

WEIGHTED MIXED WEAK-TYPE INEQUALITIES FOR MULTILINEAR FRACTIONAL OPERATORS

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ABSTRACT. The aim of this paper is to obtain mixed weak-type inequalities for multilinear fractional operators, extending results by Berra, Carena and Pradolini [*J. Math. Anal. Appl.* **479** (2019)]. We prove that, under certain conditions on the weights, there exists a constant C such that

$$\left\| \frac{\mathcal{G}_\alpha(\vec{f})}{v} \right\|_{L^{q, \infty}(\nu v^q)} \leq C \prod_{i=1}^m \|f_i\|_{L^1(u_i)},$$

where $\mathcal{G}_\alpha(\vec{f})$ is the multilinear maximal function $\mathcal{M}_\alpha(\vec{f})$ introduced by Moen [*Collect. Math.* **60** (2009)] or the multilinear fractional integral $\mathcal{I}_\alpha(\vec{f})$. As an application, a vector-valued weighted mixed inequality for $\mathcal{I}_\alpha(\vec{f})$ is provided.

1. INTRODUCTION

E. Sawyer proved in 1985 the following mixed weak-type inequality.

Theorem 1.1 ([18]). *If $u, v \in A_1$, then there is a constant C such that for all $t > 0$,*

$$uv \left\{ x \in \mathbb{R} : \frac{M(fv)(x)}{v(x)} > t \right\} \leq \frac{C}{t} \int_{\mathbb{R}} |f(x)| u(x) v(x) dx. \quad (1.1)$$

This estimate is a highly non-trivial extension of the classical weak type $(1, 1)$ inequality for the maximal operator due to the presence of the weight function v inside the distribution set. Note that if $v = 1$, this result is a well-known estimate due to C. Fefferman and E. Stein [6]. The inequality (1.1) also holds if $u \in A_1$ when $v \in A_1$; see [9].

In 2005, D. Cruz-Uribe, J. M. Martell and C. Pérez [5] extended (1.1) to \mathbb{R}^n . Furthermore, they settled that estimate for Calderón–Zygmund operators, answering affirmatively and extending a conjecture raised by E. Sawyer for the Hilbert transform [18]. The precise statement of their result is the following.

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Theorem 1.2 ([5]). *If $u, v \in A_1$, or $u \in A_1$ and $uv \in A_\infty$, then there is a constant C such that, for all $t > 0$,*

$$uv \left\{ x \in \mathbb{R}^n : \frac{|T(fv)(x)|}{v(x)} > t \right\} \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|u(x)v(x) dx, \tag{1.2}$$

where T is a Calderón–Zygmund operator with some regularity.

Quantitative versions of the previous result were obtained in [17] and also some counterparts for commutators in [1].

In [5], D. Cruz-Uribe, J. M. Martell and C. Pérez conjectured that (1.2) and (1.1) should hold for $v \in A_\infty$. This result is the most singular case, due to the fact that the A_∞ condition is the weakest possible assumption within the A_p classes.

Recently, K. Li, S. Ombrosi and C. Pérez [12] solved that conjecture. They proved the following theorem.

Theorem 1.3 ([12]). *Let $v \in A_\infty$ and $u \in A_1$. Then there is a constant C depending on the A_1 constant of u and the A_∞ constant of v such that*

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq C \|f\|_{L^{1,\infty}(uv)},$$

where T can be the Hardy–Littlewood maximal function, any Calderón–Zygmund operator or any rough singular integral.

In 2009, Lerner *et al.* [10] introduced the multi(sub)linear maximal function \mathcal{M} defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i,$$

where $\vec{f} = (f_1, \dots, f_m)$ and the supremum is taken over all cubes Q containing x .

This maximal operator is smaller than the product $\prod_{i=1}^m Mf_i$, which was the auxiliary operator used previously to estimate multilinear singular integral operators.

There is a connection between multilinear operators and mixed weak-type inequalities (see [10] or [12]). In fact, in a recent joint work with K. Li and S. Ombrosi we proved the following theorem.

