

COUPLING LOCAL AND NONLOCAL EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. We introduce two different ways of coupling local and nonlocal equations with Neumann boundary conditions in such a way that the resulting model is naturally associated with an energy functional. For these two models we prove that there is a minimizer of the resulting energy that is unique modulo adding a constant.

1. INTRODUCTION

Nonlocal models can be used to describe phenomena (including problems characterized by long-range interactions and discontinuities) that cannot be well represented by classical partial differential equations (PDE). For instance, long-range interactions effectively describe anomalous diffusion and crack formation results in material models. The fundamental difference between nonlocal models and classical local models is the fact that the latter only involve differential operators (local equations), whereas the former rely on integral operators (nonlocal equations). For general references on nonlocal models and their applications to elasticity, population dynamics, image processing, etc., we refer the reader to [6, 7, 9, 10, 11, 12, 13, 18, 21, 25, 27, 33, 35, 34, 36, 37, 38] and the book [2].

It is often the case that nonlocal effects occur only in some parts of the domain, whereas, in the remaining parts, the system can be accurately described by a local equation. The goal of coupling local and nonlocal models is to combine a local equation (a PDE) with a nonlocal one (an integral equation) acting in different parts of the domain, under the assumption that the spacial location of local and nonlocal effects can be identified in advance. In this context, one of the challenges of a coupling strategy is to provide a mathematically consistent formulation.

Along this work we consider an open bounded domain $\Omega \subset \mathbb{R}^N$. In our models it is assumed that Ω is divided into two disjoint subdomains; a local region that we

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will denote by Ω_ℓ and a nonlocal region $\Omega_{n\ell}$. Thus we have $\Omega_\ell, \Omega_{n\ell} \subset \Omega \subset \mathbb{R}^N$ with $\Omega = (\overline{\Omega_\ell} \cup \overline{\Omega_{n\ell}})^\circ$. Our main goal in this paper is to introduce two different ways of coupling a local classical PDE in Ω_ℓ with a nonlocal equation in $\Omega_{n\ell}$ in such a way that the resulting problem is naturally associated with an energy functional that is invariant under the addition of a constant (as is the usual case for Neumann boundary conditions in the literature). This paper is a continuation of [1], where we tackled the Dirichlet case.

Let us first recall some well-known facts: for the classical Laplacian (a local operator) with homogeneous Neumann boundary conditions, the model problem reads as

$$\begin{cases} \Delta u(x) = f(x), & x \in \Omega_\ell, \\ \frac{\partial u}{\partial \eta}(x) = 0, & x \in \partial\Omega_\ell. \end{cases} \quad (1.1)$$

Here, u is the unknown and f is an external source. Associated to the problem we have the natural energy

$$F_\ell(u) = \frac{1}{2} \int_{\Omega_\ell} |\nabla u(x)|^2 dx - \int_{\Omega_\ell} f(x)u(x) dx$$

that is well posed in the space $H_\ell = \{u \in H^1(\Omega_\ell) : \int_{\Omega_\ell} u(y) dy = 0\}$. Notice that we need to impose that $\int_{\Omega_\ell} f(x) dx = 0$ in order to have existence of solutions to (1.1), and in this case we get existence and uniqueness for (1.1) modulo an additive constant: there is a unique solution to (1.1) with $\int_{\Omega_\ell} u(x) dx = 0$ (that is obtained as a minimizer of the energy F_ℓ in H_ℓ), and any other solution can be obtained by adding a constant. For the proofs we refer to the textbook [20].

For a nonlocal counterpart of (1.1) in $\Omega_{n\ell}$, we need to introduce a nonnegative kernel $J : \mathbb{R}^N \mapsto \mathbb{R}$, and then we consider as a nonlocal analogous to (1.1) the following equation:

$$2 \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x)) dy = f(x), \quad x \in \Omega_{n\ell}. \quad (1.2)$$

Assuming that the kernel is symmetric, $J(z) = J(-z)$, we have the associated energy functional

$$F_{n\ell}(u) = \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)|u(y) - u(x)|^2 dy dx - \int_{\Omega_{n\ell}} f(x)u(x) dx.$$

When the kernel is in L^1 this functional can be considered in the space $H_{n\ell} = \{u \in L^2(\Omega_{n\ell}) : \int_{\Omega_{n\ell}} u(x) dx = 0\}$. As for the local case, we need to assume that $\int_{\Omega_{n\ell}} f(x) dx = 0$ and again we have existence and uniqueness for solutions to (1.2) modulo a constant (there is a unique solution to (1.1) with $\int_{\Omega_{n\ell}} u(x) dx = 0$ and it is obtained as a minimizer of $F_{n\ell}$ in $H_{n\ell}$), see [2].

Our main goal here is to present two different ways of coupling local (in Ω_ℓ) and nonlocal models (in $\Omega_{n\ell}$) in such a way that the resulting problem (in the whole Ω) has the same properties as the previous two Neumann problems; the problem is

naturally associated with an energy functional and it has a unique solution (up to an additive constant) when the external source is such that $\int_{\Omega} f(x) dx = 0$.

1.1. First model. Volumetric couplings. Let us present our first model coupling local and nonlocal equations in two disjoint subdomains $\Omega_{\ell}, \Omega_{n\ell} \subset \Omega \subset \mathbb{R}^N$. For $u : \Omega \mapsto \mathbb{R}$, we consider the local/nonlocal energy

$$E_{\text{Neu}}(u) := \int_{\Omega_{\ell}} \frac{|\nabla u(x)|^2}{2} dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)|u(y) - u(x)|^2 dy dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{\ell}} G(x,y)|u(y) - u(x)|^2 dy dx - \int_{\Omega} f(x)u(x) dx, \tag{1.3}$$

and we look for critical points (minimizers) of this energy and the corresponding equations that they satisfy.

In the functional (1.3) we can identify the local part of the energy in Ω_{ℓ} ,

$$\int_{\Omega_{\ell}} \frac{|\nabla u(x)|^2}{2} dx; \tag{1.4}$$

the nonlocal part, acting in $\Omega_{n\ell}$,

$$\frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)|u(y) - u(x)|^2 dy dx; \tag{1.5}$$

a coupling term, that involves integrals in Ω_{ℓ} and in $\Omega_{n\ell}$, and a different kernel $G(x, y)$,

$$\frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{\ell}} G(x,y)|u(y) - u(x)|^2 dy dx$$

(here $G : \Omega_{n\ell} \times \Omega_{\ell} \mapsto \mathbb{R}$ is assumed to be nonnegative but not necessarily symmetric; notice that the two variables belong to different sets); and finally, the term that involves the external source

$$\int_{\Omega} f(x)u(x) dx. \tag{1.6}$$

Now, let us state our hypothesis on the involved domains and kernels. With $J : \mathbb{R}^N \mapsto \mathbb{R}$ we denote a nonnegative measurable function that satisfies:

- (J1) *Visibility*: there exist $\delta > 0$ and $C > 0$ such that $J(z) > C$ for all z such that $\|z\| \leq 2\delta$.
- (J2) *Compactness*: the convolution type operator

$$T_J(u)(x) = \int_{\Omega_{n\ell}} J(x-y)u(y) dy$$

defines a compact operator in $L^2(\Omega_{n\ell})$.

In nonlocal models, J is a kernel that encodes the effect of a general *volumetric* nonlocal interaction inside the nonlocal part of the domain. Condition (J1) guarantees the influence of nonlocality within a horizon of size at least 2δ while (J2) is a technical requirement fulfilled, for instance, by continuous kernels, characteristic

functions, or even for L^2 kernels (this holds since these kernels produce Hilbert–Schmidt operators of the form $u \mapsto T(u)(x) := \int k(x, y)u(y) dy$ that are compact if $k \in L^2$, see [8, Chapter VI]). We also need to introduce a connectivity condition.

Definition 1.1. We say that an open set $D \subset \mathbb{R}^N$ is δ -connected, with $\delta \geq 0$, if it cannot be written as a disjoint union of two (relatively) open nontrivial sets that are at distance greater than or equal to δ .

Notice that if a set D is δ -connected, then it is δ' -connected for any $\delta' \geq \delta$. From Definition 1.1, we notice that 0-connectedness agrees with the classical notion of being connected (in particular, open connected sets are δ -connected). Definition 1.1 can be written in an equivalent way: an open set D is δ -connected if given two points $x, y \in D$, there exists a finite number of points $x_0, x_1, \dots, x_n \in D$ such that $x_0 = x$, $x_n = y$ and $\text{dist}(x_i, x_{i+1}) < \delta$.

Informally, δ -connectedness combined with (J1) says that the effect of nonlocality can travel beyond the horizon 2δ through the whole domain.

Now we can write the following assumptions on the local/nonlocal domains:

- (1) Ω_ℓ is connected and smooth (Ω_ℓ has Lipschitz boundary),
- (2) $\Omega_{n\ell}$ is δ -connected.

Concerning the kernel G involved in the coupling term, which encodes the interactions of $\Omega_{n\ell}$ with Ω_ℓ , we assume that it is given by a nonnegative and measurable function $G : \Omega_\ell \times \Omega_{n\ell} \mapsto \mathbb{R}$ such that

- (G1) there exist $\delta > 0$ and $C > 0$ such that, for any $(x, y) \in \Omega_{n\ell} \times \Omega_\ell$, $G(x, y) > C$ if $\|x - y\| \leq 2\delta$.

Finally, in order to avoid trivial couplings, we impose that Ω_ℓ and $\Omega_{n\ell}$ need to be closer than the horizon of the kernel involved in the coupling; we assume that

- (P1) $\text{dist}(\Omega_\ell, \Omega_{n\ell}) < \delta$.

Remark 1.2. Our results are valid for more general domains. In fact, we assumed that Ω_ℓ is connected and that $\Omega_{n\ell}$ is δ -connected with $\text{dist}(\Omega_\ell, \Omega_{n\ell}) < \delta$, but we can also handle the case in which Ω_ℓ has several connected components and $\Omega_{n\ell}$ has several δ -connected components as long as they are close between them. We prefer to state our results under conditions (1), (2), (G1) and (P1) just to simplify the presentation.

