

SUMMING THE LARGEST PRIME FACTOR OVER INTEGER SEQUENCES

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ABSTRACT. Given an integer $n \geq 2$, let $P(n)$ stand for its largest prime factor. We examine the behaviour of $\sum_{\substack{n \leq x \\ n \in A}} P(n)$ in the case of two sets A , namely the set of r -free numbers and the set of h -full numbers.

1. INTRODUCTION

Given an integer $n \geq 2$, let $P(n)$ stand for its largest prime factor, with $P(1) = 1$. Even though this function is very chaotic because $P(n)$ oscillates between small and large values as n varies, its average value over large intervals is more smooth and can be estimated.

The first significant estimate regarding the sum $\sum_{n \leq x} P(n)$ is due to Alladi and Erdős [1] as they proved that

$$\sum_{n \leq x} P(n) = \frac{\pi^2}{12} \frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right).$$

This result was later improved by De Koninck and Ivić [2] when they showed that, given any positive integer k , there exist computable constants c_2, \dots, c_k such that

$$\sum_{n \leq x} P(n) = \frac{\pi^2}{12} \frac{x^2}{\log x} + c_2 \frac{x^2}{\log^2 x} + \dots + c_k \frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right).$$

A natural question to ask is how the above formula changes if instead of summing $P(n)$ over all natural numbers $n \leq x$, we restrict these numbers n to a particular subset A of \mathbb{N} . For this purpose, we will consider here two large families of integers, namely the set of r -free numbers and the set of h -full numbers.

Given an integer $n \geq 2$, write its prime factorisation as $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_s^{\alpha_s}$, where $q_1 < q_2 < \dots < q_s$ are primes and $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathbb{N}$. Given fixed integers $r \geq 2$ and $h \geq 2$, we say that n is a r -free number if $\max(\alpha_1, \alpha_2, \dots, \alpha_s) \leq r - 1$, whereas

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we say that n is a h -full number if $\min(\alpha_1, \alpha_2, \dots, \alpha_s) \geq h$. We will denote by \mathbb{F}_r the set of r -free numbers; amongst these sets, the set \mathbb{F}_2 of *square-free numbers* and the set \mathbb{F}_3 of *cube-free numbers* are often mentioned in the literature. On the other hand, we will denote by \mathbb{P}_h the set of h -full numbers. Particular cases are the set \mathbb{P}_2 , known as the set of *powerful numbers* or *square-full numbers*, and the set \mathbb{P}_3 , the set of *cube-full numbers*.

In what follows we will make frequent use of the Riemann zeta function $\zeta(s)$ defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (s > 1).$$

Let $\mu_r(n)$ be the characteristic function of the r -free numbers, that is,

$$\mu_r(n) = \begin{cases} 1 & \text{if } n \text{ is } r\text{-free,} \\ 0 & \text{otherwise,} \end{cases}$$

implying in particular that its generating function is, for $s > 1$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\mu_r(n)}{n^s} &= \prod_p \left(1 + \frac{1}{p^s} + \dots + \frac{1}{p^{(r-1)s}}\right) \\ &= \frac{\prod_p \left(1 - \frac{1}{p^{rs}}\right)}{\prod_p \left(1 - \frac{1}{p^s}\right)} = \frac{\zeta(rs)}{\zeta(s)}. \end{aligned}$$

Let $\chi_h(n)$ be the characteristic function of the h -full numbers, that is,

$$\chi_h(n) = \begin{cases} 1 & \text{if } n \text{ is } h\text{-full,} \\ 0 & \text{otherwise,} \end{cases}$$

implying in particular that its generating function is, for $s > 1$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\chi_h(n)}{n^s} &= \prod_p \left(1 + \frac{1}{p^{hs}} + \frac{1}{p^{(h+1)s}} + \dots\right) \\ &= \zeta(hs) \prod_p \left(1 - \frac{1}{p^{hs}}\right) \prod_p \left(1 + \frac{1}{p^{hs}} + \frac{1}{p^{(h+1)s}} + \dots\right) \tag{1.1} \\ &= \zeta(hs) \prod_p \left(1 + \frac{1}{p^{(h+1)s}} + \frac{1}{p^{(h+2)s}} + \dots + \frac{1}{p^{(2h-1)s}}\right). \end{aligned}$$

