ON HOPF ALGEBRAS OVER BASIC HOPF ALGEBRAS OF DIMENSION 24

RONGCHUAN XIONG

ABSTRACT. We determine finite-dimensional Hopf algebras over an algebraically closed field of characteristic zero, whose Hopf coradical is isomorphic to a non-pointed basic Hopf algebra of dimension 24 and the infinitesimal braidings are indecomposable objects. In particular, we obtain families of new finite-dimensional Hopf algebras without the dual Chevalley property.

1. INTRODUCTION

Let $\Bbbk$ be an algebraically closed field of characteristic zero. It is a fundamental and difficult question in Hopf algebra theory to classify finite-dimensional ones. The research in this direction is very rich. Most of the classification results consist of Hopf algebras that are basic or have the dual Chevalley property (that is, their coradical is a subalgebra). But there are very few results on finite-dimensional Hopf algebras without the dual Chevalley property in the literature, unless examples without pointed duals were constructed in [11, 17, 24, 20, 23] via the generalized lifting method [4].

As a generalization of the *lifting method* [8], the generalized lifting method gives a technical framework to classify the Hopf algebras without the dual Chevalley property. It consists of the following steps (see [4]):

- Step 1. Classify all Hopf algebras L that are generated by a cosemisimple coalgebra.
- Step 2. Classify all connected graded Hopf algebras R in the category $^L_L\mathcal{YD}$ of left Yetter–Drinfeld modules over L.
- Step 3. Given L and R as in previous items, classify all Hopf algebras A such that $\operatorname{gr} A \cong R\sharp L$. Here A is called a *lifting* of R over L.

The method works because of the following facts. Suppose that A is a Hopf algebra over $\Bbbk$ and denote by $A_{[0]}$ the Hopf coradical of A (it is generated by the coradical A_0 of A). If $S_A(A_{[0]}) \subseteq A_{[0]}$, then the standard filtration $\{A_{[n]}\}_{n\geq 0}$, defined recursively by $A_{[n]} = A_{[n-1]} \bigwedge A_{[0]}$, is a Hopf algebra filtration. Therefore

the associated graded coalgebra $\operatorname{gr} A = \oplus_{n=0}^{\infty} A_{[n]}/A_{[n-1]}$ with $A_{[-1]} = 0$ is a Hopf algebra and so there is a connected graded braided Hopf algebra $R = \oplus_{n \geq 0} R(n)$ in $_{A_{[0]}}^{A_{[0]}}\mathcal{YD}$ such that $\operatorname{gr} A \cong R \sharp A_{[0]}$. Here R and $R(1)$ are called the *diagram* and *infinitesimal braiding* of A, respectively.

In this paper, following the work [11], we fix a Hopf algebra $\mathcal{K}_{24,1}$ of dimension 24 that is basic and generated by the coradical, and continue the study on Steps 2 and 3 in the lifting procedure.

Using the equivalence $_{\mathcal{K}_{24,1}}^{\mathcal{K}_{24,1}}\mathcal{YD} \cong {}_{\mathcal{D}(\mathcal{K}_{24,1}^{cop})}\mathcal{M}$, we determine simple Yetter–Drinfeld modules over $\mathcal{K}_{24,1}$. It turns out there exist 144 simple objects in $_{\mathcal{K}_{24,1}}^{\mathcal{K}_{24,1}}\mathcal{YD}$, among which there are 24 one-dimensional objects $\Bbbk_{\chi_{i,j,k}}$ with $(i,j,k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5}$ and 120 two-dimensional objects $V_{i,j,k,\iota}$ with $(i,j,k,\iota) \in \Lambda = \{(i,j,k,\iota) \mid i,j \in \mathbb{I}_{0,5}, k, \iota \in \mathbb{I}_{0,1}, j + 3k \not\equiv 3(\iota + 1) \bmod 6\}$; see Theorem 3.8 for details.

Now we determine finite-dimensional Nichols algebras over simple objects in $_{\mathcal{K}_{24,1}}^{\mathcal{K}_{24,1}}\mathcal{YD}$ and the liftings of their bosonizations. We first discard Nichols algebras $\mathcal{B}(V_{i,j,k,\iota})$ of infinite dimension. Using the equivalence $_{\mathcal{K}_{24,1}}^{\mathcal{K}_{24,1}}\mathcal{YD} \cong {}_{\operatorname{gr} \mathcal{A}_{24,1}}^{\operatorname{gr} \mathcal{A}_{24,1}}\mathcal{YD}$, we transport the information of $\mathcal{B}(V_{i,j,k,\iota})$ from the category $_{\mathcal{K}_{24,1}}^{\mathcal{K}_{24,1}}\mathcal{YD}$ to $_{\operatorname{gr} \mathcal{A}_{24,1}}^{\operatorname{gr} \mathcal{A}_{24,1}}\mathcal{YD}$, where $\mathcal{A}_{24,1}$ is the dual Hopf algebra of $\mathcal{K}_{24,1}$; and we prove $\mathcal{B}(V_{i,j,k,\iota}) \sharp \operatorname{gr} \mathcal{A}_{24,1}$ is infinite-dimensional by using the classification results in [15]. Then we prove the remaining ones are finite-dimensional by computing their defining relations and PBW bases in $_{\mathcal{K}_{24,1}}^{\mathcal{K}_{24,1}}\mathcal{YD}$. Finally, we study the liftings of finite-dimensional Nichols algebras following the techniques in [8, 11]. Consequently, we have the following theorem.

Theorem 1.1 (Theorems 4.7 & 5.11). *Let A be a finite-dimensional Hopf algebra over $\mathcal{K}_{24,1}$ whose infinitesimal braiding V is indecomposable in $_{\mathcal{K}_{24,1}}^{\mathcal{K}_{24,1}}\mathcal{YD}$. Then V is isomorphic either to $\Bbbk_{\chi_{i,j,k}}$ for $(i,j,k) \in \Lambda^0$ or to $V_{i,j,k,\iota}$ for $(i,j,k,\iota) \in \cup_{i=1}^{6}\Lambda^i$, and A is isomorphic to one of the following objects:*

- $\bigwedge \Bbbk_{\chi_{i,j,k}} \sharp \mathcal{K}_{24,1}$ *for $(i,j,k) \in \Lambda^0$;*
- $\mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ *for $(i,j,k,\iota) \in \cup_{i=1}^{6}\Lambda^i - \Lambda^{1*}$;*
- $\mathcal{C}_{i,j,k,\iota}(\mu)$ *for $\mu \in \Bbbk$ and $(i,j,k,\iota) \in \Lambda^{1*}$.*

The Nichols algebra $\mathcal{B}(V_{i,j,k,\iota})$ for $(i,j,k,\iota) \in \cup_{i=4}^{6}\Lambda^i$ is isomorphic as an algebra to a quantum plane. They appeared in [5] and it was shown that the braidings are of non-diagonal type. The Nichols algebra $\mathcal{B}(V_{i,j,k,\iota})$ for $(i,j,k,\iota) \in \cup_{i=1}^{3}\Lambda^i$ is an algebra of dimension 18 or 36 with no quadratic relations. They are examples of Nichols algebra of non-diagonal type, which are (up to isomorphism) arising from Nichols algebras of standard type B_2 by using the techniques in [2].

The Hopf algebras $\bigwedge \Bbbk_{\chi_{i,j,k}} \sharp \mathcal{K}_{24,1}$ with $(i,j,k) \in \Lambda^0$ are the duals of pointed Hopf algebras of dimension 48. The Hopf algebras $\mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ for $(i,j,k,\iota) \in \cup_{i=4}^{6}\Lambda^i$ are the duals of pointed Hopf algebras of dimension 96, 144 or 288. The Hopf algebras $\mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ for $(i,j,k,\iota) \in \Lambda^1$ or $\Lambda^2 \cup \Lambda^3$ are the duals of pointed Hopf algebras of dimension 432 or 864, respectively. The Hopf algebras $\mathcal{C}_{i,j,k,\iota}(\mu)$

with $\mu \neq 0$ are non-trivial liftings of $\mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ for $(i,j,k,\iota) \in \Lambda^{1*} \subset \Lambda^1$. They constitute new examples of Hopf algebras without the dual Chevalley property.

The paper is organized as follows: In section 2, we recall some basic knowledge and notations of Yetter–Drinfeld modules, Nichols algebras. In section 3, we introduce the structure of the Hopf algebra $\mathcal{K}_{24,1}$. In section 4, we determine all finite-dimensional Nichols algebras over simple objects in $^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ and present them by generators and relations. In section 5, we determine all finite-dimensional Hopf algebras over $\mathcal{K}_{24,1}$, whose infinitesimal braidings are simple objects in $^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$.

2. Preliminaries

Conventions. In the paper, the base field $\Bbbk$ is algebraically closed of characteristic zero and ξ is a primitive 6th root of unity. Let $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{I}_{k,n} := \{k, k+1, \ldots, n\}$ for $n \geq k \geq 0$.

Let H be a Hopf algebra over $\Bbbk$. Denote by $\mathcal{G}(H)$ the set of group-like elements of H. For any $g, h \in \mathcal{G}(H)$, $\mathcal{P}_{g,h}(H) = \{x \in H \mid \Delta(x) = x \otimes g + h \otimes x\}$. Our references for Hopf algebra theory are [21, 22].

2.1. **Yetter–Drinfeld modules and Hopf algebras with a projection.** Suppose that H has bijective antipode and denote by $^H_H\mathcal{YD}$ the category of left Yetter–Drinfeld modules over H. Then $^H_H\mathcal{YD}$ is braided monoidal with the braiding $c_{V,W}$ for $V, W \in {}^H_H\mathcal{YD}$ given by

$$c_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto v_{(-1)} \cdot w \otimes v_{(0)} \quad \forall\, v \in V, w \in W. \quad (2.1)$$

In particular, $c := c_{V,V}$ is a linear isomorphism satisfying the braid equation $(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$, that is, (V, c) is a braided vector space. Moreover, $^H_H\mathcal{YD}$ is rigid. The left dual V^* is defined by

$$\langle h \cdot f, v \rangle = \langle f, S(h)v \rangle, \quad f_{(-1)}\langle f_{(0)}, v \rangle = S^{-1}(v_{(-1)})\langle f, v_{(0)} \rangle.$$

If H is finite-dimensional, then by [6, Proposition 2.2.1], $^H_H\mathcal{YD} \cong {}^{H^*}_{H^*}\mathcal{YD}$ as braided monoidal categories via the functor (F, η) defined as follows: $F(V) = V$ as a vector space,

$$f \cdot v = f(S(v_{(-1)}))v_{(0)}, \quad \delta(v) = \sum_i S^{-1}(h^i) \otimes h_i \cdot v, \quad \text{and}$$
$$\eta : F(V) \otimes F(W) \to F(V \otimes W), \quad v \otimes w \mapsto w_{(-1)} \cdot v \otimes w_{(0)}, \quad (2.2)$$

where $V, W \in {}^H_H\mathcal{YD}$, $f \in H^*$, $v \in V$, $w \in W$, and $\{h_i\}$ and $\{h^i\}$ are the dual bases of H and H^*.

For a Hopf algebra $R \in {}^H_H\mathcal{YD}$ that is braided, set $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$ for the comultiplication. By the *Radford biproduct* or *bosonization* of R by H ([22]), written as $R\sharp H$, we mean a usual Hopf algebra, as a vector space, $R\sharp H = R \otimes H$, whose multiplication and comultiplication are provided by the smash product and smash coproduct, respectively:

$$(r\sharp g)(s\sharp h) = r(g_{(1)} \cdot s)\sharp g_{(2)}h, \quad \Delta(r\sharp g) = r^{(1)}\sharp(r^{(2)})_{(-1)}g_{(1)} \otimes (r^{(2)})_{(0)}\sharp g_{(2)}. \quad (2.3)$$

2.2. **Nichols algebras and skew-derivations.** Let H be a Hopf algebra with bijective antipode and $V \in {}^H_H\mathcal{YD}$. The *Nichols algebra* $\mathcal{B}(V)$ over V is a $\mathbb{N}$-graded Hopf algebra $R = \oplus_{n \geq 0} R(n)$ in ${}^H_H\mathcal{YD}$ such that

$$R(0) = \Bbbk, \quad R(1) = V, \quad R \text{ is generated as an algebra by } R(1), \quad \mathcal{P}(R) = V.$$

The Nichols algebra $\mathcal{B}(V)$ is isomorphic to $T(V)/I(V)$, where $I(V) \subset T(V)$ is the largest $\mathbb{N}$-graded ideal and coideal in ${}^H_H\mathcal{YD}$ such that $I(V) \cap V = 0$. Moreover, the ideal $I(V)$ is the kernel of the quantum symmetrizer associated to the braiding c and $\mathcal{B}(V)$ as a coalgebra and an algebra depends only on (V, c).

Remark 2.1. Suppose W is a subspace of (V, c) with $c(W \otimes W) \subset W \otimes W$, then $\dim \mathcal{B}(W) = \infty$ means $\dim \mathcal{B}(V) = \infty$. In particular, $\dim \mathcal{B}(V) = \infty$ if the braiding c has an eigenvector $v \otimes v \in V^{\otimes 2}$ with eigenvalue 1 (cf. [12]).

Remark 2.2. The Nichols algebra $\mathcal{B}(V)$ is of *diagonal type* if there is a linear basis $\{x_i, \ i \in \mathbb{I}_{1,n}\}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for some $q_{ij} \in \Bbbk$. The matrix $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}_{1,n}}$ is called the matrix of the braiding. The *generalized Dynkin diagram* of the matrix $\mathbf{q}$ is a graph with n vertices, the vertex i labeled with q_{ii}, and an arrow between the vertices i and j only if $q_{ij} q_{ji} \neq 1$, labelled with $q_{ij} q_{ji}$. Finite-dimensional Nichols algebras of diagonal type were classified by Heckenberger [15], with the help of the Weyl groupoid and generalized root systems. Their defining relations were given by Angiono [9, 10]. See [1] for a survey on Nichols algebras of diagonal type.

Let C be a coalgebra, D a subcoalgebra of C and $W \in {}^C\mathcal{M}$. Denote the largest D-subcomodule of W by

$$W(D) = \{w \in W \mid \delta(w) \in D \otimes W\}.$$

Proposition 2.3 ([16, Proposition 8.8])**.** *Let H be a Hopf algebra with bijective antipode, $N \in {}^H_H\mathcal{YD}$ and $W \in {}^{\mathcal{B}(N)\sharp H}_{\mathcal{B}(N)\sharp H}\mathcal{YD}$. Assume that W is a semisimple object in the category of $\mathbb{Z}$-graded left Yetter–Drinfeld modules over $\mathcal{B}(N)\sharp H$. Let $\mathcal{K} = \mathcal{B}(W)$ in ${}^{\mathcal{B}(N)\sharp H}_{\mathcal{B}(N)\sharp H}\mathcal{YD}$, and define $M = W(H)$. Then there is a unique isomorphism*

$$\mathcal{K}\sharp\mathcal{B}(N) \cong \mathcal{B}(M \oplus N)$$

of braided Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on $M \oplus N$.

Now we recall the standard tool, the so-called skew-derivation, for working with Nichols algebras. Let (V, c) be a (rigid) braided vector space of dimension n and $\Delta^{i,m-i} : T^m(V) \to T^i(V) \otimes T^{m-i}(V)$ the $(i, m-i)$-homogeneous component of the comultiplication $\Delta : T(V) \to T(V) \otimes T(V)$ for $m \in \mathbb{N}$ and $k \in \mathbb{I}_{0,m}$. Given $f \in V^*$, the *skew-derivation* $\partial_f \in \operatorname{End} T(V)$ is given by

$$\partial_f(v) = (f \otimes \operatorname{id})\Delta^{1,m-1}(v) : T^m(V) \to T^{m-1}(V), \quad v \in T^m(V), \ m \in \mathbb{N}. \quad (2.4)$$

Let $\{v_i\}_{1 \leq i \leq n}$ and $\{v^i\}_{1 \leq i \leq n}$ be the dual bases of V and V^*. We write $\partial_i := \partial_{v^i}$ for simplicity.

