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THREE-DIMENSIONAL C_{12}-MANIFOLDS

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Abstract. The present paper is devoted to three-dimensional C_{12}-manifolds (defined by D. Chinea and C. Gonzalez), which are never normal. We study their fundamental properties and give concrete examples. As an application, we study such structures on three-dimensional Lie groups.

1. Introduction

In [6], D. Chinea and C. Gonzalez obtained a classification of the almost contact metric manifolds, studying the space that possess the same symmetries as the covariant derivative of the fundamental 2-form. This space is decomposed into twelve irreducible components $C_1, \dots, C_{12}$. In dimension 3, the classes C_i reduce to the following classes: $|C|$ class of cosymplectic manifolds, C_5 class of β-Kenmotsu manifolds, C_6 class of α-Sasakian manifolds, C_9-manifolds and C_{12}-manifolds.

Most of the research related to almost contact metric structures is concerned with the normal structures which contain the first three classes. Regarding the C_{12} class which is not normal, only two papers address this subject. In the first one [5], the authors developed a systematic study of the curvature of the Chinea–Gonzalez class $C_5 \oplus C_{12}$ and obtain some classification theorems for those manifolds that satisfy suitable curvature conditions. This class is defined by using a certain function α and when this function vanishes the class $C_5 \oplus C_{12}$ reduces to class C_{12}. The second paper [3] contains new results on a particular three-dimensional C_{12}-manifolds with a class of concrete illustrative examples.

The present paper is devoted to three-dimensional C_{12}-manifolds. We present a detailed study of such class in dimension three and we construct a class of examples. As an application, we give all C_{12}-structures on Lie algebras of dimension 3.

First of all, we will start by introducing the basic concepts that we need in this research.

2. Almost contact manifolds

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an *almost contact metric manifold* if there exist on M a $(1,1)$-tensor field φ, a vector field ξ

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(called the structure vector field) and a 1-form η such that

$$\begin{cases}\eta(\xi)=1,\\ \varphi^2(X)=-X+\eta(X)\xi,\\ g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y)\end{cases}$$

for any vector fields X, Y on M.

In particular, in an almost contact metric manifold we also have

$$\varphi\xi=0 \qquad \text{and} \qquad \eta\circ\varphi=0.$$

The fundamental 2-form ϕ is defined by

$$\phi(X,Y)=g(X,\varphi Y).$$

It is known that the almost contact structure (φ,ξ,η) is said to be normal if and only if

$$N^{(1)}(X,Y)=N_\varphi(X,Y)+2d\eta(X,Y)\xi=0$$

for any X, Y on M, where N_φ denotes the Nijenhuis torsion of φ, given by

$$N_\varphi(X,Y)=\varphi^2[X,Y]+[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y].$$

Given an almost contact structure, one can associate in a natural manner an almost CR-structure $(\mathcal{D},\varphi|_{\mathcal{D}})$, where $\mathcal{D}:=\mathrm{Ker}(\eta)=\mathrm{Im}(\varphi)$ is the distribution of rank $2n$ transversal to the characteristic vector field ξ. If this almost CR-structure is integrable (i.e., $N_\varphi=0$) the manifold M^{2n+1} is said to be CR-integrable. It is known that normal almost contact manifolds are CR-manifolds.

For more background on almost contact metric manifolds, we recommend the references [1, 4, 9].

3. Three-dimensional C_{12}-manifolds

In the classification of D. Chinea and C. Gonzalez [6] of almost contact metric manifolds there is a class called C_{12}-manifolds which can be integrable but never normal. In this classification, C_{12}-manifolds are defined by

$$(\nabla_X\phi)(Y,Z)=\eta(X)\eta(Z)(\nabla_\xi\eta)\varphi Y-\eta(X)\eta(Y)(\nabla_\xi\eta)\varphi Z.$$

In [3] and [5], the $(2n+1)$-dimensional C_{12}-manifolds are characterized by

$$(\nabla_X\varphi)Y=\eta(X)\big(\omega(\varphi Y)\xi+\eta(Y)\varphi\psi\big) \tag{3.1}$$

for any X and Y vector fields on M, where $\omega=-\big(\nabla_\xi\xi\big)^\flat=-\nabla_\xi\eta$ and ψ is the vector field given by

$$\omega(X)=g(X,\psi)=-g(X,\nabla_\xi\xi)$$

for all X vector field on M.

Moreover, in [3] the $(2n+1)$-dimensional C_{12}-manifolds are also characterized by

$$\mathrm{d}\eta=\omega\wedge\eta, \qquad \mathrm{d}\phi=0 \qquad \text{and} \qquad N_\varphi=0.$$

Here, we emphasize that the almost C_{12}-manifolds are defined as follows.

Definition 3.1. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. M is called *almost C_{12}-manifold* if there exists a closed one-form ω which satisfies

$$\mathrm{d}\eta = \omega \wedge \eta \qquad \text{and} \qquad \mathrm{d}\phi = 0.$$

In addition, if $N_\varphi = 0$ we say that M is a C_{12}-manifold.