Finally, let us mention that estimates for the counting functions $\mathbb{F}_r(x) := \#\{n \leq x : n \in \mathbb{F}_r\}$ and $\mathbb{P}_h(x) := \#\{n \leq x : n \in \mathbb{P}_h\}$ of these two families of numbers are well-known. These are, for fixed integers $r \geq 2$ and $h \geq 2$,

$$\mathbb{F}_r(x) = \frac{1}{\zeta(r)}x + O\left(x^{1/r}\right), \tag{1.2}$$

$$\mathbb{P}_r(x) = \gamma_h x^{1/h} + O\left(x^{1/(h+1)}\right) \tag{1.3}$$

for some positive constant γ_h . For a proof of (1.2) in the simplest case, that is for $r = 2$, see Theorem 8.25 in the book of Niven, Zuckerman and Montgomery [4]; for a proof of the general case, that is for any $r \geq 2$, see the survey paper of Pappalardi [5]. For a proof of (1.3), see the paper of Ivic and Shiu [3], where in fact a much more accurate formula is proved.

2. MAIN RESULTS

For our first set A , we choose the set of r -free numbers \mathbb{F}_r . In this case we can prove the following.

Theorem 2.1. *Let $r \geq 2$ be a fixed integer. Then, given any positive integer k , there exist computable constants d_1, d_2, \dots, d_k such that*

$$\sum_{\substack{n \leq x \\ n \in \mathbb{F}_r}} P(n) = \sum_{n \leq x} \mu_r(n) P(n) = d_1 \frac{x^2}{\log x} + d_2 \frac{x^2}{\log^2 x} + \dots + d_k \frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right),$$

where in particular, in light of (1.2),

$$d_1 = d_1^{(r)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\mu_r(n)}{n^2} = \frac{\zeta(2)}{2\zeta(2r)}.$$

Remark 2.2. In the case $r = 2$, that is, the case of square-free numbers, we have

$$d_1^{(2)} = \frac{\zeta(2)}{2\zeta(4)} = \frac{15}{2\pi^2} = 0.759909\dots$$

In the case $r = 3$, that is, the case of cube-free numbers, we have

$$d_1^{(3)} = \frac{\zeta(2)}{2\zeta(6)} = \frac{315}{4\pi^4} = 0.808446\dots$$

When choosing $A = \mathbb{P}_h$, we can prove the following general result.

Theorem 2.3. *Let $h \geq 2$ be a fixed integer. Then, given any positive integer k , there exist computable constants e_1, e_2, \dots, e_k such that*

$$\sum_{\substack{n \leq x \\ n \in \mathbb{P}_h}} P(n) = e_1 \frac{x^{2/h}}{\log x} + e_2 \frac{x^{2/h}}{\log^2 x} + \dots + e_k \frac{x^{2/h}}{\log^k x} + O\left(\frac{x^{2/h}}{\log^{k+1} x}\right),$$

where

$$e_1 = \frac{h}{2} \sum_{n \in \mathbb{P}_h} \frac{1}{n^{2/h}} = \frac{h}{2} \prod_p \left(1 + \frac{1}{(p^h)^{2/h}} + \frac{1}{(p^{h+1})^{2/h}} + \dots\right).$$

Remark 2.4. In the particular case of square-full numbers, we have, in light of (1.1) with $h = 2$ and $s = 1$,

$$e_1 = \sum_{n \in \mathbb{P}_2} \frac{1}{n} = \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \dots\right) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = 1.9436\dots$$

In the case of cube-full numbers, we find

$$e_1 = \frac{3}{2} \sum_{n \in \mathbb{P}_3} \frac{1}{n^{2/3}} = \frac{3}{2} \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^{8/3}} + \frac{1}{p^{10/3}} + \dots\right) = 3.44967\dots$$

3. PRELIMINARY RESULTS

Let $\pi(x)$ stand for the number of primes not exceeding x . Using the prime number theorem in the form

$$\pi(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \frac{2!x}{\log^3 x} + \dots + \frac{(k-1)!x}{\log^k x} + O\left(\frac{x}{\log^{k+1} x}\right),$$

one can easily prove the following.

