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COORDINATE RINGS OF SOME SL_2-CHARACTER VARIETIES

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Abstract. We determine generators of the coordinate ring of SL_2-character varieties. In the case of the free group F_3 we obtain an explicit equation of the SL_2-character variety. For free groups F_k, we find transcendental generators. Finally, for the case of the 2-torus, we get an explicit equation of the SL_2-character variety and use the description to compute their E-polynomials.

1. Introduction

Let Γ be a finitely generated group and let G be an algebraic group over an algebraically closed field k. The character variety $\mathfrak{X}(\Gamma, G)$ parametrizes isomorphism classes of representations $\rho : \Gamma \to G$. Character varieties are rich objects that contain geometric information linking distant areas in mathematics. An important instance is when we take $\Gamma_g = \pi_1(\Sigma_g)$ to be the fundamental group of the compact orientable surface of genus $g \geq 1$. In this case, these character varieties are one of the three incarnations of the moduli space of Higgs bundles, as stated by the celebrated non-abelian Hodge correspondence [2, 8, 20]. For this reason, character varieties of surface groups have been widely studied, particularly regarding the computation of some algebraic invariants like their E-polynomial.

Character varieties also play a prominent role in the topology of 3-manifolds, starting with the foundational work of Culler and Shalen [3], where the authors used algebro-geometric properties of $\mathrm{SL}_2(\mathbb{C})$-character varieties to provide new proofs of remarkable results, such as Thurston's theorem that says that the space of hyperbolic structures on an acylindrical 3-manifold is compact, or the Smith conjecture [3, Corollary 5.1.4]. Character varieties of 3-manifolds allow us even to study knots $K \subset S^3$, by analyzing the character variety associated to the fundamental group of their complement, $\Gamma_K = \pi_1(S^3 - K)$. For instance, the geometry of these knot character varieties has been studied in [6, 10, 11] for trivial links (i.e. when Γ is a free group), and in [9, 15, 18, 19] for the torus knot, among others.

Fix $G = \mathrm{SL}_r$, the group of matrices of size r with trivial determinant. If we have a presentation $\Gamma = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_s \rangle$, then $\mathfrak{X}(\Gamma, G)$ parametrizes k-tuples

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$(A_1, \dots, A_k)$ of matrices in G subject to the relations $r_j(A_1, \dots, A_k) = I$. In the same vein as the isomorphism class of a semisimple matrix A is determined by the traces of its powers, $\mathrm{tr}(A^i)$, $1 \leq i \leq r-1$, the (semisimple) k-tuples $(A_1, \dots, A_k)$ are determined by the traces of suitable products of the matrices. In this paper, we focus on the group SL_2 and work out many identities with traces. This serves to find coordinates for the character variety $\mathfrak{X}(\Gamma, \mathrm{SL}_2)$. Note that there is a natural embedding $\mathfrak{X}(\Gamma, \mathrm{SL}_2) \subset \mathfrak{X}(F_k, \mathrm{SL}_2)$, where F_k is the free group of k elements, and k is the number of generators of Γ. Therefore it is natural to look initially to the case of the free group.

The structure of $\mathrm{SL}_2(\mathbb{C})$-character varieties of free groups has been well understood for some time. There is a modern treatment in [7], where historical references can be found. We thank Sean Lawton for pointing this out to us. More generally, there is an effective algorithm to compute the coordinate ring of any $\mathrm{SL}_2(\mathbb{C})$-character variety for any finitely presentable group in [1]. Here we obtain in an alternative way some of these coordinate rings, and establish connections with results of [13] about their E-polynomials. We start with the following:

Theorem 1.1. *Let $A_1, \dots, A_k \in \mathrm{SL}_2$, then the character variety $\mathfrak{X}_k = \mathfrak{X}(F_k, \mathrm{SL}_2)$ is parametrized by $T_{i_1 \dots i_p} := t_{A_{i_1} \dots A_{i_p}}$, $i_1 < \dots < i_p$ with $p = 1, 2, 3$, where $t_A = \mathrm{tr}(A)$ denotes the function on matrices defined by the trace.*

For the situation of $k = 3$, we can obtain an explicit equation. By Theorem 1.1, the coordinates of $\mathfrak{X}_3 = \mathfrak{X}(F_3, \mathrm{SL}_2)$ are given by $(x, y, z, u, v, w, P) = (t_A, t_B, t_C, t_{BC}, t_{AC}, t_{AB}, t_{ABC})$, where $(A, B, C) \in \mathfrak{X}_3$. We have the following:

Theorem 1.2. *The character variety $\mathfrak{X}_3 \subset \mathbf{k}^7$ is a hypersurface defined by the equation $P^2 = (wz + vy + ux - xyz)P - x^2 - y^2 - z^2 + uyz + vxz + wxy - uvw - u^2 - v^2 - w^2 + 4$.*

Next, we look at the case of the character varieties of a compact orientable surface Σ_g of genus $g \geq 1$. Its fundamental group is

$$\pi_1(\Sigma_g) = \left\langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle .$$

Take a conjugacy class $[\xi]$ determined by an element $\xi \in \mathrm{SL}_2$; then we define, as in [13],

$$\begin{aligned} \mathcal{M}_\xi &= \left\{ (A_1, B_1, \dots, A_g, B_g) \in (\mathrm{SL}_2)^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = \xi \right\} /\!\!/ \, \mathrm{Stab}(\xi) \\ &= \left\{ (A_1, B_1, \dots, A_g, B_g) \in (\mathrm{SL}_2)^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] \in [\xi] \right\} /\!\!/ \, \mathrm{SL}_2 \, . \end{aligned} \tag{1.1}$$

There are five different types of conjugacy classes, namely $[I], [-I], [J_+], [J_-]$ and $[\xi_t]$, where $J_\pm = \begin{pmatrix} \pm 1 & 0 \\ 1 & \pm 1 \end{pmatrix}$ are the Jordan types, and $\xi_t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $t = \lambda + \lambda^{-1}$, $\lambda \in \mathbb{C} - \{0, \pm 1\}$, are the diagonal types. For $\mathbf{k} = \mathbb{C}$, these varieties have been studied in [13] and the E-polynomials are computed in a series of papers [13, 16, 17]. For $g = 1$, we have the following result from [13, Theorem 1.1].

Theorem 1.3. *For the 2-torus $\Sigma_1 = T^2$, the E-polynomials of $\mathcal{M}_\xi$ are as follows:*

$$\begin{aligned} e(\mathcal{M}_I) &= q^2+1, \\ e(\mathcal{M}_{-I}) &= 1, \\ e(\mathcal{M}_{J_+}) &= q^2-2q-3, \\ e(\mathcal{M}_{J_-}) &= q^2+3q, \\ e(\mathcal{M}_{\xi_t}) &= q^2+4q+1, \end{aligned}$$

where $q = uv$, $e(\mathcal{M}_\xi) \in \mathbb{Z}[u, v]$.

We look at the character varieties $\mathcal{M}_\xi$ more closely by working out trace identities for commutators of two matrices. First, for matrices (A, B), we determine the equation for $(x, y, z) = (t_A, t_B, t_{AB})$,

$$F(x, y, z) = t_{[A,B]} = x^2 + y^2 + z^2 - xyz - 2,$$

which produces the character varieties

$$\mathfrak{X}_t = F^{-1}(t) = \left\{(A, B) \in (\mathrm{SL}_2)^2 \mid \mathrm{tr}([A, B]) = t\right\} /\!\!/ \mathrm{SL}_2$$

for $t \in \mathbb{C}$. Then

- $\mathfrak{X}_t = \mathcal{M}_{\xi_t}$ for $t \neq \pm 2$,
- $\mathfrak{X}_2 = \mathcal{M}_I \cup \mathcal{M}_{J_+}$,
- $\mathfrak{X}_{-2} = \mathcal{M}_{-I} \cup \mathcal{M}_{J_-}$.