Theorem 1.4 ([13]). *Let \mathcal{T} be a multilinear Calderón–Zygmund operator, $\vec{w} = (w_1, \dots, w_m)$ and $\nu = w_1^{\frac{1}{m}} \dots w_m^{\frac{1}{m}}$. Suppose that $\vec{w} \in A_{(1,\dots,1)}$ and $\nu v^{\frac{1}{m}} \in A_\infty$ or $w_1, \dots, w_m \in A_1$ and $v \in A_\infty$. Then there is a constant C such that*

$$\left\| \frac{\mathcal{T}(\vec{f})}{v} \right\|_{L^{\frac{1}{m},\infty}(\nu v^{\frac{1}{m}})} \leq C \prod_{i=1}^m \|f_i\|_{L^1(w_i)}.$$

We refer the reader to Section 2 for the definition of $A_{(1,\dots,1)}$ and more details about $A_{\vec{p}}$ weights.

The study of fractional integrals and associated maximal functions is important in harmonic analysis. We recall that the fractional integral operator or Riesz potential is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy, \quad 0 < \alpha < n,$$

and the fractional maximal function by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy, \quad 0 \leq \alpha < n,$$

where the supremum is taken over all cubes Q containing x . Note that in the case $\alpha = 0$ we recover the Hardy–Littlewood maximal operator. Properties of these operators can be found in the books by Stein [19] and Grafakos [7].

F. Berra, M. Carena and G. Pradolini [2] proved the following mixed weak-type inequality.

Theorem 1.5 ([2]). *Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and q satisfying $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If u, v are weights such that $u, v^{\frac{q}{p}} \in A_1$ or $uv^{\frac{-q}{p}} \in A_1$ and $v \in A_\infty(uv^{\frac{-q}{p}})$, then there exists a positive constant C such that for every $t > 0$*

$$uv^{\frac{q}{p}} \left\{ x \in \mathbb{R}^n : \frac{I_\alpha(fv)(x)}{v(x)} > t \right\}^{\frac{1}{q}} \leq \frac{C}{t} \left(\int_{\mathbb{R}^n} |f(x)|^p u(x)^{\frac{p}{q}} v(x) dx \right)^{\frac{1}{p}},$$

where I_α is the fractional integral or the fractional maximal function.

In the multilinear setting, a natural way to extend fractional integrals is the following.

Definition 1.6. Let α be a number such that $0 < \alpha < mn$ and let $\vec{f} = (f_1, \dots, f_m)$ be a collection of functions on \mathbb{R}^n . We define the *multilinear fractional integral* as

$$\mathcal{I}_\alpha \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \dots f_m(y_m) d\vec{y}}{(|x - y_1| + \dots + |x - y_m|)^{mn-\alpha}}.$$

K. Moen [14] introduced the multi(sub)linear maximal operator \mathcal{M}_α associated to the multilinear fractional integral \mathcal{I}_α .

Definition 1.7. For $0 \leq \alpha < mn$ and $\vec{f} = (f_1, \dots, f_m)$ as above, we define the *multi(sub)linear maximal operator* \mathcal{M}_α by

$$\mathcal{M}_\alpha \vec{f}(x) = \sup_{Q \ni x} \prod_{i=1}^m \left(\frac{1}{|Q|^{1-\frac{\alpha}{nm}}} \int_Q |f_i(y_i)| dy_i \right).$$

Observe that the case $\alpha = 0$ corresponds to the multi(sub)linear maximal function \mathcal{M} studied in [10].

At this point we present our contribution. Our first result is a counterpart of Theorem 1.5 for multilinear fractional maximal operators.

Theorem 1.8. *Let $0 \leq \alpha < mn$. Let $q = \frac{n}{mn-\alpha}$, $\vec{u}^{mq} = (u_1^{mq}, \dots, u_m^{mq})$ and $\nu = \prod_{i=1}^m u_i^q$. Suppose that $\vec{u}^{mq} \in A_{(1, \dots, 1)}$ and $\nu v^q \in A_\infty$, or $u_1^{mq}, \dots, u_m^{mq} \in A_1$ and $v^{mq} \in A_\infty$. Then there exists a constant C such that*

$$\left\| \frac{\mathcal{M}_\alpha(\vec{f})}{v} \right\|_{L^{q, \infty}(\nu v^q)} \leq C \prod_{i=1}^m \|f_i\|_{L^1(u_i)}.$$

Note that if $\alpha = 0$ then $q = \frac{1}{m}$ and we obtain Theorem 1.4 for the multi(sub)linear maximal operator \mathcal{M} .