This is useful for seeking the relations of $\mathcal{B}(V)$ due to

$$I^m(V) = \{r \in T^m(V) \mid \partial_{f_1} \partial_{f_2} \cdots \partial_{f_m}(r) = 0, \ \forall f_i \in V^*\}. \quad (2.5)$$

Furthermore, ∂_f can be defined on $\mathcal{B}(V)$ and $\underset{f \in V^*}{\cap} \ker \partial_f = \Bbbk$. For details, see [7, 2].

3. The Hopf algebra $\mathcal{K}_{24,1}$ and the category ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$

We introduce the structures of the Hopf algebra $\mathcal{K}_{24,1}$ and the category ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ of Yetter–Drinfeld modules over $\mathcal{K}_{24,1}$.

3.1. The Hopf algebra $\mathcal{K}_{24,1}$.

Definition 3.1. Let $\mathcal{K}_{24,1}$ be the algebra generated by the elements a, b, c, d, subject to the relations

$$a^6 = 1, \quad b^2 = 0, \quad c^2 = 0, \quad d^6 = 1, \quad a^2 = d^2, \quad ad = da, \quad bc = 0 = cb,$$
$$ab = \xi ba, \quad ac = \xi ca, \quad db = -\xi bd, \quad dc = -\xi cd, \quad bd = ca, \quad ba = cd. \tag{3.1}$$

$\mathcal{K}_{24,1}$ admits a Hopf algebra structure, where the coalgebra structure and antipode are given as follows:

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$
$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = d \otimes d + c \otimes b,$$
$$\epsilon(a) = 1, \quad \epsilon(b) = 0, \quad \epsilon(c) = 0, \quad \epsilon(d) = 1, \tag{3.2}$$
$$S(a) = a^{-1}, \quad S(b) = -\xi^{-1}ca^{-2}, \quad S(c) = \xi^{-1}ba^{-2}, \quad S(d) = d^{-1} = da^{-2}.$$

Remark 3.2.
(1) The set $\{a^i, da^i, ba^i, ca^i, \ i \in \mathbb{I}_{0,5}\}$ is a linear basis of $\mathcal{K}_{24,1}$.
(2) $\mathcal{G}(\mathcal{K}_{24,1}) = \Bbbk\{1, a^3, da^2, da^5\}$, $\mathcal{P}_{1,da^5}(\mathcal{K}_{24,1}) = \Bbbk\{1 - da^5, ca^5\}$ and $\mathcal{P}_{1,g}(\mathcal{K}_{24,1}) = \Bbbk\{1 - g\}$ for $g \in \Bbbk\{a^3, da^2\}$.
(3) Let $\{(a^i)^*, (ba^i)^*, (ca^i)^*, (da^i)^*, \ i \in \mathbb{I}_{0,5}\}$ be the dual basis of $\mathcal{K}_{24,1}$ and set

$$\widetilde{x} = \sum_{i=0}^{5}(ba^i)^* + (ca^i)^*, \quad \widetilde{g} = \sum_{i=0}^{5}\xi^i(a^i)^* + \xi^{i+1}(da^i)^*, \quad \widetilde{h} = \sum_{i=0}^{5}(a^i)^* - (da^i)^*.$$

The multiplication of $\mathcal{K}_{24,1}$ implies that

$$\widetilde{g}^6 = 1, \quad \widetilde{h}^2 = 1, \quad \widetilde{h}\widetilde{g} = \widetilde{g}\widetilde{h}, \quad \widetilde{g}\widetilde{x} = \widetilde{x}\widetilde{g}, \quad \widetilde{h}\widetilde{x} = -\widetilde{x}\widetilde{h},$$
$$\Delta(\widetilde{x}) = \widetilde{x} \otimes \epsilon + \widetilde{g}\widetilde{h} \otimes \widetilde{x}, \quad \Delta(\widetilde{g}) = \widetilde{g} \otimes \widetilde{g}, \quad \Delta(\widetilde{h}) = \widetilde{h} \otimes \widetilde{h}.$$

In particular, $\mathcal{G}(\mathcal{K}_{24,1}^*) \cong \mathbb{Z}_6 \otimes \mathbb{Z}_2$ with generators $\widetilde{g}$ and $\widetilde{h}$.

Let $\mathcal{A}_{24,1}$ be the Hopf algebra generated by elements g, h, x subject to the relations

$$g^6 = 1, \quad h^2 = 1, \quad gh = hg, \quad gx = xg, \quad hx = -xh, \quad x^2 = 1 - g^2,$$

where $g, h \in \mathcal{G}(\mathcal{A}_{24,1})$ and $x \in \mathcal{P}_{1,gh}(\mathcal{A}_{24,1})$. This is a pointed Hopf algebra of dimension 24, which appeared in [13] (see also [18]). We now build the Hopf algebra isomorphism $\mathcal{A}_{24,1} \cong \mathcal{K}_{24,1}^*$.

Lemma 3.3. *Let* $\psi : \mathcal{A}_{24,1} \mapsto \mathcal{K}_{24,1}^*$ *be the algebra map given by*

$$\psi(g) = \widetilde{g}, \quad \psi(h) = \widetilde{h}, \quad \psi(x) = \sqrt{1 - \xi^2}\widetilde{x}.$$

Then ψ *is a Hopf algebra isomorphism.*

Proof. By Remark 3.2 (3), ψ is a bialgebra morphism and $\psi(\mathcal{A}_{24,1})$ contains properly $\mathcal{G}(\mathcal{K}_{24,1}^*)$. Since $\dim \mathcal{G}(\mathcal{K}_{24,1}^*) = 12$, by the Nichols–Zoeller theorem, ψ must be epimorphic. The lemma follows since $\dim \mathcal{A}_{24,1} = \dim \mathcal{K}_{24,1}^* = 24$. $\qquad\square$

Remark 3.4.

(1) The set $\{g^j, g^j h, g^j x, g^j hx, \ j \in \mathbb{I}_{0,5}\}$ is a linear basis of $\mathcal{A}_{24,1}$. We have

$$\psi(g^j) = \sum_{i=0}^{5} \xi^{ij}(a^i)^* + \xi^{ij+j}(da^i)^*, \quad \psi(g^j h) = \sum_{i=0}^{5} \xi^{ij}(a^i)^* - \xi^{ij+j}(da^i)^*,$$

$$\psi(g^j x) = \sqrt{1 - \xi^2} \sum_{i=0}^{5} \xi^{ij+j}(ba^i)^* + \xi^{ij+j}(ca^i)^*,$$

$$\psi(g^j hx) = \sqrt{1 - \xi^2} \sum_{i=0}^{5} \xi^{ij+j}(ba^i)^* - \xi^{ij+j}(ca^i)^*.$$

(2) It is clear that $\operatorname{gr}\mathcal{A}_{24,1} = \mathcal{B}(X) \sharp \Bbbk[\Gamma]$, where $\Gamma \cong \mathbb{Z}_6 \times \mathbb{Z}_2$ with generators g, h and $X := \Bbbk\{x\} \in {}_{\Gamma}^{\Gamma}\mathcal{YD}$ with $g \cdot x = x, h \cdot x = -x$ and $\delta(x) = gh \otimes x$. Furthermore, by [14, Proposition 4.2], $\operatorname{gr}\mathcal{A}_{24,1} \cong (\mathcal{A}_{24,1})_\sigma$, where σ is a Hopf 2-cocycle given by

$$\sigma = \epsilon \otimes \epsilon - \zeta, \quad \text{where } \zeta(x^i g^j h^k, x^m g^n h^l) = (-1)^{mk}\delta_{2,i+m} \qquad (3.3)$$

for $i, k, m, l \in \mathbb{I}_{0,1}, j, n \in \mathbb{I}_{0,5}$. Then by [19, Theorem 2.7], ${}_{\mathcal{A}_{24,1}}^{\mathcal{A}_{24,1}}\mathcal{YD} \cong {}_{\operatorname{gr}\mathcal{A}_{24,1}}^{\operatorname{gr}\mathcal{A}_{24,1}}\mathcal{YD}$ as braided monoidal categories via the tensor functor (G, γ) defined as follows: $G(V) = V$ as vector spaces and coactions, transforming the action $\cdot$ to

$$k \cdot_\sigma v = \sigma(k_{(1)}, v_{(-2)})\sigma^{-1}(k_{(2)}v_{(-1)}S(k_{(4)}), k_{(5)})k_{(3)} \cdot v_{(0)}, \quad k \in \operatorname{gr}\mathcal{A}_{24,1},$$
$$\gamma : G(V) \otimes G(W) \to G(V \otimes W), \quad v \otimes w \mapsto \sigma(v_{(-1)}, w_{(-1)})v_{(0)} \otimes w_{(0)}. \qquad (3.4)$$

Now we give explicitly the structure of the Drinfeld double $\mathcal{D}(\mathcal{K}_{24,1}^{cop})$ of $\mathcal{K}_{24,1}^{cop}$. From now on, we set $\mathcal{D} := \mathcal{D}(\mathcal{K}_{24,1}^{cop}) := \mathcal{A}_{24,1}^{cop\,op} \otimes \mathcal{K}_{24,1}^{cop}$ for convenience. Recall that $\mathcal{D}(H^{cop}) = H^{*\,op\,cop} \otimes H^{cop}$ is a Hopf algebra with the tensor product coalgebra structure and the algebra structure given by $(p \otimes a)(q \otimes b) = p\langle q_{(3)}, a_{(1)}\rangle q_{(2)} \otimes a_{(2)}\langle q_{(1)}, S^{-1}(a_{(3)})\rangle b$.

Proposition 3.5. $\mathcal{D} = \mathcal{A}_{24,1}^{cop\,op} \otimes \mathcal{K}_{24,1}^{cop}$ *is isomorphic to the algebra generated by the elements* g, h, x, a, b, c, d, *subject to the relations in* $\mathcal{K}_{24,1}^{cop}$, *the relations in* $\mathcal{A}_{24,1}^{cop\,op}$ *and*

$$ag = ga, \quad ah = ha, \quad dg = gd, \quad dh = hd,$$
$$bg = gb, \quad bh = -hb, \quad cg = gc, \quad ch = -hc,$$

$$ax - \xi^{-1}xa = -\sqrt{1-\xi^2}\xi^{-1}(c - ghb), \quad dx + \xi^{-1}xd = -\sqrt{1-\xi^2}\xi^{-1}(ghc - b),$$
$$bx - \xi^{-1}xb = -\sqrt{1-\xi^2}\xi^{-1}(d - gha), \quad cx + \xi^{-1}xc = -\sqrt{1-\xi^2}\xi^{-1}(ghd - a).$$

3.2. **The representation of** $\mathcal{D}(\mathcal{K}_{24,1}^{cop})$. We begin this subsection by describing one-dimensional objects in $_\mathcal{D}\mathcal{M}$.

Lemma 3.6. *For* $(i,j,k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5}$, *there is a one-dimensional object* $\Bbbk_{\chi_{i,j,k}} \in {}_\mathcal{D}\mathcal{M}$, *where* $\chi_{i,j,k}$ *is given by*

$$\chi_{i,j,k}(g) = (-1)^i, \quad \chi_{i,j,k}(h) = (-1)^j, \quad \chi_{i,j,k}(x) = 0,$$
$$\chi_{i,j,k}(a) = \xi^k, \quad \chi_{i,j,k}(b) = 0, \quad \chi_{i,j,k}(c) = 0, \quad \chi_{i,j,k}(d) = (-1)^i(-1)^j\xi^k.$$

Any one-dimensional object in $_\mathcal{D}\mathcal{M}$ *is isomorphic to* $\Bbbk_{\chi_{i,j,k}}$ *for some* $(i,j,k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5}$.

Proof. Let $\chi \in \mathcal{G}(\mathcal{D}^*)$. The relations $a^6 = 1 = g^6$, $d^6 = 1 = h^2$, $b^2 = 0 = c^2$ imply that $\chi(a)^6 = 1 = \chi(g)^6$, $\chi(d)^6 = 1 = \chi(h)^2$, $\chi(b) = 0 = \chi(c)$. Then the relations $hx = -xh$ and $x^2 = 1 - g^2$ yield $\chi(x) = 0$ and $\chi(g)^2 = 1$. From the relation $bx - \xi^{-1}xb = -\sqrt{1-\xi^2}\xi^{-1}(d - gha)$, it follows that $\chi(d) = \chi(g)\chi(g)\chi(a)$. Therefore, χ is completely determined by $\chi(a)$, $\chi(g)$ and $\chi(h)$ and then $\chi = \chi_{i,j,k}$ for some $(i,j,k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5}$. Consequently, any one-dimensional $\mathcal{D}$-module is isomorphic to $\Bbbk_{\chi_{i,j,k}}$ for some $(i,j,k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5}$. It is clear that these modules are pairwise non-isomorphic in $_\mathcal{D}\mathcal{M}$. $\qquad\square$

Next, we describe two-dimensional simple objects in $_\mathcal{D}\mathcal{M}$. Let

$$\Lambda := \{(i,j,k,\iota) \mid i,j \in \mathbb{I}_{0,5}, \ k,\iota \in \mathbb{I}_{0,1}, \ j + 3k \neq 3(\iota+1) \ \text{mod} \ 6\}.$$

Clearly, $|\Lambda| = 120$.

Lemma 3.7. *For any* $(i,j,k,\iota) \in \Lambda$, *there is a 2-dimensional simple object* $V_{i,j,k,\iota} \in {}_\mathcal{D}\mathcal{M}$, *whose matrices defining the* $\mathcal{D}$-*action on a fixed basis are given by*

$$[a] = \begin{pmatrix} (-1)^{\iota+1}\xi^i & 0 \\ 0 & (-1)^{\iota+1}\xi^{i-1} \end{pmatrix}, \ [d] = \begin{pmatrix} \xi^i & 0 \\ 0 & -\xi^{i-1} \end{pmatrix}, \ [b] = \begin{pmatrix} 0 & (-1)^\iota \\ 0 & 0 \end{pmatrix},$$

$$[c] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ [g] = \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^j \end{pmatrix}, \ [h] = \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^{k+1} \end{pmatrix},$$

$$[x] = \begin{pmatrix} 0 & (1-\xi^2)^{-\frac{1}{2}}\xi^{1-i}(\xi^j(-1)^k - (-1)^\iota) \\ -\sqrt{1-\xi^2}\xi^{i-1}(\xi^j(-1)^k + (-1)^\iota) & 0 \end{pmatrix}.$$

Any two-dimensional simple object in $_\mathcal{D}\mathcal{M}$ *is isomorphic to* $V_{i,j,k,\iota}$ *for some* $(i,j,k,\iota) \in \Lambda$. *Furthermore,* $V_{i,j,k,\iota} \cong V_{p,q,r,\kappa}$ *if and only if* $(i,j,k,\iota) = (p,q,r,\kappa)$.

Proof. Let V be a simple $\mathcal{D}$-module of dimension 2. As the generators g, h, a, d commute with each other and $g^6 = h^2 = a^6 = d^6 = 1$, we may assume that the matrices defining the action on V are of the form

$$[g] = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, \ [h] = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \ [x] = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \ [a] = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},$$

$$[d] = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad [b] = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \quad [c] = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix},$$

where $a_1^6 = 1 = a_2^6$, $d_1^6 = 1 = d_2^6$, $g_1^6 = 1 = g_2^6$, $h_1^2 = 1 = h_2^2$. The relations $xh = -hx$, $bh = -hb$ and $ch = -hc$ imply that

$$x_1 = 0 = x_4, \quad (h_1 + h_2)x_2 = 0 = (h_1 + h_2)x_3,$$
$$b_1 = 0 = b_4, \quad (h_1 + h_2)b_2 = 0 = (h_1 + h_2)b_3,$$
$$c_1 = 0 = c_4, \quad (h_1 + h_2)c_2 = 0 = (h_1 + h_2)c_3.$$

We claim that $h_1 = -h_2$. Indeed, if $h_1 + h_2 \neq 0$, then $x_2 = 0 = x_3$, $b_2 = 0 = b_3$, $c_2 = 0 = c_3$, which implies that $[b]$, $[c]$, $[x]$ are zero matrices and hence V is not a simple $\mathcal{D}$-module.