Now, with all these conditions at hand we go back to our energy functional (1.3) and look for possible critical points (minimizers). Here, as usual in Neumann problems, we have to assume that

$$\int_{\Omega} f(x) dx = 0$$

and look for minimizers in the natural function space

$$H_{\text{Neu}} = \left\{ u \in L^2(\Omega) : u|_{\Omega_\ell} \in H^1(\Omega_\ell), \int_{\Omega} u(x) dx = 0 \right\}.$$

Let us state our first theorem.

Theorem 1.3. *Assume that the kernels and the domains satisfy (J1), (J2), (1), (2), (G1) and (P1). Given $f \in L^2(\Omega)$ with $\int_{\Omega} f = 0$, there exists a unique minimizer of $E_{\text{Neu}}(u)$ in H_{Neu} . The minimizer of $E_{\text{Neu}}(u)$ in H_{Neu} is a weak solution to the problem*

$$\begin{cases} -f(x) = \Delta u(x) + \int_{\Omega_{n\ell}} G(y, x)(u(y) - u(x)) \, dy, & x \in \Omega_{\ell}, \\ \frac{\partial u}{\partial \eta}(x) = 0, & x \in \partial\Omega_{\ell}, \end{cases}$$

in the local domain Ω_{ℓ} together with a nonlocal equation with a source in $\Omega_{n\ell}$,

$$-f(x) = 2 \int_{\Omega_{n\ell}} J(x - y)(u(y) - u(x)) \, dy + \int_{\Omega_{\ell}} G(x, y)(u(y) - u(x)) \, dy, \quad x \in \Omega_{n\ell}.$$

Notice that in the resulting equations the coupling terms between the local and the nonlocal regions appear as source integral terms in the corresponding equations.

The key to obtaining this result will be to prove a Poincaré–Wirtinger type inequality: there exists $c > 0$ such that

$$\begin{aligned} \int_{\Omega_{\ell}} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y)(u(y) - u(x))^2 \, dy \, dx \\ + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{\ell}} G(x, y)|u(y) - u(x)|^2 \, dy \, dx \geq c \int_{\Omega} u^2(x) \, dx \end{aligned}$$

for every function $u \in H_{\text{Neu}}$. Here the constant $c > 0$ can be estimated in terms of the parameters of the problem (the geometry of the domains and the kernels).

1.2. Second model. Mixed couplings. Now, let us present our second model. In the first model we have a nonlocal volumetric coupling between Ω_{ℓ} and $\Omega_{n\ell}$. In the second model we introduce *mixed* couplings. In mixed couplings volumetric and lower-dimensional parts can interact with each other. The mixed couplings that we introduce involve interactions of $\Omega_{n\ell}$ with a fixed smooth hypersurface

$$\Gamma \subset \overline{\Omega_{\ell}}.$$

For $u : \Omega \mapsto \mathbb{R}$, we consider the energy

$$\begin{aligned} F_{\text{Neu}}(u) := \frac{1}{2} \int_{\Omega_{\ell}} |\nabla u(x)|^2 \, dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y) (u(y) - u(x))^2 \, dy \, dx \\ + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, z) (u(x) - u(z))^2 \, d\sigma(z) \, dx - \int_{\Omega} f(x)u(x) \, dx. \end{aligned}$$

Here, we can identify again the local part of the energy acting in Ω_{ℓ} (1.4), the nonlocal part acting in $\Omega_{n\ell}$ (1.5), and the external source (1.6); but now the coupling term is different (now it involves integrals in $\Omega_{n\ell}$ and on the surface $\Gamma \subset \overline{\Omega_{\ell}}$):

$$\frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, z) (u(x) - u(z))^2 \, d\sigma(z) \, dx.$$

In this context, we assume the same conditions on J as for the first model. Concerning the coupling, a nonnegative and measurable function $G : \Gamma \times \Omega_{n\ell} \mapsto \mathbb{R}$

plays the role of the associated kernel. The following condition is analogous to the volumetric counterpart:

(G2) there exist $\delta > 0$ and $C > 0$ such that, for any $(x, y) \in \Omega_{nl} \times \Gamma$, $G(x, y) > C$ if $\|x - y\| \leq 2\delta$.

Again, to avoid trivial couplings we assume that

(P2) $\text{dist}(\Gamma, \Omega_{nl}) < \delta$.

Here again we have to assume that $\int_{\Omega} f(x) dx = 0$, and then we look for minimizers in $H_{\text{Neu}} = \{u \in L^2(\Omega) : u|_{\Omega_{\ell}} \in H^1(\Omega_{\ell}), \int_{\Omega} u(x) dx = 0\}$. As before, we can obtain the existence and uniqueness of a minimizer of $F_{\text{Neu}}(u)$ in H_{Neu} .

Theorem 1.4. *Assume that the kernels and the domains satisfy (J1), (J2), (1), (2), (G2) and (P2). Given $f \in L^2(\Omega)$ with $\int_{\Omega} f = 0$, there exists a unique minimizer of F_{Neu} in H_{Neu} . The minimizer of F_{Neu} is a weak solution to*

$$\begin{cases} -f(x) = \Delta u(x), & x \in \Omega_{\ell}, \\ \frac{\partial u}{\partial \eta}(x) = 0, & x \in \partial\Omega_{\ell} \setminus \Gamma, \\ \frac{\partial u}{\partial \eta}(x) = \int_{\Omega_{nl}} G(y, x)(u(y) - u(x)) dy, & x \in \Gamma, \end{cases}$$

and

$$-f(x) = 2 \int_{\Omega_{nl}} J(x-y)(u(y) - u(x)) dy - \int_{\Gamma} G(x, z)(u(x) - u(z)) d\sigma(z), \quad x \in \Omega_{nl}.$$

Notice that now the coupling term between the local and the nonlocal regions in the local part of the problem appears as a flux condition on Γ .

1.3. Singular kernels. We can also deal with singular kernels related to the fractional Laplacian and consider energies of the form

$$\begin{aligned} E_{\text{Neu}}(u) := & \int_{\Omega_{\ell}} \frac{|\nabla u(x)|^2}{2} dx + \frac{1}{2} \int_{\Omega_{nl}} \int_{\Omega_{nl}} \frac{C}{|x - y|^{N+2s}} |u(y) - u(x)|^2 dy dx \\ & + \frac{1}{2} \int_{\Omega_{nl}} \int_{\Omega_{\ell}} G(x, y) |u(y) - u(x)|^2 dy dx - \int_{\Omega} f(x)u(x) dx. \end{aligned}$$

Here we look for minimizers in the space

$$H_{\text{Neu}} = \left\{ u : u|_{\Omega_{\ell}} \in H^1(\Omega_{\ell}), u|_{\Omega_{nl}} \in H^s(\Omega_{nl}), \int_{\Omega} u(x) dx = 0 \right\}.$$

This case is much simpler since we have the compact embeddings $H^1(\Omega_{\ell}) \hookrightarrow L^2(\Omega_{\ell})$ and $H^s(\Omega_{nl}) \hookrightarrow L^2(\Omega_{nl})$. Also for these energies we can show the following result (that is the fractional counterpart to our previous results).

Theorem 1.5. *Given $f \in L^2(\Omega)$ with $\int_{\Omega} f = 0$, there exists a unique minimizer of $E_{\text{Neu}}(u)$ in H_{Neu} . The minimizer of $E_{\text{Neu}}(u)$ in H_{Neu} is a weak solution to the*

equation

$$\begin{cases} -f(x) = \Delta u(x) + \int_{\Omega_{n\ell}} G(x, y)(u(y) - u(x)) dy, & x \in \Omega_\ell, \\ \frac{\partial u}{\partial \eta}(x) = 0, & x \in \partial\Omega_\ell, \end{cases}$$

and a nonlocal equation with a source in $\Omega_{n\ell}$,

$$-f(x) = 2 \text{P.V.} \int_{\Omega_{n\ell}} \frac{C}{|x - y|^{N+2s}}(u(y) - u(x)) dy + \int_{\Omega_\ell} G(x, y)(u(y) - u(x)) dy,$$

$x \in \Omega_{n\ell}$.

We can also deal with mixed couplings and look for minimizers of

$$\begin{aligned} F_{\text{Neu}}(u) := & \frac{1}{2} \int_{\Omega_\ell} |\nabla u(x)|^2 dx + \frac{1}{2} \int_{\Omega_{nl}} \int_{\Omega_{nl}} \frac{C}{|x - y|^{N+2s}} (u(y) - u(x))^2 dy dx \\ & + \frac{1}{2} \int_{\Omega_{nl}} \int_{\Gamma} G(x, z) (u(x) - u(z))^2 d\sigma(z) dx - \int_{\Omega} f(x)u(x) dx. \end{aligned}$$

Theorem 1.6. *Given $f \in L^2(\Omega)$ with $\int_{\Omega} f = 0$, there exists a unique minimizer of $F_{\text{Neu}}(u)$ in H_{Neu} . The minimizer of F_{Neu} is a weak solution to*

$$\begin{cases} -f(x) = \Delta u(x), & x \in \Omega_\ell, \\ \frac{\partial u}{\partial \eta}(x) = 0, & x \in \partial\Omega_\ell \setminus \Gamma, \\ \frac{\partial u}{\partial \eta}(x) = \int_{\Omega_{nl}} G(y, x)(u(y) - u(x)) dy, & x \in \Gamma, \end{cases}$$

and

$$-f(x) = 2 \int_{\Omega_{n\ell}} \frac{C}{|x - y|^{N+2s}}(u(y) - u(x)) dy - \int_{\Gamma} G(x, z)(u(x) - u(z)) d\sigma(z),$$

$x \in \Omega_{nl}$.

