}'_1} (b) e^{\{2\}^\bullet \langle 1 \rangle \{1\}'_1} (v), \end{aligned}$$

which proves (27) for the braidings Ψ_{UU}^{-1} on U and Ψ_{WU}^\bullet between W and U .

The proof of the opposite version is similar. □

Note that we used all the lemmas of this section in the proofs of Propositions 5.8 and 5.9. As a closing remark, let us point out that the relevance of the inverse braidings is undeniable throughout this paper, not only for turning right braided vector spaces into left braided vector spaces and vice versa, but also in the definitions

of the braidings of the type Ψ_{XY}° , in the definitions of actions and coactions in Theorems 5.4 and 5.6, and in our final propositions.

6. EXAMPLES

Example 6.1. Finite-dimensional braided (co)algebras, bialgebras and Hopf algebras and their finite-dimensional (co)modules yield examples of all the structures presented [25, 30]. In the finite-dimensional situation, the non-degenerated dual spaces are obviously unique. Explicit formulas can be deduced by using the coevaluation map

$$\text{coev}_H : \mathbb{K} \longrightarrow H \otimes H', \quad \text{coev}_H(1) = \sum_{j=1}^n e_j \otimes e^j, \tag{121}$$

where $\{e_1, \dots, e_n\} \subset H$ is a linear basis and $\{e^1, \dots, e^n\} \subset H'$ its dual basis. Under the identification $\mathbb{K} \otimes H \cong H \cong H \otimes \mathbb{K}$, (121) yields $(\text{id} \otimes \text{ev}) \circ (\text{coev}_H \otimes \text{id}) = \text{id} : H \rightarrow H$ and $(\text{ev} \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_{H'}) = \text{id} : H \rightarrow H$. From the dual versions of these identities, we obtain for instance the following formulas for the induced braidings $\Psi_{H'H'}$ and $\Psi_{H'H}$:

$$g^{(1)} \otimes f^{(2)} = \sum_{j,k=1}^n g(e_j^{(1)}) f(e_k^{(2)}) e^j \otimes e^k, \quad a^{(1)} \otimes f^{(2)} = \sum_{j=1}^n f(e_j^{(1)}) a^{(2)} \otimes e^j,$$

and similar formulas for all other induced braidings. Furthermore, the coproduct (73) and the product (75) may be written in the form

$$\underline{\Delta}(f) = \sum_{j,k=1}^n f(e_j^{(1)} e_k^{(2)}) e^j \otimes e^k$$

and

$$\underline{m}(f \otimes g) = \sum_{j=1}^n f(e_{j(1)}^{(2)}) g(e_{j(2)}^{(1)}) e^j,$$

respectively. Analogous expressions can be derived for actions and coactions. For example, the coaction $\rho_R : H \rightarrow H \otimes H'$ in (103) for $V = H$ and with the multiplication as left action is given by

$$\rho_R(a) = \sum_{j=1}^n e_j^{(1)} a^{(2)} \otimes e^j.$$

Similarly, the left action $\nu_L : H \otimes H' \rightarrow H'$ in (98) for $U = H, V = H'$, and with the right H' -coaction on H' given by the coproduct $\rho_R = \underline{\Delta} : H' \rightarrow H' \otimes H'$ from (73), becomes

$$\nu_L(a \otimes f) = \sum_{j=1}^n f(e_j a) e^j.$$

Example 6.2. Graded braided (co)algebras, bi- and Hopf algebras and their graded (co)modules [27] provide examples if the spaces of homogeneous elements are finite-dimensional for all grades (i.e., if they are locally finite). The dual space may then be given as the direct sum of the duals of the spaces of homogeneous elements (i.e., as the graded dual) so that the existence of the presented structures can be deduced grade by grade from Example 6.1. In this way, we obtain a large class

of infinite-dimensional examples. With some care, these arguments can be generalized to ascending, exhaustive, locally finite filtered (co)algebras, bi- and Hopf algebras [1]. Presumably the most prominent examples are infinite-dimensional Nichols algebras generated by a finite-dimensional Yetter–Drinfeld module (see e.g. [1, 2, 4, 3, 5, 6, 8, 9]). For a convenient PBW basis, [15] may be consulted. For examples related to the upper triangular part of a quantized enveloping algebra, see [28].

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