Lemma 3.1. *Given any positive integer k , there exist $k - 1$ computable constants a_2, \dots, a_k such that*

$$\sum_{p \leq X} p = \frac{1}{2} \frac{X^2}{\log X} + a_2 \frac{X^2}{\log^2 X} + \dots + a_k \frac{X^2}{\log^k X} + O\left(\frac{X^2}{\log^{k+1} X}\right).$$

We also have the following.

Lemma 3.2. *Given fixed positive integers s and k , there exist computable constants $c_{0,s}, c_{1,s}, \dots, c_{k,s}$ such that*

$$\sum_{n \leq \exp\{\sqrt{\log x}\}} \frac{1}{n^2 \log^s(x/n)} = \frac{c_{0,s}}{\log^s x} + \frac{c_{1,s}}{\log^{s+1} x} + \dots + \frac{c_{k,s}}{\log^{s+k} x} + O\left(\frac{1}{\log^{s+k+1} x}\right). \tag{3.1}$$

On the other hand, given a fixed integer $r \geq 2$, as well as fixed integers $s, k \in \mathbb{N}$, there exist computable constants $d_{0,s}, d_{1,s}, \dots, d_{k,s}$ such that

$$\sum_{n \leq \sqrt{x}} \frac{\mu_r(n)}{n^2 \log^s(x/n)} = \frac{d_{0,s}}{\log^s x} + \frac{d_{1,s}}{\log^{s+1} x} + \dots + \frac{d_{k,s}}{\log^{s+k} x} + O\left(\frac{1}{\log^{s+k+1} x}\right). \tag{3.2}$$

Proof. We only provide the proof of (3.2) since the proof of (3.1) is similar. Since we assumed that $n \leq \sqrt{x}$, we have that $\frac{\log n}{\log x} \leq \frac{1}{2}$. Therefore, for a fixed integer $k \geq 1$ and all $y \leq \frac{1}{2}$, we may use the expansion

$$\frac{1}{1-y} = 1 + y + y^2 + \dots + y^k + O(y^{k+1})$$

to write that, in the case $s = 1$,

$$\begin{aligned} \frac{\mu_r(n)}{n^2 \log(x/n)} &= \frac{\mu_r(n)}{n^2 \log x \left(1 - \frac{\log n}{\log x}\right)} \\ &= \frac{\mu_r(n)}{n^2 \log x} \left(1 + \frac{\log n}{\log x} + \dots + \frac{\log^{k-1} n}{\log^{k-1} x} + O\left(\frac{\log^k n}{\log^k x}\right)\right) \\ &= \frac{\mu_r(n)}{n^2} \frac{1}{\log x} + \frac{\mu_r(n) \log n}{n^2} \frac{1}{\log^2 x} \\ &\quad + \dots + \frac{\mu_r(n) \log^{k-1} n}{n^2} \frac{1}{\log^k x} + O\left(\frac{1}{\log^{k+1} x}\right). \end{aligned}$$

Observing that, for each integer $j \geq 1$, the corresponding series $\sum_{n=1}^{\infty} \frac{\mu_r(n) \log^j n}{n^2}$ converges, estimate (3.2) in the case $s = 1$ easily follows. The case of an arbitrary $s \in \mathbb{N}$ can be handled similarly. \square

Lemma 3.3. *For each integer $h \geq 2$,*

$$\sum_{\substack{m \leq x \\ m \in \mathbb{P}_h}} \frac{1}{m^{2/h}} = \sum_{m \in \mathbb{P}_h} \frac{1}{m^{2/h}} + O\left(\frac{1}{x^{1/h}}\right) \tag{3.3}$$

and, for each fixed $j \in \mathbb{N}$,

$$\sum_{\substack{m \leq x \\ m \in \mathbb{P}_h}} \frac{\log^j m}{m^{2/h}} = O(1). \tag{3.4}$$

Proof. First observe that, replacing the sum $\sum_{\substack{m > x \\ m \in \mathbb{P}_h}} \frac{1}{m^{2/h}}$ by a Stieltjes integral, using integration by parts and thereafter using the bound $\mathbb{P}_h(t) = O(t^{1/h})$ guaranteed by (1.3), we obtain

$$\begin{aligned} \sum_{\substack{m > x \\ m \in \mathbb{P}_h}} \frac{1}{m^{2/h}} &= \int_x^{\infty} \frac{1}{t^{2/h}} d\mathbb{P}_h(t) = \frac{\mathbb{P}_h(t)}{t^{2/h}} \Big|_x^{\infty} + \frac{2}{h} \int_x^{\infty} t^{-\frac{2}{h}-1} \mathbb{P}_h(t) dt \\ &\ll \frac{1}{x^{2/h-1/h}} + \frac{2}{h} \int_x^{\infty} t^{1/h-1} dt \ll \frac{1}{x^{1/h}}. \end{aligned} \tag{3.5}$$