Remark 1.7. The kernel G can also be a singular kernel

$$G(x, z) = \frac{c}{|x - z|^{N+2t}}.$$

In this case one has to add the condition

$$\int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y)|u(y) - u(x)|^2 dy dx < +\infty$$

in the space H_{Neu} . The proof that there is a minimizer is similar and hence we leave the details to the reader.

Remark 1.8. When $s > 1/2$, since we have a trace theorem for $H^s(\Omega_{n\ell})$ we can also deal with surface to surface couplings and study

$$F_{\text{Neu}}(u) := \frac{1}{2} \int_{\Omega_\ell} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} \frac{C}{|x-y|^{N+2s}} (u(y) - u(x))^2 dy dx \\ + \frac{1}{2} \int_{\Gamma_\ell} \int_{\Gamma_{n\ell}} G(x, z) (u(x) - u(z))^2 d\sigma(z) d\sigma(x) - \int_{\Omega} f(x)u(x) dx.$$

Here Γ_ℓ is a smooth hypersurface in $\overline{\Omega_\ell}$ and $\Gamma_{n\ell}$ is a smooth hypersurface in $\overline{\Omega_{n\ell}}$. Notice that now we are coupling the domains via interactions on hypersurfaces. The hypothesis on the kernel G that is needed to obtain a coupling between the two domains is, as before, some strict positivity on pairs of points belonging to the coupling surfaces.

Let us end the introduction with a brief description of previous references. From a mathematical point of view, interesting problems arise from coupling local and nonlocal models, see [4, 5, 16, 17, 19, 22, 23, 26, 29] and references therein. As previous examples of coupling approaches between local and nonlocal regions, we refer the reader to [3, 4, 5, 16, 17, 14, 19, 22, 23, 24, 26, 29, 30, 31, 32], the survey [15] and references therein. Previous strategies treat the coupling condition as an optimization problem (the goal is to minimize the mismatch of the local and nonlocal solutions in a common overlapping region). Another possible strategy for coupling relies on the partitioned procedure as a general coupling strategy for heterogeneous systems: the system is divided into sub-problems in their respective sub-domains, which communicate with each other via transmission conditions. Moreover, couplings between sets of different dimension are possible. In [7] the effects of network transportation on enhancing biological invasion are studied. The proposed mathematical model consists of one equation with nonlocal diffusion in a one-dimensional domain coupled via the boundary condition with a standard reaction-diffusion in a two-dimensional domain. In [16], local and nonlocal problems were coupled through a prescribed region in which both kinds of equations overlap (the value of the solution in the nonlocal part of the domain is used as a Dirichlet boundary condition for the local part and vice-versa). This kind of coupling gives continuity of the solution in the overlapping region but does not preserve the total mass when Neumann boundary conditions are imposed. In [16] and [19], numerical schemes using local and nonlocal equations were developed and used to improve the computational accuracy when approximating a purely nonlocal problem. In [23] and [29] (see also [22, 26]), evolution problems related to energies closely related to ours are studied (here we deal with stationary problems).

The paper is organized as follows: In Section 2 we deal with our first model involving volumetric coupling and prove Theorem 1.3, while in Section 3 we prove the results concerning the second model, Theorem 1.4. Finally, in Section 4 we deal with singular kernels.

2. VOLUMETRIC COUPLING

First model. Coupling local/nonlocal problems via source terms. Our aim is to look for a minimizer of the energy

$$E_{\text{Neu}}(u) := \int_{\Omega_\ell} \frac{|\nabla u|^2}{2} + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)|u(y) - u(x)|^2 dy dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x,y)|u(y) - u(x)|^2 dy dx - \int_{\Omega} f(x)u(x) dx$$

in the space

$$H_{\text{Neu}} = \left\{ u \in L^2(\Omega) : u|_{\Omega_\ell} \in H^1(\Omega_\ell), \int_{\Omega} u = 0 \right\},$$

assuming

$$\int_{\Omega} f(x) dx = 0.$$

Let us first prove an auxiliary lemma.

Lemma 2.1. *Let D be an open δ -connected set and $u : D \rightarrow \mathbb{R}$. If*

$$\int_D \int_D J(x-y)|u(x) - u(y)|^2 dx dy = 0,$$

then $u(x) = k$ a.e. $x \in D$.

Proof. Pick $x_0 \in D$ and a ball $B_0 = B_\delta(x_0)$; we have

$$C \int_{D \cap B_0} \int_{D \cap B_0} (u(x) - u(y))^2 \leq \int_{D \cap B_0} \int_{D \cap B_0} J(x-y)(u(x) - u(y))^2 = 0$$

(since $J(x-y) > C$ for $x, y \in B_0$), hence $u(x) = k_0$ a.e. $x \in D \cap B_0$. In order to see that this property holds a.e. $x \in D$, let us introduce the set $\mathcal{M} = \{A \subset D, A \text{ open} : u(x) = k_0 \text{ a.e. } x \in A\}$ with the partial order given by \subset . Since $\mathcal{M} \neq \emptyset$, there exists a maximal open set $M \in \mathcal{M}$. If $M \subsetneq \Omega$, then we consider the set $\emptyset \neq D \setminus M$.

If $D \setminus M$ is open, we necessarily have that $\text{dist}(M, D \setminus M) < \delta$ (here we are using that D is δ -connected). If $D \setminus M$ is not open, then $\text{dist}(M, D \setminus M) = 0$ (since D is open). In either case, there exists a ball B_1 of radius δ such that $B_1 \cap D \setminus M \neq \emptyset$ and $B_1 \cap M$ has positive measure (since both B_1 and M are open sets). Arguing as before, we see that $u(x) = k_0$ a.e. $x \in B_1 \cap D$, a contradiction (since M is maximal with that property and we would have $M \subsetneq M \cup (B_1 \cap D) = M$). We see that $M = D$, and the proof is complete. \square

Lemma 2.2. *Let $u_n : \Omega \rightarrow \mathbb{R}$ be a sequence such that $u_n \rightarrow 0$ strongly in $L^2(\Omega_\ell)$ and weakly in $L^2(\Omega_{n\ell})$. If in addition*

$$\lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Omega} J(x-y)(u_n(x) - u_n(y))^2 dy dx = 0, \tag{2.1}$$

then

$$\lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} u_n(x)^2 dx = 0.$$

Proof. From (2.1), the convergence of $\{u_n\}$ and property (J2), we easily find that

$$\lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} J(x-y)u_n(x)^2 dy dx = 0. \tag{2.2}$$

Let us define

$$A_\delta^0 = \left\{ x \in \Omega_{n\ell} : \text{dist}(x, \Omega_\ell) < \delta \right\}.$$

Notice that thanks to property (P1) and to the fact that $\Omega_{n\ell}$ is open, we see that A_δ^0 is open and non-empty. In particular it has positive n -dimensional measure. For any $x \in \overline{A_\delta^0}$, we consider the continuous and *strictly* positive function $g(x) = |B_{2\delta}(x) \cap \Omega_\ell|$. Since $\overline{A_\delta^0}$ is a compact set, there exists a constant $m > 0$ such that $g(x) \geq m$ for any $x \in \overline{A_\delta^0}$. As a consequence,

$$\begin{aligned} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} J(x-y)u_n(x)^2 dy dx &\geq \int_{A_\delta^0} \int_{B_{2\delta}(x) \cap \Omega_\ell} J(x-y)u_n(x)^2 dy dx \\ &\geq mC \int_{A_\delta^0} u_n(x)^2 dx; \end{aligned}$$

and therefore, thanks to (2.2), $u_n \rightarrow 0$ in $L^2(A_\delta^0)$. In order to iterate this argument we notice that at this point we know that $u_n \rightarrow 0$ strongly in A_δ^0 and weakly in $\Omega_{n\ell} \setminus \overline{A_\delta^0}$, hence again from (2.1) we get

$$\lim_{n \rightarrow \infty} \int_{\Omega_{n\ell} \setminus \overline{A_\delta^0}} \int_{A_\delta^0} J(x-y)u_n(x)^2 dy dx = 0. \tag{2.3}$$

Since $\Omega_{n\ell}$ is δ -connected, $\text{dist}(\Omega_{n\ell} \setminus \overline{A_\delta^0}, A_\delta^0) < \delta$. Considering now

$$A_\delta^1 = \left\{ x \in \Omega_{n\ell} \setminus \overline{A_\delta^0} : \text{dist}(x, A_\delta^0) < \delta \right\},$$

and proceeding as before, we obtain, from (2.3), that $u_n \rightarrow 0$ strongly in A_δ^1 . This argument can be repeated, giving strong convergence in $L^2(A_\delta^j)$ for

$$A_\delta^j = \left\{ x \in \Omega_{n\ell} \setminus \overline{\bigcup_{0 \leq i < j} A_\delta^i} : \text{dist}\left(x, \bigcup_{0 \leq i < j} A_\delta^i\right) < \delta \right\}.$$

Since $\Omega_{n\ell}$ is bounded, we have, for a finite number $J \in \mathbb{N}$,

$$\Omega_{n\ell} = \bigcup_{0 \leq i < J} A_\delta^i,$$

and therefore the proof is complete. □

As we have mentioned in the introduction, our goal is to show that given $f \in L^2(\Omega)$ with $\int_\Omega f = 0$, there exists a unique minimizer of the energy functional $E_{\text{Neu}}(u)$ in the space H_{Neu} .

To use the direct method of calculus of variations to obtain the result we have to show that $E_{\text{Neu}}(u)$ is coercive and weakly lower semicontinuous.

To this end we prove a key result: a Poincaré–Wirtinger type inequality holds.

Lemma 2.3. *There exists $c > 0$ such that*

$$\int_{\Omega_\ell} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y)(u(y) - u(x))^2 dy dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y)|u(y) - u(x)|^2 dy dx \geq c \int_{\Omega} u^2(x) dx$$

for every function $u \in H_{\text{Neu}}$.