Now we claim that $g_1 = g_2$. Indeed, if $g_1 \neq g_2$, then the relations $gx = xg$, $bg = gb$ and $cg = gc$ yield $(g_1 - g_2)x_2 = 0 = (g_1 - g_2)x_3$, $(g_1 - g_2)b_2 = 0 = (g_1 - g_2)b_3$, $(g_1 - g_2)c_2 = 0 = (g_1 - g_2)c_3$, which implies that $[b]$, $[c]$, $[x]$ are zero matrices and hence V is not simple.

From the relations $b^2 = 0 = c^2$ and $bc = 0 = cb$, we have that

$$b_2b_3 = 0 = c_2c_3, \quad b_2c_3 = 0 = b_3c_2, \quad c_2b_3 = 0 = c_3b_2.$$

By permuting the elements of the basis, we may assume that $b_3 = 0 = c_3$. The relations $ax - \xi^{-1}xa = -\sqrt{1 - \xi^2}\xi^{-1}(c - ghb)$ and $dx + \xi^{-1}xd = -\sqrt{1 - \xi^2}\xi^{-1}(ghc - b)$ imply that

$$\begin{aligned} a_1x_2 - \xi^{-1}a_2x_2 &= -\sqrt{1 - \xi^2}\xi^{-1}(c_2 - g_1h_1b_2), \\ a_2x_3 - \xi^{-1}a_1x_3 &= -\sqrt{1 - \xi^2}\xi^{-1}(c_3 - g_2h_2b_3), \\ d_1x_2 + \xi^{-1}d_2x_2 &= -\sqrt{1 - \xi^2}\xi^{-1}(g_1h_1c_2 - b_2), \\ d_2x_3 + \xi^{-1}d_1x_3 &= -\sqrt{1 - \xi^2}\xi^{-1}(g_2h_2c_3 - b_3). \end{aligned} \tag{3.5}$$

We claim that $b_2 \neq 0$ or $c_2 \neq 0$. Suppose that $b_2 = 0 = c_2$. Then $x_2x_3 \neq 0$ and by equations (3.5) we have

$$a_1x_2 - \xi^{-1}a_2x_2 = 0, \quad a_2x_3 - \xi^{-1}a_1x_3 = 0,$$
$$d_1x_2 + \xi^{-1}d_2x_2 = 0, \quad d_2x_3 + \xi^{-1}d_1x_3 = 0.$$

Hence $a_1 - \xi^{-1}a_2 = 0$ and $a_2 - \xi^{-1}a_1 = 0$, which implies that $a_1 = 0 = a_2$, a contradiction. Thus the claim follows. We may also assume that $c_2 = 1$.

The relations $ab = \xi ba$, $ac = \xi ca$, $db = -\xi bd$ and $dc = -\xi cd$ imply $a_1 = \xi a_2$, $d_1 = -\xi d_2$. The relations $bd = ca$ and $ba = cd$ yield $b_2^2 = 1$ and $a_2 = b_2d_2$. From the relations $bx - \xi^{-1}xb = -\sqrt{1 - \xi^2}\xi^{-1}(d - gha)$ and $cx + \xi^{-1}xc = -\sqrt{1 - \xi^2}\xi^{-1}(ghd - a)$, it follows that

$$b_2x_3 = -\sqrt{1 - \xi^2}\xi^{-1}(d_1 - g_1h_1a_1), \qquad b_2x_3 = \sqrt{1 - \xi^2}(d_2 - g_2h_2a_2),$$
$$x_3 = -\sqrt{1 - \xi^2}\xi^{-1}(g_1h_1d_1 - a_1), \qquad x_3 = -\sqrt{1 - \xi^2}(g_2h_2d_2 - a_2),$$

which implies that $x_3 = -\sqrt{1 - \xi^2}\xi^{-1}(b_2 + g_1h_1)d_1$. By equations (3.5), $x_2 = (1 - \xi^2)^{-\frac{1}{2}}\xi d_1^{-1}(g_1h_1 - b_2)$.

The relations $x^2 = 1 - g^2$ and $a^2 = d^2$ imply $x_2 x_3 = 1 - g_1^2$ and $a_1^2 - d_1^2 = 0 = a_2^2 - d_2^2$. Indeed, since $a_2 = b_2 d_2$, $a_1 = \xi a_2$ and $d_2 = -\xi^{-1} d_1$, it follows that $a_1 = -b_2 d_1$ and hence $a_1^2 - d_1^2 = 0 = a_2^2 - d_2^2$. Similarly, the relation $x_2 x_3 = 1 - g_1^2$ holds.

Consequently, we have

$$[a] = \begin{pmatrix} -\lambda_4 \lambda_1 & 0 \\ 0 & -\xi^{-1}\lambda_4 \lambda_1 \end{pmatrix}, \quad [d] = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\xi^{-1}\lambda_1 \end{pmatrix}, \quad [b] = \begin{pmatrix} 0 & \lambda_4 \\ 0 & 0 \end{pmatrix},$$

$$[c] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad [g] = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad [h] = \begin{pmatrix} \lambda_3 & 0 \\ 0 & -\lambda_3 \end{pmatrix},$$

$$[x] = \begin{pmatrix} 0 & (1-\xi^2)^{-\frac{1}{2}}\xi\lambda_1^{-1}(\lambda_2\lambda_3 - \lambda_4) \\ -\sqrt{1-\xi^2}\xi^{-1}\lambda_1(\lambda_2\lambda_3 + \lambda_4) & 0 \end{pmatrix},$$

where $\lambda_1^6 = 1$, $\lambda_2^6 = 1$, $\lambda_3^2 = 1$, $\lambda_4^2 = 1$ and $\lambda_2\lambda_3 + \lambda_4 \neq 1$. Set $\lambda_1 = \xi^i$, $\lambda_2 = \xi^j$, $\lambda_3 = (-1)^k$ and $\lambda_4 = (-1)^\iota$ for some $(i, j, k, \iota) \in \Lambda$. Then $V \cong V_{i,j,k,\iota}$.

Now we claim that $V_{i,j,k,\iota} \cong V_{p,q,r,\kappa}$ if and only if $(i, j, k, \iota) = (p, q, r, \kappa)$ in Λ. Let $\Phi : V_{i,j,k,\iota} \mapsto V_{p,q,r,\kappa}$ be an isomorphism of $\mathcal{D}$-modules and $[\Phi] = (p_{i,j})_{i,j=1,2}$ the matrix of Φ in the given basis. Then $[c][\Psi] = [\Psi][c]$ and $[a][\Psi] = [\Psi][a]$, which implies that $p_{21} = 0$, $p_{11} = p_{22}$ and $(\xi^p - \xi^i)p_{11} = 0$, $(\xi^p - \xi^{i+1})p_{12} = 0$. Consequently, $\xi^i = \xi^p$ and then $p_{12} = 0$ and $[\Phi] = p_{11}I$, where I is the identity matrix. Similarly, we have $\xi^j = \xi^q$, $k = r$, $\iota = \kappa$. The claim follows. $\qquad \square$

Finally, we describe all simple objects in $_{\mathcal{D}}\mathcal{M}$ up to isomorphism.

Theorem 3.8. *There exist* 144 *simple objects in* $_{\mathcal{D}}\mathcal{M}$ *up to isomorphism, among which* 24 *one-dimensional objects are given in Lemma 3.6 and* 120 *two-dimensional simple objects are given in Lemma 3.7.*

Proof. By [21, Proposition 10.6.16], [6, Proposition 2.2.1] and Remark 3.4, $_{\mathcal{D}}\mathcal{M} \cong {}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD} \cong {}^{\mathcal{A}_{24,1}}_{\mathcal{A}_{24,1}}\mathcal{YD} \cong {}^{\text{gr}\,\mathcal{A}_{24,1}}_{\text{gr}\,\mathcal{A}_{24,1}}\mathcal{YD} \cong {}_{\mathcal{D}(\text{gr}\,\mathcal{A}_{24,1})}\mathcal{M}$. On the other hand, $\mathcal{D}(\text{gr}\,\mathcal{A})$ is isomorphic to a lifting of a quantum plane, which is generated by the elements $g_1, g_2, g_3, g_4, x_1, x_2$, subject to the relations

$$g_i g_j = g_j g_i, \quad g_{1+k}^6 = g_{2+k}^2 = 1, \quad x_k^2 = 0, \quad i, j \in \mathbb{I}_{0,5}, k \in \mathbb{I}_{0,1},$$

$$x_1 x_2 + x_2 x_1 = g_1 g_2 g_4 - 1, \quad g_i x_1 = \chi(g_i) x_1 g_i, \quad g_i x_2 = \chi^{-1}(g_i) x_2 g_i,$$

where $\chi(g_1) = 1, \chi(g_2) = \chi(g_4) = -1$, $\chi(g_3) = \xi$, $\Delta(g_i) = g_i \otimes g_i$, $\Delta(x_1) = x_1 \otimes 1 + g_1 g_2 \otimes x_1$ and $\Delta(x_2) = x_2 \otimes 1 + g_4 \otimes x_2$. Then by [3, Theorem 3.5], $\dim V < 3$ for any simple $\mathcal{D}(\text{gr}\,\mathcal{A}_{24,1})$-module V. Consequently, the proposition follows. $\qquad \square$

3.3. **The category** ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$. Using the equivalence ${}_{\mathcal{D}(\mathcal{K}_{24,1}^{cop})}\mathcal{M} \cong {}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$, we describe explicitly simple objects in ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$. Using the equivalences ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD} \cong {}^{\mathcal{A}_{24,1}}_{\mathcal{A}_{24,1}}\mathcal{YD} \cong {}^{\text{gr}\,\mathcal{A}_{24,1}}_{\text{gr}\,\mathcal{A}_{24,1}}\mathcal{YD}$, we transport the information from the category ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ to ${}^{\text{gr}\,\mathcal{A}_{24,1}}_{\text{gr}\,\mathcal{A}_{24,1}}\mathcal{YD}$.

Lemma 3.9. *Let* $\Bbbk_{\chi_{i,j,k}} = \Bbbk\{v\} \in {}_{\mathcal{D}}\mathcal{M}$ *for* $(i,j,k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5}$. *Then* $\Bbbk_{\chi_{i,j,k}} \in {}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ *with*

$$a \cdot v = \xi^k v, \quad b \cdot v = 0, \quad c \cdot v = 0, \quad d \cdot v = (-1)^{i+j}\xi^k v; \quad \delta(v) = d^j a^{3i-j} \otimes v.$$

Proof. The $\mathcal{K}_{24,1}$-action is given by the restriction of the character of $\mathcal{D}$. The coaction is of the form $\delta(v) = t \otimes v$, where $t \in G(\mathcal{K}_{24,1}) = \{1, a^3, da^2, da^{-1}\}$ such that $\langle g, t \rangle v = (-1)^i v$ and $\langle h, t \rangle v = (-1)^j v$. Therefore, $\delta(v) = d^j a^{3i-j} \otimes v$. $\qquad\square$

Corollary 3.10. *Let* $\Bbbk_{\chi_{i,j,k}} = \Bbbk\{v\} \in {}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$. *Then* $\Bbbk_{\chi_{i,j,k}} \in {}^{\operatorname{gr}\mathcal{A}_{24,1}}_{\operatorname{gr}\mathcal{A}_{24,1}}\mathcal{YD}$ *with*

$$g \cdot v = (-1)^i v, \quad h \cdot v = (-1)^j v, \quad x \cdot v = 0, \quad \delta(v) = g^{-k}h^{i+j} \otimes v.$$

Proof. Since $\mathcal{K}_{24,1} \cong \mathcal{A}_{24,1}^*$, by [6, Proposition 2.2.1], we have the equivalence ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD} \cong {}^{\mathcal{A}_{24,1}}_{\mathcal{A}_{24,1}}\mathcal{YD}$ via the functor (F, η) defined by (2.2). More precisely, by the formula (2.2), Lemma 3.9 and Remark 3.4 (1), we have $F(\Bbbk_{\chi_{i,j,k}}) = \Bbbk_{\chi_{i,j,k}} \in {}^{\mathcal{A}_{24,1}}_{\mathcal{A}_{24,1}}\mathcal{YD}$ with

$$g \cdot v = \langle g, S(d^j a^{3i-j}) \rangle v = \langle g, d^j a^{-j-3i} \rangle v = (-1)^i v,$$

$$h \cdot v = \langle h, d^j a^{-j-3i} \rangle v = (-1)^j v, \quad x \cdot v = \langle x, d^j a^{-j-3i} \rangle v = 0,$$

$$\delta(v) = \sum_i S^{-1}((a^i)^*) \otimes a^i \cdot v + S^{-1}((da^i)^*) \otimes da^i \cdot v$$

$$= S^{-1}(g^k h^{i+j}) \otimes v = g^{-k}h^{i+j} \otimes v.$$

Then by Remark 3.4 (2), we have ${}^{\mathcal{A}_{24,1}}_{\mathcal{A}_{24,1}}\mathcal{YD} \cong {}^{\operatorname{gr}\mathcal{A}_{24,1}}_{\operatorname{gr}\mathcal{A}_{24,1}}\mathcal{YD}$ via the functor (G, γ) defined by the formulae (3.3)–(3.4), and then $GF(\Bbbk_{\chi_{i,j,k}}) = \Bbbk_{\chi_{i,j,k}} \in {}^{\operatorname{gr}\mathcal{A}_{24,1}}_{\operatorname{gr}\mathcal{A}_{24,1}}\mathcal{YD}$ with the module structure given by

$$h \cdot_\sigma v = \sigma(h_{(1)}, v_{(-2)})\sigma^{-1}(h_{(2)}v_{(-1)}S(h_{(4)}), h_{(5)})h_{(3)} \cdot v_{(0)}$$

$$= \sigma(h, g^{-k}h^{i+j})\sigma^{-1}(g^{-k}h^{i+j}, h)h \cdot v = h \cdot v = (-1)^j v,$$

$$g \cdot_\sigma v = \sigma(g, g^{-k}h^{i+j})\sigma^{-1}(gg^{-k}h^{i+j}S(g), g)g \cdot v$$

$$= \sigma(g, g^{-k}h^{i+j})\sigma^{-1}(g^{-k}h^{i+j}, g)g \cdot v = g \cdot v = (-1)^i v,$$

$$x \cdot_\sigma v = \sigma(x_{(1)}, v_{(-2)})\sigma^{-1}(x_{(2)}v_{(-1)}S(x_{(4)}), x_{(5)})x_{(3)} \cdot v_{(0)}$$

$$= \sigma(x, g^{-k}h^{i+j})\sigma^{-1}(g^{-k}h^{i+j}, 1)1 \cdot v + \sigma(gh, g^{-k}h^{i+j})\sigma^{-1}(xg^{-k}h^{i+j}, 1)1 \cdot v$$

$$\quad + \sigma(gh, g^{-k}h^{i+j})\sigma^{-1}(ghg^{-k}h^{i+j}, 1)x \cdot v$$

$$\quad + \sigma(gh, g^{-k}h^{i+j})\sigma^{-1}(ghg^{-k}h^{i+j}S(x), 1)gh \cdot v$$

$$\quad + \sigma(gh, g^{-k}h^{i+j})\sigma^{-1}(ghg^{-k}h^{i+j}S(gh), x)gh \cdot v$$

$$= \sigma(x, g^{-k}h^{i+j})\sigma^{-1}(g^{-k}h^{i+j}, 1)v + \sigma(gh, g^{-k}h^{i+j})\sigma^{-1}(xg^{-k}h^{i+j}, 1)v$$

$$\quad + \sigma(gh, g^{-k}h^{i+j})\sigma^{-1}(g^{1-k}h^{i+j+1}, 1)x \cdot v$$

$$\quad + \sigma(gh, g^{-k}h^{i+j})\sigma^{-1}(g^{1-k}h^{i+j+1}S(x), 1)gh \cdot v$$

$$+\, \sigma(gh, g^{-k}h^{i+j})\sigma^{-1}(g^{-k}h^{i+j}, x)gh \cdot v$$
$$=\, x \cdot v = 0. \qquad \qquad \square$$