Using (3.5), it follows that

$$\sum_{\substack{m \leq x \\ m \in \mathbb{P}_h}} \frac{1}{m^{2/h}} = \sum_{m \in \mathbb{P}_h} \frac{1}{m^{2/h}} - \sum_{\substack{m > x \\ m \in \mathbb{P}_h}} \frac{1}{m^{2/h}} = \sum_{m \in \mathbb{P}_h} \frac{1}{m^{2/h}} + O\left(\frac{1}{x^{1/h}}\right),$$

thus completing the proof of (3.3).

The proof of (3.4) can easily be established using the same technique. \square

4. PROOF OF THEOREM 2.1

First observe that

$$\begin{aligned} M(x) &:= \sum_{n \leq x} \mu_r(n) P(n) = \sum_{p \leq x} p \sum_{\substack{mp \leq x \\ P(m) < p}} \mu_r(m) \\ &= \sum_{p \leq \sqrt{x}} p \sum_{\substack{m \leq x/p \\ P(m) < p}} \mu_r(m) + \sum_{\sqrt{x} < p \leq x} p \sum_{\substack{m \leq x/p \\ P(m) < p}} \mu_r(m) \\ &= M_1(x) + M_2(x). \end{aligned} \tag{4.1}$$

It is trivial that

$$M_1(x) \leq \sum_{p \leq \sqrt{x}} p \cdot \frac{x}{p} = x \pi(\sqrt{x}) \ll \frac{x^{3/2}}{\log x}. \tag{4.2}$$

To estimate $M_2(x)$, first observe that if $p > \sqrt{x}$, we have that $x/p < p$, in which case the condition $P(m) < p$ appearing in the definition of $M_2(x)$ can be dropped. Hence, inverting the sum over p with the sum over m , and then using Lemma 3.1 with $X = \sqrt{x}$, we obtain

$$\begin{aligned}
 M_2(x) &= \sum_{\sqrt{x} < p \leq x} p \sum_{m \leq x/p} \mu_r(m) = \sum_{m \leq \sqrt{x}} \mu_r(m) \sum_{\sqrt{x} < p \leq x/m} p \\
 &= \sum_{m \leq \sqrt{x}} \mu_r(m) \sum_{p \leq x/m} p - \sum_{m \leq \sqrt{x}} \mu_r(m) \sum_{p \leq \sqrt{x}} p \\
 &= \sum_{m \leq \sqrt{x}} \mu_r(m) \sum_{p \leq x/m} p + O\left(\frac{x^{3/2}}{\log x}\right) \\
 &= M_3(x) + O\left(\frac{x^{3/2}}{\log x}\right).
 \end{aligned} \tag{4.3}$$

Again using Lemma 3.1, but this time with $X = x/m$, and thereafter using formula (3.2) of Lemma 3.2, we obtain

$$\begin{aligned}
 M_3(x) &= \sum_{m \leq \sqrt{x}} \mu_r(m) \left\{ \frac{1}{2} \frac{(x/m)^2}{\log(x/m)} + a_2 \frac{(x/m)^2}{\log^2(x/m)} \right. \\
 &\quad \left. + \cdots + a_k \frac{(x/m)^2}{\log^k(x/m)} + O\left(\frac{(x/m)^2}{\log^{k+1}(x/m)}\right) \right\} \\
 &= \frac{1}{2} \sum_{m=1}^{\infty} \frac{\mu_r(m)}{m^2} \frac{x^2}{\log x} + d_2 \frac{x^2}{\log^2 x} \\
 &\quad + \cdots + d_k \frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right),
 \end{aligned} \tag{4.4}$$

where we took the liberty to replace $\sum_{m \leq \sqrt{x}} \frac{\mu_r(m)}{m^2}$ by $\sum_{m=1}^{\infty} \frac{\mu_r(m)}{m^2}$, a justified move since

$$\begin{aligned}
 \sum_{m \leq \sqrt{x}} \frac{\mu_r(m)}{m^2} &= \sum_{m=1}^{\infty} \frac{\mu_r(m)}{m^2} - \sum_{m > \sqrt{x}} \frac{\mu_r(m)}{m^2} \\
 &= \sum_{m=1}^{\infty} \frac{\mu_r(m)}{m^2} + O\left(\int_{\sqrt{x}}^{\infty} \frac{dt}{t^2}\right) \\
 &= \sum_{m=1}^{\infty} \frac{\mu_r(m)}{m^2} + O\left(\frac{1}{\sqrt{x}}\right).
 \end{aligned}$$