Proof. First, let us point out that when G is in convolution form, $G(x, y) = G(x - y)$ and $\Omega_\ell \subset \Omega$ is a smooth convex subdomain, we can use a result from [23]. In this case the norm in H_{Neu} bounds a pure nonlocal seminorm. There exists $c > 0$ such that

$$\int_{\Omega_\ell} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega} J(x - y)(u(y) - u(x))^2 dy dx \geq c \int_{\Omega} \int_{\Omega} J(x - y)(u(y) - u(x))^2 dy dx,$$

for every $u \in \mathcal{H} := \{u \in L^2(\Omega) : u|_{\Omega_\ell} \in H^1(\Omega_\ell)\}$. Then the desired Poincaré–Wirtinger inequality follows from an analogous inequality for the purely nonlocal energy: there exists $c > 0$ such that

$$\int_{\Omega} \int_{\Omega} J(x - y)(u(y) - u(x))^2 dy dx \geq c \int_{\Omega} u^2(x) dx$$

for every function $u \in \mathcal{H}$ such that $\int_{\Omega} u = 0$ (see [2]).

To obtain the Poincaré–Wirtinger inequality in the general case we argue by contradiction. Assume that there is a sequence u_n such that

$$\int_{\Omega_\ell} |\nabla u_n|^2(x) dx \rightarrow 0, \tag{2.4}$$

$$\frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y)|u_n(y) - u_n(x)|^2 dy dx \rightarrow 0, \tag{2.5}$$

$$\frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y)|u_n(y) - u_n(x)|^2 dy dx \rightarrow 0, \tag{2.6}$$

$$\int_{\Omega} |u_n|^2(x) dx = 1 \tag{2.7}$$

and

$$\int_{\Omega} u_n(x) dx = 0. \tag{2.8}$$

Since the L^2 norm is bounded, we can extract a subsequence such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\Omega_\ell);$$

then, from (2.4) we get that there is a constant k_1 such that

$$u_n \rightarrow k_1 \quad \text{strongly in } H^1(\Omega_\ell).$$

Now, from (2.5) (using again that the L^2 norm is bounded) we can refine the subsequence to obtain that

$$u_n \rightharpoonup k_2 \quad \text{weakly in } L^2(\Omega_{n\ell})$$

for some constant k_2 .

From (2.6) we conclude that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y) |k_1 - k_2|^2 dy dx \\ & \leq \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y) |u_n(y) - u_n(x)|^2 dy dx = 0, \end{aligned}$$

and hence

$$k_1 = k_2.$$

We have

$$|\Omega_\ell|k_1 = \lim_n \int_{\Omega_\ell} u_n(x) dx$$

and

$$|\Omega_{n\ell}|k_2 = \lim_n \int_{\Omega_{n\ell}} u_n(x) dx.$$

Then, adding the previous identities and using that $k_1 = k_2$, from (2.8), we obtain

$$k_1 = k_2 = 0.$$

From the strong convergence in $H^1(\Omega_\ell)$ we obtain

$$\lim_n \int_{\Omega_\ell} |u_n|^2(x) dx = 0,$$

and hence from (2.7) we get

$$\int_{\Omega_{n\ell}} |u_n|^2(x) dx \rightarrow 1.$$

This contradicts the result in [2] since there exists $c > 0$ such that

$$\begin{aligned} 0 &= \lim_n \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y) (u_n(y) - u_n(x))^2 dy dx \\ &\geq c \lim_n \int_{\Omega_{n\ell}} \left| u_n(x) - \int_{\Omega_{n\ell}} u_n \right|^2 dx = 1. \end{aligned}$$

This contradiction finishes the proof. □

The proof of Lemma 2.3 is made by contradiction and therefore it does not provide an estimate for the constant. In what follows we present an alternative proof that provides an explicit constant in terms of the relevant quantities, the geometries of the involved domains Ω_ℓ and $\Omega_{n\ell}$, and the kernels J and G (in fact, the constant depends on the parameter δ and $\min J, \min G$ for points (x, y) such that $|x - y| < 2\delta$). In order to do so, we introduce the following remark.

Remark 2.4 (Geometric structure of bounded δ -connected sets). Let D be a bounded δ -connected set. Boundedness implies that it is possible to find a *finite* collection \mathcal{C} of open sets, each of them of diameter less than or equal to $\delta/2$, such that

$$D = \bigcup_{B \in \mathcal{C}} B.$$

We introduce a tree structure in \mathcal{C} in the following way: for a fixed $B \in \mathcal{C}$, which will be called the root of the tree, we set $B_0 = B$. Now we pick $B_1 \neq B_0, B_1 \in \mathcal{C}$ such that $\text{dist}(B_0, B_1) < \delta$ (if more than one set has this property, we choose any of them). Now we proceed in the same way, picking $B_2 \in \mathcal{C} \setminus \{B_0, B_1\}$ such that $\text{dist}(B_1, B_2) < \delta$ and repeat the procedure until we reach a set B_k such that either $\emptyset = \mathcal{C} \setminus \{B_0, B_1, \dots, B_k\}$ or $\emptyset \neq \mathcal{C} \setminus \{B_0, B_1, \dots, B_k\}$ and all the elements in $\mathcal{C} \setminus \{B_0, B_1, \dots, B_k\}$ are at distance greater than or equal to δ from B_k . In the first case we are done and we introduce a (total) order in \mathcal{C} given by $B_i < B_j$ if B_j was chosen later than B_i . In the second case, we call the totally ordered set \mathcal{C}_1^0 obtained a branch with root B_0 . Then we continue with further branches. First, repeating our procedure, we check if with the remaining elements $\mathcal{C} \setminus \mathcal{C}_1^0$ it is possible to get another branch \mathcal{C}_2^0 with the same root B_0 . When all branches with root B_0 are exhausted, we look within the set of the remaining elements for potential branches with root B_1, B_2 , etc. until we reach B_{k-1} . At this point, and iteratively, all elements of all generated branches are tested as roots of new branches. Since \mathcal{C} is finite, this procedure reaches an end in a finite number of steps yielding a partially ordered set \mathcal{T} . Naturally $\mathcal{T} \subset \mathcal{C}$; we claim that $\mathcal{T} = \mathcal{C}$. Otherwise, we consider the non-empty open sets $D_1 = \cup_{B \in \mathcal{C} \setminus \mathcal{T}} B, D_2 = \cup_{B \in \mathcal{T}} B$ and then $D = D_1 \cup D_2$ with $\text{dist}(D_1, D_2) > \delta$, a contradiction since D is δ -connected.

In \mathcal{T} we define the natural *partial* order and \mathcal{T} is called a δ -tree. Notice that for two *consecutive* elements $B < \tilde{B}$, we have $\text{diam}(B, \tilde{B}) < 2\delta$. In particular, from our hypothesis on the kernel J , this implies that $J(x - y) > 0$ for $x \in B, y \in \tilde{B}$.

For further use we introduce the following definitions: we let the degree of a root be the number of branches emerging from that root, while the degree of \mathcal{T} is the maximum degree of all the roots belonging to \mathcal{T} . Finally, the length of a branch is the cardinal of that branch excluding its root.

Remark 2.5. In simple cases it is possible to easily characterize potential tree structures on domains. In a square $D \subset \mathbb{R}^2$ (or a cube in \mathbb{R}^N) we can build \mathcal{T} with a single branch. The cardinal of \mathcal{T} is bounded by $\frac{8^2|D|}{\delta^2}$ (or by $\frac{8^N|D|}{\delta^N}$ in \mathbb{R}^N). Moreover, even rather complicated structures can be covered by simple trees if the scale related to δ is large enough as it is shown in Figures 1 and 2.

With this structure at hand, let us prove the following lemma.

Lemma 2.6. *Let $B^* = \{B_0, B_1, \dots, B_N\}$ be a finite collection of open sets $B_i \subset \mathbb{R}^n$ such that $\text{diam}(B_i \cup B_{i+1}) < 2\delta$. Then, for any $u : \cup_j B_j \rightarrow \mathbb{R}$, it holds that*

$$\sum_{k=0}^N \int_{B_k} |u(x)|^2 dx \leq \left(\sum_{k=0}^N 2^k \frac{|B_k|}{|B_0|} \right) \int_{B_0} u^2 dx + \frac{1}{m_J} \sum_{i=1}^N 2^i \left(\sum_{k=i}^N \frac{|B_k|}{|B_{k-i}||B_{k-i+1}|} \int_{B_{k-i}} \int_{B_{k-i+1}} J(x-y)(u(x) - u(y))^2 dx dy \right),$$

where

$$0 < m_J = \inf_{|z| < 2\delta} J(z).$$

Proof. Since

$$\int_{B_1} |u(x)|^2 dx = \frac{1}{|B_0|} \int_{B_0} \int_{B_1} |u(x)|^2 dx dy,$$

we have

$$\int_{B_1} |u(x)|^2 dx \leq \frac{2}{|B_0|} \int_{B_0} \int_{B_1} |u(x) - u(y)|^2 dx dy + 2 \frac{|B_1|}{|B_0|} \int_{B_0} |u(y)|^2 dx dy,$$

and similarly

$$\int_{B_k} |u(x)|^2 dx \leq \frac{2}{|B_{k-1}|} \int_{B_{k-1}} \int_{B_k} |u(x) - u(y)|^2 dx dy + 2 \frac{|B_k|}{|B_{k-1}|} \int_{B_{k-1}} |u(y)|^2 dx dy.$$