Lemma 3.11. *Let $V_{i,j,k,\iota} = \Bbbk\{v_1, v_2\} \in {}_{\mathcal{D}}\mathcal{M}$ for any $(i, j, k, \iota) \in \Lambda$. Then $V_{i,j,k,\iota} \in {}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ with the module structure given by*

$$a \cdot v_1 = (-1)^{\iota+1}\xi^i v_1, \quad b \cdot v_1 = 0, \quad c \cdot v_1 = 0, \quad d \cdot v_1 = \xi^i v_1,$$
$$a \cdot v_2 = (-1)^{\iota+1}\xi^{i-1}v_2, \quad b \cdot v_2 = (-1)^{\iota}v_1, \quad c \cdot v_2 = v_1, \quad d \cdot v_2 = -\xi^{i-1}v_2;$$

and the following comodule structure:

(1) *for $k = 0$,*

$$\delta(v_1) = a^j \otimes v_1 + (1 - \xi^2)^{-\frac{1}{2}}x_2 ba^{j-1} \otimes v_2,$$
$$\delta(v_2) = da^{j-1} \otimes v_2 + (1 - \xi^2)^{-\frac{1}{2}}x_1 ca^{j-1} \otimes v_1;$$

(2) *for $k = 1$,*

$$\delta(v_1) = da^{j-1} \otimes v_1 + (1 - \xi^2)^{-\frac{1}{2}}x_2 ca^{j-1} \otimes v_2,$$
$$\delta(v_2) = a^j \otimes v_2 + (1 - \xi^2)^{-\frac{1}{2}}x_1 ba^{j-1} \otimes v_1,$$

where $x_1 = (1 - \xi^2)^{-\frac{1}{2}}\xi^{1-i}(\xi^j(-1)^k - (-1)^{\iota})$ and $x_2 = -\sqrt{1 - \xi^2}\xi^{i-1}(\xi^j(-1)^k + (-1)^{\iota})$.

Proof. Let $\{h_i\}_{i \in \mathbb{I}_{1,24}}$ and $\{h^i\}_{i \in \mathbb{I}_{1,24}}$ be the dual bases of $\mathcal{K}_{24,1}$ and $\mathcal{K}^*_{24,1}$. The $\mathcal{K}_{24,1}$-action is given by the restriction of the $\mathcal{D}$-action and the $\mathcal{K}_{24,1}$-comodule structure is given by $\delta(v) = \sum_{i=1}^{24} h_i \otimes h^i \cdot v$ for any $v \in V_{i,j,k}$. By Lemma 3.3 and Remark 3.4, we have

$$(g^l)^* = \frac{1}{12}\sum_{i=0}^{5}\xi^{-il}a^i + \xi^{-(i+1)l}da^i, \quad (g^l h)^* = \frac{1}{12}\sum_{i=0}^{5}\xi^{-il}a^i - \xi^{-(i+1)l}da^i,$$

$$(g^l x)^* = \frac{1}{12\sqrt{1-\xi^2}}\sum_{i=0}^{5}\xi^{-(i+1)l}ba^i + \xi^{-(i+1)l}ca^i,$$

$$(g^l hx)^* = \frac{1}{12\sqrt{1-\xi^2}}\sum_{i=0}^{5}\xi^{-(i+1)l}ba^i - \xi^{-(i+1)l}ca^i.$$

Then the lemma follows by direct computation. $\qquad \qquad \square$

Remark 3.12. $V^*_{i,j,k,\iota} \cong V_{-i+4,-j,k+1,\iota+1}$ for all $(i, j, k, \iota) \in \Lambda$.

In this section, for $(i, j, k, \iota) \in \Lambda$, set

$$x_1 := (1 - \xi^2)^{-\frac{1}{2}}\xi^{1-i}(\xi^j(-1)^k - (-1)^{\iota}),$$
$$x_2 := -\sqrt{1 - \xi^2}\xi^{i-1}(\xi^j(-1)^k + (-1)^{\iota}).$$

Corollary 3.13. *Let* $V_{i,j,k,\iota} = \Bbbk\{v_1, v_2\} \in {}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ *for* $(i,j,k,\iota) \in \Lambda$. *Then* $V_{i,j,k,\iota} \in {}^{\mathrm{gr}\,\mathcal{A}_{24,1}}_{\mathrm{gr}\,\mathcal{A}_{24,1}}\mathcal{YD}$ *with the module structure given by*

$$g \cdot v_1 = \xi^{-j}v_1, \quad h \cdot v_1 = (-1)^k v_1, \quad x \cdot v_1 = (-1)^{k+1}x_2\xi^{-j}v_2,$$
$$g \cdot v_2 = \xi^{-j}v_2, \quad h \cdot v_2 = (-1)^{k+1}v_2, \quad x \cdot v_2 = 0;$$

and the comodule structure given as follows:

(1) *for* $\iota = 0$,

$$\delta(v_1) = g^{-3-i}h \otimes v_1, \quad \delta(v_2) = g^{-2-i} \otimes v_2 + \xi^{1-i}(1-\xi^2)^{-\frac{1}{2}}g^{-3-i}hx \otimes v_1;$$

(2) *for* $\iota = 1$,

$$\delta(v_1) = g^{-i} \otimes v_1, \quad \delta(v_2) = g^{-i+1}h \otimes v_2 - \xi^{1-i}(1-\xi^2)^{-\frac{1}{2}}g^{-i}x \otimes v_1.$$

Proof. Similar to the proof of Corollary 3.10, using the equivalence ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD} \cong {}^{\mathcal{A}_{24,1}}_{\mathcal{A}_{24,1}}\mathcal{YD}$ via the functor (F, η) defined by (2.2), by Lemma 3.11, we have $F(V_{i,j}) \in {}^{\mathcal{A}_{24,1}}_{\mathcal{A}_{24,1}}\mathcal{YD}$ with the comodule structure given in the corollary and the module structure given by

$$g \cdot v_1 = \xi^{-j}v_1, \quad h \cdot v_1 = (-1)^k v_1, \quad x \cdot v_1 = (-1)^{k+1}x_2\xi^{-j}v_2,$$
$$g \cdot v_2 = \xi^{-j}v_2, \quad h \cdot v_2 = (-1)^{k+1}v_2, \quad x \cdot v_2 = (-1)^{k+1}x_1\xi^{-j}v_1;$$

Then using the equivalence ${}^{\mathcal{A}_{24,1}}_{\mathcal{A}_{24,1}}\mathcal{YD} \cong {}^{\mathrm{gr}\,\mathcal{A}_{24,1}}_{\mathrm{gr}\,\mathcal{A}_{24,1}}\mathcal{YD}$ via the functor (G, γ) defined by the formulae (3.3)–(3.4), by a direct and tedious computation we get $GF(V_{i,j,k,\iota}) = V_{i,j,k,\iota} \in {}^{\mathrm{gr}\,\mathcal{A}_{24,1}}_{\mathrm{gr}\,\mathcal{A}_{24,1}}\mathcal{YD}$ with the structure given in the corollary. $\qquad\square$

Finally, we describe the braiding of simple objects in ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$.

Lemma 3.14. *Let* $\Bbbk_{\chi_{i,j,k}} = \Bbbk\{v\} \in {}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ *for* $(i,j,k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5}$. *Then the braiding of* $\Bbbk_{\chi_{i,j,k}}$ *is given by* $c(v \otimes v) = (-)^{i(j+k)+j}v \otimes v$.

Proof. By the formula (2.1) of the braiding in ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ and Lemma 3.9, we have

$$c(v \otimes v) = d^j a^{3i-j} \cdot v \otimes v = \xi^{(3i-j)k}(-1)^{(i+j)j}\xi^{jk}v \otimes v = (-1)^{ik+ij+j^2}v \otimes v.$$

$$\square$$

For $(i,j,k,\iota) \in \Lambda$, we set

$$a_{12} := [(-1)^{(\iota+1)j}\xi^{ij} + (-1)^{(j-1)\iota}\xi^{(i+2)j}],$$
$$a_{11} := \frac{1}{1-\xi^2}(-1)^{\iota(j-1)}\xi^{(j-2)i+2j-1}[\xi^j - (-1)^\iota],$$
$$b_{12} := [(-1)^{(j-1)(\iota+1)}\xi^{ij} + (-1)^{j\iota}\xi^{(i+2)j}],$$
$$b_{11} := \frac{1}{1-\xi^2}(-1)^{j\iota}\xi^{2j+2+ij+4i}[\xi^j + (-1)^\iota].$$

Lemma 3.15. *Let $V_{i,j,k,\iota} = \Bbbk\{v_1, v_2\} \in {}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ for $(i, j, k, \iota) \in \Lambda$. Then the braiding of $V_{i,j,k,\iota}$ is given as follows:*

(1) *If $k = 0$, then*

$$c\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix}\right)$$
$$= \begin{bmatrix} (-1)^{(\iota+1)j}\xi^{ij}v_1 \otimes v_1 & (-1)^{j\iota}\xi^{(i+2)j}v_2 \otimes v_1 + a_{12}v_1 \otimes v_2 \\ (-1)^{(\iota+1)(j-1)}\xi^{ij}v_1 \otimes v_2 & (-1)^{(j-1)\iota}\xi^{(i+2)j}v_2 \otimes v_2 + a_{11}v_1 \otimes v_1 \end{bmatrix}.$$

(2) *If $k = 1$, then*

$$c\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix}\right)$$
$$= \begin{bmatrix} (-1)^{(j-1)(\iota+1)}\xi^{ij}v_1 \otimes v_1 & (-1)^{(j-1)\iota}\xi^{(i+2)j}v_2 \otimes v_1 + b_{12}v_1 \otimes v_2 \\ (-1)^{(\iota+1)j}\xi^{ij}v_1 \otimes v_2 & (-1)^{j\iota}\xi^{(i+2)j}v_2 \otimes v_2 + b_{11}v_1 \otimes v_1 \end{bmatrix}.$$

Proof. If $k = 0$, then by Lemma 3.11 and the formula (2.1), we have

$$c(v_1 \otimes v_1) = a^j \cdot v_1 \otimes v_1 + (1 - \xi^2)^{-\frac{1}{2}} x_2 ba^{j-1} \cdot v_1 \otimes v_2 = (-1)^{(\iota+1)j}\xi^{ij}v_1 \otimes v_1,$$

$$c(v_1 \otimes v_2) = (-1)^{j\iota}\xi^{(i+2)j}v_2 \otimes v_1 + a_{12}v_1 \otimes v_2,$$

$$c(v_2 \otimes v_1) = da^{j-1} \cdot v_1 \otimes v_2 + (1 - \xi^2)^{-\frac{1}{2}} x_1 ca^{j-1} \cdot v_1 \otimes v_1$$
$$= (-1)^{(\iota+1)(j-1)}\xi^{ij}v_1 \otimes v_2,$$

$$c(v_2 \otimes v_2) = (-1)^{(j-1)\iota}\xi^{(i+2)j}v_2 \otimes v_2 + a_{11}v_1 \otimes v_1.$$

Similarly, we can obtain the matrix of the braiding associated with $V_{i,j,1,\iota}$. □

4. Nichols algebras in ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$

We determine all finite-dimensional Nichols algebras over simple objects in ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$. Let

$$\Lambda^0 := \{(i, j, k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5} \mid i(j + k) + j \equiv 1 \mod 2\}.$$

We shall show that finite-dimensional Nichols algebras over one-dimensional objects in ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ are parametrized by Λ^0.

Proposition 4.1. *Let $\Bbbk_{\chi_{i,j,k}} = \Bbbk\{v\}$ for $(i, j, k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5}$. Then*

$$\mathcal{B}(\Bbbk_{\chi_{i,j,k}}) = \begin{cases} \bigwedge \Bbbk_{\chi_{i,j,k}}, & (i, j, k) \in \Lambda^0; \\ \Bbbk[v], & otherwise. \end{cases}$$

Proof. It follows directly from Lemma 3.14 and Remark 2.1. □

Let $V_{i,j,k,\iota} = \Bbbk\{v_1, v_2\} \in {}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ for $(i, j, k, \iota) \in \Lambda$. Then using the equivalence ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD} \cong {}^{\operatorname{gr}\mathcal{A}_{24,1}}_{\operatorname{gr}\mathcal{A}_{24,1}}\mathcal{YD}$, we have $V_{i,j,k,\iota} \in {}^{\operatorname{gr}\mathcal{A}_{24,1}}_{\operatorname{gr}\mathcal{A}_{24,1}}\mathcal{YD}$, that is, $V_{i,j,k,\iota} \in {}^{\mathcal{B}(X)\sharp\Bbbk[\Gamma]}_{\mathcal{B}(X)\sharp\Bbbk[\Gamma]}\mathcal{YD}$ with the structure given by Corollary 3.13.

By Proposition 2.3 (see also [2, Theorem 1.1]), we have $\mathcal{B}(V_{i,j,k,\iota})\sharp\mathcal{B}(X) \cong \mathcal{B}(X \oplus X_{i,j,k,\iota})$ in $_{\Gamma}^{\Gamma}\mathcal{YD}$ which is the identity on $X \oplus X_{i,j,k,\iota}$, where $X_{i,j,k,\iota} = \Bbbk\{v_1\} \in {}_{\Gamma}^{\Gamma}\mathcal{YD}$ with

$$g \cdot v_1 = \xi^{-j}v_1, \quad h \cdot v_1 = (-1)^k v_1, \quad \delta(v_1) = g^{-3(\iota-1)-i}h^{\iota-1} \otimes v_1.$$

It is clear that $\mathcal{B}(X \oplus X_{i,j,k,\iota})$ is of diagonal type with the generalized Dynkin diagram given by

$$\underset{x}{\overset{-1}{\circ}} \xrightarrow{\quad (-1)^{k+\iota-1}\xi^{-j} \quad} \underset{v_1}{\overset{(-)^{(k+j)(\ell-1)}\xi^{ij}}{\circ}} \ .$$

Now we show that infinite-dimensional Nichols algebras over two-dimensional simple objects in $_{\mathcal{K}_{24,1}}^{\mathcal{K}_{24,1}}\mathcal{YD}$ are parametrized by the following subsets:

$$\Lambda^{0*} := \{(i,j,k,\iota) \in \Lambda \mid 3(k+j)(\iota-1) + ij \text{ or }$$
$$3(k+\iota) + 3(k+j)(\iota-1) + (i-1)j \equiv 0 \mod 6\};$$
$$\Lambda^{0**} := \{(i,j,k,\iota) \in \Lambda \mid i = 5, j \in \{1,5\}, k+\iota+1 \equiv 0 \mod 2\},$$

and present finite-dimensional ones by generators and relations.