Remark. If in Theorem 1.8 we take $m = 1$ we get that $\frac{1}{q} = 1 - \frac{\alpha}{n}$ and the hypothesis on the weights reduces to $u^q \in A_1$ and $v \in A_\infty$. Then we recover Theorem 1.5 in the case $p = 1$ for a more general class of weights v . The weight u^q in Theorem 1.8 plays the role of the weight u in Theorem 1.5.

By extrapolation arguments, we can extend this result to multilinear fractional integrals. The theorem below was essentially obtained in [16]; however, for the sake of completeness, we will give a complete proof in Appendix A.

Theorem 1.9 ([16]). *Let $0 < \alpha < mn$. Let $q = \frac{n}{mn-\alpha}$, $\vec{u}^{mq} = (u_1^{mq}, \dots, u_m^{mq}) \in A_{(1, \dots, 1)}$, $v^q \in A_\infty$ and set $\nu = \prod_{i=1}^m u_i^q$. Then there exists a constant C such that*

$$\left\| \frac{\mathcal{I}_\alpha(\vec{f})}{v} \right\|_{L^{q, \infty}(\nu v^q)} \leq C \left\| \frac{\mathcal{M}_\alpha(\vec{f})}{v} \right\|_{L^{q, \infty}(\nu v^q)}.$$

Finally, as a consequence of Theorem 1.8 and Theorem 1.9, we obtain the main result of this paper.

Theorem 1.10. *Let $0 < \alpha < mn$. Let $q = \frac{n}{mn-\alpha}$, $\vec{u}^{mq} = (u_1^{mq}, \dots, u_m^{mq})$ and $\nu = \prod_{i=1}^m u_i^q$. Suppose that $\vec{u}^{mq} \in A_{(1, \dots, 1)}$ and $\nu v^q \in A_\infty$, or $u_1^{mq}, \dots, u_m^{mq} \in A_1$ and $v^{mq} \in A_\infty$. Then there exists a constant C such that*

$$\left\| \frac{\mathcal{I}_\alpha(\vec{f})}{v} \right\|_{L^{q, \infty}(\nu v^q)} \leq C \prod_{i=1}^m \|f_i\|_{L^1(u_i)}.$$

The rest of the article is organized as follows. In Section 2 we recall the definition of the A_p and $A_{\vec{p}}$ classes of weights. Section 3 is devoted to the proof of Theorem 1.8. In Section 4, as an application of Theorem 1.10, we obtain a vector-valued extension of the mixed weighted inequalities for multilinear fractional integrals. We end this paper with an appendix, in which we give a proof of Theorem 1.9.

2. PRELIMINARIES

By a *weight* we mean a non-negative locally integrable function defined on \mathbb{R}^n such that $0 < w(x) < \infty$ almost everywhere. We recall that a weight w belongs to the class A_p , introduced by B. Muckenhoupt [15], $1 < p < \infty$, if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(y) dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{1-p'} dy \right)^{p-1} < \infty,$$

where p' is the conjugate exponent of p defined by the equation $\frac{1}{p} + \frac{1}{p'} = 1$. A weight w belongs to the A_1 class if there exists a constant C such that

$$\frac{1}{|Q|} \int_Q w(y) dy \leq C \inf_Q w.$$

Since the A_p classes are increasing with respect to p , it is natural to define the A_∞ class of weights by $A_\infty = \cup_{p \geq 1} A_p$.

In 2009, Lerner *et al.* [10] showed that there is a way to define an analogue of the Muckenhoupt A_p classes for multiple weights.

Definition 2.1. Let m be a positive integer. Let $1 \leq p_1, \dots, p_m < \infty$. We denote by p the number given by $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and by \vec{P} the vector $\vec{P} = (p_1, \dots, p_m)$.

Definition 2.2. Let $1 \leq p_1, \dots, p_m < \infty$. Given $\vec{w} = (w_1, \dots, w_m)$, set

$$\nu_{\vec{w}} = \prod_{i=1}^m w_i^{\frac{p}{p_i}}.$$

We say that \vec{w} satisfies the $A_{\vec{P}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q w_i^{1-p'_i} \right)^{\frac{1}{p'_i}} < \infty.$$

When $p_i = 1$, $\left(\frac{1}{|Q|} \int_Q w_i^{1-p'_i} \right)^{\frac{1}{p'_i}}$ is understood as $(\inf_Q w_i)^{-1}$. Then we will say that $\vec{w} \in A_{(1, \dots, 1)}$ if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \right)^{\frac{1}{p}} \prod_{i=1}^m (\inf_Q w_i)^{-1} < \infty.$$

The multilinear $A_{\vec{P}}$ condition has the following characterization in terms of the linear A_p classes.

