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THE JOHN–NIRENBERG INEQUALITY FOR ORLICZ–LORENTZ SPACES IN A PROBABILISTIC SETTING

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ABSTRACT. The John–Nirenberg inequality is widely studied in the field of mathematical analysis and probability theory. In this paper we study a new type of the John–Nirenberg inequality for Orlicz–Lorentz spaces in a probabilistic setting. To be precise, let $0 < q \leq \infty$ and Φ be an N-function with some proper restrictions. We prove that if the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then $BMO_{\Phi,q} = BMO$, with equivalent (quasi)-norms. The result is new, which improves previous work on martingale Hardy theory.

1. Introduction

One of the most important properties of BMO spaces (spaces of functions satisfying a bounded mean oscillation) is the so-called John–Nirenberg inequality, which was originally proved by John and Nirenberg in [14]. It was later extended to the probabilistic context by Garsia and Herz in [3, 8]. In this paper, we deal with the John–Nirenberg inequality for the new type of BMO spaces in probability theory.

Before describing our main results, we recall the classical John–Nirenberg inequality in probability theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_n\}_{n\geq 0}$ be a non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\big(\bigcup_{n\geq 0} \mathcal{F}_n\big)$. The expectation operator and the conditional expectation operator with respect to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence of $f = (f_n)_{n\geq 0}$ of random variables is said to be a martingale if f_n is $\mathcal{F}_n$-measurable, $\mathbb{E}(|f_n|) < \infty$ and $\mathbb{E}_n(f_{n+1}) = f_n$ for each $n \geq 0$. The spaces BMO_p, $1 \leq p < \infty$, are defined as

$$BMO_p = \left\{ f \in L_p : \|f\|_{BMO_p} = \sup_{n\geq 0} \big\|\mathbb{E}_n(|f - f_n|^p)\big\|_{L_\infty}^{1/p} < \infty \right\},$$

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where the $f_n = \mathbb{E}_n(f)$. Let $\mathcal{T}$ be the set of all stopping times with respect to $\{\mathcal{F}_n\}_{n\geq 0}$. It is easy to check that (see [8, 19, 25])

$$\|f\|_{BMO_p} = \sup_{\tau\in\mathcal{T}} \frac{\|(f - f^\tau)\chi_{\{\tau<\infty\}}\|_{L_p}}{\mathbb{P}(\tau < \infty)^{1/p}}.$$

Note that $BMO_2 = BMO$. Based mainly on duality ($(H_1)^* = BMO$), the John–Nirenberg inequality plays an important role in classical analysis and martingale theory.

The well-known John–Nirenberg inequality (one of the most important theorems in martingale theory, see [8, 25]) says that if the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then $BMO_p = BMO$ with respect to these norms. That is,

$$\|f\|_{BMO} \lesssim \|f\|_{BMO_p} \lesssim \|f\|_{BMO}, \qquad 1 \leq p < \infty. \tag{1.1}$$

Here, the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is said to be regular if there exists $\mathcal{R} > 1$ such that

$$f_n \leq \mathcal{R} f_{n-1} \qquad \forall\, n \geq 1$$

holds for all non-negative martingales $f = (f_n)_{n\geq 0}$ adapted to $\{\mathcal{F}_n\}_{n\geq 0}$. The reader is referred to [19, 22, 25] for more information about martingale theory and regularity. Now the probabilistic version of the John–Nirenberg inequality has been extended to various known function spaces, such as the rearrangement invariant Banach function space [2, 26], the Lebesgue space with variable exponents [13], the non-commutative Lebesgue space [10, 15], the Lorentz space [9, 12, 17], and the Iwaniec–Sbordone space [5]. It is worth noting that these spaces are Banach function spaces.

In this paper, we will continue to answer whether the John–Nirenberg inequality is true for the non-Banach function spaces. Our purpose is to establish the John–Nirenberg inequality in the probabilistic version of Orlicz–Lorentz spaces $L_{\Phi,q}$ (where $0 < q \leq \infty$ and Φ is an N-function), introduced in [6] (see section 2). Our main result, stated informally, reads as follows.

Theorem 1.1. *Let $0 < q \leq \infty$ and Φ be an N-function with some proper restrictions. If the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then*

$$BMO_{\Phi,q} = BMO.$$

For the precise statement see Theorem 3.2 in section 3, where we also define the class $BMO_{\Phi,q}$. In order to prove theorem above, we need to discover more properties of the Orlicz–Lorentz spaces associated with $0 < q \leq \infty$ and N-function Φ. Such properties (see section 2) improve the properties of classical Lebesgue and Lorentz spaces.

Throughout this paper, we denote by C an absolute positive constant that is independent of the main parameters involved but whose value may differ from line to line. The notation $f \lesssim g$ stands for the inequality $f \leq Cg$. If we write $f \approx g$, we mean $f \lesssim g \lesssim f$.

2. Preliminaries

In this section, we give some preliminaries necessary for the whole paper.