As a consequence, calling $\Delta = u(x) - u(y)$, we get

$$\int_{B_k} |u(x)|^2 dx \leq \sum_{i=1}^k \frac{2^i |B_k|}{|B_{k-i}||B_{k-i+1}|} \int_{B_{k-i}} \int_{B_{k-i+1}} \Delta^2 dx dy + 2^k \frac{|B_k|}{|B_0|} \int_{B_0} u^2 dx,$$

and therefore we obtain

$$\begin{aligned} \sum_{k=0}^N \int_{B_k} |u(x)|^2 dx &\leq \left(\sum_{k=0}^N 2^k \frac{|B_k|}{|B_0|} \right) \int_{B_0} u^2 dx \\ &\quad + \sum_{k=1}^N \left(\sum_{i=1}^k \frac{2^i |B_k|}{|B_{k-i}||B_{k-i+1}|} \int_{B_{k-i}} \int_{B_{k-i+1}} \Delta^2 dx dy \right) \\ &= \left(\sum_{k=0}^N 2^k \frac{|B_k|}{|B_0|} \right) \int_{B_0} u^2 dx \\ &\quad + \sum_{i=1}^N 2^i \left(\sum_{k=i}^N \frac{|B_k|}{|B_{k-i}||B_{k-i+1}|} \int_{B_{k-i}} \int_{B_{k-i+1}} \Delta^2 dx dy \right). \end{aligned}$$

Hence, the lemma follows by using that $1 \leq J(x-y)/m_J$ for any x, y belonging to consecutive sets B_i . □

The previous lemma in particular says that in a branch of a tree structure of sets (as the ones described in Remark 2.4), the L^2 norm of a function defined on that branch can be bounded in terms of the L^2 norm on the root of the branch plus an energy involving the nonlocal operator J along the branch. Taking into account that in a tree \mathcal{T} roots of branches are members of previous branches, it is clear that the L^2 norm of a function defined on \mathcal{T} can be bounded in a similar fashion in terms of the L^2 norm on the root of the tree and the nonlocal energy. Moreover, if needed, explicit constants can be tracked back along the tree structure. For the sake of clarity, we do not state the next result with such a generality, although we provide below some simple examples with explicit constants.

Corollary 2.7. *For any $u \in L^2(\Omega_{n\ell})$, we consider a tree structure \mathcal{T} in $\Omega_{n\ell}$ with root B ; then there exists a constant $C = C(\mathcal{T}, m_J)$ such that*

$$\int_{\Omega_{n\ell}} u^2 dx \leq C \left(\int_B u^2 dx + \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(x) - u(y))^2 dx dy \right).$$

Proof. Immediate from Lemma 2.6. □

Remark 2.8 (Explicit constants). Notice that in some cases an explicit expression for $C(\mathcal{T}, m_J)$ can be obtained. If $\Omega_{n\ell}$ is a cube in \mathbb{R}^N (see Remark 2.5), then we can take \mathcal{T} with a single branch of length bounded by $\frac{8^N |\Omega_{n\ell}|}{\delta^N}$. In this case, using Lemma 2.6 and taking into account that $|B_i| = |B| = c_N \frac{\delta^N}{2^N}$ is constant (here c_N is the measure of the unit ball in \mathbb{R}^N), together with the fact that

$$\sum_{k=i}^N 2^k = 2^i \sum_{k=0}^{N-i} 2^k = 2^i (2^{N-i+1} - 1) \leq 2^{N+1},$$

we can obtain the bound

$$\int_{\Omega_{n\ell}} u^2 dx \leq 2^{\frac{8^N |\Omega_{n\ell}|}{\delta^N} + 1} \left(\int_B u^2 dx + \frac{2^N}{c_N \delta^N m_J} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(x) - u(y))^2 dx dy \right).$$

Notice that the obtained constant deteriorates as $\delta \rightarrow 0$ or $m_J \rightarrow 0$.

Now, we are ready to present a proof of the Poincaré–Wirtinger inequality where the constant can be tracked.

Lemma 2.9. *The optimal constant $C^* > 0$ such that*

$$\int_{\Omega_\ell} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x))^2 dy dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x,y)|u(y) - u(x)|^2 dy dx \geq C^* \int_{\Omega} u^2(x) dx$$

for every function $u \in H_{\text{Neu}}$ (that is, for $u \in \mathcal{H}$ with $\int_{\Omega} u = 0$) can be estimated as

$$C^* \geq c^*(J, G, \Omega_\ell, \mathcal{T}, \delta, A, B)$$

with A, B two open sets $A \subset \Omega_\ell$, $B \subset \Omega_{n\ell}$ such that $\text{diam}(A \cup B) < 2\delta$ and \mathcal{T} a δ -tree structure of $\Omega_{n\ell}$ with root B .

Proof. We aim to obtain a bound of the form

$$\begin{aligned} & \inf_{r \in \mathbb{R}} \int_{\Omega} |u - r|^2(x) \, dx \\ & \leq \frac{1}{c^*} \left[\int_{\Omega_\ell} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y)(u(y) - u(x))^2 \, dy \, dx \right. \\ & \qquad \qquad \qquad \left. + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y)|u(y) - u(x)|^2 \, dy \, dx \right] \end{aligned}$$

for functions $u \in L^2(\Omega)$ such that $u|_{\Omega_\ell} \in H^1(\Omega_\ell)$.

From our hypothesis on G and the assumption $\text{dist}(\Omega_\ell, \Omega_{n\ell}) < \delta$, there exist two sets $A \subset \Omega_\ell$ and $B \subset \Omega_{n\ell}$ such that

$$|x - y| \leq 2\delta \quad \forall x \in A, y \in B,$$

and therefore

$$G(x, y) \geq m_G = \min_{|x-y|<2\delta} G(x, y) \quad \forall x \in A, y \in B.$$

Moreover, reducing the size of B if necessary, we may assume that $\text{diam}(B) < \delta/2$, and we define \mathcal{T} a δ -tree on $\Omega_{n\ell}$ with root B .

Let us take $r \in \mathbb{R}$ such that

$$\int_A (u(x) - r) \, dx = 0.$$

Then we have, for $v = u - r$,

$$\begin{aligned} & \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y)|v(y) - v(x)|^2 \, dy \, dx \\ & \geq \frac{1}{2} \int_B \int_A G(x, y)|v(y) - v(x)|^2 \, dy \, dx \qquad (2.9) \\ & \geq m_G \frac{|A|}{2} \int_B |v(x)|^2 \, dx, \end{aligned}$$

where we are using that

$$\int_B |v(x)|^2 \, dx = \frac{1}{|A|^2} \int_B \left| \int_A (v(x) - v(y)) \, dy \right|^2 \, dx \leq \frac{1}{|A|} \int_B \int_A |v(x) - v(y)|^2 \, dy \, dx.$$

Now, since

$$0 < \sigma(\Omega_\ell, A) = \inf_{v \in H^1(\Omega_\ell), \int_A v \, dx = 0} \frac{\int_{\Omega_\ell} |\nabla v|^2(x) \, dx}{\int_{\Omega_\ell} v^2(x) \, dx}$$

for a suitable Poincaré constant $\sigma(\Omega_\ell, A)$, we have for the local region

$$\int_{\Omega_\ell} v^2(x) dx \leq \frac{1}{\sigma(\Omega_\ell, A)} \int_{\Omega_\ell} |\nabla v|^2(x) dx.$$

On the other hand, Corollary 2.7 says that

$$\int_{\Omega_{n\ell}} v^2 dx \leq C(\mathcal{T}, m, J) \left(\int_B v^2 dx + \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y) |v(x) - v(y)|^2 dx dy \right),$$

and the lemma follows by collecting the last two inequalities together with (2.9). \square

Remark 2.10. The constant $\sigma(\Omega_\ell, A)$ can be difficult to characterize even in the case $A = \Omega_\ell$ (for which we just write $\sigma(\Omega_\ell)$). For some elementary domains, such as cubes or balls in \mathbb{R}^N , explicit computations of $\sigma(\Omega_\ell)$ can be done. Moreover, in particular circumstances and for simple geometries, some bounds are easy to find. Notably, for convex domains, it is known that

$$\frac{\pi^2}{\text{diam}^2(\Omega)} \leq \sigma(\Omega_\ell),$$

see [28]. On the other hand, a simple argument relates $\sigma(\Omega_\ell, A)$ with $\sigma(\Omega_\ell)$ as follows: assume that $u \in H^1(\Omega_\ell)$, $\frac{1}{|A|} \int_A u(x) dx = 0$ and write

$$\begin{aligned} \bar{u}_{\Omega_\ell} &= \frac{1}{|\Omega_\ell|} \int_{\Omega_\ell} u(x) dx \\ &= \int_{\Omega_\ell} u(x) \left(\frac{1}{|\Omega_\ell|} - \frac{\chi_A(x)}{|A|} \right) dx \\ &= \int_{\Omega_\ell} (u(x) - \bar{u}_{\Omega_\ell}) \left(\frac{1}{|\Omega_\ell|} - \frac{\chi_A(x)}{|A|} \right) dx; \end{aligned}$$

then, Schwartz’s inequality yields

$$\bar{u}_{\Omega_\ell}^2 \leq \frac{|\Omega_\ell|}{|A|^2} \int_{\Omega_\ell} (u(x) - \bar{u}_{\Omega_\ell})^2 dx \leq \frac{|\Omega_\ell|}{|A|^2 \sigma(\Omega_\ell)} \int_{\Omega_\ell} |\nabla u|^2(x) dx.$$

Hence, we have

$$\int_{\Omega_\ell} u(x)^2 dx \leq 2 \int_{\Omega_\ell} (u(x) - \bar{u}_{\Omega_\ell})^2 dx + 2 \int_{\Omega_\ell} \bar{u}_{\Omega_\ell}^2 dx \leq 2 \frac{1 + \left(\frac{|\Omega_\ell|}{|A|}\right)^2}{\sigma(\Omega_\ell)} \int_{\Omega_\ell} |\nabla u|^2(x) dx,$$

and, as a consequence, we conclude that

$$\frac{\sigma(\Omega_\ell)}{2 \left(1 + \left(\frac{|\Omega_\ell|}{|A|}\right)^2 \right)} \leq \sigma(\Omega_\ell, A).$$

Remark 2.11. For two adjacent square domains $\Omega_{n\ell} = [-1, 0] \times [0, 1]$, $\Omega_\ell = [0, 1] \times [0, 1]$ of unitary side, Remarks 2.8 and 2.10 give an explicit estimate of the constant.