We study the geometry of the character varieties $\mathfrak{X}_t$, and recover the results of Theorem 1.3. More specifically:

Theorem 1.4. *Let $t \in \mathbb{C}$. We have the following:*

- *For $t \neq \pm 2$, the character variety $\mathfrak{X}_t \subset \mathbb{C}^3$ is a smooth surface, and $e(\mathfrak{X}_t) = q^2 + 4q + 1$.*
- *For $t = 2$, the character variety $\mathfrak{X}_2 \subset \mathbb{C}^3$ has 4 ordinary double points. We have $\mathcal{M}_{J_+} \subset \mathcal{M}_I$, $\mathfrak{X}_2 = \mathcal{M}_I$, and $e(\mathfrak{X}_2) = q^2 + 1$.*
- *For $t = -2$, the character variety $\mathfrak{X}_{-2} \subset \mathbb{C}^3$ has only one singular point which is an ordinary double point. We have $\mathfrak{X}_{-2} = \mathcal{M}_{-I} \sqcup \mathcal{M}_{J_-}$, and $e(\mathfrak{X}_{-2}) = q^2 + 3q + 1$.*

2. Moduli of representations and character varieties

Let Γ be a finitely presented group, and let $G < \mathrm{GL}_r$ be an algebraic group over an algebraically closed field $\mathbf{k}$. A *representation* of Γ in G is a homomorphism $\rho : \Gamma \to G$. Consider a presentation $\Gamma = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_s \rangle$. Then ρ is determined by the k-tuple $(A_1, \ldots, A_k) = (\rho(x_1), \ldots, \rho(x_k))$ subject to the relations $r_j(A_1, \ldots, A_k) = I$, $1 \leq j \leq s$. The space of representations is

$$R(\Gamma, G) = \mathrm{Hom}(\Gamma, G) = \left\{(A_1, \ldots, A_k) \in G^k \mid r_j(A_1, \ldots, A_k) = I,\ 1 \leq j \leq s\right\}. \tag{2.1}$$

Therefore $R(\Gamma, G)$ is an affine algebraic set.

We say that two representations ρ and ρ' are *equivalent* if there exists $P \in G$ such that $\rho'(g) = P^{-1}\rho(g)P$ for every $g \in G$. This corresponds to a change of basis in $\mathbf{k}^r$, as $G < \mathrm{GL}_r$ (the change of basis is in G, so it respects the structure that the group G determines). Note that the action of G descends to an action of the projective group $\mathrm{P}G < \mathrm{PGL}_r$ on $R(\Gamma, G)$. The *moduli space of representations* is the GIT quotient

$$\mathfrak{M}(\Gamma, G) = R(\Gamma, G) /\!\!/ G.$$

Recall that by definition of GIT quotient for an affine variety, if we write $R(\Gamma, G) = \operatorname{Spec} \mathcal{O}$, then $\mathfrak{M}(\Gamma, G) = \operatorname{Spec} \mathcal{O}^G$.

Suppose from now on that $G = \mathrm{SL}_r$. A representation ρ is *reducible* if there exists some proper subspace $V \subset \mathbf{k}^r$ such that, for all $g \in G$, we have $\rho(g)(V) \subset V$; otherwise ρ is *irreducible*. If ρ is reducible, then let $V \subset \mathbf{k}^r$ be an invariant subspace, and consider a complement $\mathbf{k}^r = V \oplus W$. Let $\rho_1 = \rho|_V$ and let ρ_2 be the induced representation on the quotient space $W = \mathbf{k}^r/V$. Then we can write $\rho = \begin{pmatrix} \rho_1 & 0 \\ f & \rho_2 \end{pmatrix}$, where $f : \Gamma \to \operatorname{Hom}(W, V)$. Take $P_t = \begin{pmatrix} t\, I_V & 0 \\ 0 & I_W \end{pmatrix}$, where $k = \dim V$. Then $P_t^{-1} \rho P_t = \begin{pmatrix} \rho_1 & 0 \\ tf & \rho_2 \end{pmatrix} \to \rho' = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$, when $t \to 0$. Therefore ρ and ρ' define the same point in the quotient $\mathfrak{M}(\Gamma, G)$. Repeating this, we can substitute any representation ρ by some $\tilde{\rho} = \bigoplus \rho_i$, where all ρ_i are irreducible representations. We call this process *semisimplification*, and $\tilde{\rho}$ a semisimple representation; also ρ and $\tilde{\rho}$ are called S-equivalent. The space $\mathfrak{M}(\Gamma, G)$ parametrizes semisimple representations [14, Theorem 1.28].

Given a representation $\rho : \Gamma \to G$, we define its *character* as the map $\chi_\rho : \Gamma \to \mathbf{k}$, $\chi_\rho(g) = \operatorname{tr} \rho(g)$. Note that two equivalent representations ρ and ρ' have the same character. There is a character map $\chi : R(\Gamma, G) \to \mathbf{k}^\Gamma$, $\rho \mapsto \chi_\rho$, whose image

$$\mathfrak{X}(\Gamma, G) = \chi(R(\Gamma, G))$$

is called the *character variety* of Γ. Let us give $\mathfrak{X}(\Gamma, G)$ the structure of an algebraic variety. The traces χ_ρ span a subring $B \subset A$. Clearly $B \subset A^G$. As A is noetherian, and G is a linear algebraic group over an algebraically closed field, we have that A^G is a finitely generated algebra. Therefore, since Γ is finitely presented, B is a finitely generated $\mathbf{k}$-algebra. Hence there exists a collection $g_1, \ldots, g_a$ of elements of G such that χ_ρ is determined by $\chi_\rho(g_1), \ldots, \chi_\rho(g_a)$ for any ρ. Such collection gives a map

$$\bar{\chi} : R(\Gamma, G) \to \mathbf{k}^a, \qquad \bar{\chi}(\rho) = (\chi_\rho(g_1), \ldots, \chi_\rho(g_a)),$$

and $\mathfrak{X}(\Gamma, G) \cong \bar{\chi}(R(\Gamma, G))$. This endows $\mathfrak{X}(\Gamma, G)$ with the structure of an algebraic variety, which is independent of the chosen collection. The natural algebraic map

$$\mathfrak{M}(\Gamma, G) \to \mathfrak{X}(\Gamma, G)$$

is an isomorphism (see [11, Chapter 1]). This is the same as to say that $B = A^G$, that is, the ring of invariant polynomials is generated by characters.

2.1. **Hodge structures and E-polynomials.** Later we need the notion of E-polynomial, which is an invariant of a complex algebraic variety constructed as an Euler characteristic of its Hodge numbers. We introduce the basic definitions. Here the ground field is $\mathbf{k} = \mathbb{C}$. A pure Hodge structure of weight k consists of a finite dimensional complex vector space H with a real structure, and a decomposition $H = \bigoplus_{k=p+q} H^{p,q}$ such that $H^{q,p} = \overline{H^{p,q}}$, the bar meaning complex conjugation on H. A Hodge structure of weight k gives rise to the so-called Hodge filtration, which is a descending filtration $F^p = \bigoplus_{s \geq p} H^{s,k-s}$. We define $\mathrm{Gr}^p_F(H) := F^p/F^{p+1} = H^{p,k-p}$.

A mixed Hodge structure consists of a finite dimensional complex vector space H with a real structure, an ascending (weight) filtration $\cdots \subset W_{k-1} \subset W_k \subset \cdots \subset H$ (defined over $\mathbb{R}$) and a descending (Hodge) filtration F such that F induces a pure Hodge structure of weight k on each $\mathrm{Gr}^W_k(H) = W_k/W_{k-1}$. We define $H^{p,q} := \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q}(H)$ and write $h^{p,q}$ for the *Hodge number* $h^{p,q} := \dim H^{p,q}$.