Finally, gathering (4.2), (4.3) and (4.4) in (4.1) completes the proof of Theorem 2.1.

5. PROOF OF THEOREM 2.3

First, we write

$$\begin{aligned}
 U(x) &:= \sum_{\substack{n \leq x \\ n \in \mathbb{P}_h}} P(n) = \sum_{p \leq x^{1/h}} p \sum_{\substack{mp^h \leq x \\ m \in \mathbb{P}_h \\ P(m) \leq p}} 1 = \sum_{p \leq x^{1/h}} p \sum_{\substack{m \leq x/p^h \\ m \in \mathbb{P}_h \\ P(m) \leq p}} 1 \\
 &= \sum_{p \leq x^{1/(h+1)}} p \sum_{\substack{m \leq x/p^h \\ m \in \mathbb{P}_h \\ P(m) \leq p}} 1 + \sum_{x^{1/(h+1)} < p \leq x^{1/h}} p \sum_{\substack{m \leq x/p^h \\ m \in \mathbb{P}_h \\ P(m) \leq p}} 1 \quad (5.1) \\
 &= U_1(x) + U_2(x).
 \end{aligned}$$

It follows from estimate (1.3) that

$$\begin{aligned}
 U_1(x) &\leq \sum_{p \leq x^{1/(h+1)}} p \sum_{\substack{m \leq x/p^h \\ m \in \mathbb{P}_h}} 1 = \sum_{p \leq x^{1/(h+1)}} p \mathbb{P}_h \left(\frac{x}{p^h} \right) \\
 &\ll \sum_{p \leq x^{1/(h+1)}} p \frac{x^{1/h}}{p} = x^{1/h} \pi(x^{1/(h+1)}) \ll \frac{x^{\frac{2h+1}{h(h+1)}}}{\log x}. \quad (5.2)
 \end{aligned}$$

To evaluate $U_2(x)$, first observe that, for $p > x^{1/(h+1)}$, we have that $\frac{x}{p^h} < p$, implying that in this case the condition $P(m) \leq p$ appearing in the second sum defining $U_2(x)$ can be dropped and therefore that

$$\begin{aligned}
 U_2(x) &= \sum_{x^{1/(h+1)} < p \leq x^{1/h}} p \sum_{\substack{mp^h \leq x \\ m \in \mathbb{P}_h}} 1 = \sum_{p \leq x^{1/h}} p \sum_{\substack{mp^h \leq x \\ m \in \mathbb{P}_h}} 1 - \sum_{p \leq x^{1/(h+1)}} p \sum_{\substack{mp^h \leq x \\ m \in \mathbb{P}_h}} 1 \\
 &= \sum_{p \leq x^{1/h}} p \sum_{\substack{mp^h \leq x \\ m \in \mathbb{P}_h}} 1 + O \left(\frac{x^{\frac{2h+1}{h(h+1)}}}{\log x} \right) \quad (5.3) \\
 &= T(x) + O \left(\frac{x^{\frac{2h+1}{h(h+1)}}}{\log x} \right) = T(x) + O \left(\frac{x^{2/h}}{\log^{k+1} x} \right),
 \end{aligned}$$

where we made use of (5.2) and the fact that $\frac{2h+1}{h(h+1)} < \frac{2}{h}$.