On the other hand, in [7] the author proved that, for an arbitrary 3-dimensional almost contact metric manifold $(M^3, \varphi, \xi, \eta, g)$, we have

$$\begin{cases} (1) \quad (\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi, \\ (2) \quad \mathrm{d}\phi = (\operatorname{div} \xi)\eta \wedge \phi, \\ (3) \quad \mathrm{d}\eta = \eta \wedge (\nabla_\xi \eta) + \frac{1}{2}\big(\operatorname{tr}_g(\varphi \nabla \xi)\big)\phi. \end{cases}$$

Then, for any 3-dimensional almost C_{12}-manifold $(M^3, \varphi, \xi, \eta, g)$ we get

$$(\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi \tag{3.2}$$

and

$$\operatorname{div} \xi = \operatorname{tr}_g(\varphi \nabla \xi) = 0.$$

Now we shall introduce another possible sufficient and necessary condition of the integrability of almost C_{12}-manifolds.

Proposition 3.2. *The almost C_{12}-structure (φ, ξ, η, g) is integrable if and only if, for all X and Y vector fields on M, we have*

$$\big(\nabla_{\varphi X} \varphi\big)Y - \varphi\big(\nabla_X \varphi\big)Y = -g(\nabla_X \xi, Y)\xi - \eta(X)\big(\omega(Y)\xi - \eta(Y)\psi\big). \tag{3.3}$$

Proof. We know that

$$N_\varphi(X, Y) = (\varphi \nabla_Y \varphi - \nabla_{\varphi Y} \varphi)X - (\varphi \nabla_X \varphi - \nabla_{\varphi X} \varphi)Y.$$

Suppose that $N_\varphi = 0$ and put

$$\begin{aligned} T(X, Y, Z) &= g\big(\varphi(\nabla_X \varphi)Y - (\nabla_{\varphi X} \varphi)Y, Z\big) \\ &= -g\big((\nabla_X \varphi)Y, \varphi Z\big) - g\big((\nabla_{\varphi X} \varphi)Y, Z\big). \end{aligned}$$

One can easily get

$$T(X, Y, Z) = T(Y, X, Z). \tag{3.4}$$

On the other hand, using formulas

$$\nabla_X(\varphi Y) = (\nabla_X \varphi)Y + \varphi \nabla_X Y \quad \text{and} \quad g(\varphi X, Y) = -g(X, \varphi Y),$$

we can get

$$g\big((\nabla_X \varphi)Y, Z\big) = -g\big(Y, (\nabla_X \varphi)Z\big),$$

and by straightforward computation we have

$$T(X, Y, Z) = -T(X, Z, Y) + g(\nabla_X \xi, Y)\eta(Z) + g(\nabla_X \xi, Z)\eta(Y). \tag{3.5}$$

Now, using formulas (3.4) and (3.5) we obtain

$$\begin{aligned}
T(X,Y,Z) &= T(Y,X,Z)\\
&= -T(Y,Z,X) + g(\nabla_Y\xi, X)\eta(Z) + g(\nabla_Y\xi, Z)\eta(X)\\
&= T(Z,X,Y) - g(\nabla_Z\xi, X)\eta(Y) - g(\nabla_Z\xi, Y)\eta(X)\\
&\quad + g(\nabla_Y\xi, X)\eta(Z) + g(\nabla_Y\xi, Z)\eta(X)\\
&= -T(X,Y,Z) + g(\nabla_X\xi, Y)\eta(Z) + g(\nabla_X\xi, Z)\eta(Y)\\
&\quad - g(\nabla_Z\xi, X)\eta(Y) - g(\nabla_Z\xi, Y)\eta(X)\\
&\quad + g(\nabla_Y\xi, X)\eta(Z) + g(\nabla_Y\xi, Z)\eta(X),
\end{aligned}$$

which implies

$$2T(X,Y,Z) = \big(g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)\big)\eta(Z) + 2\mathrm{d}\eta(X,Z)\eta(Y) + 2\mathrm{d}\eta(Y,Z)\eta(X).$$

Since the structure is almost C_{12}-structure, we have

$$\begin{aligned}
2\mathrm{d}\eta(X,Y) &= g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X)\\
&= \omega(X)\eta(Y) - \eta(X)\omega(Y),
\end{aligned}$$

therefore

$$T(X,Y,Z) = g(\nabla_X\xi, Y)\eta(Z) + \eta(X)\big(\omega(Y)\eta(Z) - \eta(Y)\omega(Z)\big),$$

which gives our formula (3.3). The proof of the converse is direct. □

We summarize all the above in the following main theorem.

Theorem 3.3. *Let $(M^3, \varphi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. M is a C_{12}-manifold if and only if*

$$\nabla_X\xi = -\eta(X)\psi,$$

where $\psi = -\nabla_\xi\xi$.

Proof. Suppose that $\nabla_X\xi = -\eta(X)\psi$ for all X vector field on M. From (3.2), we get

$$(\nabla_X\varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big),$$

with $\omega(X) = g(\psi, X)$.

Conversely, assuming that $(M^3, \varphi, \xi, \eta, g)$ is a C_{12}-manifold, this is equivalent to

$$(\nabla_X\varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big).$$

Setting $Y = \xi$ gives

$$-\varphi\nabla_X\xi = \eta(X)\varphi\psi,$$

and hence

$$\nabla_X\xi = \eta(X)\varphi^2\psi = -\eta(X)\psi.$$ □

The following proposition provides another characterization of 3-dimensional C_{12}-manifolds.