First, we have

$$\int_{\Omega_\ell} v^2(x) dx \leq \frac{1}{\sigma(\Omega_\ell, A)} \int_{\Omega_\ell} |\nabla v|^2(x) dx \leq \frac{[\text{diam}(\Omega)]^2}{2\pi^2 \left(1 + \left(\frac{|\Omega_\ell|}{|A|}\right)^2\right)} \int_{\Omega_\ell} |\nabla v|^2(x) dx.$$

On the other hand, from Remark 2.8 we get

$$\begin{aligned} \int_{\Omega_{n\ell}} u^2 dx &\leq 2^{\frac{8^N |\Omega_{n\ell}|}{\delta^N} + 1} \left(\int_B u^2 dx + \frac{2^N}{c_N \delta^N m_J} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(x) - u(y))^2 dx dy \right). \end{aligned}$$

Therefore, for the particular case of two adjacent unit squares, we have that the best constant in the Poincaré type inequality

$$\begin{aligned} \int_{\Omega_\ell} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x))^2 dy dx &+ \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x,y)|u(y) - u(x)|^2 dy dx \\ &\geq C^* \int_{\Omega} u^2(x) dx \quad \forall u \in H_{\text{Neu}} \end{aligned}$$

can be estimated as

$$C^* \geq c \delta^N e^{c \delta^{-N}} \min\{m_J, m_G\}$$

for δ small enough (here c is a constant independent of δ).

Now we are ready to proceed with the proof of the existence and uniqueness of minimizers of $E_{\text{Neu}}(u)$ in H_{Neu} .

Theorem 2.12. *Given $f \in L^2(\Omega)$ with $\int_{\Omega} f = 0$, there exists a unique minimizer of $E_{\text{Neu}}(u)$ in H_{Neu} .*

Proof of Theorem 2.12. From the previous Poincaré–Wirtinger type inequality we obtain

$$\begin{aligned} E_{\text{Neu}}(u) &= \int_{\Omega_\ell} \frac{|\nabla u(x)|^2}{2} dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega} J(x-y)|u(y) - u(x)|^2 dy dx \\ &\quad + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x,y)|u(y) - u(x)|^2 dy dx - \int_{\Omega} f(x)u(x) dx \\ &\geq c \int_{\Omega} u^2(x) dx - \int_{\Omega} f(x)u(x) dx, \end{aligned}$$

from where it follows that $E_{\text{Neu}}(u)$ is bounded below and coercive in H_{Neu} . Hence, existence of a minimizer follows by the direct method of calculus of variations. Just take a minimizing sequence u_n and extract a subsequence that converges weakly

in $L^2(\Omega)$. Then, we have

$$0 = \lim_{n \rightarrow \infty} \int_{\Omega} u_n(x) \, dx = \int_{\Omega} u(x) \, dx,$$

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x)u_n(x) \, dx = \int_{\Omega} f(x)u(x) \, dx$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega_{\ell}} \frac{|\nabla u(x)|^2}{2} \, dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega} J(x-y)|u(y) - u(x)|^2 \, dy \, dx \\ & \quad + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{\ell}} G(x,y)|u(y) - u(x)|^2 \, dy \, dx \\ & \geq \int_{\Omega_{\ell}} |\nabla u(x)|^2 \, dx + \frac{\alpha}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x))^2 \, dy \, dx \\ & \quad + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{\ell}} G(x,y)|u(y) - u(x)|^2 \, dy \, dx. \end{aligned}$$

Then, we obtain that $u \in H_{\text{Neu}}$ and that u is a minimizer:

$$E_{\text{Neu}}(u) = \min_{v \in H} E_{\text{Neu}}(v).$$

Uniqueness of minimizers follows from the strict convexity of the functional $E_{\text{Neu}}(u)$. □

Associated with this energy we have an equation in Ω .

Lemma 2.13. *The minimizer of $E_{\text{Neu}}(u)$ in H_{Neu} is a weak solution to the following problem: a local equation with a source in Ω_{ℓ} ,*

$$\begin{cases} -f(x) = \Delta u(x) + \int_{\Omega_{n\ell}} G(x,y)(u(y) - u(x)) \, dy, & x \in \Omega_{\ell}, \, t > 0, \\ \frac{\partial u}{\partial \eta}(x) = 0, & x \in \partial\Omega_{\ell}, \end{cases} \tag{2.10}$$

and a nonlocal equation with a source in $\Omega_{n\ell}$,

$$-f(x) = 2 \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x)) \, dy + \int_{\Omega_{\ell}} G(x,y)(u(y) - u(x)) \, dy, \quad x \in \Omega_{n\ell}. \tag{2.11}$$

Proof. Let u be the minimizer of $E_{\text{Neu}}(u)$ in H_{Neu} ; then for every smooth φ with $\int_{\Omega} \varphi = 0$ and every $t \in \mathbb{R}$, we have

$$E_{\text{Neu}}(u + t\varphi) - E_{\text{Neu}}(u) \geq 0.$$

Therefore, we have

$$\frac{\partial}{\partial t} E_{\text{Neu}}(u + t\varphi)|_{t=0} = 0,$$

that is,

$$\int_{\Omega} f\varphi = \int_{\Omega_{\ell}} \nabla u \nabla \varphi + \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x))(\varphi(y) - \varphi(x)) dy dx + \int_{\Omega_{n\ell}} \int_{\Omega_{\ell}} G(x,y)(u(y) - u(x))(\varphi(y) - \varphi(x)) dy dx.$$

Using that the kernel J is symmetric and Fubini's theorem we obtain

$$\int_{\Omega} f\varphi = \int_{\Omega_{\ell}} \nabla u \nabla \varphi - 2 \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x)) dy \varphi(x) dx + \int_{\Omega_{n\ell}} \int_{\Omega_{\ell}} G(x,y)(u(y) - u(x)) dx \varphi(y) dy - \int_{\Omega_{n\ell}} \int_{\Omega_{\ell}} G(x,y)(u(y) - u(x)) dy \varphi(x) dx,$$

from where it follows that u is a weak solution to (2.10) and (2.11). □

Connections with probability theory. A probabilistic interpretation of this model in terms of particle systems runs as follows. Take an exponential clock that controls the jumps of the particles. In the local region Ω_{ℓ} , the particles move according to Brownian motion (with a reflexion on the boundary) and when the clock rings, a new position is sorted according to the kernel $G(x, \cdot)$. Then they jump if the new position is in the nonlocal region $\Omega_{n\ell}$, while if the sorted position lies in Ω_{ℓ} , they just continue moving by Brownian motion ignoring the ring of the clock. In the nonlocal region $\Omega_{n\ell}$, the particles stay still until the clock rings and then they jump using the kernel $J(x - \cdot)$ or the kernel $G(x, \cdot)$ to select the new position in the whole Ω . Notice that the total number of particles inside the domain Ω remains constant in time.

The minimizer to our functional $E_{\text{Neu}}(u)$ gives the stationary distribution of particles provided that there is an external source f (that adds particles where $f > 0$ and removes particles where $f < 0$).

3. MIXED COUPLING

Second model. Coupling local/nonlocal problems via flux terms. Our aim now is to look for a scalar problem with an energy that combines local and nonlocal terms acting in different subdomains Ω_{ℓ} and $\Omega_{n\ell}$ of Ω , but now the coupling is made balancing the fluxes across a prescribed hypersurface Γ .

Recall from the introduction that we look for a minimizer of the energy

$$F_{\text{Neu}}(u) := \frac{1}{2} \int_{\Omega_{\ell}} |\nabla u(x)|^2 dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x))^2 dy dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x,z)(u(x) - u(z))^2 d\sigma(z) dx - \int_{\Omega} f(x)u(x) dx$$

in

$$H_{\text{Neu}} = \left\{ u \in L^2(\Omega) : u \in H^1(\Omega_\ell), \int_{\Omega} u(x) dx = 0 \right\},$$

assuming that

$$\int_{\Omega} f(x) dx = 0.$$

As before, we first prove a lemma that gives coercivity for our functional.

Lemma 3.1. *There exists a constant C such that*

$$\begin{aligned} \int_{\Omega_\ell} \frac{|\nabla u(x)|^2}{2} dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x))^2 dy dx \\ + \frac{1}{2} \int_{\Omega_{nl}} \int_{\Gamma} G(x,z)(u(x) - u(z))^2 d\sigma(z) dx \geq C \int_{\Omega} |u(x)|^2 dx \end{aligned}$$

for every u in H_{Neu} .

Proof. Arguing by contradiction, we assume that there is a sequence $u_n \in H_{\text{Neu}}$ such that

$$\int_{\Omega} |u_n(x)|^2 dx = 1$$

and

$$\begin{aligned} \int_{\Omega_\ell} \frac{|\nabla u_n(x)|^2}{2} dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u_n(y) - u_n(x))^2 dy dx \\ + \frac{1}{2} \int_{\Omega_{nl}} \int_{\Gamma} G(x,z)(u_n(x) - u_n(z))^2 d\sigma(z) dx \rightarrow 0. \end{aligned}$$

Then we have that

$$\begin{aligned} \int_{\Omega_\ell} |\nabla u_n(x)|^2 dx \rightarrow 0, \\ \int_{\Omega_{nl}} \int_{\Gamma} G(x,z)(u_n(x) - u_n(z))^2 d\sigma(z) dx \rightarrow 0, \end{aligned}$$

and

$$\int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u_n(y) - v_n(x))^2 dy dx \rightarrow 0.$$

Since

$$\int_{\Omega} |u_n(x)|^2 dx = 1,$$

we obtain that u_n is bounded in $L^2(\Omega_\ell)$ and then, using that

$$\int_{\Omega_\ell} \frac{|\nabla u_n(x)|^2}{2} dx \rightarrow 0,$$

we get that there exists a constant k_1 such that, along a subsequence,

$$u_n \rightarrow k_1$$

strongly in $H^1(\Omega_\ell)$. Hence, using the trace theorem on Γ , $H^1(\Omega_\ell) \hookrightarrow L^2(\Gamma)$, we obtain

$$u_n \rightarrow k_1$$

strongly $L^2(\Gamma)$.