Let Z be any quasi-projective algebraic variety (possibly non-smooth or non-compact). The cohomology groups $H^k(Z)$ and the cohomology groups with compact support $H^k_c(Z)$ are endowed with mixed Hodge structures [4]. We define the *Hodge numbers* of Z by $h^{k,p,q}_c(Z) = h^{p,q}(H^k_c(Z)) = \dim \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} H^k_c(Z)$. The E-polynomial is defined as

$$e(Z) := \sum_{p,q,k} (-1)^k h^{k,p,q}_c(Z) u^p v^q.$$

The key property of Hodge–Deligne polynomials that permits their calculation is that they are additive for stratifications of Z. If Z is a complex algebraic variety and $Z = \bigsqcup_{i=1}^n Z_i$, where all Z_i are locally closed in Z, then $e(Z) = \sum_{i=1}^n e(Z_i)$. Also $e(X \times Y) = e(X)e(Y)$.

When $h^{k,p,q}_c = 0$ for $p \neq q$, the polynomial $e(Z)$ depends only on the product uv. This will happen in all the cases that we shall investigate here. In this situation, it is conventional to use the variable $q = uv$. Basic cases are $e(\mathbb{C}) = q$, $e(\mathbb{C}^r) = q^r$, $e(\mathbb{P}^r) = q^r + \cdots + q^2 + q + 1$.

3. The character variety for free groups

Now we focus on the case of a free group. Let $\Gamma = F_k := \langle x_1, x_2, \ldots, x_k \rangle$ be the free group generated by k elements. Then, the space of representations of F_k in SL_r is just

$$\mathrm{Hom}(F_k, \mathrm{SL}_r) = (\mathrm{SL}_r)^k = \{(A_1, A_2, \ldots, A_k) \mid A_i \in \mathrm{SL}_r\},$$

the space of k-tuples of matrices in SL_r. The moduli space of k-tuples of matrices up to conjugation is

$$\mathfrak{M}(F_k, \mathrm{SL}_r) = (\mathrm{SL}_r)^k /\!\!/ \mathrm{SL}_r\, .$$

As we said in Section 2, this is isomorphic to the character variety $\mathfrak{X}(F_k, \mathrm{SL}_r)$. This implies that there are finitely many $g_1, \ldots, g_a \in F_k$ such that a character $\chi_\rho \in \mathfrak{X}(F_k, \mathrm{SL}_r)$ is determined by $\chi_\rho(g_1), \ldots, \chi_\rho(g_a)$. Set $\rho(x_i) = A_i \in \mathrm{SL}_r$, and also $\operatorname{tr}(A) = t_A$ for the trace of a matrix A. For an element $g_j = x_{i_{j1}} \dots x_{i_{j\ell_j}} \in F_k$, we have

$$\chi_\rho(g) = \operatorname{tr}\rho(g) = \operatorname{tr}(\rho(x_{i_{j1}}) \cdots \rho(x_{i_{j\ell_j}})) = \operatorname{tr}(A_{i_{j1}} \cdots A_{i_{j\ell_j}}) = t_{A_{i_{j1}} \cdots A_{i_{j\ell_j}}}.$$

This implies that $\mathfrak{X}(F_k, \mathrm{SL}_r)$ is parametrized by the above traces for $j = 1, \dots, a$, that is,

$$\begin{aligned} \mathfrak{M}(F_k, \mathrm{SL}_r) &\longrightarrow \mathfrak{X}(F_k, \mathrm{SL}_r) \subset \mathbf{k}^a \\ (A_1, \dots, A_k) &\mapsto \left(t_{A_{i_{11}} \cdots A_{i_{1\ell_1}}}, \dots, t_{A_{i_{a1}} \cdots A_{i_{a\ell_a}}}\right) \end{aligned} \tag{3.1}$$

is a parametrization of the character variety.

Proposition 3.1. *If $k \geq 2$, the dimension of the character variety $\mathfrak{X}(F_k, \mathrm{SL}_r)$ is*

$$\dim \mathfrak{X}(F_k, \mathrm{SL}_r) = (r^2 - 1)(k - 1).$$

If $k = 1$, then

$$\dim \mathfrak{X}(F_1, \mathrm{SL}_r) = r - 1.$$

Proof. Let us assume that $k \geq 2$. The action of SL_r on irreducible representations has finite stabilizer, so the action has generic orbits of dimension $\dim \mathrm{SL}_r$. This means that

$$\dim \mathfrak{X}(F_k, \mathrm{SL}_r) = \dim(\mathrm{SL}_r)^k - \dim \mathrm{SL}_r = (k - 1)\dim \mathrm{SL}_r = (r^2 - 1)(k - 1).$$

On the other hand, if $k = 1$, the character variety $\mathfrak{X}(F_1, \mathrm{SL}_r) = \mathrm{SL}_r /\!\!/ \mathrm{SL}_r$ is canonically isomorphic to $\mathbf{k}^{r-1}$, as it is proved in [19], so it has dimension $r - 1$. □

From now on we focus on the case of rank 2, that is, the group SL_2. We want to determine how many traces are needed in (3.1). First, we demonstrate some useful matrix identities.

Lemma 3.2. *Let $P, Q \in \mathrm{SL}_2$. Then the following holds:*

$$QP = (t_{PQ} - t_P t_Q)I + t_P Q + t_Q P - PQ.$$

Proof. First of all, let us recall that $t_{A^{-1}} = t_A$ for every $A \in \mathrm{SL}_2$. On the other hand, the relation $t_{AB} = t_{BA}$ holds for every pair of square matrices A, B of the same size. Given $A \in \mathrm{SL}_2$, the following relation is given by the characteristic polynomial of A,

$$A^2 = t_A A - I \tag{3.2}$$

and therefore, the following holds:

$$A^{-1} = t_A I - A. \tag{3.3}$$

By (3.3), we can write $(PQ)^{-1}$ as

$$Q^{-1}P^{-1} = (PQ)^{-1} = t_{PQ} I - PQ.$$

Applying (3.3) to Q^{-1} and P^{-1} on the above equation we obtain

$$(t_Q I - Q)(t_P I - P) = t_{PQ} I - PQ,$$

therefore $QP = (t_{PQ} - t_P t_Q)I + t_P Q + t_Q P - PQ$, as required. □

Proposition 3.3. *Let $A, B, C, P, Q \in \mathrm{SL}_2$. Then the following statements hold:*

(i) $t_I = 2$.
(ii) $t_{AB} = t_{BA}$.
(iii) $t_{A^2} = t_A^2 - 2$.
(iv) $t_{ABAB} = t_{AB}^2 - 2$.
(v) $t_{PBAQ} = t_{PQ}t_{AB} - t_{PQ}t_At_B + t_At_{PBQ} + t_Bt_{PAQ} - t_{PABQ}$.
(vi) $t_{PA^2Q} = t_At_{PAQ} - t_{PQ}$.
(vii) $t_{PA^{-1}Q} = t_At_{PQ} - t_{PAQ}$.
(viii) $t_{ABC} = t_At_{BC} + t_Bt_{AC} + t_Ct_{AB} - t_At_Bt_C - t_{ACB}$.