Inverting the two sums appearing in the definition of $T(x)$, we can rewrite $T(x)$ as follows:

$$\begin{aligned}
 T(x) &= \sum_{\substack{m \leq x \\ m \in \mathbb{P}_h}} \sum_{p^h \leq x/m} p = \sum_{\substack{m \leq x \\ m \in \mathbb{P}_h}} \sum_{p \leq (x/m)^{1/h}} p \\
 &= \sum_{\substack{m \leq \exp\{\sqrt{\log x}\} \\ m \in \mathbb{P}_h}} \sum_{p \leq (x/m)^{1/h}} p + \sum_{\substack{\exp\{\sqrt{\log x}\} < m \leq p^{1/h} \\ m \in \mathbb{P}_h}} \sum_{p \leq (x/m)^{1/h}} p \quad (5.4) \\
 &= T_1(x) + T_2(x).
 \end{aligned}$$

Again, using the bound $\mathbb{P}_h(t) \ll t^{1/h}$ ensured by estimate (1.3), we have, arguing as we did in Lemma 3.3,

$$\begin{aligned}
 T_2(x) &\leq \sum_{\substack{\exp\{\sqrt{\log x}\} < m \leq p^{1/h} \\ m \in \mathbb{P}_h}} \left(\frac{x}{m}\right)^{2/h} = x^{2/h} \sum_{\substack{\exp\{\sqrt{\log x}\} < m \leq p^{1/h} \\ m \in \mathbb{P}_h}} \frac{1}{m^{2/h}} \\
 &= x^{2/h} \int_{\exp\{\sqrt{\log x}\}}^x t^{-2/h} d\mathbb{P}_h(t) \\
 &\ll x^{2/h} \left(t^{-2/h} t^{1/h} \Big|_{\exp\{\sqrt{\log x}\}}^x + \int_{\exp\{\sqrt{\log x}\}}^x t^{-\frac{2}{h}-1} t^{1/h} dt \right) \\
 &\ll x^{2/h} \cdot \frac{1}{t^{1/h}} \Big|_{\exp\{\sqrt{\log x}\}}^x \ll \frac{x^{2/h}}{\exp\{\frac{1}{h}\sqrt{\log x}\}} \ll \frac{x^{2/h}}{\log^{k+1} x}.
 \end{aligned} \tag{5.5}$$

Making use of Lemma 3.1 with $X = (x/m)^{1/h}$ and thereafter of formula (3.1) of Lemma 3.2, we obtain

$$\begin{aligned}
 T_1(x) &= \sum_{\substack{m \leq \exp\{\sqrt{\log x}\} \\ m \in \mathbb{P}_h}} \left\{ \frac{1}{2} \frac{(x/m)^{2/h}}{\frac{1}{h} \log(x/m)} + a_2 \frac{(x/m)^{2/h}}{\frac{1}{h^2} \log^2(x/m)} \right. \\
 &\quad \left. + \dots + a_k \frac{(x/m)^{2/h}}{\frac{1}{h^k} \log^k(x/m)} + O\left(\frac{(x/m)^{2/h}}{\log^{k+1}(x/m)}\right) \right\} \\
 &= e_1 \frac{x^{2/h}}{\log x} + e_2 \frac{x^{2/h}}{\log^2 x} + \dots + e_k \frac{x^{2/h}}{\log^k x} + O\left(\frac{x^{2/h}}{\log^{k+1} x}\right),
 \end{aligned} \tag{5.6}$$

where we used the fact that, in light of estimate (3.3) of Lemma 3.3,

$$\begin{aligned}
 \frac{h x^{2/h}}{2 \log x} \sum_{\substack{m \leq \exp\{\sqrt{\log x}\} \\ m \in \mathbb{P}_h}} \frac{1}{m^{2/h}} &= \frac{h x^{2/h}}{2 \log x} \left(\sum_{\substack{m=1 \\ m \in \mathbb{P}_h}}^{\infty} \frac{1}{m^{2/h}} - \sum_{\substack{m > \exp\{\sqrt{\log x}\} \\ m \in \mathbb{P}_h}} \frac{1}{m^{2/h}} \right) \\
 &= \frac{x^{2/h}}{\log x} \frac{h}{2} \sum_{\substack{m=1 \\ m \in \mathbb{P}_h}}^{\infty} \frac{1}{m^{2/h}} \left(1 + O\left(\frac{1}{x^{1/h}}\right) \right)
 \end{aligned}$$

and where we used estimate (3.4) of Lemma 3.3 to manage the other coefficients e_i appearing in (5.6).

Finally, gathering estimates (5.2), (5.3), (5.4), (5.5) and (5.6) in (5.1) completes the proof of Theorem 2.3.

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