Proposition 3.4. *Let* $(M^3, \varphi, \xi, \eta, g)$ *be a 3-dimensional almost contact metric manifold.* M *is a* C_{12}*-manifold if and only if*

$$\nabla_{\varphi X}\xi = 0.$$

Proof. It is sufficient to prove that $\nabla_{\varphi X}\xi = 0$ and $\nabla_X \xi = -\eta(X)\psi$ are equivalent with $\psi = -\nabla_\xi \xi$. Suppose that $\nabla_X\xi = -\eta(X)\psi$, so it is easy to see that $\nabla_{\varphi X}\xi = 0$.

Conversely, suppose that $\nabla_{\varphi X}\xi = 0$ and replacing X by φX using the formula $\varphi^2 X = -X + \eta(X)\xi$, we obtain $\nabla_X\xi = \eta(X)\nabla_\xi\xi$. This completes the proof. ☐

In [3], the authors studied the 3-dimensional unit C_{12}-manifold, i.e. the case where ψ is a unit vector field. We will deal here with the general case, i.e. ψ is not necessarily unitary. For that, taking $V = \mathrm{e}^{-\rho}\psi$ where $\mathrm{e}^{\rho} = |\psi|$, we get immediately that $\{\xi, V, \varphi V\}$ is an orthonormal frame. We refer to this basis as *fundamental basis*.

Using this frame, one can get the following:

Proposition 3.5. *For any* C_{12}*-manifold, for all vector fields* X *on* M *we have*

(1) $\nabla_X\xi = -\mathrm{e}^{\rho}\eta(X)V$,
(2) $\nabla_\xi V = \mathrm{e}^{\rho}\xi$,
(3) $\nabla_V V = \varphi V(\rho)\varphi V$,
(4) $\nabla_\xi \varphi V = 0$,
(5) $\nabla_V \varphi V = -\varphi V(\rho) V$.

Proof. For the first, using (3.1) for $Y = \xi$ we get

$$(\nabla_X\varphi)\xi = \eta(X)\varphi\psi = \mathrm{e}^{\rho}\eta(X)\varphi V;$$

knowing that $(\nabla_X\varphi)Y = \nabla_X\varphi Y - \varphi\nabla_X Y$ and applying φ we obtain

$$\nabla_X\xi = \mathrm{e}^{\rho}\eta(X)\varphi^2 V = -\mathrm{e}^{\rho}\eta(X)V.$$

For the second, we have

$$2\mathrm{d}\omega(\xi, X) = 0 \;\Leftrightarrow\; g(\nabla_\xi\psi, X) = g(\nabla_X\psi, \xi) = -g(\psi, \nabla_X\xi) = \mathrm{e}^{2\rho}\eta(X),$$

which gives

$$\nabla_\xi\psi = \mathrm{e}^{2\rho}\xi \tag{3.6}$$

and then

$$\nabla_\xi V = \nabla_\xi(\mathrm{e}^{-\rho}\psi) = -\xi(\rho)V + \mathrm{e}^{\rho}\xi.$$

On the other hand, we have

$$\xi(\rho) = \tfrac{1}{2}\mathrm{e}^{-2\rho}\xi(\mathrm{e}^{2\rho}) = \tfrac{1}{2}\mathrm{e}^{-2\rho}\xi\big(g(\psi,\psi)\big) = \mathrm{e}^{-2\rho}g(\nabla_\xi\psi, \psi) = 0,$$

because of (3.6). Then,

$$\nabla_\xi V = \mathrm{e}^{\rho}\xi.$$

For $\nabla_V V$, we have

$$2\mathrm{d}\omega(\psi, X) = 0 \;\Leftrightarrow\; g(\nabla_\psi\psi, X) = g(\nabla_X\psi, \psi) = \tfrac{1}{2}Xg(\psi,\psi) = \mathrm{e}^{2\rho}g(\operatorname{grad}\rho, X),$$

i.e. $\nabla_\psi\psi = \mathrm{e}^{2\rho}\operatorname{grad}\rho$, which gives $\nabla_V V = \operatorname{grad}\rho - V(\rho)V$.

Also, we have

$$\operatorname{grad}\rho = \xi(\rho)\xi + V(\rho)V + \varphi V(\rho)\varphi V = V(\rho)V + \varphi V(\rho)\varphi V;$$

then,

$$\nabla_V V = \varphi V(\rho)\varphi V.$$

For the rest, just use the formula $\nabla_X \varphi Y = (\nabla_X \varphi)Y + \varphi \nabla_X Y$ noting that

$$(\nabla_V \varphi)X = (\nabla_{\varphi V}\varphi)X = 0. \qquad \square$$

It remains to calculate $\nabla_{\varphi V} V$ and $\nabla_{\varphi V}\varphi V$. For that, we have the following lemma.

Lemma 3.6. *For any* 3*-dimensional* C_{12}*-manifold, we have*

(1) $\nabla_{\varphi V} V = (-\mathrm{e}^{\rho} + \operatorname{div} V)\varphi V$,
(2) $\nabla_{\varphi V}\varphi V = (\mathrm{e}^{\rho} - \operatorname{div} V)V$.