Proof. (i) and (ii) Immediate.
(iii) Since $A \in \mathrm{SL}_2$, the result follows from Equation (3.2) taking traces.
(iv) It follows from (iii) by just observing that $ABAB = (AB)^2$.
(v) Use the formula of Lemma 3.2 multiplying on the left by P and on the right by Q to get

$$PBAQ = P((t_{AB} - t_At_B)I + t_AB + t_BA - AB)Q \tag{3.4}$$

and take traces to obtain the sought formula.
(vi) Start with Equation (3.2), and multiply on the left by P and on the right by Q, to get

$$PA^2Q = t_APAQ - PQ.$$

Finally, taking traces we get the required formula.
(vii) From Equation (3.3), we get

$$PA^{-1}Q = t_APQ - PAQ$$

and take traces.
(viii) In (v), take $P = I$, $Q = C$ to get $t_{BAC} = t_Ct_{AB} - t_Ct_At_B + t_At_{BC} + t_Bt_{AC} - t_{ABC}$, as the claimed formula. □

Theorem 3.4. *Let $A_1, \dots, A_k \in \mathrm{SL}_2$, take a monomial $x = A_{i_1}^{\alpha_1} \dots A_{i_m}^{\alpha_m}$, with $\alpha_j \in \mathbb{Z}$ and $1 \le i_1, \dots, i_m \le k$. Then t_x has a (polynomial) expression in terms of*

$$T_{i_1 \dots i_p} := t_{A_{i_1} \dots A_{i_p}}, \qquad 1 \le i_1 < \dots < i_p \le k.$$

Therefore the ring of functions of $\mathfrak{X}_k = \mathfrak{X}(F_k, \mathrm{SL}_2) = \operatorname{Spec} \mathcal{O}_{\mathfrak{X}_k}$ is given as

$$\mathcal{O}_{\mathfrak{X}_k} = \mathbf{k}\big[\{T_{i_1 \dots i_p}\}_{1 \le i_1 < \dots < i_p \le k}\big]/\mathcal{I}$$

for some ideal $\mathcal{I}$ of relations.

Proof. If $\alpha_j < 0$, we use Proposition 3.3 (vii) to write t_x in terms of traces of monomials in which $\alpha_j \ge 0$. Now, if $\alpha_j \ge 2$, we use Proposition 3.3 (vi) to write t_x in terms of traces of monomials in which α_j is smaller. Repeating we can reach an expression with $\alpha_j = 0, 1$. Doing this for all indices, we finally get a polynomial expression in terms of $t_{A_{i_1} \dots A_{i_p}}$, $1 \le i_1, \dots, i_p \le k$, where $i_j \ne i_{j+1}$. That is, in the monomial two consecutive matrices are distinct.

Now suppose that $i_j > i_{j+1}$. Then we use Proposition 3.3 (v) to get an expression in which the traces appearing have either less number of matrices, or $A_{i_j}, A_{i_{j+1}}$ are swapped. In the first case, we can work by induction on the number of matrices involved to get to the result (note that the other operations do not increase the number of matrices in a given monomial). In the second case, now we get an expression $PA_{i_{j+1}}A_{i_j}Q$ with $i_{j+1} < i_j$. If now there are two consecutive matrices repeated (that is, a square), we use Proposition 3.3 (vi) again. Otherwise, we have managed to reorder two matrices. We can permute the matrices with this process until i_1 is the lowest index, so that $i_1 < i_2, \dots, i_p$. We continue in this fashion until $i_1 < i_2 < \cdots < i_p$. □

The number of monomials of the form $A_{i_1} \dots A_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq k$ described in Theorem 3.4 is

$$\sum_{p=1}^{k} \binom{k}{p} = 2^k - 1.$$

Corollary 3.5. *For $k = 2$, the character variety $\mathfrak{X}_2 = \mathfrak{X}(F_2, \mathrm{SL}_2)$ is isomorphic to $\mathbf{k}^3$, and it is parametrized by (t_A, t_B, t_{AB}) for $(A, B) \in \mathfrak{X}_2$.*

Proof. We have by Proposition 3.1 that $\dim \mathfrak{X}(F_2, \mathrm{SL}_2) = 3$. By Theorem 3.4, the traces t_A, t_B, t_{AB} parametrize. Therefore we have the result. □

4. Equation of the character variety $\mathfrak{X}(F_3, \mathrm{SL}_2)$

Since a general algorithm to compute the ideal $\mathcal{I}$ described in Theorem 3.4 is unknown, let us start by looking at the free group generated by three elements, F_3. The aim of this section is to study the character variety

$$\mathfrak{X}_3 = \mathfrak{X}(F_3, \mathrm{SL}_2) = \{(A, B, C) \mid A, B, C \in \mathrm{SL}_2\} /\!\!/ \mathrm{SL}_2 \,.$$

By Proposition 3.1, we have that $\dim \mathfrak{X}(F_3, \mathrm{SL}_2) = 6$. By Theorem 3.4, the traces

$$t_A,\, t_B,\, t_C,\, t_{AB},\, t_{AC},\, t_{BC},\, t_{ABC}$$

generate the ring of functions of $\mathfrak{X}(F_3, \mathrm{SL}_2)$. These are 7 variables, hence there is an embedding

$$\mathfrak{X}(F_3, \mathrm{SL}_2) \subset \mathbf{k}^7,$$

and the character variety is a hypersurface defined by a single equation. To find such equation, we work as follows. For the sake of clarity, let us set the following variables:

$$\begin{aligned} x &= t_A, & y &= t_B, & z &= t_C, \\ u &= t_{BC}, & v &= t_{AC}, & w &= t_{AB}, \\ P &= t_{ABC}. \end{aligned}$$

Now we complete Theorem 1.2.

Theorem 4.1. *The character variety $\mathfrak{X}_3 \subset \mathbf{k}^7$ is a hypersurface defined by the equation*

$$P^2 = (wz+vy+ux-xyz)P-x^2-y^2-z^2+uyz+vxz+wxy-uvw-u^2-v^2-w^2+4.$$

Proof. Since $A, B, C \in \mathrm{SL}_2$, by Proposition 3.3 (iv) the following holds:

$$t_{ABCABC} = t_{ABC}^2 - 2.$$

Then

$$\begin{aligned}
t_{ABCABC} &= t_{ABBC}(t_{AC} - t_A t_C) + t_A t_{ABCBC} + t_C t_{ABABC} - t_{ABACBC} \\
&\qquad\qquad \text{(Prop. 3.3 (v))} \\
&= t_{ABBC}(t_{AC} - t_A t_C) + t_A\big(t_{ABC}(t_{BC} - t_B t_C) + t_B t_{ABCC} + t_C t_{ABBC} \\
&\quad - t_{ABBCC}\big) + t_C\big(t_{ABC}(t_{AB} - t_A t_B) + t_B t_{AABC} + t_A t_{ABBC} \\
&\quad - t_{AABBC}\big) - \big(t_{ACBC}(t_{AB} - t_A t_B) + t_B t_{AACBC} + t_A t_{ABCBC} \\
&\quad - t_{AABCBC}\big) \qquad \text{(Prop. 3.3 (v))} \\
&= (t_B t_{ABC} - t_{AC})(t_{AC} - t_A t_C) + t_A\big(t_{ABC}(t_{BC} - t_B t_C) + t_B(t_C t_{ABC} \\
&\quad - t_{AB}) + t_C(t_B t_{ABC} - t_{AC}) - (t_B t_C t_{ABC} - t_B t_{AB} - t_C t_{AC} + t_A)\big) \\
&\quad + t_C\big(t_{ABC}(t_{AB} - t_A t_B) + t_B t_A t_{ABC} - t_B t_{BC} + t_A(t_B t_{ABC} - t_{AC}) \\
&\quad - (t_A t_B t_{ABC} - t_A t_{AC} - t_B t_{BC} + t_C)\big) \\
&\quad - \big(t_{ACBC}(t_{AB} - t_A t_B) + t_B t_A t_{ACBC} - t_B t_{CBC} \\
&\quad + t_A t_{ABCBC} - t_A t_{ABCBC} + t_{BCBC}\big) \qquad \text{(Prop. 3.3 (vi))} \\
&= -z^2 - x^2 + vxz - v^2 + (wz + vy + ux - xyz)P \\
&\quad - \big(t_{ACBC} t_{AB} - t_B t_{CBC} + t_{BCBC}\big) \\
&= -z^2 - x^2 + vxz \quad - v^2 + (wz + vy + ux - xyz)P \\
&\quad - \Big(\big(t_{AC}(t_{BC} - t_B t_C) + t_C t_{ABC} + t_B t_{ACC} - t_{ABCC}\big)t_{AB} \\
&\quad - t_B\big(t_C(t_{BC} - t_B t_C) + t_B t_{CC} + t_C t_{BC} - t_{BCC}\big) \qquad \text{(Prop. 3.3 (v))} \\
&\quad + t_{BC}^2 - 2\Big) \qquad \text{(Prop. 3.3 (iv))} \\
&= -z^2 - x^2 + vxz - v^2 + (wz + vy + ux - xyz)P \\
&\quad - \Big(\big(t_{AC}(t_{BC} - t_B t_C) + t_C t_{ABC} + t_B(t_{AC} t_C - t_A) - t_{ABC} t_C + t_{AB}\big)t_{AB} \\
&\quad - t_B\big(t_C(t_{BC} - t_B t_C) + t_B(t_C^2 - 2) + t_C t_{BC} - t_{BC} t_C + t_B\big) \\
&\quad + t_{BC}^2 - 2\Big) \qquad \text{(Prop. 3.3 (vi))} \\
&= -x^2 - y^2 - z^2 + uyz + vxz + wxy - uvw \\
&\quad - u^2 - v^2 - w^2 + 2 + (wz + vy + ux - xyz)P.
\end{aligned}$$