Lemma 4.2. *Let* $(i,j,k,\iota) \in \Lambda^{0*} \cup \Lambda^{0**}$*. Then* $\dim \mathcal{B}(V_{i,j,k,\iota}) = \infty$*.*

Proof. It suffices to show that $\dim \mathcal{B}(X \oplus X_{i,j,k,\iota}) = \infty$. If $(i,j,k,\iota) \in \Lambda^{0*}$, then the Dynkin diagram is

$$\underset{x}{\overset{-1}{\circ}} \xrightarrow{\quad (-1)^{k+\iota-1}\xi^{-j} \quad} \underset{v_1}{\overset{(-)^{(k+j)(\iota-1)}\xi^{ij}}{\circ}} \ .$$

If $(i,j,k,\iota) \in \Lambda^{0**}$, then the Dynkin diagram of $X \oplus X_{i,j,k,\iota}$ is given by

$$\underset{x}{\overset{-1}{\circ}} \xrightarrow{\quad \xi^{-j} \quad} \underset{v_1}{\overset{\xi^{-j}}{\circ}} \ .$$

These diagrams do not appear in [15, Table 1], that is, they have infinite root systems. Therefore, $\dim \mathcal{B}(X \oplus X_{i,j,k,\iota}) = \infty$. $\qquad\square$

Proposition 4.3. *Let* $\Lambda^1 = \{(i,j,k,\iota) \in \Lambda \mid 3(k+\iota-1) - j \equiv \pm 1 \mod 6,$ $3(k+j)(\iota-1)+ij \equiv 3\pm 1 \mod 6\}$*. The Nichols algebra* $\mathcal{B}(V_{i,j,k,\iota})$ *for* $(i,j,k,\iota) \in \Lambda^1$ *is generated by* v_1, v_2*, subject to the relations*

$$v_1^3 = 0, \quad \xi^{2j}v_1^2v_2 + (-1)^{\iota}\xi^{-2j}v_1v_2v_1 + v_2v_1^2 = 0, \tag{4.1}$$

$$v_1v_2^2 + (-1)^{\iota}v_2v_1v_2 + v_2^2v_1 = 0, \tag{4.2}$$

$$\frac{(-1)^{\iota}(1-\xi^{2j})\xi^4}{1+\xi^5}v_1^2v_2 + \frac{(1-\xi^{-2j})\xi^4}{1+\xi^5}v_1v_2v_1 + v_2^3 = 0. \tag{4.3}$$

Proof. By Lemma 3.15, the braiding of $V_{i,j,k,\iota}$ is given by

$$c\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix}\right) = \begin{bmatrix} \xi^{2j}v_1 \otimes v_1 & (-1)^{\iota}\xi^{-2j}v_2 \otimes v_1 - v_1 \otimes v_2 \\ (-1)^{\iota+1}\xi^{2j}v_1 \otimes v_2 & \xi^{-2j}v_2 \otimes v_2 + Av_1 \otimes v_1 \end{bmatrix},$$

where $A = \frac{[(-1)^\iota \xi^{-2j} + (-1)^{\iota+1}\xi^{2j}]\xi^4}{1+\xi^5}$. Since $c(v_1 \otimes v_1) = \xi^{2j}v_1 \otimes v_1$, it follows by the formulae (2.3) and (2.4) that

$$\partial_1(v_1^3) = (v^1 \otimes \mathrm{id}^{\otimes 2})\Delta^{1,2}(v_1^3) = (1 + \xi^{2j} + \xi^{4j})v_1^2 = 0,$$
$$\partial_2(v_1^3) = (v^2 \otimes \mathrm{id}^{\otimes 2})\Delta^{1,2}(v_1^3) = 0.$$

Similarly, we obtain

$$\partial_1(v_1^2 v_2) = v_1 v_2 + (-1)^{\iota+1}\xi^{-2j}v_2 v_1, \quad \partial_2(v_1^2 v_2) = \xi^{2j}v_1^2;$$
$$\partial_1(v_1 v_2 v_1) = (-1)^{\iota+1}\xi^{-2j}v_1 v_2, \quad \partial_2(v_1 v_2 v_1) = (-1)^\iota \xi^{-2j}v_1^2;$$
$$\partial_1(v_2 v_1^2) = (-1)^\iota v_2 v_1, \quad \partial_2(v_2 v_1^2) = v_1^2;$$
$$\partial_1(v_2^3) = (1+\xi^{4j})Av_1 v_2 + (-1)^{\iota+1}\xi^{2j}Av_2 v_1, \quad \partial_2(v_2^3) = 0;$$
$$\partial_1(v_1 v_2^2) = A\xi^{2j}v_1^2 - \xi^{4j}v_2^2, \quad \partial_2(v_1 v_2^2) = (-)^{\iota+1}v_1 v_2;$$
$$\partial_1(v_2^2 v_1) = Av_1^2 + \xi^{4j}v_2^2, \quad \partial_2(v_2^2 v_1) = -\xi^{2j}v_2 v_1;$$
$$\partial_1(v_2 v_1 v_2) = (-1)^\iota \xi^{-2j}Av_1^2, \quad \partial_2(v_2 v_1 v_2) = v_1 v_2 + (-1)^\iota \xi^{2j}v_2 v_1.$$

It is easy to verify that $\partial_1(r) = 0 = \partial_2(r)$ for any relation r given in (4.1)–(4.3). Then by (2.5), the quotient $\mathfrak{B}$ of $T(V_{i,j,k,\iota})$ by the relations (4.1)–(4.3) projects onto $\mathcal{B}(V_{i,j,k,\iota})$. We claim that $I = \Bbbk\{v_1^i(v_2 v_1)^j v_2^k, \ i,k \in \mathbb{I}_{0,2}, j \in \mathbb{I}_{0,1}\}$ is a left ideal. Indeed, from (4.1)–(4.3) and $(v_2 v_1)^2 = (-1)^\iota v_1^2 v_2^2$, we have $v_1 I, v_2 I \subset I$. Hence I linearly generates $\mathfrak{B}$ since $1 \in I$.

We claim that $\dim \mathcal{B}(V_{i,j,k,\iota}) \geq 18 = |I|$. Indeed, the Dynkin diagram of $X \oplus X_{i,j,k,\iota}$ is $\overset{-1}{\circ} \underline{\quad -\xi^{2j} \quad} \overset{\xi^{2j}}{\circ}$. Since $j \notin \{0,3\}$, it is of standard type B_2. By [9, 10], $\dim \mathcal{B}(X \oplus X_{i,j,k,\iota}) = 36$. The claim follows since $\dim \mathcal{B}(V_{i,j,k,\iota}) \geq \frac{1}{2}\dim \mathcal{B}(X \oplus X_{i,j,k,\iota})$. Consequently, $\mathcal{B} \cong \mathcal{B}(V_{i,j,k,\iota})$. $\qquad\square$

Proposition 4.4. *Let* $\Lambda^2 = \{(i,j,k,\iota) \in \Lambda \mid 3(k+\iota-1) - j \equiv \pm 2 \mod 6,\ 3(k+j)(\iota-1)+ij \equiv \pm 2 \mod 6\}$. *The Nichols algebra* $\mathcal{B}(V_{i,j,k,\iota})$ *for* $(i,j,k,\iota) \in \Lambda^2$ *is generated by* v_1, v_2, *subject to the relations*

$$v_1^3 = 0, \quad \xi^{2j}v_1^2 v_2 + (-1)^k \xi^j v_1 v_2 v_1 + v_2 v_1^2 = 0, \tag{4.4}$$
$$v_2^2 v_1 + [(-1)^\iota \xi^{2j} + (-1)^k \xi^j]v_2 v_1 v_2 - v_1 v_2^2 = 0, \quad v_2^6 = 0. \tag{4.5}$$

Proof. The braiding of $V_{i,j,k,\iota}$ is given by

$$c\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix}\right)$$
$$= \begin{bmatrix} \xi^{2j}v_1 \otimes v_1 & (-1)^k \xi^j v_2 \otimes v_1 + [\xi^{2j}+(-1)^{k+\iota}\xi^j]v_1 \otimes v_2 \\ (-1)^{\iota+1}\xi^{2j}v_1 \otimes v_2 & (-1)^{k+\iota}\xi^j v_2 \otimes v_2 - (-1)^\iota \frac{\xi}{1-\xi^2}v_1 \otimes v_1 \end{bmatrix}.$$

Then a direct computation shows that the relations (4.4) and (4.5) are zero in $\mathcal{B}(V_{i,j,k,\iota})$ being annihilated by ∂_1, ∂_2 and hence the quotient $\mathfrak{B}$ of $T(V_{i,j,k,\iota})$ by the relations (4.4) and (4.5) projects onto $\mathcal{B}(V_{i,j,k,\iota})$. We claim that $I = \Bbbk\{v_1^i(v_2 v_1)^j v_2^k, i \in \mathbb{I}_{0,2}, j \in \mathbb{I}_{0,1}, k \in \mathbb{I}_{0,5}\}$ is a left ideal. Indeed, from (4.4),

(4.5) and $(v_2v_1)^2 = -[(-1)^\iota\xi^{2j} + (-1)^k\xi^j]v_1^2v_2^2 - 2(v_1v_2)^2$, it is easy to show that $v_1I, v_2I \subset I$. Hence I linearly generates $\mathfrak{B}$ since $1 \in I$.

We claim that $\dim \mathcal{B}(V_{i,j,k,\iota}) \geq 36 = |I|$. Indeed, the Dynkin diagram of $X \oplus X_{i,j,k,\iota}$ is $\overset{-1}{\underset{\circ}{}} \xrightarrow{\ \xi^{2j}\ } \overset{\xi^{2j}}{\underset{\circ}{}}$. Since $j \notin \{0,3\}$, it is of standard type B_2 and by [9, 10], $\dim \mathcal{B}(X \oplus X_{i,j,k,\iota}) = 72$. The claim follows since $\dim \mathcal{B}(V_{i,j,k,\iota}) \geq \frac{1}{2}\dim \mathcal{B}(X \oplus X_{i,j,k,\iota})$. Consequently, $\mathcal{B} \cong \mathcal{B}(V_{i,j,k,\iota})$. □

Proposition 4.5. *Let* $\Lambda^3 = \{(i,j,k,\iota) \in \Lambda \mid 3(k+\iota-1) - j \equiv \mp 2 \mod 6, 3(k+j)(\iota-1)+ij \equiv \pm 1 \mod 6\}$. *The Nichols algebra* $\mathcal{B}(V_{i,j,k,\iota})$ *for* $(i,j,k,\iota) \in \Lambda^3$ *is generated by* v_1, v_2, *subject to the relations*

$$v_1^6 = 0, \quad (-1)^{k+\iota}\xi^{-j}v_1^2v_2 + [(-1)^\iota + (-1)^k\xi^{-j}]v_1v_2v_1 + v_2v_1^2 = 0,$$

$$(-)^{\iota+1}\frac{1}{3}(\xi + \xi^2)v_1^3 + v_1v_2^2 + (-1)^\iota v_2v_1v_2 + v_2^2v_1 = 0, \tag{4.6}$$

$$\frac{(-1)^\iota\xi^{1-2j}}{1+\xi^5}v_1^2v_2 + \frac{(-1)^{k+\iota}\xi^{1-j}}{1+\xi^5}v_1v_2v_1 + v_2^3 = 0.$$

Proof. The braiding of $V_{i,j,k,\iota}$ is given by

$$c\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix}\right)$$
$$= \begin{bmatrix} (-1)^{k+\iota}\xi^{-j}v_1 \otimes v_1 & (-1)^\iota\xi^{-2j}v_2 \otimes v_1 + [\xi^{-2j} + (-1)^{k+\iota}\xi^{-j}]v_1 \otimes v_2 \\ (-1)^{k+1}\xi^{-j}v_1 \otimes v_2 & \xi^{-2j}v_2 \otimes v_2 + (-1)^\iota\frac{\xi}{1-\xi^2}v_1 \otimes v_1 \end{bmatrix}.$$

Then a direct computation shows that the relations (4.6) are zero in $\mathcal{B}(V_{i,j,k,\iota})$ being annihilated by ∂_1, ∂_2 and hence the quotient $\mathfrak{B}$ of $T(V_{i,j,k,\iota})$ by the relations (4.6) projects onto $\mathcal{B}(V_{i,j,k,\iota})$. We claim that $I = \Bbbk\{v_1^i(v_2v_1)^jv_2^k, i \in \mathbb{I}_{0,5}, j \in \mathbb{I}_{0,1}, k \in \mathbb{I}_{0,2}\}$ is a left ideal. Indeed, from (4.6) and $(v_2v_1)^2 = -(\xi^{2j} + \xi^{2j+1})v_1^2v_2^2 - 2\xi^{2j}(v_1v_2)^2$, it is easy to show that $v_1I, v_2I \subset I$. Hence I linearly generates $\mathfrak{B}$ since clearly $1 \in I$.

We claim that $\dim \mathcal{B}(V_{i,j,k,\iota}) \geq 36 = |I|$. Indeed, the generalized Dynkin diagram of $\mathcal{B}(X \oplus X_{i,j,k,\iota})$ is $\overset{-1}{\underset{\circ}{}} \xrightarrow{\ (-1)^{k+\iota+1}\xi^{-j}\ } \overset{(-1)^{k+\iota}\xi^{-j}}{\underset{\circ}{}}$. Since $j \notin \{0,3\}$, it is of standard type B_2. By [9, 10], $\dim \mathcal{B}(X \oplus X_{i,j,k,\iota}) = 72$. The claim follows since $\dim \mathcal{B}(V_{i,j,k,\iota}) \geq \frac{1}{2}\dim \mathcal{B}(X \oplus X_{i,j,k,\iota})$. Consequently, $\mathcal{B} \cong \mathcal{B}(V_{i,j,k,\iota})$. □

Proposition 4.6.

(1) Let $(i,j,k,\iota) \in \Lambda^4 = \{(i,3,0,1), (i,3,1,0), i \in \{1,3,5\}\}$. *The Nichols algebra* $\mathcal{B}(V_{i,j,k,\iota})$ *is generated as an algebra by* v_1, v_2, *subject to the relations*

$$v_1^2 = 0, \quad v_1v_2 + (-1)^\iota v_2v_1 = 0, \quad v_2^2 = 0. \tag{4.7}$$

(2) Let $\Lambda^5 = \{(i,j,k,\iota) \in \Lambda - \Lambda^4 \mid 3(k+j)(\iota-1)+ij \equiv 3 \mod 6\}$. *The Nichols algebra* $\mathcal{B}(V_{i,j,k,\iota})$ *for* $(i,j,k,\iota) \in \Lambda^5$ *is generated by* v_1, v_2, *subject to the relations*

$$v_1^2 = 0, \quad v_1v_2 + (-1)^k\xi^{5j}v_2v_1 = 0, \quad v_2^N = 0, \quad N = \mathrm{ord}((-1)^{k+\iota-1}\xi^{-j}) \in \{3,6\}.$$

(3) *Let $\Lambda^6 = \{(i,j,k,\iota) \in \Lambda - \Lambda^4 \mid 3(k+\iota) + 3(k+j)(\iota-1) + (i-1)j \equiv 3$ mod 6\}. The Nichols algebra $\mathcal{B}(V_{i,j,k,\iota})$ for $(i,j,k,\iota) \in \Lambda^6$ is generated by v_1, v_2, subject to the relations*

$$v_1 v_2 + (-1)^\iota v_2 v_1 = 0, \quad v_2^2 + (1-\xi^2)^{-1}\xi^{2+4i}(-1)^\iota v_1^2 = 0, \quad v_1^N = 0,$$

where $N = \mathrm{ord}((-1)^{\iota+1+k}\xi^j) \in \{3,6\}$.

Proof. Assume that $(i,j,k,\iota) \in \Lambda^4$. The braiding of $V_{i,j,k,\iota} = \{v_1, v_2\}$ is given by

$$c\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix}\right) = \begin{bmatrix} -v_1 \otimes v_1 & (-1)^{\iota-1} v_2 \otimes v_1 - 2 v_1 \otimes v_2 \\ (-1)^\iota v_1 \otimes v_2 & -v_2 \otimes v_2 \end{bmatrix}.$$

Then a direct computation shows that $v_1^2, v_2^2, v_1 v_2 + (-1)^\iota v_2 v_1 \in \mathcal{P}(T(V_{i,j,k,\iota}))$. Indeed, we have

$$\Delta(v_i^2) = v_i^2 \otimes 1 + v_i \otimes v_i + c(v_i \otimes v_i) + 1 \otimes v_i^2 = v_i^2 \otimes 1 + 1 \otimes v_i^2; \quad i \in \mathbb{I}_{1,2},$$
$$\Delta(v_1 v_2 + (-1)^\iota v_2 v_1) = (v_1 v_2 + (-1)^\iota v_2 v_1) \otimes 1 + 1 \otimes (v_1 v_2 + (-1)^\iota v_2 v_1).$$

Therefore, the quotient $\mathfrak{B}$ of $T(V_{i,j,k,\iota})$ by the relations (4.7) projects onto $\mathcal{B}(V_{i,j,k,\iota})$. From (4.6), it is easy to show that $I = \Bbbk\{v_1^i v_2^j, i, j \in \mathbb{I}_{0,1}\}$ is a left ideal and linearly generates $\mathfrak{B}$.