Theorem 2.3 ([10, Theorem 3.6]). *Let $\vec{w} = (w_1, \dots, w_m)$ and $1 \leq p_1, \dots, p_m < \infty$. Then $\vec{w} \in A_{\vec{P}}$ if and only if*

$$\begin{cases} w_i^{1-p'_i} \in A_{mp'_i}, & i = 1, \dots, m, \\ \nu_{\vec{w}} \in A_{mp}, \end{cases}$$

where the condition $w_i^{1-p'_i} \in A_{mp'_i}$ in the case $p_i = 1$ is understood as $w_i^{\frac{1}{m}} \in A_1$.

A more general result can be found in [11, Lemma 3.2].

Observe that in the particular case where every $p_i = 1$ we have $p = \frac{1}{m}$. By Theorem 2.3, given $\vec{w} = (w_1, \dots, w_m)$, we have that the following statements hold:

- If $\vec{w} = (w_1, \dots, w_m) \in A_{(1, \dots, 1)}$ then $\nu_{\vec{w}} = w_1^{\frac{1}{m}} \dots w_m^{\frac{1}{m}} \in A_1$.
- If $\vec{w} \in A_{(1, \dots, 1)}$ then $w_i^{\frac{1}{m}} \in A_1$ for all $i = 1, \dots, m$.

Observe that $\vec{w} \in A_{(1, \dots, 1)}$ does not imply that $w_i \in A_1$ for every $i = 1, \dots, m$. We can see this with a simple counterexample. Let $m = 2$ and consider the weights $w_1 = 1$ and $w_2 = \frac{1}{|x|}$. Then $\vec{w} \in A_{(1, \dots, 1)}$, $w_1^{\frac{1}{2}}, w_2^{\frac{1}{2}} \in A_1$, $w_1 \in A_1$, but $w_2 \notin A_1$.

3. PROOF OF THEOREM 1.8

In order to prove Theorem 1.8 we need the following pointwise estimate for \mathcal{M}_α in terms of the multilinear maximal operator \mathcal{M} . This is a multilinear version of Lemma 4 in [2], and to prove it, we follow a similar approach to the one that is used there.

Lemma 3.1. *Let $q = \frac{n}{mn-\alpha}$. Then*

$$\mathcal{M}_\alpha(f_1, \dots, f_m)(x) \leq \mathcal{M}(f_1 u_1^{1-mq}, \dots, f_m u_m^{1-mq})^{\frac{1}{mq}}(x) \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i u_i \right)^{\frac{\alpha}{mn}}.$$

Proof. Let us fix $x \in \mathbb{R}^n$ and let Q be a cube containing x . Applying Hölder's inequality with $\frac{1}{1-\frac{\alpha}{mn}}$ and $\frac{mn}{\alpha}$ we obtain

$$\begin{aligned} \prod_{i=1}^m \left(\frac{1}{|Q|^{1-\frac{\alpha}{mn}}} \int_Q f_i \right) &= \prod_{i=1}^m \left(\frac{1}{|Q|^{1-\frac{\alpha}{mn}}} \int_Q f_i^{1-\frac{\alpha}{mn}} f_i^{\frac{\alpha}{mn}} u_i^{\frac{1}{mq}-1} u_i^{\frac{mq-1}{mq}} \right) \\ &\leq \prod_{i=1}^m \left[\left(\frac{1}{|Q|} \int_Q f_i u_i^{1-mq} \right)^{\frac{1}{mq}} \left(\int_Q f_i u_i \right)^{\frac{\alpha}{mn}} \right] \\ &= \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q f_i u_i^{1-mq} \right)^{\frac{1}{mq}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i u_i \right)^{\frac{\alpha}{mn}} \\ &\leq \mathcal{M}(f_1 u_1^{1-mq}, \dots, f_m u_m^{1-mq})^{\frac{1}{mq}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i u_i \right)^{\frac{\alpha}{mn}}. \end{aligned}$$

□

Now we have all the tools that we need to prove Theorem 1.8.