□

Theorem 4.1 can be rewritten as the following equality with traces for triples of matrices $A, B, C \in \mathrm{SL}_2$,

$$\begin{aligned} t_{ABC}^2 = (t_A t_{BC} + t_B t_{AC} + t_C t_{AB} - t_A t_B t_C) t_{ABC} - t_A^2 - t_B^2 - t_C^2 \\ + t_A t_B t_{AB} + t_A t_C t_{AC} + t_B t_C t_{BC} - t_{AB}^2 - t_{AC}^2 - t_{BC}^2 - t_{AB} t_{AC} t_{BC} + 4. \end{aligned} \tag{4.1}$$

Corollary 4.2. *The variety $\mathfrak{X}_3 = \mathfrak{X}(F_3, \mathrm{SL}_2)$ is a ramified double cover of the plane $\mathbf{k}^6$. The variables $t_A, t_B, t_C, t_{AB}, t_{AC}, t_{BC}$ are transcendental generators and the ring of functions $\mathcal{O}_{\mathfrak{X}_3}$ is a degree 2 extension of $\mathbf{k}[t_A, t_B, t_C, t_{AB}, t_{AC}, t_{BC}]$.*

By Theorem 4.1, $\mathfrak{X}_3$ is a double cover over $\mathbf{k}^6$ ramified over $V(\Delta)$, where $\Delta = Y^2 - 4X$ is the discriminant, with X, Y defined as in (5.2). This is a sextic in $\mathbf{k}^6$. The singularities of $\mathfrak{X}_3 \subset \mathbf{k}^7$ are at the points $(x, y, z, u, v, w) \in V(\Delta)$, $P = \frac{1}{2}X$, which are singular points of $V(\Delta)$. The singular locus of $\mathfrak{X}_3$ is determined in [5], and it is equal to the reducible locus $\mathfrak{X}_3^{\mathrm{red}}$, which consists of representations (A, B, C) which can be put, in a suitable basis, as

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}.$$

This is a 3-dimensional subspace. Equivalently, A, B, C are pairwise commuting, which by Equation (6.1), amounts to the equations

$$x^2 + y^2 + w^2 = xyw - 2,\ x^2 + z^2 + v^2 = xzv - 2,\ y^2 + z^2 + u^2 = yzu - 2,\ P = \frac{1}{2}X.$$

5. Generators of the ring of $\mathfrak{X}(F_k, \mathrm{SL}_2)$ for $k \geq 4$

Now we give an expression for the trace of matrices that extends Theorem 3.4 to products of more than four matrices.

Theorem 5.1. *Let $A, B, C, D \in \mathrm{SL}_2$. The trace of $ABCD$ can be expressed as a polynomial expression in terms of t_A, t_B, t_C, t_{AB}, t_{AC}, t_{AD}, t_{BC}, t_{BD}, t_{CD}, t_{ABC}, t_{ABD}, t_{ACD}, and t_{BCD}. More specifically,*

$$\begin{aligned} t_{ABCD} = \frac{1}{2}\big(t_A t_{BCD} + t_B t_{ACD} + t_C t_{ABD} + t_D t_{ABC} + t_{AD} t_{BC} - t_{AC} t_{BD} \\ + t_{AB} t_{CD} - t_{AD} t_B t_C - t_{BC} t_A t_D - t_{AB} t_C t_D - t_{CD} t_A t_B + t_A t_B t_C t_D\big). \end{aligned} \tag{5.1}$$

Proof. By Proposition 3.3 (v), we have that

$$\begin{aligned} t_{ABCD} &= t_{AD}(t_{BC} - t_B t_C) + t_B t_{ACD} + t_C t_{ABD} - t_{ACBD}, \\ t_{ACBD} &= t_{CBDA} = t_{CA}(t_{BD} - t_B t_D) + t_B t_{CDA} + t_D t_{CBA} - t_{CDBA}, \\ t_{CDBA} &= t_{DBAC} = t_{DC}(t_{BA} - t_B t_A) + t_B t_{DAC} + t_A t_{DBC} - t_{DABC}. \end{aligned}$$

Substituting the expression of each equation into the previous one, and using the cyclicity of the traces, namely $t_{DABC} = t_{ABCD}$, we obtain

$$\begin{aligned} t_{ABCD} = t_{AD}(t_{BC} - t_B t_C) + t_B t_{ACD} + t_C t_{ABD} - \big(t_{CA}(t_{BD} - t_B t_D) \\ + t_B t_{CDA} + t_D t_{CBA}\big) + t_{DC}(t_{BA} - t_B t_A) + t_B t_{DAC} + t_A t_{DBC} - t_{ABCD}, \end{aligned}$$

and hence

$$\begin{aligned}
t_{ABCD} &= \frac{1}{2}\Big(t_{AD}(t_{BC}-t_Bt_C)+t_Bt_{ACD}+t_Ct_{ABD}-\big(t_{AC}(t_{BD}-t_Bt_D)+t_Bt_{ACD}\\
&\quad +t_Dt_{ACB}\big)+t_{CD}(t_{AB}-t_At_B)+t_Bt_{ACD}+t_At_{BCD}\Big)\\
&= \frac{1}{2}\Big(t_{AD}(t_{BC}-t_Bt_C)+t_Bt_{ACD}+t_Ct_{ABD}-t_{AC}(t_{BD}-t_Bt_D)-t_Bt_{ACD}\\
&\quad -t_D\big(t_At_{BC}+t_Bt_{AC}+t_Ct_{AB}-t_At_Bt_C-t_{ABC}\big)+t_{CD}(t_{AB}-t_At_B)\\
&\quad +t_Bt_{ACD}+t_At_{BCD}\Big) \qquad \text{(Prop. 3.3 (viii))}\\
&= \frac{1}{2}\Big(t_At_{BCD}+t_Bt_{ACD}+t_Ct_{ABD}+t_Dt_{ABC}+t_{AD}t_{BC}-t_{AC}t_{BD}\\
&\quad +t_{AB}t_{CD}-t_{AD}t_Bt_C-t_{BC}t_At_D-t_{AB}t_Ct_D-t_{CD}t_At_B+t_At_Bt_Ct_D\Big)\\
&\qquad \text{(Simplifying).}
\end{aligned}$$

□

As a consequence, the ring of functions of $\mathfrak{X}_k=\mathfrak{X}(F_k,\mathrm{SL}_2)$ is generated by traces of the product of at most three matrices. This completes the proof of Theorem 1.1.