2.1. **N-functions.** Let us first recall the definition of N-function. An *N-function* is a continuous and convex function $\Phi : [0,\infty) \longrightarrow \mathbb{R}$ such that $\Phi(s) > 0$, $s > 0$, $\Phi(s)/s \longrightarrow 0$ as $s \longrightarrow 0$, and $\Phi(s)/s \longrightarrow \infty$ as $s \longrightarrow \infty$. It is well known that an N-function Φ has the representation

$$\Phi(s) = \int_0^s \phi(t)\,dt,$$

where $\phi : [0,\infty) \longrightarrow \mathbb{R}$ is continuous from the right, non-decreasing such that $\phi(s) > 0$, $s > 0$, $\phi(0) = 0$ and $\phi(s) \longrightarrow \infty$ for $s \longrightarrow \infty$.

Associated to ϕ we have the function $\psi : [0,\infty) \longrightarrow \mathbb{R}$ defined by

$$\psi(t) = \sup\{s : \phi(s) \leq t\},$$

which has the same aforementioned properties of ϕ. We will call ψ the generalized inverse of ϕ. The N-function Ψ defined by

$$\Psi(t) = \int_0^t \psi(s)\,ds$$

is called the *complementary* N-function of Φ.

We have the following relationship between an N-function and its complementary function.

Proposition 2.1 (See [23]). *If Φ is an N-function and Ψ is the complementary of Φ, then*

$$t < \Phi^{-1}(t)\Psi^{-1}(t) \leq 2t \qquad \forall\, t > 0,$$

where Φ^{-1} and Ψ^{-1} denote the inverse function of Φ and Ψ, respectively.

2.2. **Orlicz spaces.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and f be an $\mathcal{F}$-measurable function defined on Ω. The distribution function of f is the function $\lambda_s(f)$ given by

$$\lambda_s(f) = \mathbb{P}\big(\{\omega \in \Omega : |f(\omega)| > s\}\big), \quad s \geq 0.$$

Denote by f^* the decreasing rearrangement of f, defined by

$$f^*(t) = \inf\{s \geq 0 : \lambda_s(f) \leq t\}, \quad t \geq 0,$$

with the convention that $\inf \emptyset = \infty$.

Definition 2.2. Let Φ be an increasing function. The *Orlicz space* $L_\Phi := L_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ is the set of all $\mathcal{F}$-measurable functions f satisfying $\mathbb{E}\big(\Phi(c|f|)\big) < \infty$ for some $c > 0$ and

$$\|f\|_{L_\Phi} = \inf\big\{c > 0 : \mathbb{E}\big(\Phi\big(|f|/c\big)\big) \leq 1\big\},$$

where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$.

If $\Phi(t) = t^p$ $(1 < p < \infty)$, then L_Φ is the usual Lebesgue space L_p. In this case we denote $\|\cdot\|_{L_p}$ by $\|\cdot\|_{L_\Phi}$. For the N-function Φ, the functional $\|\cdot\|_{L_\Phi}$ is a norm and thereby $(L_\Phi, \|\cdot\|_{L_\Phi})$ is a Banach space. By a simple calculation, one can check that for any $A \in \mathcal{F}$, $\mathbb{P}(A) > 0$,

$$\|\chi_A\|_{L_\Phi} = \frac{1}{\Phi^{-1}\left(\frac{1}{\mathbb{P}(A)}\right)}.$$

Recall the Hölder inequality on Orlicz spaces, which is analogous to the case of classical Lebesgue spaces:

Proposition 2.3 (See [23]). *Let Φ be an N-function and Ψ be the complementary function of Φ. There exists an absolute constant $C \geq 1$ depending only on Φ and Ψ such that if $f \in L_\Phi$ and $g \in L_\Psi$, we have*

$$\mathbb{E}(fg) \leq C\|f\|_{L_\Phi}\|g\|_{L_\Psi}.$$

Hardy, Littlewood and Pólya extended the above result to the more general case as follows.

Proposition 2.4 (See [7]). *Let $\Phi_i : [0, \infty) \longrightarrow \mathbb{R}$, $i = 1, 2, 3$, be N-functions such that*

$$\Phi_3^{-1}(t) = \Phi_1^{-1}(t)\Phi_2^{-1}(t) \qquad \forall\, t \geq 0.$$

There exists an absolute constant $C \geq 1$ depending only on Φ_1 and Φ_2 such that if $f \in L_{\Phi_1}$ and $g \in L_{\Phi_2}$, we have

$$\|fg\|_{L_{\Phi_3}} \leq C\|f\|_{L_{\Phi_1}}\|g\|_{L_{\Phi_2}}.$$

The lower and upper Simonenko indices of N-function Φ are respectively defined as

$$p_\Phi = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \qquad \text{and} \qquad q_\Phi = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

Clearly, $1 \leq p_\Phi \leq q_\Phi \leq \infty$. Simonenko introduced these indices in [24]. Moreover, Mao and Ren [23] prove that, if Φ is an N-function with $1 < p_\Phi \leq q_\Phi < \infty$ and Ψ is the complementary of Φ, then the lower and upper Simonenko indices of Ψ satisfy $1 < p_\Psi \leq q_\Psi < \infty$.