Proof of Theorem 1.8. By applying Lemma 3.1 and Theorem 1.4, we get

$$\begin{aligned} & \nu v^q \left\{ x \in \mathbb{R}^n : \frac{\mathcal{M}_\alpha(\vec{f})(x)}{v(x)} > \lambda \right\}^{\frac{1}{q}} \\ & \leq u_1^q \dots u_m^q v^q \left\{ x \in \mathbb{R}^n : \frac{\mathcal{M}(f_1 u_1^{1-mq}, \dots, f_m u_m^{1-mq})^{\frac{1}{mq}}(x)}{v(x)} > \frac{\lambda}{\prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i u_i \right)^{\frac{\alpha}{mn}}} \right\}^{\frac{1}{q}} \\ & = u_1^q \dots u_m^q v^q (E_\lambda) \\ & = (u_1^{mq})^{\frac{1}{m}} \dots (u_m^{mq})^{\frac{1}{m}} (v^{mq})^{\frac{1}{m}} (E_\lambda)^{m \frac{1}{mq}} \\ & \leq \frac{C}{\lambda} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i u_i \right)^{\frac{\alpha}{mn}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i u_i^{1-mq} u_i^{mq} \right)^{\frac{1}{mq}} \\ & = \frac{C}{\lambda} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i u_i \right) \\ & = \frac{C}{\lambda} \prod_{i=1}^m \|f_i\|_{L^1(u_i)}, \end{aligned}$$

where

$$E_\lambda = \left\{ x \in \mathbb{R}^n : \frac{\mathcal{M}(f_1 u_1^{1-mq}, \dots, f_m u_m^{1-mq})(x)}{v^{mq}(x)} > \left(\frac{\lambda}{\prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i u_i \right)^{\frac{\alpha}{mn}}} \right)^{mq} \right\}^{\frac{1}{q}}.$$

□

4. A VECTOR-VALUED EXTENSION OF THEOREM 1.10

Recently, D. Carando, M. Mazzitelli and S. Ombrosi [4] obtained a generalization of the Marcinkiewicz–Zygmund inequalities to the context of multilinear operators. We recall one of the results in that work that extends previously known results from [8] and [3].

Theorem 4.1 ([4]). *Let $0 < p, q_1, \dots, q_m < r < 2$ or $r = 2$ and $0 < p, q_1, \dots, q_m < \infty$ and, for each $1 \leq i \leq m$, consider $\{f_{k_i}^i\}_{k_i} \subset L^{q_i}(\mu_i)$. Let S be a multilinear operator such that $S: L^{q_1}(\mu_1) \times \dots \times L^{q_m}(\mu_m) \rightarrow L^{p, \infty}(\nu)$. Then, there exists a constant $C > 0$ such that*

$$\left\| \left(\sum_{k_1, \dots, k_m} |S(f_{k_1}^1, \dots, f_{k_m}^m)|^r \right)^{\frac{1}{r}} \right\|_{L^{p, \infty}(\nu)} \leq C \|S\|_{weak} \prod_{i=1}^m \left\| \left(\sum_{k_i} |f_{k_i}^i|^r \right)^{\frac{1}{r}} \right\|_{L^{q_i}(\mu_i)}.$$

As a consequence of this theorem and Theorem 1.10 we obtain the following mixed weighted vector valued inequality for a multilinear fractional operator \mathcal{I}_α .

Corollary 4.2. *Let $S(\vec{f}) = \frac{\mathcal{I}_\alpha(\vec{f})}{v}$, where \mathcal{I}_α is a multilinear fractional operator. Let $q = \frac{n}{mn-\alpha}$, $\vec{u}^{mq} = (u_1^{mq}, \dots, u_m^{mq})$ and $\nu = \prod_{i=1}^m u_i^q$. Suppose that $\vec{u}^{mq} \in A_{(1, \dots, 1)}$ and $\nu v^q \in A_\infty$, or $u_1^{mq}, \dots, u_m^{mq} \in A_1$ and $v^{mq} \in A_\infty$. For each $1 \leq i \leq m$,*

consider $\{f_{k_i}^i\}_{k_i} \subset L^1(u_i)$. Then, there exists a constant $C > 0$ such that

$$\left\| \left(\sum_{k_1, \dots, k_m} |S(f_{k_1}^1, \dots, f_{k_m}^m)|^r \right)^{\frac{1}{r}} \right\|_{L^{q, \infty}(\nu v^q)} \leq C \prod_{i=1}^m \left\| \left(\sum_{k_i} |f_{k_i}^i|^r \right)^{\frac{1}{r}} \right\|_{L^1(u_i)}.$$

Observe that under the hypothesis of Corollary 4.2, S satisfies $S: L^1(u_1) \times \dots \times L^1(u_m) \rightarrow L^{q, \infty}(\nu v^q)$. So we are under the hypothesis of Theorem 4.1.