We claim that $\dim \mathcal{B}(V_{i,j,k,\iota}) \geq 4 = |I|$. Indeed, the diagram is $\overset{-1}{\underset{x}{\circ}} \overset{-1}{\rule{1.5cm}{0.4pt}} \overset{-1}{\underset{v_1}{\circ}}$.

It is of Cartan type A_2. By [9, 10], $\dim \mathcal{B}(X \oplus X_{i,j,k,\iota}) = 8$. The claim follows since $\dim \mathcal{B}(V_{i,j,k,\iota}) = \frac{1}{2} \dim \mathcal{B}(X \oplus X_{i,j,k,\iota})$. Consequently, $\mathfrak{B} \cong \mathcal{B}(V_{i,j,k,\iota})$.

The proof follows for $(i,j,k,\iota) \in \Lambda^5$ or Λ^6 the same lines as for $(i,j,k,\iota) \in \Lambda^4$. In these cases, the generalized Dynkin diagram of $X \oplus X_{i,j,k,\iota}$ is given by

$$\overset{-1}{\underset{x}{\circ}} \overset{(-1)^{k+\iota-1}\xi^{-j}}{\rule{2cm}{0.4pt}} \overset{-1}{\underset{v_1}{\circ}} \quad \text{or} \quad \overset{-1}{\underset{x}{\circ}} \overset{(-1)^{\iota+1+k}\xi^{-j}}{\rule{2.5cm}{0.4pt}} \overset{(-1)^{\iota+1+k}\xi^j}{\underset{v_1}{\circ}} \ .$$

They are of standard A_2 type. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Theorem 4.7. *Let V be a simple object in $\,^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$ such that $\dim \mathcal{B}(V) < \infty$. Then V is isomorphic either to $\Bbbk_{\chi_{i,j,k}}$ for $(i,j,k) \in \Lambda^0$ or to $V_{i,j,k,\iota}$ for $(i,j,k,\iota) \in \cup_{i=1}^6 \Lambda^i$.*

Proof. By Theorem 3.8, V is isomorphic to $\Bbbk_{\chi_{i,j,k}}$ for $(i,j,k) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,1} \times \mathbb{I}_{0,5}$ or $V_{i,j,k,\iota}$ for $(i,j,k,\iota) \in \Lambda$. If $\dim V = 1$, by Proposition 4.1, $V \cong \Bbbk_{\chi_{i,j,k}}$ for $(i,j,k) \in \Lambda^0$. Observe that $\Lambda = \Lambda^{0*} \cup \Lambda^{0**} \cup \cup_{i=1}^6 \Lambda^i$. If $\dim V = 2$, then by Propositions 4.3–4.6, $V \cong V_{i,j,k,\iota}$ for $(i,j,k,\iota) \in \cup_{i=1}^6 \Lambda^i$. $\qquad\qquad \square$

Remark 4.8.

(1) $|\Lambda^1| = 12$, $|\Lambda^2| = 8 = |\Lambda^3|$, $|\Lambda^4| = 6$, $|\Lambda^5| = 12 = |\Lambda^6|$.

(2) The Nichols algebras $\mathcal{B}(V_{i,j,k,\iota})$ with $(i,j,k,\iota) \in \Lambda^4 \cup \Lambda^5 \cup \Lambda^6$ have already appeared in [5]. They are isomorphic to quantum planes as algebras. They can be recovered, up to isomorphism, by using the techniques in [2]. Indeed, from the proofs of Proposition 4.6, they are arising from Nichols algebras of Cartan type A_2 or standard type A_2.

(3) The Nichols algebra $\mathcal{B}(V_{i,j,k,\iota})$ for $(i,j,k,\iota) \in \cup_{i=1}^{3}\Lambda^i$ is an algebra of dimension 18 or 36 with no quadratic relations. They appeared in [23] (see also [24, 20]) with different parameters. Indeed, from the proofs of Propositions 4.3–4.6, [23, Remark 4.21] and [20, Proposition 5.9], up to isomorphism, they are arising from Nichols algebras of standard type B_2 whose Dynkin diagram is $\overset{-1}{\circ}\!\!\underline{\quad\overset{q^{-2}}{}\quad}\!\!\overset{q}{\circ}$ ($q \in \Bbbk^\times - \{1,-1\}$ and $q^4 \neq 1$), with different matrices of the braiding.

5. Hopf algebras over $\mathcal{K}_{24,1}$

We determine all finite-dimensional Hopf algebras over $\mathcal{K}_{24,1}$, whose infinitesimal braidings are simple objects in $_{\mathcal{K}_{24,1}}^{\mathcal{K}_{24,1}}\mathcal{YD}$. We first define four families of Hopf algebras $\mathcal{C}_{i,j,k,\iota}(\mu)$ for $(i,j,k,\iota) \in \Lambda^{1*}$ and show that they are indeed the liftings of $\mathcal{B}(V_{i,j,k,\iota})\sharp\mathcal{K}_{24,1}$. Here $\Lambda^{1*} = \{(2,j_1,0,0),(2,j_2,1,0), j_1 = 2,4, j_2 = 1,5\} \subset \Lambda^1$.

Definition 5.1. For $j \in \{2,4\}$ and $\mu \in \Bbbk$, let $\mathcal{C}_{2,j,0,0}(\mu)$ be the algebra generated by a,b,c,d,v_1,v_2, subject to the relations (3.1) and the following ones:

$$av_1 = \xi^5 v_1 a, \quad av_2 = \xi^4 v_2 a + v_1 c, \quad bv_1 = \xi^5 v_1 b, \quad bv_2 = \xi^4 v_2 b + v_1 d, \quad (5.1)$$

$$cv_1 = \xi^2 v_1 c, \quad cv_2 = \xi^4 v_2 c + v_1 a, \quad dv_1 = \xi^2 v_1 d, \quad dv_2 = \xi^4 v_2 d + v_1 b, \quad (5.2)$$

$$v_1^3 = 0, \quad \xi^{2j} v_1^2 v_2 + \xi^{-2j} v_1 v_2 v_1 + v_2 v_1^2 = 0, \quad (5.3)$$

$$\frac{(1-\xi^{2j})\xi^4}{1+\xi^5} v_1^2 v_2 + \frac{(1-\xi^{4j})\xi^4}{1+\xi^5} v_1 v_2 v_1 + v_2^3 = \mu(1 - da^{-1}), \quad (5.4)$$

$$v_1 v_2^2 + v_2 v_1 v_2 + v_2^2 v_1 = -2\mu\xi^4 ba^{-1}. \quad (5.5)$$

$\mathcal{C}_{2,j,0,0}(\mu)$ admits a Hopf algebra structure, where the comultiplication is given by (3.2) and

$$\begin{aligned}
\Delta(v_1) &= v_1 \otimes 1 + a^j \otimes v_1 - \xi(1+\xi^j)ba^{j-1} \otimes v_2, \\
\Delta(v_2) &= v_2 \otimes 1 + da^{j-1} \otimes v_2 - (1-\xi^2)^{-1}\xi^{-1}(1-\xi^j)ca^{j-1} \otimes v_1.
\end{aligned} \quad (5.6)$$

Remark 5.2. It is clear that $\mathcal{C}_{2,j,0,0}(0) \cong \mathcal{B}(V_{2,j,0,0})\sharp\mathcal{K}_{24,1}$ and $\mathcal{C}_{2,j,0,0}(\mu)$ with $\mu \neq 0$ is not isomorphic to $\mathcal{C}_{2,j,0,0}(0)$ for $j \in \{2,4\}$.

Definition 5.3. For $j \in \{1,5\}$ and $\mu \in \Bbbk$, let $\mathcal{C}_{2,j,1,0}(\mu)$ be the algebra generated by a,d,c,d,v_1,v_2, subject to the relations (3.1), (5.1)–(5.3) and the following ones:

$$\frac{(1-\xi^{2j})\xi^4}{1+\xi^5} v_1^2 v_2 + \frac{(1-\xi^{4j})\xi^4}{1+\xi^5} v_1 v_2 v_1 + v_2^3 = \mu(1 - a^3), \quad (5.7)$$

$$v_1 v_2^2 + v_2 v_1 v_2 + v_2^2 v_1 = -2\mu\xi^4 ca^2. \quad (5.8)$$

$\mathcal{C}_{2,j,1,0}(\mu)$ admits a Hopf algebra structure, where the comultiplication is given by (3.2) and

$$\begin{aligned}
\Delta(v_1) &= v_1 \otimes 1 + da^{j-1} \otimes v_1 - \xi(1-\xi^j)ca^{j-1} \otimes v_2, \\
\Delta(v_2) &= v_2 \otimes 1 + a^j \otimes v_2 - (1-\xi^2)^{-1}\xi^{-1}(1+\xi^j)ba^{j-1} \otimes v_1.
\end{aligned} \quad (5.9)$$

Remark 5.4. It is clear that $\mathcal{C}_{2,j,1,0}(0) \cong \mathcal{B}(V_{2,j,1,0})\sharp\mathcal{K}_{24,1}$ and $\mathcal{C}_{2,j,1,0}(\mu)$ with $\mu \neq 0$ is not isomorphic to $\mathcal{C}_{2,j,1,0}(0)$ for $j \in \{1,5\}$.

Lemma 5.5. *A linear basis of* $\mathcal{C}_{i,j,k,\iota}(\mu)$ *for* $(i,j,k,\iota) \in \Lambda^{1*}$ *is given by*

$$\{v_2^i(v_1v_2)^j v_1^k d^\mu c^\nu b^l a^m,\ i,k \in \mathbb{I}_{0,2},\ j,\mu,\nu,l,\mu+\nu+l \in \mathbb{I}_{0,1},\ m \in \mathbb{I}_{0,5}\}.$$

Proof. We prove the assertion for $\mathcal{C}_{2,j,1,0}(\mu)$, the proof for $\mathcal{C}_{2,j,0,0}(\mu)$ being completely analogous. We write $v_{12} := v_1v_2$ for short. By the diamond lemma, it suffices to show that all overlap ambiguities are resolvable, that is, the ambiguities can be reduced to the same expression by different substitution rules with the order $v_2 < v_1v_2 < v_1 < d < c < b < a$. Here we verify that the overlapping pair $(fv_2)v_2^2 = f(v_2^3)$ for $f \in \{a,\ b,\ c,\ d\}$ is resolvable:

$$(av_2)v_2^2 = (\xi^4 v_2 a + v_1 c)v_2^2 = \xi^2 v_2^2 a v_2 + \xi^4(v_1v_2 + v_2v_1)cv_2 + v_1^2 av_2$$

$$= \xi^2 v_2^2(\xi^4 v_2 a + v_1 c) + \xi^4(v_1v_2 + v_2v_1)(\xi^4 v_2 c + v_1 a) + v_1^2(\xi^4 v_2 a + v_1 c)$$

$$= v_2^3 a + \xi^2 v_2^2 v_1 c + \xi^2 v_1 v_2^2 c + \xi^4 v_1 v_2 v_1 a + \xi^2 v_2 v_1 v_2 c + \xi^4 v_2 v_1^2 a$$

$$\quad + \xi^4 v_1^2 v_2 a + v_1^3 c$$

$$= v_2^3 a + \xi^4 v_1 v_2 v_1 a + \xi^4 v_2 v_1^2 a + \xi^4 v_1^2 v_2 a + \xi^2(v_2^2 v_1 + v_1 v_2^2 + v_2 v_1 v_2)c$$

$$= v_2^3 a + \xi^4 v_1 v_2 v_1 a + \xi^4 v_2 v_1^2 a + \xi^4 v_1^2 v_2 a$$

$$= \frac{(1-\xi^{2j})\xi}{1+\xi^5}av_1^2 v_2 + \frac{(1-\xi^{4j})\xi}{1+\xi^5}av_1 v_2 v_1 + \mu a(1-a^3) = a(v_2^3).$$

$$(bv_2)v_2^2 = (\xi^4 v_2 b + v_1 d)v_2^2 = \xi^2 v_2^2 bv_2 + \xi^4 v_2 v_1 dv_2 + \xi^4 v_1 v_2 dv_2 + v_1^2 bv_2$$

$$= v_2^3 b + \xi^2 v_2^2 v_1 d + \xi^2 v_2 v_1 v_2 d + \xi^4 v_2 v_1^2 b + \xi^2 v_1 v_2^2 d + \xi^4 v_1 v_2 v_1 b$$

$$\quad + \xi^4 v_1^2 v_2 b + v_1^3 d$$

$$= v_2^3 b + \xi^4 v_1 v_2 v_1 b + \xi^4 v_2 v_1^2 b + \xi^4 v_1^2 v_2 b + \xi^2(v_2^2 v_1 + v_1 v_2^2 + v_2 v_1 v_2)d$$

$$= v_2^3 b + \xi^4 v_1 v_2 v_1 b + \xi^4 v_2 v_1^2 b + \xi^4 v_1^2 v_2 b - 2\mu cda^2$$

$$= \frac{(1-\xi^{2j})\xi}{1+\xi^5}bv_1^2 v_2 + \frac{(1-\xi^{4j})\xi}{1+\xi^5}bv_1 v_2 v_1 + \mu b(1-a^3) = b(v_2^3).$$

$$(dv_2)v_2^2 = (\xi^4 v_2 d + v_1 b)v_2^2 = \xi^2 v_2^2 dv_2 + \xi^4(v_1v_2 + v_2v_1)bv_2 + v_1^2 dv_2$$

$$= v_2^3 d + \xi^2 v_2^2 v_1 b + \xi^2 v_1 v_2^2 b + \xi^4 v_1 v_2 v_1 d + \xi^2 v_2 v_1 v_2 b + \xi^4 v_2 v_1^2 d$$

$$\quad + \xi^4 v_1^2 v_2 d + v_1^3 b$$

$$= v_2^3 d + \xi^4 v_1 v_2 v_1 d + \xi^4 v_2 v_1^2 d + \xi^4 v_1^2 v_2 d + \xi^2(v_2^2 v_1 + v_1 v_2^2 + v_2 v_1 v_2)b + v_1^3 b$$

$$= \frac{(1-\xi^{2j})\xi}{1+\xi^5}dv_1^2 v_2 + \frac{(1-\xi^{4j})\xi}{1+\xi^5}dv_1 v_2 v_1 + \mu d(1-a^3) = d(v_2^3).$$

$$(cv_2)v_2^2 = (\xi^4 v_2 c + v_1 a)v_2^2$$

$$= v_2^3 c + \xi^4 v_2 v_1^2 c + \xi^4 v_1 v_2 v_1 c + \xi^4 v_1^2 v_2 c + \xi^2(v_2^2 v_1 + v_2 v_1 v_2 + v_1 v_2^2)a + v_1^3 a$$

$$= \frac{(1-\xi^{2j})\xi}{1+\xi^5}cv_1^2 v_2 + \frac{(1-\xi^{4j})\xi}{1+\xi^5}cv_1 v_2 v_1 + \mu c(1-a^3) = c(v_2^3).$$