Proposition 2.5. *Let Φ be an N-function with $q_\Phi < \infty$. Then $\frac{\Phi(t)}{t^{p_\Phi}}$ is increasing on $(0, \infty)$ and $\frac{\Phi(t)}{t^{q_\Phi}}$ is decreasing on $(0, \infty)$.*

The above property of the indices of N-function Φ will be used in what follows. It is classical and can be found in [5, 11].

2.3. **Orlicz–Lorentz spaces.** Let $0 < q \le \infty$ and $\Phi : [0,\infty) \longrightarrow [0,\infty)$ be an increasing function such that $\Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = \infty$. The Orlicz–Lorentz space $L_{\Phi,q}(\Omega, \mathcal{F}, \mathbb{P})$ consists of the $\mathcal{F}$-measurable functions f with finite (quasi)-norm $\|f\|_{L_{\Phi,q}}$ given by

$$\|f\|_{L_{\Phi,q}} = \begin{cases} \left(q \int_0^\infty \left(t\|\chi_{\{|f|>t\}}\|_{L_\Phi}\right)^q \frac{dt}{t}\right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup\limits_{t>0} t \left\|\chi_{\{|f|>t\}}\right\|_{L_\Phi} & \text{if } q = \infty. \end{cases}$$

These spaces are the generalizations of classical Lorentz spaces $L_{p,q}$ and they coincide with $L_{p,q}$ when $\Phi(t) = t^p$ for $0 < p < \infty$. Moreover, if $\Phi(t) = t^q$ for $0 < q < \infty$, then $L_{\Phi,q}$ is the usual Lebesgue space L_q. The following fundamental properties of the functional $\|\cdot\|_{L_{\Phi,q}}$ were proved in [6]:

(1) $\|f\|_{L_{\Phi,q}} \ge 0$, and $\|f\|_{L_{\Phi,q}} = 0$ if and only if $f = 0$;
(2) $\|\lambda \cdot f\|_{L_{\Phi,q}} = |\lambda| \cdot \|f\|_{L_{\Phi,q}}$ for any $\lambda \in \mathbb{C}$;
(3) $\|f + g\|_{L_{\Phi,q}} \le C\big(\|f\|_{L_{\Phi,q}} + \|g\|_{L_{\Phi,q}}\big)$;
(4) $\|\chi_A\|_{L_{\Phi,q}} = \|\chi_A\|_{L_\Phi} = \frac{1}{\Phi^{-1}\left(\frac{1}{\mathbb{P}(A)}\right)}$ for any $A \in \mathcal{F}$ and $\mathbb{P}(A) > 0$.

Studies on the theory of Orlicz–Lorentz spaces can be found in [16, 21, 20, 18]. Next we shall present more properties of Orlicz–Lorentz spaces with N-function, which are new and useful for the main results in the paper.

Proposition 2.6. *Let $0 < q \le \infty$ and Φ be an N-function with $q_\Phi < \infty$. Then $\|f\|_{L_{\Phi,q}}$ and*

$$|||f|||_{L_{\Phi,q}} = \begin{cases} \left(q \int_0^1 \left(\frac{1}{\Phi^{-1}(1/t)} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup\limits_{t>0} \frac{1}{\Phi^{-1}(1/t)} f^*(t) & \text{if } q = \infty \end{cases}$$

are equivalent (quasi)-norms.

Proof. For any measurable function f, there exists a sequence of non-negative simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \uparrow |f|$ a.e. Moreover, $d_{f_n} \uparrow d_f$ and $f_n^* \uparrow f^*$. Therefore, by using Lebesgue's monotone convergence theorem, it suffices to establish that the quasi-norm defined as $|||f|||_{\Phi,q}$ is equivalent to $\|f\|_{\Phi,q}$ for non-negative simple functions.

Now let

$$f(\omega) = \sum_{i=1}^N \alpha_i \chi_{A_i}(\omega),$$

where $\{A_i\}_{i=1}^N$ is a family of disjoint measurable sets and $\{\alpha_j\}_{j=1}^N \subseteq \mathbb{R}$ satisfy $0 \le \alpha_j \le \alpha_i$ for $1 \le i \le j \le N$. For any $t \ge 0$, we have

$$\lambda_t(f) = \sum_{j=1}^N \beta_j \chi_{[\alpha_{j+1},\alpha_j)}(t),$$

where $\alpha_{N+1}=0$ and $\beta_j=\sum_{i=1}^{j}\mathbb{P}(A_i)$ for $1\le j\le N$. Also, one can see that

$$f^*(t)=\sum_{j=1}^{N}a_j\chi_{[\beta_{j-1},\beta_j)}(t),$$

where $\beta_0=0$.