Proof. Since $\{\xi, V, \varphi V\}$ is an orthonormal frame,

$$\nabla_{\varphi V} V = a\,\xi + b\,V + c\,\varphi V.$$

Using Proposition 3.5, we have

$$a = g(\nabla_{\varphi V} V, \xi) = -g(V, \nabla_{\varphi V}\xi) = 0$$

and $b = g(\nabla_{\varphi V} V, V) = 0$. To get the component c, we have

$$\begin{aligned}\operatorname{div} V &= g(\nabla_\xi V, \xi) + g(\nabla_{\varphi V}\psi, \varphi V)\\ &= \mathrm{e}^{\rho} + g(\nabla_{\varphi\psi}\psi, \varphi\psi) \;\Leftrightarrow\; g(\nabla_{\varphi V} V, \varphi V) = -\mathrm{e}^{\rho} + \operatorname{div} V;\end{aligned}$$

then,

$$\nabla_{\varphi V} V = (-\mathrm{e}^{\rho} + \operatorname{div} V)\varphi V.$$

Applying φ with (3.1), we obtain

$$\nabla_{\varphi V}\varphi V = (\mathrm{e}^{\rho} - \operatorname{div} V)V. \qquad \square$$

According to Proposition 3.5 and Lemma 3.6, the 3-dimensional C_{12}-manifold is completely controllable. That is:

Corollary 3.7. *For any* C_{12}*-manifold, we have*

$$\begin{array}{lll}\nabla_\xi \xi = -\mathrm{e}^{\rho} V, & \nabla_\xi V = \mathrm{e}^{\rho}\xi, & \nabla_\xi \varphi V = 0,\\ \nabla_V \xi = 0, & \nabla_V V = \varphi V(\rho)\varphi V, & \nabla_V \varphi V = -\varphi V(\rho) V,\\ \nabla_{\varphi V}\xi = 0, & \nabla_{\varphi V} V = (-\mathrm{e}^{\rho} + \operatorname{div} V)\varphi V, & \nabla_{\varphi V}\varphi V = (\mathrm{e}^{\rho} - \operatorname{div} V)V.\end{array}$$

To clarify these notions, we give the following class of examples.

Example 3.8. We denote the Cartesian coordinates in a 3-dimensional Euclidean space $M = \mathbb{R}^3$ by (x, y, z) and define a symmetric tensor field g by

$$g = \mathrm{e}^{2f}\begin{pmatrix}\alpha^2 + \beta^2 & 0 & -\beta\\ 0 & \alpha^2 & 0\\ -\beta & 0 & 1\end{pmatrix},$$

where $f = f(y) \neq$ const, $\beta = \beta(x)$ and $\alpha = \alpha(x, y) \neq 0$ everywhere are functions on $\mathbb{R}^3$ with $f' = \frac{\partial f}{\partial y}$. Further, we define an almost contact metric (φ, ξ, η) on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\beta & 0 \end{pmatrix}, \quad \xi = \mathrm{e}^{-f} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \eta = \mathrm{e}^{f}(-\beta, 0, 1).$$

The fundamental 1-form η and the 2-form ϕ have the forms

$$\eta = \mathrm{e}^{f}(dz - \beta dx) \qquad \text{and} \qquad \phi = -2\alpha^2 \mathrm{e}^{2f} dx \wedge dy,$$

and hence

$$\mathrm{d}\eta = f'\mathrm{e}^{f}\big(\beta dx \wedge dy + dy \wedge dz\big) = f' dy \wedge \eta,$$
$$\mathrm{d}\phi = 0.$$

By a direct computation the nontrivial components of $N^{(1)\ i}_{kj}$ are given by

$$N^{(1)\ 3}_{12} = \beta f', \quad N^{(1)\ 3}_{23} = f' \neq 0.$$

But, for all $i, j, k \in \{1, 2, 3\}$,

$$(N_\varphi)^i_{kj} = 0,$$

implying that (φ, ξ, η) becomes integrable non-normal. We have $\omega = f' dy$, i.e. $\mathrm{d}\omega = 0$ and knowing that ω is the g-dual of ψ, i.e. $\omega(X) = g(X, \psi)$, we have immediately that

$$\psi = \frac{f'}{\alpha^2} \mathrm{e}^{-2f} \frac{\partial}{\partial y}.$$

Thus, (φ, ξ, η, g) is a 3-parameter family of C_{12} structure on $\mathbb{R}^3$.

Notice that

$$|\psi|^2 = \omega(\psi) = g(\psi, \psi) = \frac{f'^2}{\alpha^2} \mathrm{e}^{-2f}$$

implies that $V = \frac{\mathrm{e}^{-f}}{\alpha} \frac{\partial}{\partial y}$ is a unit vector field; then

$$\left\{ \xi = \mathrm{e}^{-f} \frac{\partial}{\partial z},\ V = \frac{\mathrm{e}^{-f}}{\alpha} \frac{\partial}{\partial y},\ \varphi V = \frac{\mathrm{e}^{-f}}{\alpha} \left(\frac{\partial}{\partial x} + \beta \frac{\partial}{\partial z} \right) \right\}$$

form an orthonormal basis. To verify the result in formula (3.1), the components of the Levi-Civita connection corresponding to g are given by

$$\begin{aligned}
&\nabla_\xi \xi = -\frac{f'\mathrm{e}^{-f}}{\alpha} V, && \nabla_\xi V = \frac{f'\mathrm{e}^{-f}}{\alpha} \xi, && \nabla_\xi \varphi V = 0, \\
&\nabla_V \xi = 0, && \nabla_V V = -\frac{\mathrm{e}^{-f}}{\alpha^2} \alpha_1 \varphi V, && \nabla_V \varphi V = -\varphi \nabla_V V, \\
&\nabla_{\varphi V} \xi = 0, && \nabla_{\varphi V} V = \frac{\mathrm{e}^{-f}}{\alpha^2} (f'\alpha + \alpha_2) \varphi V, && \nabla_{\varphi V} \varphi V = \varphi \nabla_{\varphi V} V,
\end{aligned}$$

where $\alpha_i = \frac{\partial \alpha}{\partial x_i}$. Then, one can easily check that, for all $i, j \in \{1, 2, 3\}$,

$$(\nabla_{e_i} \varphi) e_j = \nabla_{e_i} \varphi e_j - \varphi \nabla_{e_i} e_j = \eta(e_i)\big(\omega(\varphi e_j)\xi + \eta(e_j)\varphi\psi\big).$$