5. APPENDIX A. PROOF OF THEOREM 1.9

In order to prove Theorem 1.9 we will need two known results. The first one is due to K. Moen.

Theorem 5.1 ([14, Theorem 3.1]). *Suppose that $0 < \alpha < mn$; then for every $w \in A_\infty$ and all $0 < s < \infty$ we have*

$$\int_{\mathbb{R}^n} |\mathcal{I}_\alpha \vec{f}(x)|^s w(x) dx \leq C \int_{\mathbb{R}^n} \mathcal{M}_\alpha \vec{f}(x)^s w(x) dx$$

for all functions \vec{f} with f_i bounded with compact support.

The second result we will rely upon is due to D. Cruz-Uribe, J. M. Martell and C. Pérez.

Theorem 5.2 ([5, Theorem 1.7]). *Let \mathcal{F} be a family of pairs of functions that satisfies that there exists a number $p_0, 0 < p_0 < \infty$, such that, for all $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w(x) dx$$

for all $(f, g) \in \mathcal{F}$ such that the left hand side is finite, and with C depending only on $[w]_{A_\infty}$. Then, for all weights u, v such that $u \in A_1$ and $v \in A_\infty$, we have that

$$\|fv^{-1}\|_{L^{1, \infty}(uv)} \leq C \|gv^{-1}\|_{L^{1, \infty}(uv)} \quad (f, g) \in \mathcal{F}.$$

Having those results at our disposal we proceed as follows. First of all observe that if $\vec{u}^{mq} = (u_1^{mq}, \dots, u_m^{mq}) \in A_{(1, \dots, 1)}$, then $\nu = u_1^q \dots u_m^q \in A_1$. Then, by

Theorem 5.1 and Theorem 5.2,

$$\begin{aligned}
\left\| \frac{\mathcal{I}_\alpha(\vec{f})}{v} \right\|_{L^{q,\infty}(\nu v^q)}^q &= \sup_{\lambda>0} \lambda^q \left(\nu v^q \left\{ x \in \mathbb{R}^n : \left| \frac{\mathcal{I}_\alpha(\vec{f})(x)}{v(x)} \right| > \lambda \right\} \right) \\
&= \sup_{\lambda>0} \lambda^q \left(\nu v^q \left\{ x \in \mathbb{R}^n : \left| \frac{T(\vec{f})(x)}{v(x)} \right|^q > \lambda^q \right\} \right) \\
&= \sup_{t>0} t \left(\nu v^q \left\{ x \in \mathbb{R}^n : \left| \frac{T(\vec{f})(x)}{v(x)} \right|^q > t \right\} \right) \\
&= \left\| \left(\frac{\mathcal{I}_\alpha(\vec{f})}{v} \right)^q \right\|_{L^{1,\infty}(\nu v^q)} \\
&\leq C \left\| \left(\frac{\mathcal{M}_\alpha(\vec{f})}{v} \right)^q \right\|_{L^{1,\infty}(\nu v^q)} \\
&= \sup_{\lambda>0} \lambda \left(\nu v^q \left\{ x \in \mathbb{R}^n : \left(\frac{\mathcal{M}_\alpha(\vec{f})(x)}{v(x)} \right)^q > \lambda \right\} \right) \\
&= \sup_{t>0} t^q \left(\nu v^q \left\{ x \in \mathbb{R}^n : \left(\frac{\mathcal{M}_\alpha(\vec{f})(x)}{v(x)} \right)^q > t^q \right\} \right) \\
&= \sup_{t>0} t^q \left(\nu v^q \left\{ x \in \mathbb{R}^n : \frac{\mathcal{M}_\alpha(\vec{f})(x)}{v(x)} > t \right\} \right) \\
&= \left\| \frac{\mathcal{M}_\alpha(\vec{f})}{v} \right\|_{L^{q,\infty}(\nu v^q)}^q.
\end{aligned}$$

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