We first consider the case of $q=\infty$. Since $\Phi^{-1}(t)$ is increasing on $(0,\infty)$, we get

$$\begin{aligned}\|f\|_{L_{\Phi,\infty}}&=\sup_{t>0}t\|\chi_{\{|f|>t\}}\|_{L_\Phi}=\sup_{t>0}\frac{t}{\Phi^{-1}\big(1/\lambda_t(f)\big)}\\&=\sup_{t>0}\sum_{j=1}^{N}\frac{t}{\Phi^{-1}(1/\beta_j)}\chi_{[\alpha_{j+1},\alpha_j)}(t)=\max_{1\le j\le N}\frac{\alpha_j}{\Phi^{-1}(1/\beta_j)}\end{aligned}$$

and

$$\begin{aligned}|||f|||_{L_{\Phi,\infty}}&=\sup_{t>0}\frac{1}{\Phi^{-1}(1/t)}f^*(t)=\sup_{t>0}\sum_{j=1}^{N}\frac{\alpha_j}{\Phi^{-1}(1/t)}\chi_{[\beta_{j-1},\beta_j)}(t)\\&=\max_{1\le j\le N}\frac{\alpha_j}{\Phi^{-1}(1/\beta_j)},\end{aligned}$$

which implies

$$|||f|||_{L_{\Phi,\infty}}=\|f\|_{L_{\Phi,\infty}}.$$

Now we consider the case of $0<q<\infty$. It follows from the Abel transformation that

$$\begin{aligned}|||f|||^q_{L_{\Phi,q}}&=q\sum_{i=1}^{N}\alpha_i^q\int_{\beta_{i-1}}^{\beta_i}\left(\frac{1}{\Phi^{-1}(1/t)}\right)^q\frac{dt}{t}=\sum_{i=1}^{N}\alpha_i^q\big(K(\beta_i)-K(\beta_{i-1})\big)\\&=\sum_{i=1}^{N}\big(\alpha_i^q-\alpha_{i+1}^q\big)K(\beta_i),\end{aligned}$$

where

$$K(t)=q\int_0^t\left(\frac{1}{\Phi^{-1}(1/s)}\right)^q\frac{ds}{s}.$$

It follows from Proposition 2.5 that $\frac{t^{1/q_\Phi}}{\Phi^{-1}(t)}$ is decreasing on $(0,\infty)$. This implies that

$$\begin{aligned} K(\beta_i) &= q\int_0^{\beta_i}\left(\frac{1}{\Phi^{-1}(1/t)}\right)^q\frac{dt}{t} = q\int_{1/\beta_i}^{\infty}\left(\frac{1}{\Phi^{-1}(t)}\right)^q\frac{dt}{t} \\ &= q\int_{1/\beta_i}^{\infty}\left(\frac{t^{1/q_\Phi}}{\Phi^{-1}(t)}\right)^q t^{-(1+q/q_\Phi)}\,dt \\ &\le q\left(\frac{\beta_i^{-1/q_\Phi}}{\Phi^{-1}(1/\beta_i)}\right)^q\int_{1/\beta_i}^{\infty} t^{-(1+q/q_\Phi)}\,dt \\ &= q_\Phi\left(\frac{1}{\Phi^{-1}(1/\beta_i)}\right)^q. \end{aligned}$$

The convexity of Φ and $\Phi(0)=0$ imply that $\frac{t}{\Phi^{-1}(t)}$ is increasing on $(0,\infty)$. This means that

$$\begin{aligned} K(\beta_i) &= q\int_{1/\beta_i}^{\infty}\left(\frac{t}{\Phi^{-1}(t)}\right)^q t^{-(1+q)}\,dt \\ &\ge q\left(\frac{1/\beta_i}{\Phi^{-1}(1/\beta_i)}\right)^q\int_{1/\beta_i}^{\infty} t^{-(1+q)}\,dt \\ &= \left(\frac{1}{\Phi^{-1}(1/\beta_i)}\right)^q. \end{aligned}$$

Hence we get

$$\begin{aligned} \sum_{i=1}^{N}\left(\alpha_i^q-\alpha_{i+1}^q\right)\left(\frac{1}{\Phi^{-1}(1/\beta_i)}\right)^q &\le |||f|||_{L_{\Phi,q}}^q \\ &\le q_\Phi\sum_{i=1}^{N}\left(\alpha_i^q-\alpha_{i+1}^q\right)\left(\frac{1}{\Phi^{-1}(1/\beta_i)}\right)^q. \end{aligned} \tag{2.1}$$

Moreover, we have

$$\begin{aligned} \|f\|_{L_{\Phi,q}}^q &= q\int_0^\infty \left(t\|\chi_{\{|f|>t\}}\|_{L_\Phi}\right)^q\frac{dt}{t} = q\int_0^\infty\left(\frac{t}{\Phi^{-1}\big(1/\lambda_t(f)\big)}\right)^q\frac{dt}{t} \\ &= q\sum_{i=1}^{N}\int_{\alpha_{i+1}}^{\alpha_i}\left(\frac{t}{\Phi^{-1}(1/\beta_i)}\right)^q\frac{dt}{t} \\ &= \sum_{i=1}^{N}\left(\alpha_i^q-\alpha_{i+1}^q\right)\left(\frac{1}{\Phi^{-1}(1/\beta_i)}\right)^q. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), one can see that

$$\|f\|_{L_{\Phi,q}} \le |||f|||_{L_{\Phi,q}} \le q_\Phi^{1/q}\|f\|_{L_{\Phi,q}}.$$