Now we argue in the nonlocal part $\Omega_{n\ell}$. Since u_n is bounded in $L^2(\Omega_{n\ell})$ and

$$\int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u_n(y) - u_n(x))^2 dy dx \rightarrow 0,$$

we have that (extracting another subsequence if necessary)

$$u_n \rightharpoonup u$$

weakly in $L^2(\Omega_{n\ell})$ and therefore,

$$\begin{aligned} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u(y) - u(x))^2 dy dx \\ \leq \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x-y)(u_n(y) - u_n(x))^2 dy dx = 0. \end{aligned}$$

Hence, we get that

$$u \equiv k_2$$

in $\Omega_{n\ell}$ (here we need to assume that $\Omega_{n\ell}$ is δ -connected).

From the weak convergence of u_n to u in $L^2(\Omega_{n\ell})$ and the strong convergence of u_n to u in $L^2(\Gamma)$, we obtain

$$\begin{aligned} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x,z)(k_2 - k_1)^2 d\sigma(z) dx \\ \leq \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x,z)(u_n(x) - u_n(z))^2 d\sigma(z) dx = 0. \end{aligned}$$

Hence, from the fact that

$$\int_{\Omega_{n\ell}} \int_{\Gamma} G(x,z) d\sigma(z) dx > 0,$$

we obtain

$$k_1 = k_2.$$

Now, from the fact that $u_n \in H_{\text{Neu}}$, we have

$$\int_{\Omega} u_n(x) dx = 0,$$

and then, passing to the limit, we obtain

$$\int_{\Omega_\ell} k_1 + \int_{\Omega_{n\ell}} k_2 = 0.$$

Hence, as $k_1 = k_2$, we conclude that

$$k_1 = k_2 = 0.$$

Up to now we have that $u_n \rightarrow 0$ strongly in $H^1(\Omega_\ell)$ and $u_n \rightarrow 0$ weakly in $L^2(\Omega_{n\ell})$. Then, as

$$1 = \int_{\Omega} |u_n(x)|^2 dx = \int_{\Omega_\ell} |u_n(x)|^2 dx + \int_{\Omega_{n\ell}} |u_n(x)|^2 dx,$$

we get that

$$\int_{\Omega_{n\ell}} |u_n(x)|^2 dx \rightarrow 1.$$

Now, we go back to

$$\int_{\Omega_{n\ell}} \int_{\Gamma} G(x, z) (u_n(x) - u_n(z))^2 d\sigma(z) dx \rightarrow 0$$

and we obtain

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, z) (u_n(x) - u_n(z))^2 d\sigma(z) dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, z) |u_n(x)|^2 d\sigma(z) dx \\ &\quad + \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, z) |u_n(z)|^2 d\sigma(z) dx \\ &\quad - \lim_{n \rightarrow \infty} 2 \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, z) u_n(x) u_n(z) d\sigma(z) dx. \end{aligned}$$

Since $u_n \rightarrow 0$ strongly in $L^2(\Gamma)$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, z) |u_n(z)|^2 d\sigma(z) dx = 0$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, z) u_n(x) u_n(z) d\sigma(z) dx = 0.$$

Therefore, we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} |u_n(x)|^2 \int_{\Gamma} G(x, z) dy dx = 0. \quad (3.1)$$

On the other hand, we have

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y) (u_n(y) - u_n(x))^2 dy dx \\ &= \lim_{n \rightarrow \infty} 2 \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y) |u_n(x)|^2 dy dx \\ &\quad - \lim_{n \rightarrow \infty} 2 \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y) u_n(y) u_n(x) dy dx. \end{aligned}$$

Now we use that $v_n \rightarrow 0$ weakly in $L^2(\Omega_{n\ell})$ and that J is continuous to obtain

$$\int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y) u_n(y) dy \rightarrow 0$$

strongly in $L^2(\Omega_{n\ell})$ and then we obtain

$$- \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y) u_n(y) u_n(x) dy dx = 0.$$

Hence, we get

$$\lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} |v_n(x)|^2 \int_{\Omega_{n\ell}} J(x - y) dy dx = 0. \tag{3.2}$$

From (3.1) and (3.2) and the fact that there exists a constant c such that

$$\int_{\Omega_{n\ell}} J(x - y) dy + \int_{\Gamma} G(x, z) d\sigma(z) \geq c > 0,$$

we conclude that

$$\begin{aligned} 0 < c &= c \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} |u_n(x)|^2 dx \\ &\leq \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} |u_n(x)|^2 \left(\int_{\Omega_{n\ell}} J(x - y) dy + \int_{\Gamma} G(x, z) d\sigma(z) \right) dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} |u_n(x)|^2 \int_{\Omega_{n\ell}} J(x - y) dy dx \\ &\quad + \lim_{n \rightarrow \infty} \int_{\Omega_{n\ell}} |u_n(x)|^2 \int_{\Gamma} G(x, z) d\sigma(z) dx = 0, \end{aligned}$$

a contradiction. □

Similarly to Lemma 2.3, Lemma 3.1 does not provide information regarding the constant c .

Lemma 3.2. *The optimal constant $c > 0$ such that*

$$\begin{aligned} \int_{\Omega_\ell} \frac{|\nabla u(x)|^2}{2} dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} J(x - y) (u(y) - u(x))^2 dy dx \\ + \frac{1}{2} \int_{\Omega_{nl}} \int_{\Gamma} G(x, z) (u(x) - u(z))^2 d\sigma(z) dx \geq c \int_{\Omega} |u(x)|^2 dx \end{aligned}$$

for every function $u \in H_{\text{Neu}}$ (that is, for u in the natural space with $\int_{\Omega} u = 0$) can be estimated as

$$c \geq c^*(J, G, \Omega_\ell, \mathcal{T}, \delta, \Gamma_A, B)$$

with $\Gamma_A \subset \Omega_\ell$, $B \subset \Omega_{n\ell}$ such that $\text{diam}(\Gamma_A \cup B) < 2\delta$ and \mathcal{T} a δ -tree structure of $\Omega_{n\ell}$ with root at B .

Proof. The proof uses the same ideas as the ones of Lemma 2.9. The only significant modifications are the following: First, from our hypothesis on G and the assumption $\text{dist}(\Gamma_A, \Omega_{n\ell}) < \delta$, there exist two sets $\Gamma_A \subset \Gamma$ and $B \subset \Omega_{n\ell}$ such that

$$|x - y| \leq 2\delta \quad \forall x \in \Gamma_A, \forall y \in B.$$

Take $r \in \mathbb{R}$ such that

$$\int_{\Gamma_A} (u(z) - r) d\sigma(z) = 0.$$

Then we have, for $v = u - r$ and calling $|\Gamma_A|$ the $(N - 1)$ -dimensional measure of Γ_A , the following variant of (2.9):

$$\begin{aligned} & \frac{1}{2} \int_{\Omega_{nl}} \int_{\Gamma} G(x, z) (u(x) - u(z))^2 \, d\sigma(z) \, dx \\ & \geq \frac{1}{2} \int_B \int_{\Gamma_A} G(x, z) (u(x) - u(z))^2 \, d\sigma(z) \, dx \\ & \geq m_G \frac{|\Gamma_A|}{2} \int_B |v(x)|^2 \, dx, \end{aligned}$$

where we are using that

$$\begin{aligned} \int_B |v(x)|^2 \, dx &= \frac{1}{|\Gamma_A|^2} \int_B \left| \int_{\Gamma_A} (v(x) - v(z)) \, d\sigma(z) \right|^2 \, dx \\ &\leq \frac{1}{|A|} \int_B \int_{\Gamma_A} |v(x) - v(z)|^2 \, d\sigma(z) \, dx. \end{aligned}$$

Using now that

$$0 < \sigma(\Omega_\ell, \Gamma_A) = \inf_{v \in H^1(\Omega_\ell), \int_{\Gamma_A} v \, d\sigma(z) = 0} \frac{\int_{\Omega_\ell} |\nabla v|^2(x) \, dx}{\int_{\Omega_\ell} v^2(x) \, dx}$$

for a suitable Poincaré constant $\sigma(\Omega_\ell, \Gamma_A)$, the proof follows exactly as in Lemma 2.9. □

As before, we are ready to obtain the existence and uniqueness of a minimizer of $F_{\text{Neu}}(u)$ in H_{Neu} .

Theorem 3.3. *Given $f \in L^2(\Omega)$ with $\int_{\Omega} f = 0$, there exists a unique minimizer of $F_{\text{Neu}}(u)$ in H_{Neu} .*

Proof of Theorem 3.3. The previous lemma implies that $F_{\text{Neu}}(u)$ is bounded below and coercive. Hence, the existence of a minimizer follows just considering a minimizing sequence (u_n) and extracting a subsequence that converges weakly in $H^1(\Omega_\ell) \cap L^2(\Omega_{nl})$. The uniqueness of a minimizer follows from the strict convexity of the functional $F_{\text{Neu}}(u)$. □

Now, we find the equations verified by the minimizer.