Now, we denote by R the curvature tensor and by S the Ricci curvature. From [5, Corollary 3.1], one can get the following:

Corollary 3.9. *For any 3-dimensional C_{12}-manifold, we have*

$$\begin{aligned} R(X,Y)\xi &= -2\mathrm{d}\eta(X,Y)\psi - \eta(Y)\nabla_X\psi + \eta(X)\nabla_Y\psi, \\ R(X,\xi)Y &= \omega(X)\big(\omega(Y)\xi - \eta(Y)\psi\big) + g(\nabla_X\psi, Y)\xi - \eta(Y)\nabla_X\psi, \\ S(X,\xi) &= -\eta(X)\operatorname{div}\psi. \end{aligned} \tag{3.7}$$

By use of (3.7), we have

$$R(\xi,\psi)\xi = -\omega(\psi)\psi - \nabla_\psi\psi.$$

Therefore

$$g(R(\xi,\psi)\psi,\xi) = -\omega(\psi)^2 - g(\nabla_\psi\psi,\psi).$$

Thus we have

Proposition 3.10. *On 3-dimensional C_{12}-manifolds, the sectional curvature of the plane section spaned by $\{\xi,\psi\}$ is $-\omega(\psi)^2 - g(\nabla_\psi\psi,\psi)$ and if ψ is unitary the sectional curvature is -1.*

Recall that the conformal curvature tensor vanishes in a 3-dimensional Riemannian manifold, therefore we get (see [2])

$$\begin{aligned} R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ - \frac{r}{2}\big(g(Y,Z)X - g(X,Z)Y\big), \end{aligned} \tag{3.8}$$

where r is the scalar curvature. In the following theorem, we obtain an expression for the Ricci operator in a 3-dimensional C_{12}-manifold.

Theorem 3.11. *In a 3-dimensional C_{12}-manifold, the Ricci operator is given by*

$$QX = (\operatorname{div}\psi)X + (\mathrm{e}^\rho - 2\operatorname{div}\psi)\eta(X)\xi - \omega(X)\psi - \nabla_X\psi - \frac{r}{2}\varphi^2X, \tag{3.9}$$

where Q is the Ricci operator defined by

$$S(X,Y) = g(QX,Y). \tag{3.10}$$

Proof. For a 3-dimensional C_{12}-manifold, from (3.7) and (3.8) we have

$$R(X,\xi)\xi = QX + (\operatorname{div}\psi)X - 2(\operatorname{div}\psi)\eta(X)\xi + \frac{r}{2}\varphi^2X, \tag{3.11}$$

and from formula (3.7) we get

$$R(X,\xi)\xi = -\omega(X)\psi - \nabla_X\psi + \mathrm{e}^{2\rho}\eta(X)\xi. \tag{3.12}$$

In view of (3.11) and (3.12), we obtain our formula. ☐

Corollary 3.12. *In a 3-dimensional C_{12}-manifold, the Ricci tensor and the curvature tensor are given respectively by*

$$\begin{aligned} S(X,Y) = \Big(\frac{r}{2} + \operatorname{div}\psi\Big)g(X,Y) + \big(\mathrm{e}^{2\rho} - 2\operatorname{div}\psi - \frac{r}{2}\big)\eta(X)\eta(Y) \\ - \omega(X)\omega(Y) - g(\nabla_X\psi, Y), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} R(X,Y)Z = & \left(\mathrm{e}^{2\rho} - 2\operatorname{div}\psi - \frac{r}{2}\right)\eta(Z)\big(\eta(Y)X - \eta(X)Y\big) \\ & - g(Y,Z)\Big(\omega(X)\psi + \nabla_X\psi - \Big(2\operatorname{div}\psi + \frac{r}{2}\Big)X\Big) \\ & + g(X,Z)\Big(\omega(Y)\psi + \nabla_Y\psi - \Big(2\operatorname{div}\psi + \frac{r}{2}\Big)Y\Big) \\ & + \left(\mathrm{e}^{2\rho} - 2\operatorname{div}\psi - \frac{r}{2}\right)\Big(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big)\xi \\ & - \omega(Z)\big(\omega(Y)X - \omega(X)Y\big) + g(\nabla_X\psi, Z)Y - g(\nabla_Y\psi, Z)X. \end{aligned} \tag{3.14}$$

Proof. Equation (3.13) follows from (3.9) and (3.10). Using (3.9) and (3.13) in (3.8), the curvature tensor in a 3-dimensional C_{12}-manifold is given by (3.14). □

4. C_{12}-structures on three-dimensional Lie groups

An almost contact metric structure (φ, ξ, η, g) on a connected Lie group G is said to be left invariant if g is left invariant and if the left multiplication map $L_a : G \to G$, $L_a(x) = a.x$ has the properties

$$\varphi \circ L_a = L_a \circ \varphi \quad \text{and} \quad L_a(\xi) = \xi \qquad \text{for all } a \in G.$$