This completes the proof. □

Note that if we consider the special N-function $\Phi(t) = t^p$ ($t \in [0,\infty)$, $1 < p < \infty$), then $p_\Phi = q_\Phi = p < \infty$ (see [1]). From Proposition 2.6, we obtain the following fact:

Corollary 2.7. *Let $1 < p < \infty$ and $0 < q \le \infty$. The Lorentz spaces $\big(L_{p,q}, \|\cdot\|_{L_{p,q}}\big)$ are equivalent to $\big(L_{p,q}, |||\cdot|||_{L_{p,q}}\big)$. That is,*

$$\|\cdot\|_{L_{p,q}} \approx |||\cdot|||_{L_{p,q}}.$$

Using Proposition 2.6, we have the following embedding relationships among these Orlicz–Lorentz spaces:

Proposition 2.8. *Let $0 < q < p \le \infty$ and Φ be an N-function with $q_\Phi < \infty$. Then $L_{\Phi,q}$ is a subspace of $L_{\Phi,p}$, i.e.,*

$$\|f\|_{L_{\Phi,p}} \lesssim \|f\|_{L_{\Phi,q}} \qquad \forall\, f \in L_{\Phi,q}.$$

Proof. Let $f \in L_{\Phi,q}$. For $p = \infty$, it follows from $\frac{t}{\Phi^{-1}(t)}$ being increasing on $(0,\infty)$ and Proposition 2.6 that

$$\begin{aligned}\frac{1}{\Phi^{-1}\big(1/t\big)} f^*(t) &\le \left(q\int_0^t \left(\frac{1}{\Phi^{-1}\big(1/s\big)} f^*(t)\right)^q \frac{ds}{s}\right)^{1/q}\\ &\le \left(q\int_0^t \left(\frac{1}{\Phi^{-1}\big(1/s\big)} f^*(s)\right)^q \frac{ds}{s}\right)^{1/q}\\ &\le |||f|||_{L_{\Phi,q}} \le q_\Phi^{1/q}\|f\|_{L_{\Phi,q}}.\end{aligned}$$

Taking the supremum over all $t > 0$ for these inequalities, we hence obtain

$$\|f\|_{L_{\Phi,\infty}} \le q_\Phi^{1/q}\|f\|_{L_{\Phi,q}}.$$

Finally, when $p < \infty$, it follows from Proposition 2.6 that

$$\begin{aligned}\|f\|_{L_{\Phi,p}} &\approx \left(p\int_0^\infty \left(\frac{1}{\Phi^{-1}\big(1/s\big)} f^*(s)\right)^{p-q+q} \frac{ds}{s}\right)^{1/p} \lesssim |||f\,|||_{L_{\Phi,q}}^{q/p}\, \big\|f\big\|_{L_{\Phi,\infty}}^{(p-q)/p}\\ &\lesssim \|f\|_{L_{\Phi,q}}.\end{aligned}$$

This completes the proof. □

We present the following Hölder-type inequality for Orlicz–Lorentz spaces.

Proposition 2.9. *Let Φ be an N-function with $1 < p_\Phi \le q_\Phi < \infty$ and Ψ be the complementary of Φ.*

(i) If $1 < q \le \infty$, $f \in L_{\Phi,q}$ and $g \in L_{\Psi,q'}$, then we have

$$\mathbb{E}(fg) \le C\|f\|_{L_{\Phi,q}}\|g\|_{L_{\Psi,q'}},$$

where q' satisfies $1/q + 1/q' = 1$.

(ii) If $0 < q \le 1$, $f \in L_{\Phi,q}$ and $g \in L_{\Psi,\infty}$, then we have

$$\mathbb{E}(fg) \le C\|f\|_{L_{\Phi,q}}\|g\|_{L_{\Psi,\infty}}.$$

In order to prove the Hölder-type inequality for Orlicz–Lorentz spaces, we need the famous Hardy inequality as follows:

Lemma 2.10 (See [1])**.** *For f and g measurable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we have*

$$\mathbb{E}(fg) = \int_\Omega fg\, d\mathbb{P} \le \int_0^\infty f^*(t) g^*(t)\, dt.$$

Now we prove Proposition 2.9:

Proof. Since the N-function Φ satisfies the condition $1 < p_\Phi \le q_\Phi < \infty$, we have $1 < p_\Psi \le q_\Psi < \infty$ for N-function Ψ. Therefore $\|\cdot\|_{\Phi,q}$ and $\|\cdot\|_{\Psi,q}$ are equivalent to $|||\cdot|||_{\Phi,q}$ and $|||\cdot|||_{\Psi,q}$, respectively. According to Proposition 2.1, we have

$$\frac{t}{2} \le \frac{1}{\Phi^{-1}\big(1/t\big)} \frac{1}{\Psi^{-1}\big(1/t\big)} < t. \tag{2.3}$$