One can also show that the remaining overlaps

$$\{(fv_1)v_1^2, f(v_1^3)\}, \quad \{(fv_{12})v_{12}, f(v_{12}^2)\}, \quad \{(v_1v_{12})v_{12}, v_1(v_{12}^2)\},$$

$$\{(v_1^3)v_{12}, v_1^2(v_1v_{12})\}, \quad \{(v_1^3)v_2, v_1^2(v_1v_2)\}, \quad \{v_1(v_2^3), (v_1v_2)v_2^2\},$$

$$\{v_{12}(v_2^3), (v_{12}v_2)v_2^2\}, \quad \{(v_{12}^2)v_2, v_{12}(v_{12}v_2)\}, \quad \{v_i^3 v_i^r, v_i^r v_i^3\}, \quad \{v_{12}^2 v_{12}^h, v_{12}^h v_{12}^2\}$$

are resolvable by using the defining relations. Here we omit the details since it is tedious but straightforward. □

Lemma 5.6. *For $(i,j,k,\iota) \in \Lambda^{1*}$, $\mathrm{gr}\,\mathcal{C}_{i,j,k,\iota}(\mu) \cong \mathcal{B}(V_{i,j,k,\iota})\sharp\mathcal{K}_{24,1}$.*

Proof. Let $\mathfrak{C}_0$ be the Hopf subalgebra of $\mathcal{C}_{i,j,k,\iota}(\mu)$ generated by the simple sub-coalgebra $\Bbbk\{a,b,c,d\}$. By Lemma 5.5, $\dim \mathfrak{C}_0 = 24$. It is clear that $\mathfrak{C}_0 \cong \mathcal{K}_{24,1}$. Let $\mathfrak{C}_n = \mathfrak{C}_{n-1} + \mathcal{K}_{24,1}\{y^i(xy)^j x^k, i+2j+k = n, i,k \in \mathbb{I}_{0,2}, j \in \mathbb{I}_{0,1}\}$ for $n \in \mathbb{I}_{0,6}$. A direct computation shows that $\{\mathfrak{C}_n\}_{n\in\mathbb{I}_{0,6}}$ is a coalgebra filtration of $\mathcal{C}_{i,j,k,\iota}(\mu)$ and hence the coradical $(\mathcal{C}_{i,j,k,\iota}(\mu))_0 \subset \mathfrak{C}_0 \cong \mathcal{K}_{24,1}$, which implies that $(\mathcal{C}_{i,j,k,\iota}(\mu))_{[0]} \cong \mathcal{K}_{24,1}$ and $\mathrm{gr}\,\mathcal{C}_{i,j,k,\iota}(\mu) \cong R_{i,j,k,\iota}\sharp\mathcal{K}_{24,1}$, where $R_{i,j,k,\iota}$ is a connected Hopf algebra in $^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$. Since $V_{i,j,k,\iota} \subset \mathcal{P}(R_{i,j,k,\iota})$ by definition and $\dim R_{i,j,k,\iota} = 18 = \dim \mathcal{B}(V_{i,j,k,\iota})$ by Lemma 5.5, it follows that $R_{i,j,k,\iota} \cong \mathcal{B}(V_{i,j,k,\iota})$ and consequently, $\mathrm{gr}\,\mathcal{C}_{i,j,k,\iota}(\mu) \cong \mathcal{B}(V_{i,j,k,\iota})\sharp\mathcal{K}_{24,1}$. □

Proposition 5.7. *Let A be a finite-dimensional Hopf algebra over $\mathcal{K}_{24,1}$ such that $\mathrm{gr}\,A \cong \mathcal{B}(V_{i,j,k,\iota})\sharp\mathcal{K}_{24,1}$ for $(i,j,k,\iota) \in \Lambda^{1*}$. Then $A \cong \mathcal{C}_{i,j,k,\iota}(\mu)$ for some $\mu \in \Bbbk$.*

Proof. Let $X = \xi^{2j}v_1^2 v_2 + \xi^{-2j}v_1 v_2 v_1 + v_2 v_1^2$, $Y = \frac{(1-\xi^{2j})\xi^4}{1+\xi^5}v_1^2 v_2 + \frac{(1-\xi^{4j})\xi^4}{1+\xi^5}v_1 v_2 v_1 + v_2^3$ and $Z = v_1 v_2^2 + v_2 v_1 v_2 + v_2^2 v_1$ for simplicity. Assume that $(i,j,k,\iota) = (2,j,0,0)$ for some $j \in \{2,4\}$. Note that $\mathrm{gr}\,A \cong \mathcal{B}(V_{i,j,k,\iota})\sharp\mathcal{K}_{24,1}$. By (5.6), a direct computation shows that

$$\Delta(v_1^3) = v_1^3 \otimes 1 + 1 \otimes v_1^3 + (1-\xi^2)^{-\frac{1}{2}}x_2 ba^{-1} \otimes X,$$

$$\Delta(X) = X \otimes 1 + da^{-1} \otimes X, \quad \Delta(Y) = Y \otimes 1 + da^{-1} \otimes Y,$$

$$\Delta(Z) = Z \otimes 1 + 1 \otimes Z + (1-\xi^2)^{-\frac{1}{2}}x_1 ba^{-1} \otimes X - 2\xi^j(1-\xi^2)^{-\frac{1}{2}}x_2 ba^{-1} \otimes Y.$$

It follows that $X, Y \in \mathcal{P}_{1,da^{-1}}(A) = \mathcal{P}_{1,da^{-1}}(\mathcal{K}_{24,1}) = \Bbbk\{1-da^{-1}, ca^{-1}\}$, that is, $X = \alpha_1(1-da^{-1}) + \alpha_2 ca^{-1}, Y = \beta_1(1-da^{-1}) + \beta_2 ca^{-1}$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Bbbk$. Then

$$\Delta(v_1^3 + \alpha_1(1-\xi^2)^{-\frac{1}{2}}x_2 ba^{-1}) = (v_1^3 + \alpha_1(1-\xi^2)^{-\frac{1}{2}}x_2 ba^{-1}) \otimes 1$$
$$+ 1 \otimes (v_1^3 + \alpha_1(1-\xi^2)^{-\frac{1}{2}}x_2 ba^{-1}) + (1-\xi^2)^{-\frac{1}{2}}x_2 ba^{-1} \otimes \alpha_2 ca^{-1}.$$

If the relation $v_1^3 = 0$ admits a non-trivial deformation, then $v_1^3 \in A_{[2]}$. Since $av_1^3 = -v_1^3 a$, $bv_1^3 = -v_1^3 b$, $cv_1^3 = v_1^3 c$ and $dv_1^3 = v_1^3 c$, a tedious computation on $A_{[2]}$ shows that $v_1^3 = 0$ must hold in A. Therefore, the last equation holds only if $\alpha_1 = 0 = \alpha_2$, which implies that $X = 0$ in A. Similarly, we have that $Y = \beta_1(1-da^{-1})$ and $Z = 2\beta_1\xi^j(1-\xi^2)^{-\frac{1}{2}}x_2 ba^{-1} = -2\beta_1\xi^4 ba^{-1}$. Therefore, the defining relations of $\mathcal{C}_{2,j,0,0}(\beta_1)$ hold in A and hence there is a surjective Hopf

algebra morphism from $\mathcal{C}_{2,j,0,0}(\beta_1)$ to A. By Lemma 5.5, $\dim A = \dim \mathcal{C}_{2,j,0,0}(\beta_1)$ and hence $A \cong \mathcal{C}_{2,j,0,0}(\beta_1)$.

Assume that $(i, j, k, \iota) = (2, j, 1, 0)$ for some $j \in \{1, 5\}$. By (5.9), a direct computation shows that

$$\Delta(v_1^3) = v_1^3 \otimes 1 + da^2 \otimes v_1^3 + (1 - \xi^2)^{-\frac{1}{2}} x_2 ca^2 \otimes X,$$

$$\Delta(X) = X \otimes 1 + a^3 \otimes X, \quad \Delta(Y) = Y \otimes 1 + a^3 \otimes Y,$$

$$\Delta(Z) = Z \otimes 1 + da^2 \otimes Z + (1 - \xi^2)^{-\frac{1}{2}} x_1 ca^2 \otimes X + 2\xi^j (1 - \xi^2)^{-\frac{1}{2}} x_2 ca^2 \otimes Y.$$

It follows that $X = \alpha_1(1 - a^3)$, $Y = \alpha_2(1 - a^3)$ for some $\alpha_1, \alpha_2 \in \Bbbk$. Then $v_1^3 + \alpha_1(1 - \xi^2)^{-\frac{1}{2}} x_2 ca^2 \in \mathcal{P}_{1,da^2}(A) = \mathcal{P}_{1,da^2}(\mathcal{K}_{24,1})$, that is, $v_1^3 + \alpha_1(1 - \xi^2)^{-\frac{1}{2}} x_2 ca^2 = \alpha_3(1 - da^2)$ for some $\alpha_3 \in \Bbbk$. Since $av_1^3 = -v_1^3 a$, $bv_1^3 = -v_1^3 b$, $cv_1^3 = v_1^3 c$ and $dv_1^3 = v_1^3 c$, it follows that $\alpha_1 = 0 = \alpha_3$ and hence $v_1^3 = 0 = X$ in A. Then $Z + 2\alpha_2 \xi^j (1 - \xi^2)^{-\frac{1}{2}} x_2 ca^2 \in \mathcal{P}_{1,da^2}(A) = \mathcal{P}_{1,da^2}(\mathcal{K}_{24,1})$, that is, $Z + 2\alpha_2 \xi^j (1 - \xi^2)^{-\frac{1}{2}} x_2 ca^2 = \alpha_4(1 - da^2)$ for some $\alpha_4 \in \Bbbk$. Since $aZ = \xi Za$, it follows that $\alpha_4 = 0$ and hence $Z = -2\alpha_2 \xi^j (1 - \xi^2)^{-\frac{1}{2}} x_2 ca^2 = -2\alpha_2 \xi^4 ca^2$. Therefore, there is a surjective Hopf algebra morphism from $\mathcal{C}_{2,j,1,0}(\alpha_2)$ to A. Since $\dim A = \dim \mathcal{C}_{2,j,1,0}(\alpha_2)$ by Lemma 5.5, $A \cong \mathcal{C}_{2,j,1,0}(\alpha_2)$. □

Proposition 5.8. *Let A be a finite-dimensional Hopf algebra over $\mathcal{K}_{24,1}$ such that $\mathrm{gr}\, A \cong \mathcal{B}(V) \sharp \mathcal{K}_{24,1}$, where V is isomorphic either to $\Bbbk_{\chi_{i,j,k}}$ for $(i, j, k) \in \Lambda^0$ or to $V_{i,j,k,\iota}$ for $(i, j, k, \iota) \in \Lambda^4$. Then $A \cong \mathrm{gr}\, A$.*

Proof. Assume that $V \cong \Bbbk_{\chi_{i,j,k}}$ for $(i, j, k) \in \Lambda^0$. Since $\Delta(v) = v \otimes 1 + d^j a^{3i-j} \otimes v$, we have $\Delta(v^2) = v^2 \otimes 1 + 1 \otimes v^2$ and hence $v^2 = 0$ in A. Consequently, $A \cong \mathrm{gr}\, A$.

Assume that $V \cong V_{i,j,k,\iota}$ for $(i, j, k, \iota) \in \Lambda^4$. A direct computation shows that

$$\Delta(v_1^2) = v_1^2 \otimes 1 + 1 \otimes v_1^2 + (-1)^\iota (1 - \xi^2)^{-\frac{1}{2}} x_2 ba^{-1} \otimes (v_1 v_2 + (-1)^\iota v_1 v_2), \quad (5.10)$$

$$\Delta(v_2^2) = v_2^2 \otimes 1 + 1 \otimes v_2^2, \quad (5.11)$$

$$\Delta(v_1 v_2 + (-1)^\iota v_1 v_2) = (v_1 v_2 + (-1)^\iota v_1 v_2) + da^{-1} \otimes (v_1 v_2 + (-1)^\iota v_1 v_2). \quad (5.12)$$

It follows by (5.11) that $v_2^2 = 0$ in A and by (5.12) that $v_1 v_2 + (-1)^\iota v_2 v_1 \in \mathcal{P}_{1,da^{-1}}(A) = \mathcal{P}_{1,da^{-1}}(\mathcal{K}_{24,1})$, that is, $v_1 v_2 + (-1)^\iota v_2 v_1 = \alpha_1(1 - da^{-1}) + \alpha_2 ca^{-1}$ for some $\alpha_1, \alpha_2 \in \Bbbk$. Let $X := v_1^2 + (-1)^\iota \alpha_1(1 - \xi^2)^{-\frac{1}{2}} x_2 ba^{-1}$ for short. Then (5.10) can be rewritten as

$$\Delta(X) = X \otimes 1 + 1 \otimes X + (-1)^\iota (1 - \xi^2)^{-\frac{1}{2}} x_2 ba^{-1} \otimes \alpha_2 ca^{-1}. \quad (5.13)$$

If the relation $v_1^2 = 0$ admits non-trivial deformations, then $v_1^2 \in A_{[1]}$. Since $av_1^2 = \xi^{2i} v_1^2 a$, $bv_1^2 = \xi^{2i} v_1^2 b$, $dv_1^2 = \xi^{2i} v_1^2 d$ and $cv_1^2 = \xi^{2i} v_1^2 c$, a direct computation on $A_{[1]}$ shows that $v_1^2 = 0$ in A and hence (5.13) holds only if $\alpha_1 = 0 = \alpha_2$, which implies that $v_1 v_2 + (-1)^\iota v_2 v_1 = 0$ in A. □

Proposition 5.9. *Let A be a finite-dimensional Hopf algebra over $\mathcal{K}_{24,1}$ such that $\mathrm{gr}\, A \cong \mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ for $(i, j, k, \iota) \in \Lambda^1 \cup \Lambda^2 \cup \Lambda^3 - \Lambda^{1*}$. Then $A \cong \mathrm{gr}\, A$.*

Proof. Assume that $(i, j, k, \iota) \in \Lambda^1 - \Lambda^{1*}$. Let $X = \xi^{2j}v_1^2 v_2 + (-1)^\iota \xi^{-2j} v_1 v_2 v_1 + v_2 v_1^2$, $Y = \frac{(-1)^\iota (1-\xi^{2j})\xi^4}{1+\xi^5} v_1^2 v_2 + \frac{(1-\xi^{4j})\xi^4}{1+\xi^5} v_1 v_2 v_1 + v_2^3$ and $Z = v_1 v_2^2 + (-1)^\iota v_2 v_1 v_2 + v_2^2 v_1$ for simplicity. If $k = 0$, then we have

$$\Delta(v_1^3) = v_1^3 \otimes 1 + a^{3j} \otimes v_1^3 + (1-\xi^2)^{-\frac{1}{2}} x_2 ba^{3j-1} \otimes X, \tag{5.14}$$

$$\Delta(X) = X \otimes 1 + da^{3j-1} \otimes X, \quad \Delta(Y) = Y \otimes 1 + da^{3j-1} \otimes Y, \tag{5.15}$$

$$\Delta(Z) = Z \otimes 1 + a^{3j} \otimes Z - (\xi^{-j} + \xi^j)(1-\xi^2)^{-\frac{1}{2}} x_1 ba^{3j-1} \otimes X \\ + 2(-1)^{j+1}\xi^j (1-\xi^2)^{-\frac{1}{2}} x_2 ba^{3j-1} \otimes Y. \tag{5.16}$$

If $j \in \{2, 4\}$, then $i - 5 = k = \iota = 0$ and by (5.15), $X, Y \in \mathcal{P}_{1,da^{-1}}(A) = \mathcal{P}_{1,da^{-1}}(\mathcal{K}_{24,1}) = \Bbbk\{1 - da^{-1}, ca^{-1}\}$, that is, $X = \alpha_1(1 - da^{-1}) + \alpha_2 ca^{-1}$, $Y = \beta_1(1-da^{-1}) + \beta_2 ca^{-1}$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Bbbk$. Set $r := v_1^3 + \alpha_1(1-\xi^2)^{-\frac{1}{2}} x_2 ba^{-1}$. Then

$$\Delta(r) = r \otimes 1 + 1 \otimes r + (1-\xi^2)^{-\frac{1}{2}} x_2 ba^{-1} \otimes \alpha_2 ca^{-1}. \tag{5.17}$$

If the relation $v_1^3 = 0$ admits non-trivial deformations, then $v_1^3 \in A_{[2]}$. Since $av_1^3 = v_1^3 a$, $bv_1^3 = v_1^3 b$, $cv_1^3 = -v_1^3 c$ and $dv_1^3 = -v_1^3 d$, a tedious computation on $A_{[2]}$ shows that $v_1^3 = 0$ must hold in A, which implies that (5.17) holds only if $\alpha_1 = 0 = \alpha_2$ and hence $X = 0$ in A. Similarly, $Y = \beta_1(1 - da^{-1})$ and $Z = -2\beta_1 \xi^j (1-\xi^2)^{-\frac{1}{2}} x_2 ba^{-1}$. Since $aY = -Ya$ and $ab = \xi ba$, it follows that $\beta_1 = 0$. Consequently, $A \cong \text{gr } A$.