Let $\mathfrak{g}$ be an odd-dimensional Lie algebra. An almost contact metric structure on $\mathfrak{g}$ is a quadruple (φ, ξ, η, g), where η is a one-form, φ is an endomorphism of $\mathfrak{g}$ and $\xi \in \mathfrak{g}$ such that

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all vector fields X, Y and g is a positive definite compatible inner product on $\mathfrak{g}$. It is also convenient to use defining relations for the structures on Lie algebras. For instance, an almost contact metric structure (φ, ξ, η, g) on a Lie algebra $\mathfrak{g}$ is said to be a C_{12}-structure if and only if

$$\nabla_X\xi = -\eta(X)\psi = \eta(X)\nabla_\xi\xi \tag{4.1}$$

for all X vector field in $\mathfrak{g}$.

Let G be a connected Lie group of dimension 3, endowed with a left invariant almost contact metric structure (φ, ξ, η, g) and let $\mathfrak{g} \cong T_eG$ be the corresponding Lie algebra of G. If $\{e_1, e_2, e_3\}$ is an orthonormal basis on $\mathfrak{g}$ then

$$\varphi e_i = \sum_j \varphi_i^j e_j \qquad \text{and} \qquad \xi = ae_1 + be_2 + ce_3,$$

where φ_i^j and a, b, c are constants such that $a^2 + b^2 + c^2 = 1$.

A classification of the Lie algebras of dimension three is found in [8], where Patera et al. list the nine classes of three-dimensional and twelve classes of four-dimensional Lie algebras. Here is the list of non-abelian three-dimensional algebras along with their defining Lie bracket equations.

Name	**Structure equations**		
$A_{3,1}$	$[e_2, e_3] = e_1$		
$A_{3,2}$	$[e_1, e_3] = e_1$	$[e_2, e_3] = e_1 + e_2$	
$A_{3,3}$	$[e_1, e_3] = e_1$	$[e_2, e_3] = e_2$	
$A_{3,4}$	$[e_1, e_3] = e_1$	$[e_2, e_3] = -e_2$	
$A^{\lambda}_{3,5}$	$[e_1, e_3] = e_1$	$[e_2, e_3] = \lambda e_2 \quad (0 < \|\lambda\| < 1)$	
$A_{3,6}$	$[e_1, e_3] = -e_2$	$[e_2, e_3] = e_1$	
$A^{\lambda}_{3,7}$	$[e_1, e_3] = -\lambda e_1 - e_2$	$[e_2, e_3] = e_1 + \lambda e_2 \quad (\lambda > 0)$	
$A_{3,8}$	$[e_1, e_2] = e_1$	$[e_1, e_3] = -2e_2$	$[e_2, e_3] = e_3$
$A_{3,9}$	$[e_1, e_2] = e_3$	$[e_1, e_3] = -e_2$	$[e_2, e_3] = e_1$

We will investigate the existence of C_{12}-structures on each $A_{3,i}$ and it is sufficient here to find ξ and ψ. From (4.1), we conclude that the existence of the C_{12}-structure is equivalent to

$$\nabla_{e_i}\xi = g(\xi, e_i)\nabla_\xi \xi$$

for any $i \in \{1, 2, 3\}$ or equivalently,

$$\begin{cases} \nabla_{e_1}\xi = a\nabla_\xi \xi \\ \nabla_{e_2}\xi = b\nabla_\xi \xi \\ \nabla_{e_3}\xi = c\nabla_\xi \xi. \end{cases} \tag{4.2}$$

In other words, the existence of C_{12}-structures requires the existence of the constants a, b and c provided that $\nabla_\xi \xi \neq 0$.

The algebra $A_{3,1}$. By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{lll} \nabla_{e_1}e_1 = 0 & \nabla_{e_1}e_2 = -\frac{1}{2}e_3 & \nabla_{e_1}e_3 = \frac{1}{2}e_2 \\ \nabla_{e_2}e_1 = -\frac{1}{2}e_3 & \nabla_{e_2}e_2 = 0 & \nabla_{e_2}e_3 = \frac{1}{2}e_1 \\ \nabla_{e_3}e_1 = \frac{1}{2}e_2 & \nabla_{e_3}e_2 = -\frac{1}{2}e_1 & \nabla_{e_3}e_3 = 0. \end{array}$$

By a simple computation using the covariant derivatives of the basis elements, one can get

$$\nabla_{e_1}\xi = \begin{pmatrix} 0 \\ \frac{c}{2} \\ -\frac{b}{2} \end{pmatrix}, \ \nabla_{e_2}\xi = \begin{pmatrix} \frac{c}{2} \\ 0 \\ -\frac{a}{2} \end{pmatrix}, \ \nabla_{e_3}\xi = \begin{pmatrix} -\frac{b}{2} \\ \frac{a}{2} \\ 0 \end{pmatrix} \text{ and } \nabla_\xi \xi = \begin{pmatrix} 0 \\ ac \\ -ab \end{pmatrix}.$$

With the help of system (4.2), we obtain

$$a = b = c = 0.$$

Then, there exists no C_{12}-structure on $A_{3,1}$.