Corollary 5.2. *Let $A_1,\dots,A_k\in\mathrm{SL}_2$. Take a monomial $x=A_{i_1}^{\alpha_1}\dots A_{i_m}^{\alpha_m}$, with $1\le i_1,\dots,i_m\le k$ and $\alpha_j\in\mathbb{Z}$. Then t_x has a (polynomial) expression in terms of*

$$T_{i_1\dots i_p}:=t_{A_{i_1}\dots A_{i_p}},\quad 1\le i_1<\dots<i_p\le k,$$

with $p\le 3$. Therefore

$$\mathcal{O}_{\mathfrak{X}}=\mathbf{k}\big[\{T_{i_1\dots i_p}\}_{1\le i_1<\dots<i_p\le k,1\le p\le 3}\big]/\mathcal{I}$$

for some ideal $\mathcal{I}$ of relations.

Proof. By Theorem 3.4, $T_{i_1\dots i_p}$, with $1\le p\le k$, gives generators of the ring of all t_x. Now by Theorem 5.1, $t_{A_{i_1}\dots A_{i_4}}$ is expressible in terms of all traces of one, two and three matrices among $A_{i_1},\dots,A_{i_4}$. In general, for $p\ge 4$,

$$t_{A_{i_1}\dots A_{i_p}}=t_{A_{i_1}A_{i_2}A_{i_3}(A_{i_4}\cdots A_{i_p})}$$

is expressible in terms of the traces of products of one, two and three matrices among A_{i_1},A_{i_2},A_{i_3}, and $Q:=A_{i_4}\cdots A_{i_p}$. These are traces of products of less than $p-1$ matrices. By induction, we get the result. □

In virtue of Corollary 5.2, the number of generators of the ring $\mathcal{O}_{\mathfrak{X}_k}$ is

$$k+\binom{k}{2}+\binom{k}{3}.$$

If we look at the case $k=4$, to parametrize

$$\mathfrak{X}_4=\mathfrak{X}(F_4,\mathrm{SL}_2)=\{(A,B,C,D)\mid A,B,C,D\in\mathrm{SL}_2\}/\!\!/\,\mathrm{SL}_2$$

Corollary 5.2 says that we need the traces

$$t_A,\,t_B,\,t_C,\,t_D,\,t_{AB},\,t_{AC},\,t_{AD},\,t_{BC},\,t_{BD},\,t_{CD},\,t_{ABC},\,t_{ABD},\,t_{ACD},\,t_{BCD},$$

giving an embedding $\mathfrak{X}(F_4, \mathrm{SL}_2) \subset \mathbf{k}^{14}$. By Proposition 3.1, $\dim \mathfrak{X}(F_4, \mathrm{SL}_2) = 9$, so five of the above traces are algebraically dependent on the other ones. Letting

$$X(x,y,z,u,v,w) := wz + vy + ux - xyz,$$
$$Y(x,y,z,u,v,w) := -x^2 - y^2 - z^2 + uyz + vxz + wxy - uvw - u^2 - v^2 - w^2 + 4, \tag{5.2}$$

we have the equation

$$t_{ABC}^2 = X(t_A, t_B, t_C, t_{BC}, t_{AC}, t_{AB}) \cdot t_{ABC} + Y(t_A, t_B, t_C, t_{BC}, t_{AC}, t_{AB}), \tag{5.3}$$

and similarly for the others t_{ABD}, t_{ACD}, t_{BCD}. This gives four algebraically dependent variables.

Proposition 5.3. *The trace t_{CD} is algebraically dependent with t_A, t_B, t_C, t_D, t_{AB}, t_{AC}, t_{BC}, t_{AD}, and t_{BD}. Therefore t_A, t_B, t_C, t_D, t_{AB}, t_{AC}, t_{BC}, t_{AD}, t_{BD} are transcendental generators of $\mathcal{O}_{\mathfrak{X}_4}$.*

Proof. Clearly there is an algebraic dependence relation between all these variables, as the transcendental degree of the field that they generate is 9. The variables t_A, t_B, t_C, t_D are clearly algebraically independent. Therefore, there is one of the other variables that depends algebraically on the rest. Permuting the order of the matrices, we can assume that it is t_{CD}. □

It is not easy to find out an explicit algebraic equation satisfied by t_{CD} in Proposition 5.3. This can be done as follows. Consider the element $t_{ABCD} = t_{(AB)CD}$ and apply Equation (4.1) to get

$$\begin{aligned} t_{ABCD}^2 = {} & X(t_{AB}, t_C, t_D, t_{CD}, t_{ABD}, t_{ABC}) t_{ABCD} \\ & + Y(t_{AB}, t_C, t_D, t_{CD}, t_{ABD}, t_{ABC}), \end{aligned} \tag{5.4}$$

with the expressions X, Y appearing in (5.2). Now use Equation (5.1) to substitute t_{ABCD} in the above. This gives an equation involving $t_A, \ldots, t_D, t_{AB}, \ldots, t_{CD}$ and $t_{ABC}, t_{ABD}, t_{ACD}$, t_{BCD}. Using Theorem 4.1 we have algebraic equations for $t_{ABC}, \ldots, t_{BCD}$ in terms of the traces t_A, t_B, t_C, t_D, t_{AB}, t_{AC}, t_{AD}, t_{BC}, t_{BD}, t_{CD}. This will yield an equation involving all required traces. Note that we can also work out an equation like (5.4) for $t_{ABCD} = t_{A(BC)D}$ or $t_{ABCD} = t_{AB(CD)}$ or $t_{ABCD} = t_{(DA)BC}$. This can serve to eliminate t_{ABCD}.

Corollary 5.4. *In $\mathfrak{X}_k = \mathfrak{X}(F_k, \mathrm{SL}_2)$, we have parameters for $(A_1, \ldots, A_k)$ given by t_{A_i}, $t_{A_iA_j}$, $t_{A_iA_jA_k}$, $i < j < k$. The parameters*

$$t_{A_1}, t_{A_2}, t_{A_1A_2}, \quad \textit{and} \quad t_{A_j}, t_{A_1A_j}, t_{A_2A_j}, \ j \geq 3,$$

are transcendental generators of $\mathcal{O}_{\mathfrak{X}_k}$.

Proof. First, by Proposition 3.1 the dimension of $\mathfrak{X}_k$ is $3(k-1)$. Now $t_{A_1}, t_{A_2}, t_{A_1A_2}$ generate $\mathcal{O}_{\mathfrak{X}_2}$ by Corollary 3.5. For $k = 3$, Corollary 4.2 says that t_{A_1}, t_{A_2}, t_{A_3}, $t_{A_1A_2}$, $t_{A_1A_3}$, $t_{A_2A_3}$ are transcendental generators of $\mathcal{O}_{\mathfrak{X}_3}$. For $k \geq 4$, we use Proposition 5.3 applied to (A_1, A_2, A_i, A_j) to get an algebraic equation for $t_{A_iA_j}$ in terms of t_{A_1}, t_{A_2}, t_{A_i}, t_{A_j}, $t_{A_1A_2}$, $t_{A_1A_i}$, $t_{A_1A_j}$, $t_{A_2A_i}$, $t_{A_2A_j}$. Therefore the given set of traces are transcendental generators. There cannot be less than they are because $\dim \mathfrak{X}_k = 3k - 3$, which is the number of parameters in the list. □

6. Character variety of the 2-torus

Now we are going to focus on the 2-torus T^2 and the space of representations of its fundamental group $\Gamma = \pi_1(T^2) = \langle x, y \mid [x, y] = 1\rangle$ in SL_2. By the general description in (2.1), we have that the character variety of a finitely generated group embeds as a subvariety of the character variety of the free group F_k, where k is the number of generators of the group. In this situation,

$$\mathfrak{X}_{T^2} = \mathfrak{X}(T^2, \mathrm{SL}_2) \subset \mathfrak{X}(F_2, \mathrm{SL}_2) = \mathbf{k}^3,$$

the last equality by Corollary 3.5. Then $\mathfrak{X}_{T^2}$ is parametrized by (t_A, t_B, t_{AB}), and there will be an equation describing this variety. To find it, we work out a relation for the trace of a commutator.