Applying Lemma 2.10 and (2.3), we have

$$\mathbb{E}(fg) \le \int_0^\infty f^*(t) g^*(t)\, dt \le 2 \int_0^\infty \frac{1}{\Phi^{-1}\big(1/t\big)} f^*(t) \frac{1}{\Psi^{-1}\big(1/t\big)} g^*(t) \frac{dt}{t}. \tag{2.4}$$

(i) Combining the Hölder inequality and inequality (2.4), we obtain, for $1 < q < \infty$,

$$\begin{aligned}\mathbb{E}(fg) &\le 2 \bigg(\int_0^\infty \Big[\frac{1}{\Phi^{-1}\big(1/t\big)} f^*(t) \Big]^q \frac{dt}{t} \bigg)^{1/q} \bigg(\int_0^\infty \Big[\frac{1}{\Psi^{-1}\big(1/t\big)} g^*(t) \Big]^{q'} \frac{dt}{t} \bigg)^{1/q'} \\ &= 2\, |||\, f \,|||_{L_{\Phi,q}}\, |||g|||_{L_{\Psi,q'}} \approx \|f\|_{L_{\Phi,q}} \|g\|_{L_{\Psi,q'}}.\end{aligned}$$

Moreover, for $q = \infty$,

$$\begin{aligned}\mathbb{E}(fg) &\le 2 \int_0^\infty \sup_{t>0} \bigg(\frac{1}{\Phi^{-1}\big(1/t\big)} f^*(t) \bigg) \frac{1}{\Psi^{-1}\big(1/t\big)} g^*(t) \frac{dt}{t} \\ &= 2\, |||\, f \,|||_{L_{\Phi,\infty}}\, |||g|||_{L_{\Psi,1}} \approx \|f\|_{L_{\Phi,\infty}} \|g\|_{L_{\Psi,1}}.\end{aligned}$$

(ii) Combining Proposition 2.8 and (2.4), we have

$$\begin{aligned}\mathbb{E}(fg) &\le 2 \int_0^\infty \frac{1}{\Phi^{-1}\big(1/t\big)} f^*(t) \sup_{t>0} \bigg(\frac{1}{\Psi^{-1}\big(1/t\big)} g^*(t) \bigg) \frac{dt}{t} \\ &= 2\, |||\, f \,|||_{L_{\Phi,1}}\, |||g|||_{L_{\Psi,\infty}} \\ &\approx \|f\|_{L_{\Phi,1}} \|g\|_{L_{\Psi,\infty}} \\ &\lesssim \|f\|_{L_{\Phi,q}} \|g\|_{L_{\Psi,\infty}}.\end{aligned}$$

This completes the proof. □

In particular, if $\Phi(t) = t^p$ for $t \in [0, \infty)$ in Theorem 2.9, we obtain the Hölder inequality for classical Lorentz spaces.

Corollary 2.11. *Let $1 < p < \infty$ and $0 < q \le \infty$; then the following statements hold:*

(i) If $1 < q \le \infty$, $f \in L_{p,q}$ and $g \in L_{p',q'}$, then we have

$$\mathbb{E}(fg) \le C \|f\|_{L_{p,q}} \|g\|_{L_{p',q'}},$$

where $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$.

(ii) If $0 < q \leq 1$, $f \in L_{p,q}$ *and* $g \in L_{p',\infty}$, *then we have*

$$\mathbb{E}(fg) \leq C\|f\|_{L_{p,q}}\|g\|_{L_{p',\infty}}.$$

Remark 2.12. According to [4], the generalized Hölder inequality for classical Lorentz spaces also holds, i.e.,

$$\|fg\|_{L_{p,q}} \leq C\|f\|_{L_{p_1,q_1}}\|g\|_{L_{p_2,q_2}} \qquad (f \in L_{p_1,q_1},\ g \in L_{p_2,q_2}),$$

where $0 < p, p_1, p_2 < \infty$ and $0 < q, q_1, q_2 \leq \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$.

3. The John–Nirenberg inequality

In this section, we prove the new John–Nirenberg inequality for Orlicz–Lorentz spaces with N-function in a probabilistic setting. We first introduce the generalized BMO associated with Orlicz–Lorentz space $L_{\Phi,q}$.