Lemma 3.4. *The minimizer of F_{Neu} is a weak solution to*

$$\begin{cases} f(x) = \Delta u(x), & x \in \Omega_\ell, \\ \frac{\partial u}{\partial \eta}(x) = 0, & x \in \partial\Omega_l \cap \partial\Omega, \\ \frac{\partial u}{\partial \eta}(x) = \int_{\Omega_{nl}} G(y, x)(u(y) - u(x)) \, dy, & x \in \Gamma, \end{cases} \tag{3.3}$$

and

$$f(x) = 2 \int_{\Omega_{nl}} J(x - y)(u(y) - u(x)) dy - \int_{\Gamma} G(x, z)(u(x) - u(z)) d\sigma(z), \quad x \in \Omega_{nl}. \tag{3.4}$$

Proof. Let u be the minimizer of $F_{\text{Neu}}(u)$ in H_{Neu} ; then, for every smooth φ with $\int_{\Omega} \varphi = 0$ and every $t \in \mathbb{R}$, we have

$$F_{\text{Neu}}(u + t\varphi) - F_{\text{Neu}}(u) \geq 0.$$

Therefore, we get

$$\frac{\partial}{\partial t} F_{\text{Neu}}(u + t\varphi) \Big|_{t=0} = 0,$$

that is,

$$\begin{aligned} \int_{\Omega} f\varphi &= \int_{\Omega_{\ell}} \nabla u \nabla \varphi + \int_{\Omega_{nl}} \int_{\Omega_{nl}} J(x - y)(u(y) - u(x))(\varphi(y) - \varphi(x)) dy dx \\ &\quad + \int_{\Omega_{nl}} \int_{\Gamma} G(x, y)(u(y) - u(x))\varphi(y) d\sigma(y) dx \\ &\quad - \int_{\Omega_{nl}} \int_{\Gamma} G(x, y)(u(y) - u(x))\varphi(x) d\sigma(y) dx. \end{aligned}$$

Using that the kernel J is symmetric, we obtain

$$\begin{aligned} \int_{\Omega} f\varphi &= \int_{\Omega_{\ell}} \nabla u \nabla \varphi - 2 \int_{\Omega_{nl}} \int_{\Omega_{nl}} J(x - y)(u(y) - u(x)) dy \varphi(x) dx \\ &\quad - \int_{\Gamma} \int_{\Omega_{nl}} G(y, x)(u(y) - u(x)) dy \varphi(x) d\sigma(x) \\ &\quad - \int_{\Omega_{nl}} \int_{\Gamma} G(x, y)(u(y) - u(x)) d\sigma(y) \varphi(x) dx, \end{aligned}$$

that is, u is a weak solution to (3.3) and (3.4). □

Probability interpretation. A probabilistic interpretation of this model in terms of particle systems runs as follows. In the local region Ω_{ℓ} , the particles move according to Brownian motion (with a reflexion on the boundary of Ω_{ℓ} , $\partial\Omega_{\ell}$) and when they arrive to the surface Γ , they pass to the nonlocal region to a point selected using the kernel G . In the nonlocal region, take an exponential clock that controls the jumps of the particles, and when the clock rings, a new position is sorted according to the kernel $J(x - \cdot)$ (for the movements inside Ω_{nl}) or according to $G(x, \cdot)$ for jumping back to the local region entering at a point in the surface Γ . The total number of particles inside the domain Ω remains constant in time.

The minimizer to our functional $F_{\text{Neu}}(u)$ gives the stationary distribution of particles (u for the particles inside the local region Ω_{ℓ} and v for particles in the nonlocal part Ω_{nl}) provided that there is an external source f (that adds particles where $f > 0$ and removes particles where $f < 0$).

Notice that here the coupling term appears as a flux boundary condition for the local part of the problem.

4. SINGULAR KERNELS

First, let us deal with

$$E_{\text{Neu}}(u) := \int_{\Omega_\ell} \frac{|\nabla u(x)|^2}{2} dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} \frac{C}{|x-y|^{N+2s}} |u(y) - u(x)|^2 dy dx \\ + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y) |u(y) - u(x)|^2 dy dx - \int_{\Omega} f(x) u(x) dx.$$

We look for minimizers in the space

$$H_{\text{Neu}} = \left\{ u : u|_{\Omega_\ell} \in H^1(\Omega_\ell), u|_{\Omega_{n\ell}} \in H^s(\Omega_{n\ell}), \int_{\Omega} u = 0 \right\}.$$

In fact, now the key lemma (the Poincaré–Wirtinger type inequality) is simpler.

Lemma 4.1. *There exists $c > 0$ such that*

$$\int_{\Omega_\ell} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} \frac{C}{|x-y|^{N+2s}} (u(y) - u(x))^2 dy dx \\ + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y) |u(y) - u(x)|^2 dy dx \geq c \int_{\Omega} u^2(x) dx$$

for every function $u \in H_{\text{Neu}}$.

Proof. As before, we argue by contradiction. Assume that there is a sequence u_n such that

$$\int_{\Omega_\ell} |\nabla u_n|^2(x) dx \rightarrow 0, \\ \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} \frac{C}{|x-y|^{N+2s}} |u_n(y) - u_n(x)|^2 dy dx \rightarrow 0, \quad (4.1) \\ \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y) |u_n(y) - u_n(x)|^2 dy dx \rightarrow 0, \\ \int_{\Omega} |u_n|^2(x) dx = 1,$$

and

$$\int_{\Omega} u_n(x) dx = 0.$$

Since the L^2 norm is bounded, we can extract a subsequence such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\Omega_\ell),$$

and

$$u_n \rightharpoonup u \quad \text{weakly in } H^s(\Omega_{n\ell}).$$

Then, from the compact embeddings $H^1(\Omega_\ell) \hookrightarrow L^2(\Omega_\ell)$ and $H^s(\Omega_\ell) \hookrightarrow L^2(\Omega_\ell)$ and (2.4) and (4.1), we get that there are constants k_1, k_2 such that

$$u_n \rightarrow k_1 \quad \text{strongly in } L^2(\Omega_\ell)$$

and

$$u_n \rightarrow k_2 \quad \text{strongly in } L^2(\Omega_{n\ell}).$$

From (2.6) we conclude that

$$\begin{aligned} \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y) |k_1 - k_2|^2 dy dx \\ \leq \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_\ell} G(x, y) |u_n(y) - u_n(x)|^2 dy dx = 0, \end{aligned}$$

and hence

$$k_1 = k_2.$$

Now, from (2.8), we obtain

$$k_1 = k_2 = 0.$$

Hence we get

$$\int_{\Omega} |u_n|^2(x) dx = \int_{\Omega_\ell} |u_n|^2(x) dx + \int_{\Omega_{n\ell}} |u_n|^2(x) dx \rightarrow 0.$$

This contradicts that

$$\int_{\Omega} |u_n|^2(x) dx \rightarrow 1,$$

and ends the proof. □

Now we are ready to proceed with the proof of Theorem 1.5.

Proof of Theorem 1.5. From the previous Poincaré–Wirtinger type inequality we obtain

$$E_{\text{Neu}}(u) \geq c \int_{\Omega} u^2(x) dx - \int_{\Omega} f(x)u(x) dx,$$

from where it follows that $E_{\text{Neu}}(u)$ is bounded below and coercive in H . Hence, existence of a minimizer follows by the direct method of calculus of variations. Just take a minimizing sequence u_n and extract a subsequence that converges weakly in $L^2(\Omega)$. Then, using the lower semicontinuity of the seminorms we obtain that $u \in H_{\text{Neu}}$ and that u is a minimizer:

$$E_{\text{Neu}}(u) = \min_{v \in H} E_{\text{Neu}}(v).$$

The uniqueness of minimizers follows from the strict convexity of the functional $E_{\text{Neu}}(u)$.

Associated with this energy we have an equation in Ω that can be obtained exactly as before. □

Now let us move to mixed couplings,

$$\begin{aligned} F_{\text{Neu}}(u) := \frac{1}{2} \int_{\Omega_t} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_{n_l}} \int_{\Omega_{n_t}} \frac{C}{|x - y|^{N+2s}} (u(y) - u(x))^2 dy dx \\ + \frac{1}{2} \int_{\Omega_{n_l}} \int_{\Gamma} G(x, z) (u(x) - u(z))^2 d\sigma(z) dx - \int_{\Omega} f(x)u(x) dx. \end{aligned}$$

The key lemma follows as before.

Lemma 4.2. *There exists $c > 0$ such that*

$$\int_{\Omega_\ell} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Omega_{n\ell}} \frac{C}{|x-y|^{N+2s}} (u(y) - u(x))^2 dy dx + \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, y) |u(y) - u(x)|^2 d\sigma(y) dx \geq c \int_{\Omega} u^2(x) dx$$

for every function $u \in H_{\text{Neu}}$.

Proof. The proof runs as before, arguing by contradiction. The only minor point is that, after proving that there is a subsequence u_n such that

$$u_n \rightarrow k_1 \quad \text{strongly in } L^2(\Omega_\ell)$$

and

$$u_n \rightarrow k_2 \quad \text{strongly in } L^2(\Omega_{n\ell}),$$

we have to use that

$$\begin{aligned} \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, y) |k_1 - k_2|^2 d\sigma(y) dx \\ \leq \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega_{n\ell}} \int_{\Gamma} G(x, y) |u_n(y) - u_n(x)|^2 d\sigma(y) dx = 0 \end{aligned}$$

to obtain

$$k_1 = k_2.$$

The rest of the arguments until reaching a contradiction are exactly as before. \square

Now we are ready to proceed with the proof of Theorem 2.12.

Proof of Theorem 2.12. From the previous Poincaré–Wirtinger type inequality we obtain

$$E_{\text{Neu}}(u) \geq c \int_{\Omega} u^2(x) dx - \int_{\Omega} f(x)u(x) dx,$$

from where it follows that $E_{\text{Neu}}(u)$ is bounded below and coercive in H . Hence, the existence of a minimizer follows by the direct method of calculus of variations. Just take a minimizing sequence u_n and extract a subsequence that converges weakly in $L^2(\Omega)$. Then, using the lower semicontinuity of the seminorms we obtain that $u \in H_{\text{Neu}}$ and that u is a minimizer:

$$E_{\text{Neu}}(u) = \min_{v \in H} E_{\text{Neu}}(v).$$

The uniqueness of minimizers follows from the strict convexity of the functional $E_{\text{Neu}}(u)$.

Associated with this energy we have an equation in Ω that can be obtained exactly as before. \square

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