Lemma 6.1. *For matrices $A, B \in \mathrm{SL}_2$, we have*

$$t_{[A,B]} = t_A^2 + t_B^2 + t_{AB}^2 - t_A t_B t_{AB} - 2. \tag{6.1}$$

Proof. We compute

$$\begin{aligned}
[A,B] &= ABA^{-1}B^{-1} = AB(t_A I - A)B^{-1} && \text{(by Eqn. (3.3))}\\
&= t_A A - ABAB^{-1}\\
&= t_A A - A\big((t_{AB} - t_A t_B)I + t_A B + t_B A - AB\big)B^{-1} && \text{(by Eqn. (3.4))}\\
&= t_A A - (t_{AB} - t_A t_B)AB^{-1} - t_A A - t_B A^2 B^{-1} + A^2\\
&= -(t_{AB} - t_A t_B)A(t_B I - B) - t_B(t_A A - I)(t_B I - B)\\
&\qquad + t_A A - I && \text{(by Eqn. (3.2))}\\
&= -t_{AB} t_B A + t_A t_B^2 A + t_{AB} AB - t_A t_B AB - t_A t_B^2 A + t_B^2 I\\
&\qquad + t_A t_B AB - t_B B + t_A A - I\\
&= -t_{AB} t_B A + t_{AB} AB + t_B^2 I - t_B B + t_A A - I.
\end{aligned} \tag{6.2}$$

Taking traces,

$$t_{[A,B]} = -t_{AB} t_B t_A + t_{AB}^2 + 2t_B^2 - t_B^2 + t_A^2 - 2,$$

producing the result. □

From now on, we fix the ground field $\mathbf{k} = \mathbb{C}$. Take a conjugacy class $[\xi] = \mathrm{SL}_2 \cdot \xi$ determined by an element $\xi \in \mathrm{SL}_2$ (the action of SL_2 by conjugation). We have the *twisted* moduli space of representations as defined in (1.1),

$$\mathcal{M}_\xi = \{(A, B) \in (\mathrm{SL}_2)^2 \mid [A, B] \in [\xi]\} /\!\!/ \mathrm{SL}_2 .$$

There are five different types of conjugacy classes. We have $[I]$, $[-I]$, $[J_+]$, $[J_-]$, and $[\xi_t]$, where $J_\pm = \begin{pmatrix} \pm 1 & 0 \\ 1 & \pm 1 \end{pmatrix}$ are the Jordan types, and $\xi_t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $t = \lambda + \lambda^{-1}$, $\lambda \in \mathbb{C} - \{0, \pm 1\}$, are the diagonal types. Consider the trace map

$$\mathrm{tr} : \mathrm{SL}_2 \longrightarrow \mathbb{C}$$

and note that

$$\begin{aligned} \mathrm{tr}^{-1}(t) &= [\xi_t], \qquad t \in \mathbb{C} - \{\pm 2\}, \\ \mathrm{tr}^{-1}(2) &= [I] \sqcup [J_+], \\ \mathrm{tr}^{-1}(-2) &= [-I] \sqcup [J_-]. \end{aligned}$$

Using the variables $x = t_A$, $y = t_B$, $z = t_{AB}$, Lemma 6.1 gives the function

$$F(x, y, z) = t_{[A,B]} = x^2 + y^2 + z^2 - xyz - 2. \tag{6.3}$$

Then we have the following *twisted character varieties*:

$$\mathfrak{X}_t = F^{-1}(t) = \{(A, B) \in \mathrm{SL}_2 \mid \mathrm{tr}([A, B]) = t\} /\!\!/ \mathrm{SL}_2$$

for $t \in \mathbb{C}$. Then

- $\mathfrak{X}_t = \mathcal{M}_{\xi_t}$ for $t \neq \pm 2$,
- $\mathfrak{X}_2 = \mathcal{M}_I \cup \mathcal{M}_{J_+}$,
- $\mathfrak{X}_{-2} = \mathcal{M}_{-I} \cup \mathcal{M}_{J_-}$.

Remark 6.2. Note the symmetry of equation (6.3). This is given by the change of generators $(A, B) \mapsto (AB, B^{-1})$, which changes $(t_A, t_B, t_{AB}) \mapsto (t_{AB}, t_B, t_A)$.

We study the geometry of the character varieties $\mathfrak{X}_t$, and recover results of Theorem 1.3. Now we prove Theorem 1.4.

Theorem 6.3. *Let $t \in \mathbb{C}$. We have the following:*

- *For $t \neq \pm 2$, the character variety $\mathfrak{X}_t \subset \mathbb{C}^3$ is a smooth surface. The E-polynomial of $\mathfrak{X}_t$ is $e(\mathfrak{X}_t) = q^2 + 4q + 1$.*
- *For $t = 2$, the character variety $\mathfrak{X}_2 \subset \mathbb{C}^3$ has 4 ordinary double points. Moreover, $\mathcal{M}_{J_+} \subset \mathcal{M}_I$, $\mathfrak{X}_2 = \mathcal{M}_I$, and $e(\mathfrak{X}_2) = q^2 + 1$.*
- *For $t = -2$, the character variety $\mathfrak{X}_{-2} \subset \mathbb{C}^3$ has only one singular point which is an ordinary double point. Now $\mathfrak{X}_{-2} = \mathcal{M}_{-I} \sqcup \mathcal{M}_{J_-}$, and $e(\mathfrak{X}_{-2}) = q^2 + 3q + 1$.*

Proof. We start analyzing the singular points of $\{F(x, y, z) = t\}$. We compute the derivatives of F,

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (2x - yz, 2y - xz, 2z - xy).$$

For a singular point, we have $2x = yz$, $2y = xz$, $2z = xy$. From this, we get $x^2 = y^2 = z^2 = \frac{1}{2}xyz$ and hence $F = \frac{1}{2}xyz - 2$.

- For $t = -2$, we have $x^2 = y^2 = z^2 = \frac{1}{2}xyz = 0$, so there is a singular point $(x, y, z) = (0, 0, 0)$. The leading term of F is $x^2 + y^2 + z^2$, hence the point is an ordinary double point.
- For $t = 2$, we have $x^2 = y^2 = z^2 = \frac{1}{2}xyz = 4$. Therefore, the singular points are $(2, 2, 2)$, $(2, -2, -2)$, $(-2, 2, -2)$ and $(-2, -2, 2)$. Let us focus on one of them, say $(2, 2, 2)$; then the Hessian of F is

$$H_F(2, 2, 2) = \begin{pmatrix} 2 & -z & -y \\ -z & 2 & -x \\ -y & -x & 2 \end{pmatrix}\Bigg|_{(2,2,2)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix},$$

which is non-degenerate, hence it is an ordinary double point. The other singular points are similar.

- For $t \neq \pm 2$, we have $x^2 = y^2 = z^2 = \frac{1}{2}xyz = 2 + t \neq 0$. Then $x = \pm y$, $x = \pm z$, hence $2x = yz = \pm x^2$, and $x = \pm 2$. This implies that $x^2 = 4 = 2 + t$, hence $t = 2$. So for $t \neq \pm 2$, the surface $\mathfrak{X}_t$ is smooth.

To proceed, consider the completion of $V(F-t) \subset \mathbb{C}^3$ in the projective space $\mathbb{P}^3$. This is given by the homogeneous polynomial

$$\hat{F}_t = x^2u + y^2u + z^2u - xyz - (2+t)u^3$$

for projective coordinates $[x, y, z, u]$. We compute the derivatives

$$\left(\frac{\partial \hat{F}_t}{\partial x}, \frac{\partial \hat{F}_t}{\partial y}, \frac{\partial \hat{F}_t}{\partial z}, \frac{\partial \hat{F}_t}{\partial u}\right) = (2xu - yz, 2yu - xz, 2zu - xy, x^2 + y^2 + z^2 - 3(2+t)u^2)$$

and look at a point at infinity, that is, $u = 0$. Then the above derivatives reduce to $(-yz, -xz, -xy, x^2 + y^2 + z^2)$. This cannot vanish, because for this to be zero, two coordinates should vanish, and hence $x^2 + y^2 + z^2 \neq 0$. This means that $V(\hat{F}_t)$ is smooth at the points at infinity.