Definition 3.1. Let Φ be an N-function and $0 < q \leq \infty$. We define the BMO associated with Orlicz–Lorentz space $L_{\Phi,q}$ as

$$BMO_{\Phi,q} = \big\{f \in L_{\Phi,q} : \|f\|_{BMO_{\Phi,q}} < \infty\big\},$$

where

$$\|f\|_{BMO_{\Phi,q}} = \sup_{\tau\in\mathcal{T}} \frac{\|(f - f^{\tau})\chi_{\{\tau<\infty\}}\|_{L_{\Phi,q}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{\Phi,q}}}.$$

Note that if $\Phi(t) = t^p$, $0 < p < \infty$, then $BMO_{\Phi,q}$ becomes $BMO_{p,q}$ introduced in [17]. Moreover, if $\Phi(t) = t^q$, $0 < q < \infty$, then $BMO_{\Phi,q}$ can be reduced to BMO_q. Now we present the main result in this paper.

Theorem 3.2. *Let* Φ *be an* N*-function with* $1 < p_\Phi \leq q_\Phi < \infty$ *and* $0 < q \leq \infty$. *If the stochastic basis* $\{\mathcal{F}_n\}_{n\geq 0}$ *is regular, then*

$$BMO_{\Phi,q} = BMO$$

with equivalent (quasi)-norms.

Proof. According to (1.1), it is sufficient to prove

$$BMO_{\Phi,q} = BMO_1$$

with equivalent (quasi)-norms. Let $f \in BMO_{\Phi,q}$; then $f \in L_{\Phi,q}$. If $1 < q \le \infty$, then Proposition 2.9 gives

$$\begin{aligned} \|f\|_{BMO_1} &= \sup_{\nu\in\mathcal{T}} \frac{\|f-f^\nu\|_{L_1}}{\mathbb{P}(\nu<\infty)} \\ &= \sup_{\nu\in\mathcal{T}} \frac{\|(f-f^\nu)\chi_{\{\nu<\infty\}}\|_{L_1}}{\mathbb{P}(\nu<\infty)} \\ &\lesssim \sup_{\nu\in\mathcal{T}} \frac{\|f-f^\nu\|_{L_{\Phi,q}}\|\chi_{\{\nu<\infty\}}\|_{L_{\Psi,q'}}}{\mathbb{P}(\nu<\infty)} \\ &= \sup_{\nu\in\mathcal{T}} \frac{\|f-f^\nu\|_{L_{\Phi,q}}}{\mathbb{P}(\nu<\infty)\Psi^{-1}\left(\frac{1}{\mathbb{P}(\nu<\infty)}\right)}, \end{aligned} \tag{3.1}$$

where Ψ is the complementary of Φ and q' satisfies $1/q' + 1/q = 1$.

Using Proposition 2.1, we have

$$\mathbb{P}(\nu<\infty)\Psi^{-1}\left(\frac{1}{\mathbb{P}(\nu<\infty)}\right) \ge \frac{1}{\Phi^{-1}\left(\frac{1}{\mathbb{P}(\nu<\infty)}\right)} = \|\chi_{\{\nu<\infty\}}\|_{L_\Phi}. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$\begin{aligned} \|f\|_{BMO_1} &\lesssim \sup_{\nu\in\mathcal{T}} \frac{\|f-f^\nu\|_{L_{\Phi,q}}\|\chi_{\{\nu<\infty\}}\|_{L_{\Psi,q'}}}{\mathbb{P}(\nu<\infty)} \\ &\approx \sup_{\nu\in\mathcal{T}} \frac{\|f-f^\nu\|_{L_{\Phi,q}}}{\|\chi_{\{\nu<\infty\}}\|_{L_\Phi}} \\ &= \sup_{\nu\in\mathcal{T}} \frac{\|f-f^\nu\|_{L_{\Phi,q}}}{\|\chi_{\{\nu<\infty\}}\|_{L_{\Phi,q}}} \\ &= \|f\|_{BMO_{\Phi,q}}. \end{aligned} \tag{3.3}$$

When $0 < q \le 1$, Proposition 2.8 and (3.3) give

$$\begin{aligned} \|f\|_{BMO_1} &\lesssim \sup_{\nu\in\mathcal{T}} \frac{\|f-f^\nu\|_{L_{\Phi,2}}}{\|\chi_{\{\nu<\infty\}}\|_{L_\Phi}} \\ &\lesssim \sup_{\nu\in\mathcal{T}} \frac{\|f-f^\nu\|_{L_{\Phi,q}}}{\|\chi_{\{\nu<\infty\}}\|_{L_\Phi}} \\ &= \sup_{\nu\in\mathcal{T}} \frac{\|f-f^\nu\|_{L_{\Phi,q}}}{\|\chi_{\{\nu<\infty\}}\|_{L_{\Phi,q}}} \\ &= \|f\|_{BMO_{\Phi,q}}. \end{aligned} \tag{3.4}$$