The algebra $A_{3,2}$**.** By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{lll} \nabla_{e_1}e_1=-e_3 & \nabla_{e_1}e_2=-\frac{1}{2}e_3 & \nabla_{e_1}e_3=e_1+\frac{1}{2}e_2\\ \nabla_{e_2}e_1=-\frac{1}{2}e_3 & \nabla_{e_2}e_2=-e_3 & \nabla_{e_2}e_3=\frac{1}{2}e_1+e_2\\ \nabla_{e_3}e_1=\frac{1}{2}e_2 & \nabla_{e_3}e_2=-\frac{1}{2}e_1 & \nabla_{e_3}e_3=0. \end{array}$$

One can get

$$\nabla_{e_1}\xi=\begin{pmatrix} c\\ \frac{c}{2}\\ -a-\frac{b}{2}\end{pmatrix},\quad \nabla_{e_2}\xi=\begin{pmatrix} \frac{c}{2}\\ c\\ -\frac{a}{2}-b\end{pmatrix},\quad \nabla_{e_3}\xi=\begin{pmatrix} -\frac{b}{2}\\ \frac{a}{2}\\ 0\end{pmatrix}$$

and

$$\nabla_{\xi}\xi=\begin{pmatrix} ac\\ ac+bc\\ -a^2-b^2-ab\end{pmatrix}.$$

With the help of system 4.2, we get

$$a=b=c=0.$$

Then, there exists no C_{12}-structure on $A_{3,2}$.

The algebra $A_{3,3}$**.** By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{lll} \nabla_{e_1}e_1=-e_3 & \nabla_{e_1}e_2=0 & \nabla_{e_1}e_3=e_1\\ \nabla_{e_2}e_1=0 & \nabla_{e_2}e_2=-e_3 & \nabla_{e_2}e_3=e_2\\ \nabla_{e_3}e_1=0 & \nabla_{e_3}e_2=0 & \nabla_{e_3}e_3=0. \end{array}$$

One can get

$$\nabla_{e_1}\xi=\begin{pmatrix} c\\ 0\\ -a\end{pmatrix},\ \nabla_{e_2}\xi=\begin{pmatrix} 0\\ c\\ -b\end{pmatrix},\ \nabla_{e_3}\xi=\begin{pmatrix} 0\\ 0\\ 0\end{pmatrix}\ \text{and}\ \nabla_{\xi}\xi=\begin{pmatrix} ac\\ bc\\ -a^2-b^2\end{pmatrix}.$$

With the help of system 4.2, we get an infinite number of solutions of the form

$$c=0 \qquad \text{with } a^2+b^2=1,$$

i.e.,

$$\xi=ae_1\pm\sqrt{1-a^2}e_2, \qquad \text{with } a\in[-1,+1] \text{ and } \psi=e_3.$$

Then, there exists an infinite number of C_{12}-structures on $A_{3,3}$.

The algebra $A_{3,4}$**.** By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{lll} \nabla_{e_1}e_1=-e_3 & \nabla_{e_1}e_2=0 & \nabla_{e_1}e_3=e_1\\ \nabla_{e_2}e_1=0 & \nabla_{e_2}e_2=e_3 & \nabla_{e_2}e_3=-e_2\\ \nabla_{e_3}e_1=0 & \nabla_{e_3}e_2=0 & \nabla_{e_3}e_3=0. \end{array}$$

Therefore, we obtain

$$\nabla_{e_1}\xi = \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix}, \ \nabla_{e_2}\xi = \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}, \ \nabla_{e_3}\xi = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \nabla_{\xi}\xi = \begin{pmatrix} ac \\ bc \\ -a^2-b^2 \end{pmatrix}.$$

With the help of system 4.2, we get four solutions of the form

$$(a,b,c) \in \{(1,0,0); (-1,0,0); (0,1,0); (0,-1,0)\},$$

i.e.,

$$(\xi,\psi) \in \{(e_1,e_3), (-e_1,e_3), (e_2,e_3), (-e_2,e_3)\}.$$

So, there exists an infinite number of C_{12}-structures on $A_{3,4}$ with

$$(\xi,\psi) \in \{(e_1,e_3), (-e_1,e_3), (e_2,e_3), (-e_2,e_3)\} \quad \text{and} \quad \varphi e_i = \sum_j \varphi_i^j e_j.$$

The algebra $A_{3,5}^{\lambda}$. By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{lll} \nabla_{e_1}e_1 = -e_3 & \nabla_{e_1}e_2 = 0 & \nabla_{e_1}e_3 = e_1 \\ \nabla_{e_2}e_1 = 0 & \nabla_{e_2}e_2 = -\lambda e_3 & \nabla_{e_2}e_3 = \lambda e_2 \\ \nabla_{e_3}e_1 = 0 & \nabla_{e_3}e_2 = 0 & \nabla_{e_3}e_3 = 0. \end{array}$$

Therefore, we obtain

$$\nabla_{e_1}\xi = \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix}, \ \nabla_{e_2}\xi = \begin{pmatrix} 0 \\ \lambda c \\ -\lambda b \end{pmatrix}, \ \nabla_{e_3}\xi = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \nabla_{\xi}\xi = \begin{pmatrix} ac \\ \lambda bc \\ -a^2-\lambda b^2 \end{pmatrix}.$$

Replacing in the system 4.2, we get four solutions of the form

$$(a,b,c) \in \{(1,0,0); (-1,0,0); (0,1,0); (0,-1,0)\},$$

i.e.,

$$(\xi,\psi) \in \{(e_1,e_3), (-e_1,e_3), (e_2,\lambda e_3), (-e_2,\lambda e_3)\}.$$

Then, there exists an infinite number of C_{12}-structures on $A_{3,5}^{\lambda}$with $0 < \lambda < 1$.