Now take $t \neq \pm 2$. Then $V = V(\hat{F}_t) \subset \mathbb{P}^3$ is a smooth surface of degree 3. By [12, Example 9.11], we have the Hodge numbers of V to be $h^{1,0} = h^{0,1} = 0$, $h^{2,0} = h^{0,2} = 0$ and $h^{1,1} = 7$. Hence, the E-polynomial is $e(V) = q^2 + 7q + 1$, where $q = uv$. Now, the intersection $V_\infty := V \cap \{u = 0\} = \ell_1 \cup \ell_2 \cup \ell_3$ consists of 3 lines and has E-polynomial $e(V_\infty) = \sum e(\ell_i) - \sum e(\ell_i \cap \ell_j) = 3(q+1) - 3 = 3q$. Therefore

$$e(\mathfrak{X}_t) = e(V) - e(V_\infty) = q^2 + 4q + 1.$$

Let $f : \mathrm{SL}_2^2 \to \mathrm{SL}_2$, $f(A, B) = [A, B]$. As in [13, Section 4] we write $X_0 = f^{-1}(I)$, $X_1 = f^{-1}(-I)$, $X_2 = f^{-1}([J_+])$, $X_3 = f^{-1}([J_-])$, so that $f^{-1}(\mathrm{tr}^{-1}(2)) = X_0 \sqcup X_2$ and $f^{-1}(\mathrm{tr}^{-1}(-2)) = X_1 \sqcup X_3$. By [13, Section 4.3], the representations of

$$\bar{X}_2 = \left\{(A, B) \mid [A, B] = J_+ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right\}$$

are of the form $A = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$ and $B = \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix}$. Conjugating by $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, we get the matrices

$$A_t = \begin{pmatrix} a & 0 \\ tb & a^{-1} \end{pmatrix}, \quad B_t = \begin{pmatrix} x & 0 \\ ty & x^{-1} \end{pmatrix}, \quad [A_t, B_t] = J_{t,+} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Taking $t \to 0$, we get in the limit representations in X_0. Therefore X_2 is contained in the closure of X_0. This implies that $\mathcal{M}_{J_+} \subset \mathcal{M}_I$. Hence $\mathfrak{X}_2 = \mathcal{M}_I$.

To compute the E-polynomial note that $X = V(\hat{F}_2) \subset \mathbb{P}^3$ appears as a degeneration of $V(\hat{F}_t)$ when $t \to 2$. Such degeneration produces four singularities which are ordinary double points. Each of them reduces the Betti number b_2 by one, hence $b_2(X) = 3$. Therefore the E-polynomial of X is $e(X) = q^2 + 3q + 1$. Now

$$e(\mathfrak{X}_2) = e(X) - e(V_\infty) = q^2 + 1.$$

Finally, we look at the case $t=-2$. By [13, Section 4.2], $\mathcal{M}_{-I}$ consists of one point, which has representative $A=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $B=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, with $(t_A,t_B,t_{AB})=(0,0,0)$, the singular point of $\mathfrak{X}_{-2}$. Regarding $\mathcal{M}_{J_-}$, according to [13, Section 4.4], the matrices of

$$\bar{X}_3=\left\{(A,B)\mid [A,B]=J_-=\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}\right\}$$

are of the form $A=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B=\begin{pmatrix} x & z \\ y & w \end{pmatrix}$ with $z=2(x+w)$, $c=-2(a+d)$, $cy+2dw+bz=0$. This implies that it cannot be $(t_A,t_B)=(z,c)=(0,0)$. So $\mathcal{M}_{J_-}=\mathfrak{X}_{-2}-\mathcal{M}_{-I}$, and hence $\mathfrak{X}_{-2}=\mathcal{M}_{J_-}\sqcup\mathcal{M}_{-I}$.

To compute the E-polynomial note that $Y=V(\hat{F}_{-2})\subset\mathbb{P}^3$ appears as a degeneration of $V(\hat{F}_t)$ when $t\to -2$. Such degeneration produces one ordinary double point, hence $b_2(Y)=6$ and $e(Y)=q^2+6q+1$. Thus

$$e(\mathfrak{X}_{-2})=e(Y)-e(V_\infty)=q^2+3q+1=e(\mathcal{M}_{J_-})+e(\mathcal{M}_{-I}). \qquad \square$$

Note that the results of the E-polynomials of Theorem 6.3 agree with those of Theorem 1.3.

7. Character variety of the genus 2 surface

We look at the character variety

$$\mathcal{M}_2=\mathfrak{X}(\pi_1(\Sigma_2),\mathrm{SL}_2)=\{(A,B,C,D)\in(\mathrm{SL}_2)^4\mid [A,B]\,[C,D]=I\}\,/\!\!/\,\mathrm{SL}_2\,.$$

The dimension of $\mathcal{M}_g=\mathfrak{X}(\pi_1(\Sigma_g),\mathrm{SL}_2)$, for the orientable compact surface Σ_g of genus g, is $\dim\mathcal{M}_g=6g-6$. Therefore $\dim\mathcal{M}_2=6$.

Proposition 7.1. *The ring $\mathcal{O}_{\mathcal{M}_2}$ has transcendental generators t_A, t_B, t_C, t_D, t_{AB}, t_{AC}.*

Proof. We have that $\mathcal{M}_2\subset\mathfrak{X}_4=\mathfrak{X}(F_4,\mathrm{SL}_2)$ and $\dim\mathfrak{X}_4=9$, generated by t_A, t_B, t_C, t_D, t_{AB}, t_{AC}, t_{BC}, t_{AD}, t_{BD}, where t_{CD} is algebraically dependent on the previous ones, by Corollary 5.4. Using $[A,B]=[C,D]^{-1}$, we have $t_{[A,B]}=t_{[C,D]}$, and using Equation (6.1),

$$t_A^2+t_B^2+t_{AB}^2-t_At_Bt_{AB}=t_C^2+t_D^2+t_{CD}^2-t_Ct_Dt_{CD},$$

thereby t_{CD} is algebraically dependent on t_A, t_B, t_C, t_D, t_{AB}.

Using Equation (6.2), we get

$$t_{[A,B]C}=-t_{AB}t_Bt_{AC}+t_{AB}t_{ABC}+t_B^2t_C-t_Bt_{BC}+t_At_{AC}-t_C.$$

From $[A,B]\,[C,D]=I$, we rewrite $[A,B]C=DCD^{-1}$ which implies that

$$-t_{AB}t_Bt_{AC}+t_{AB}t_{ABC}+t_B^2t_C-t_Bt_{BC}+t_At_{AC}-t_C=t_C.$$

Using (5.3), which is an algebraic dependence of t_{ABC} on t_A, t_B, t_C, t_{AB}, t_{AC}, t_{BC}, and, unravelling the above, we get an algebraic equation, and then we isolate t_{BC} as algebraically dependent on t_A, t_B, t_C, t_{AB}, t_{AC}.

Now we use the equation $[A,B]\,[C,D] = I$ to get $[C,D]A = BAB^{-1}$, and working as before we get an algebraic dependence of t_{AD} on t_A, t_C, t_D, t_{CD}, t_{AC}. But using the dependence of t_{CD} on t_{AB}, we get that t_{AD} is algebraically dependent on t_A, t_B, t_C, t_D, t_{AB}, t_{AC}.

Finally take $[A,B]\,[C,D] = I$, to rewrite $D^{-1}[A,B] = CD^{-1}C^{-1}$, and work analogously to get an algebraic dependence of t_{BD} in terms of t_A, t_B, t_D, t_{AB}, t_{AD}. Using the previous paragraph, we get that t_{BD} is algebraic dependent on t_A, t_B, t_C, t_D, t_{AB}, t_{AC}. □

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