If $j \in \{1, 5\}$, then $i - 2 = k = \iota - 1 = 0$ and by (5.15), $X, Y \in \mathcal{P}_{1,da^2}(A) = \mathcal{P}_{1,da^2}(\mathcal{K}_{24,1}) = \Bbbk\{1 - da^2\}$, that is, $X = \alpha_1(1 - da^2)$ and $Y = \alpha_2(1 - da^2)$ for some $\alpha_1, \alpha_2 \in \Bbbk$. Moreover, $v_1^3 + \alpha_1(1-\xi^2)^{-\frac{1}{2}} x_2 ba^2 \in \mathcal{P}_{1,a^3}(A) = \mathcal{P}_{1,a^3}(\mathcal{K}_{24,1})$, which implies that $v_1^3 + \alpha_1(1-\xi^2)^{-\frac{1}{2}} x_2 ba^2 = \alpha_3(1 - a^3)$. Since $dv_1^3 = v_1^3 d$ and $cv_1^3 = v_1^3 c$, it follows that $\alpha_1 = 0 = \alpha_3$ and hence $X = 0 = v_1^3$ in A. Then a direct computation shows that $Z + 2\alpha_2 \xi^j (1-\xi^2)^{-\frac{1}{2}} x_2 ba^2 \in \mathcal{P}_{1,a^3}(A) = \mathcal{P}_{1,a^3}(\mathcal{K}_{24,1})$, that is, $Z + 2\alpha_2 \xi^j (1-\xi^2)^{-\frac{1}{2}} x_2 ba^2 = \alpha_4(1 - a^3)$ for some $\alpha_4 \in \Bbbk$. Since $aZ = \xi^4 Za$, $bZ = \xi^4 Zb$, $cZ = \xi^4 Zc$ and $dZ = \xi^4 Zd$, it follows that $\alpha_2 = 0 = \alpha_4$. Consequently, $\text{gr } A \cong A$.

If $k = 1$, then $(i, j, k, \iota) \in \{(2, j_1, 1, 1), (5, j_1, 1, 1) \mid j_1 = 2, 4\}$ and we have that

$$\Delta(v_1^3) = v_1^3 \otimes 1 + da^{3j-1} \otimes v_1^3 + (1-\xi^2)^{-\frac{1}{2}} x_2 ca^{3j-1} \otimes X,$$

$$\Delta(X) = X \otimes 1 + a^{3j} \otimes X,$$

$$\Delta(Y) = Y \otimes 1 + a^{3j} \otimes Y,$$

$$\Delta(Z) = Z \otimes 1 + da^{3j-1} \otimes Z + (\xi^{-j} + \xi^j)(1-\xi^2)^{-\frac{1}{2}} x_1 ca^{3j-1} \otimes X \\ + 2(-1)^{j+1}\xi^j (1-\xi^2)^{-\frac{1}{2}} x_2 ca^{3j-1} \otimes Y.$$

It follows that $X = 0 = Y$ in A and $v_1^3, Z \in \mathcal{P}_{1,da^{-1}}(A) = \mathcal{P}_{1,da^{-1}}(\mathcal{K}_{24,1})$, that is, $v_1^3 = \alpha_1(1 - da^{-1}) + \alpha_2 ca^{-1}$ and $Z = \beta_1(1 - da^{-1}) + \beta_2 ca^{-1}$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Bbbk$. Since $av_1^3 = (-1)^i v_1^3 a$, $bv_1^3 = (-1)^i v_1^3 b$, $cv_1^3 = (-1)^i v_1^3 c$ and $dv_1^3 = (-1)^i v_1^3 d$, it follows that $\alpha_1 = 0 = \alpha_2$. Since $aZ = (-1)^i \xi^4 Za$, $bZ = (-1)^i \xi^4 Zb$, $cZ = (-1)^i \xi^4 Zc$ and $dZ = (-1)^i \xi^4 Zd$, it follows that $\beta_1 = 0 = \beta_2$. Consequently, $A \cong \text{gr } A$.

The proof for $(i, j, k, \iota) \in \Lambda^2 \cup \Lambda^3$ follows the same lines as for $(i, j, k, \iota) \in \Lambda^1$. $\square$

Proposition 5.10. *Let A be a finite-dimensional Hopf algebra over $\mathcal{K}_{24,1}$ such that* $\operatorname{gr} A \cong \mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ *for $(i, j, k, \iota) \in \Lambda^5 \cup \Lambda^6$. Then $A \cong \operatorname{gr} A$.*

Proof. Assume that $(i, j, k, \iota) \in \Lambda^5$. Observe that $i \in \{0, 3\}$. If the relation $v_1^2 = 0$ admits non-trivial deformations, then $v_1^2 \in A_{[1]}$. Hence there exist some elements $\alpha_{p,q,r}, \beta_{p,q,r}, \gamma_{p,q,r}, \lambda_{p,q,r} \in \Bbbk$ with $p + q, p, q \in \mathbb{I}_{0,1}, r \in \mathbb{I}_{0,5}$ such that

$$v_1^2 = \sum \alpha_{p,q,r} v_1^p v_2^q a^r + \beta_{p,q,r} v_1^p v_2^q d a^r + \gamma_{p,q,r} v_1^p v_2^q b a^r + \lambda_{p,q,r} v_1^p v_2^q c a^r.$$

Since $a v_1^2 = v_1^2 a, b v_1^2 = v_1^2 b, c v_1^2 = v_1^2 c, d v_1^2 = v_1^2 d$, it follows that $\alpha_{p,q,r} = \beta_{p,q,r} = \gamma_{p,q,r} = \lambda_{p,q,r} = 0$ with $p + q, p, q \in \mathbb{I}_{0,1}, r \in \mathbb{I}_{0,5}$ and hence $v_1^2 = 0$ in A. Similarly, we have that $v_1 v_2 + \xi^{5j} v_2 v_1 = 0$ since

$$a(v_1 v_2 + (-1)^k \xi^{5j} v_2 v_1) = \xi^{-1}(v_1 v_2 + (-1)^k \xi^{5j} v_2 v_1)a,$$
$$b(v_1 v_2 + (-1)^k \xi^{5j} v_2 v_1) = \xi^{-1}(v_1 v_2 + (-1)^k \xi^{5j} v_2 v_1)b,$$
$$c(v_1 v_2 + (-1)^k \xi^{5j} v_2 v_1) = -\xi^{-1}(v_1 v_2 + (-1)^k \xi^{5j} v_2 v_1)c,$$
$$d(v_1 v_2 + (-1)^k \xi^{5j} v_2 v_1) = -\xi^{-1}(v_1 v_2 + (-1)^k \xi^{5j} v_2 v_1)d.$$

If $N = 6$, then the relation $v_2^6 = 0$ must hold in A since $\Delta(v_2^6) = v_2^6 \otimes 1 + 1 \otimes v_2^6$. If $N = 3$, then the relation $v_2^3 = 0$ must hold in A since $a v_2^3 = (-1)^{i+\iota} v_2^3 a, b v_2^3 = (-1)^{i+\iota} v_2^3 b, c v_2^3 = (-1)^i v_2^3 c, d v_2^3 = (-1)^i v_2^3 d$ and $v_2^3 \in \mathcal{P}_{1, da^{3j-1}}(A)$ or $\mathcal{P}_{1, a^3}(A)$. Consequently, $A \cong \operatorname{gr} A$.

The proof for $(i, j, k, \iota) \in \Lambda^6$ follows the same lines as for $(i, j, k, \iota) \in \Lambda^5$. $\square$

Finally, we give the classification of finite-dimensional Hopf algebras over $\mathcal{K}_{24,1}$ whose infinitesimal braidings are indecomposable objects in ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$.

Theorem 5.11. *Let A be a finite-dimensional Hopf algebra over $\mathcal{K}_{24,1}$ whose infinitesimal braiding V is indecomposable in ${}^{\mathcal{K}_{24,1}}_{\mathcal{K}_{24,1}}\mathcal{YD}$. Then A is isomorphic to one of the following objects:*

- *$\bigwedge \Bbbk_{\chi_{i,j,k}} \sharp \mathcal{K}_{24,1}$ for $(i, j, k) \in \Lambda^0$;*
- *$\mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ for $(i, j, k, \iota) \in \cup_{i=1}^6 \Lambda^i - \Lambda^{1*}$;*
- *$\mathcal{C}_{i,j,k,\iota}(\mu)$ for $\mu \in \Bbbk$ and $(i, j, k, \iota) \in \Lambda^{1*}$.*

Proof. Since $A_{[0]} \cong \mathcal{K}_{24,1}$, $\operatorname{gr} A \cong R \sharp \mathcal{K}_{24,1}$. By [2, Theorem 1.3], V is simple and $R \cong \mathcal{B}(V)$. Then by Theorem 4.7, V is isomorphic either to $\Bbbk_{\chi_{i,j,k}}$ for $(i, j, k) \in \Lambda^0$ or to $V_{i,j,k,\iota}$ for $(i, j, k, \iota) \in \cup_{i=1}^6 \Lambda^i$. The theorem follows by Propositions 5.7–5.10. The Hopf algebras from different families are pairwise non-isomorphic since the diagrams are not isomorphic as Yetter–Drinfeld modules over $\mathcal{K}_{24,1}$. $\square$

Remark 5.12.
- $\bigwedge \Bbbk_{\chi_{i,j,k}} \sharp \mathcal{K}_{24,1}$ with $(i, j, k) \in \Lambda^0$ are basic Hopf algebras of dimension 48.
- For $(i, j, k, \iota) \in \Lambda^4$, Λ^5 and Λ^6, $\mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ are basic Hopf algebras of dimension 96, 144 and 288, respectively.

- For $(i, j, k, \iota) \in \Lambda^1$ and $\Lambda^2 \cup \Lambda^3$, $\mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ are basic Hopf algebras of dimension 432 and 864, respectively.
- $\mathcal{C}_{i,j,k,\iota}(\mu)$ with $\mu \neq 0$ are non-trivial liftings of $\mathcal{B}(V_{i,j,k,\iota}) \sharp \mathcal{K}_{24,1}$ for $(i, j, k, \iota) \in \Lambda^{1*} \subset \Lambda^1$. They constitute new examples of Hopf algebras without the dual Chevalley property.

Acknowledgment

References

[1] N. Andruskiewitsch and I. Angiono, On finite dimensional Nichols algebras of diagonal type, *Bull. Math. Sci.* **7** no. 3 (2017), 353–573. DOI   MR   Zbl

[2] N. Andruskiewitsch and I. Angiono, On Nichols algebras over basic Hopf algebras, *Math. Z.* **296** no. 3-4 (2020), 1429–1469. DOI   MR   Zbl

[3] N. Andruskiewitsch and M. Beattie, Irreducible representations of liftings of quantum planes, in *Lie Theory and Its Applications in Physics V*, World Scientific, River Edge, NJ, 2004, pp. 414–423. DOI   MR   Zbl

[4] N. Andruskiewitsch and J. Cuadra, On the structure of (co-Frobenius) Hopf algebras, *J. Noncommut. Geom.* **7** no. 1 (2013), 83–104. DOI   MR   Zbl

[5] N. Andruskiewitsch and J. M. J. Giraldi, Nichols algebras that are quantum planes, *Linear Multilinear Algebra* **66** no. 5 (2018), 961–991. DOI   MR   Zbl

[6] N. Andruskiewitsch and M. Graña, Braided Hopf algebras over non abelian finite groups, *Bol. Acad. Nac. Cienc. (Córdoba)* **63** (1999), 45–78. MR   Zbl

[7] N. Andruskiewitsch, I. Heckenberger, and H.-J. Schneider, The Nichols algebra of a semisimple Yetter-Drinfeld module, *Amer. J. Math.* **132** no. 6 (2010), 1493–1547. MR   Zbl

[8] N. Andruskiewitsch and H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3, *J. Algebra* **209** no. 2 (1998), 658–691. DOI   MR   Zbl

[9] I. Angiono, On Nichols algebras of diagonal type, *J. Reine Angew. Math.* **683** (2013), 189–251. DOI   MR   Zbl

[10] I. Angiono, A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems, *J. Eur. Math. Soc. (JEMS)* **17** no. 10 (2015), 2643–2671. DOI   MR   Zbl

[11] G. A. García and J. M. J. Giraldi, On Hopf algebras over quantum subgroups, *J. Pure Appl. Algebra* **223** no. 2 (2019), 738–768. DOI   MR   Zbl

[12] M. Graña, A freeness theorem for Nichols algebras, *J. Algebra* **231** no. 1 (2000), 235–257. DOI   MR   Zbl

[13] M. Graña, On Nichols algebras of low dimension, in *New Trends in Hopf Algebra Theory (La Falda, 1999)*, Contemp. Math. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 111–134. DOI   MR   Zbl

[14] L. GRUNENFELDER and M. MASTNAK, *Pointed Hopf algebras as cocycle deformations*, 2010. arXiv:1010.4976 [math.QA].

[15] I. HECKENBERGER, Classification of arithmetic root systems, *Adv. Math.* **220** no. 1 (2009), 59–124. DOI MR Zbl

[16] I. HECKENBERGER and H.-J. SCHNEIDER, Yetter-Drinfeld modules over bosonizations of dually paired Hopf algebras, *Adv. Math.* **244** (2013), 354–394. DOI MR Zbl

[17] N. HU and R. XIONG, On families of Hopf algebras without the dual Chevalley property, *Rev. Un. Mat. Argentina* **59** no. 2 (2018), 443–469. DOI MR Zbl

[18] L. KROP and D. E. RADFORD, Finite-dimensional Hopf algebras of rank one in characteristic zero, *J. Algebra* **302** no. 1 (2006), 214–230. DOI MR Zbl

[19] S. MAJID and R. OECKL, Twisting of quantum differentials and the Planck scale Hopf algebra, *Comm. Math. Phys.* **205** no. 3 (1999), 617–655. DOI MR Zbl

[20] O. MÁRQUEZ, D. BAGIO, J. M. J. GIRALDI, and G. A. GARCÍA, Finite-dimensional Nichols algebras over dual Radford algebras, *J. Algebra Appl.* **20** no. 1 (2021), Paper No. 2140001, 39 pp. DOI MR Zbl

[21] S. MONTGOMERY, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conference Series in Mathematics 82, American Mathematical Society, Providence, RI, 1993. DOI MR Zbl

[22] D. E. RADFORD, *Hopf Algebras*, Series on Knots and Everything 49, World Scientific, Hackensack, NJ, 2012. MR Zbl

[23] R. XIONG, On Hopf algebras over the unique 12-dimensional Hopf algebra without the dual Chevalley property, *Comm. Algebra* **47** no. 4 (2019), 1516–1540. DOI MR Zbl

[24] R. XIONG and N. HU, Classification of finite-dimensional Hopf algebras over dual Radford algebras, *Bull. Belg. Math. Soc. Simon Stevin* **28** no. 5 (2022), 633–688. DOI MR Zbl

Rongchuan Xiong
School of Computer Science and Artificial Intelligence, Changzhou University, Changzhou 213164, China
rcxiong@foxmail.com