On the other hand, let $f \in BMO_1$. It is easy to see that

$$BMO_1 = BMO \subseteq L_{q_\Phi,q},$$

i.e., $f \in L_{q_\Phi,q}$. Indeed, $BMO \subseteq L_{q_\Phi+1} \subseteq L_{q_\Phi,q}$. We first consider the case of $0<q<\infty$. It follows from Proposition 2.4 that for any stopping time $\tau \in \mathcal{T}$,

$$\begin{aligned}
\|f-f^\tau\|_{L_{\Phi,q}} &= \|(f-f^\tau)\chi_{\{\tau<\infty\}}\|_{L_{\Phi,q}} \\
&= \Big(\int_0^\infty \lambda^q \|\chi_{\{|(f-f^\nu)\chi_{\{\nu<\infty\}}|>\lambda\}}\|_{L_\Phi}^q \frac{d\lambda}{\lambda}\Big)^{1/q} \\
&= \Big(\int_0^\infty \lambda^q \big(\|\chi_{\{|(f-f^\nu)\chi_{\{\nu<\infty\}}|>\lambda\}}\chi_{\{\nu<\infty\}}\|_{L_\Phi}\big)^q \frac{d\lambda}{\lambda}\Big)^{1/q} \\
&\lesssim \Big(\int_0^\infty \lambda^q \big(\|\chi_{\{|(f-f^\nu)\chi_{\{\nu<\infty\}}|>\lambda\}}\|_{L_{q_\Phi}}\|\chi_{\{\nu<\infty\}}\|_{L_{\Phi_1}}\big)^q \frac{d\lambda}{\lambda}\Big)^{1/q} \\
&= \|(f-f^\tau)\chi_{\{\tau<\infty\}}\|_{L_{q_\Phi,q}}\|\chi_{\{\nu<\infty\}}\|_{L_{\Phi_1}},
\end{aligned}$$

where

$$\Phi_1^{-1}(t) = \Phi^{-1}(t)\cdot t^{-1/q_\Phi}.$$

Hence, combining this with Hölder's inequality for classical Lorentz spaces (see Remark 2.12), we obtain

$$\begin{aligned}
\|f\|_{BMO_{\Phi,q}} &= \sup_{\tau\in\mathcal{T}} \frac{\|f-f^\tau\|_{L_{\Phi,q}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{\Phi,q}}} \\
&\lesssim \sup_{\tau\in\mathcal{T}} \frac{\|(f-f^\tau)\chi_{\{\tau<\infty\}}\|_{L_{q_\Phi,q}}\|\chi_{\{\nu<\infty\}}\|_{L_{\Phi_1}}}{\|\chi_{\{\tau<\infty\}}\|_{L_\Phi}} \\
&= \sup_{\tau\in\mathcal{T}} \frac{\|(f-f^\tau)\chi_{\{\tau<\infty\}}\|_{L_{q_\Phi,q}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{q_\Phi}}} \\
&\le \sup_{\tau\in\mathcal{T}} \frac{\|(f-f^\tau)\chi_{\{\tau<\infty\}}\|_{L_{r,r}}\|\chi_{\{\tau<\infty\}}\|_{L_{s,u}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{q_\Phi}}} \\
&= \sup_{\tau\in\mathcal{T}} \frac{\|(f-f^\tau)\chi_{\{\tau<\infty\}}\|_{L_r}}{\|\chi_{\{\tau<\infty\}}\|_{L_r}} \\
&= \|f\|_{BMO_r},
\end{aligned}$$

where the real constant $r>\max\{q,q_\Phi\}$, $0<s<\infty$, $0<u<\infty$ and

$$\begin{cases} \dfrac{1}{q_\Phi} = \dfrac{1}{r}+\dfrac{1}{s}, \\[2mm] \dfrac{1}{q} = \dfrac{1}{r}+\dfrac{1}{u}. \end{cases}$$

According to the John–Nirenberg inequality for BMO (see (1.1)), one can get

$$\|f\|_{BMO_{\Phi,q}} \lesssim \|f\|_{BMO_r} \lesssim \|f\|_{BMO_1}, \qquad 0<q<\infty. \tag{3.5}$$

If $q=\infty$, it follows from Proposition 2.8 and (3.5) that

$$\begin{aligned}\|f\|_{BMO_{\Phi,\infty}} &= \sup_{\tau\in\mathcal{T}} \frac{\|f-f^\tau\|_{L_{\Phi,\infty}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{\Phi,\infty}}} \le \sup_{\tau\in\mathcal{T}} \frac{\|f-f^\tau\|_{L_{\Phi,q_\Phi}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{\Phi,\infty}}} \\ &= \|f\|_{BMO_{\Phi,q_\Phi}} \lesssim \|f\|_{BMO_1}.\end{aligned} \tag{3.6}$$

Thus, combining (3.3), (3.4), (3.5) with (3.6), we have

$$BMO_{\Phi,q} = BMO_1, \qquad 0<q\le\infty.$$

Hence the result is proved. This completes the proof of the theorem. □

Theorem 3.2 improves the results from [17]. That is, if we consider the case $\Phi(t)=t^p$ for $t\in[0,\infty)$ in Theorem 3.2, we get the following result:

Corollary 3.3. *Let $1<p<\infty$, $0<q\le\infty$. If the stochastic basis $\{\mathcal{F}_n\}_{n\ge0}$ is regular, then*

$$BMO_{p,q} = BMO$$

with equivalent (quasi)-norms.

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