The algebra $A_{3,6}$. By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{lll} \nabla_{e_1}e_1 = 0 & \nabla_{e_1}e_2 = 0 & \nabla_{e_1}e_3 = 0 \\ \nabla_{e_2}e_1 = 0 & \nabla_{e_2}e_2 = 0 & \nabla_{e_2}e_3 = 0 \\ \nabla_{e_3}e_1 = e_2 & \nabla_{e_3}e_2 = -e_1 & \nabla_{e_3}e_3 = 0. \end{array}$$

One can get

$$\nabla_{e_1}\xi = \nabla_{e_2}\xi = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \nabla_{e_3}\xi = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix} \quad \text{and} \quad \nabla_{\xi}\xi = \begin{pmatrix} -bc \\ ac \\ 0 \end{pmatrix}.$$

From system 4.2, we get $a = b = 0$ and $c \in \mathbb{R}$ this implies $\nabla_{\xi}\xi = 0$. Then, there exists no C_{12}-structure on $A_{3,6}$.

The algebra $A_{3,7}^{\lambda}$. By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{lll} \nabla_{e_1} e_1 = \lambda e_3 & \nabla_{e_1} e_2 = 0 & \nabla_{e_1} e_3 = -\lambda e_1 \\ \nabla_{e_2} e_1 = 0 & \nabla_{e_2} e_2 = -\lambda e_3 & \nabla_{e_2} e_3 = \lambda e_2 \\ \nabla_{e_3} e_1 = e_2 & \nabla_{e_3} e_2 = -e_1 & \nabla_{e_3} e_3 = 0. \end{array}$$

One can get

$$\nabla_{e_1}\xi = \lambda \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix}, \quad \nabla_{e_2}\xi = \lambda \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}, \quad \nabla_{e_3}\xi = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$$

and

$$\nabla_{\xi}\xi = \begin{pmatrix} -c(a\lambda + b) \\ c(a + b\lambda) \\ \lambda(a^2 - b^2) \end{pmatrix}.$$

From 4.2, we get

$$a = b = c = 0.$$

Then, there exists no C_{12}-structure on $A_{3,7}^{\lambda}$.

The algebra $A_{3,8}$. By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{lll} \nabla_{e_1} e_1 = -e_2 & \nabla_{e_1} e_2 = e_1 + e_3 & \nabla_{e_1} e_3 = -e_2 \\ \nabla_{e_2} e_1 = e_3 & \nabla_{e_2} e_2 = 0 & \nabla_{e_2} e_3 = -e_1 \\ \nabla_{e_3} e_1 = e_2 & \nabla_{e_3} e_2 = -e_1 - e_3 & \nabla_{e_3} e_3 = e_2. \end{array}$$

One can get

$$\nabla_{e_1}\xi = -\nabla_{e_3}\xi = \begin{pmatrix} b \\ -a - c \\ b \end{pmatrix}, \quad \nabla_{e_2}\xi = \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix} \quad \text{and} \quad \nabla_{\xi}\xi = \begin{pmatrix} b(a - 2c) \\ -a^2 + c^2 \\ b(2a - c) \end{pmatrix}.$$

From 4.2, we obtain the system

$$a^2 = b^2 = \frac{1}{3} \quad \text{and} \quad c = -a,$$

which gives four solutions;

$$(a, b, c) \in \left\{ \tfrac{1}{\sqrt{3}}(1, 1, -1); \tfrac{1}{\sqrt{3}}(1, -1, -1); \tfrac{1}{\sqrt{3}}(-1, 1, 1); \tfrac{1}{\sqrt{3}}(-1, -1, 1) \right\}.$$

So, there exists an infinite number of C_{12}-structures on $A_{3,8}$.

The algebra $A_{3,9}$. By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{lll} \nabla_{e_1}e_1 = 0 & \nabla_{e_1}e_2 = \frac{1}{2}e_3 & \nabla_{e_1}e_3 = -\frac{1}{2}e_2 \\ \nabla_{e_2}e_1 = -\frac{1}{2}e_3 & \nabla_{e_2}e_2 = 0 & \nabla_{e_2}e_3 = \frac{1}{2}e_1 \\ \nabla_{e_3}e_1 = \frac{1}{2}e_2 & \nabla_{e_3}e_2 = \frac{1}{2}e_1 & \nabla_{e_3}e_3 = 0. \end{array}$$

By a simple computation using the covariant derivatives of the basis elements, one can get

$$\nabla_{e_1}\xi = \begin{pmatrix} 0 \\ -\frac{c}{2} \\ \frac{b}{2} \end{pmatrix}, \ \nabla_{e_2}\xi = \begin{pmatrix} \frac{c}{2} \\ 0 \\ -\frac{a}{2} \end{pmatrix}, \ \nabla_{e_3}\xi = \begin{pmatrix} -\frac{b}{2} \\ \frac{a}{2} \\ 0 \end{pmatrix} \text{ and } \nabla_{\xi}\xi = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\nabla_\xi \xi = 0$, there exists no C_{12}-structure on $A_{3,9}$.

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