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WARPED PRODUCT LIGHTLIKE SUBMANIFOLDS WITH A SLANT FACTOR

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Abstract. In the present study, we investigate a new type of warped products on manifolds with indefinite metrics, namely, warped product lightlike submanifolds of indefinite Kaehler manifolds with a slant factor. First, we show that indefinite Kaehler manifolds do not admit any proper warped product semi-slant lightlike submanifolds of the type $N_T \times_\lambda N_\theta$, $N_\theta \times_\lambda N_T$, $N_\perp \times_\lambda N_\theta$ and $N_\theta \times_\lambda N_\perp$, where N_T is a holomorphic submanifold, $N_\perp$ is a totally real submanifold and N_θ is a proper slant submanifold. Then, we study warped product semi-slant lightlike submanifolds of the type $B \times_\lambda N_\theta$, where $B = N_T \times N_\perp$, of an indefinite Kaehler manifold. Following this, we give one non-trivial example for this kind of warped products of indefinite Kaehler manifolds. Then, we establish a geometric estimate for the squared norm of the second fundamental form involving the Hessian of warping function λ for this class of warped products. Finally, we present a sharp geometric inequality for the squared norm of second fundamental form of warped product semi-slant lightlike submanifolds of the type $B \times_\lambda N_\theta$.

1. Introduction

The geometry of warped product manifolds was firstly introduced by Bishop and O'Neill [2] in 1969. However, the theory of warped products became popular among mathematicians and physicists in the beginning of the 21st century, when the notion of CR-warped product submanifolds of Kaehler manifolds was introduced by Chen in [6], where he showed the non-existence of warped product CR-submanifolds of the type $N_\perp \times_\lambda N_T$, where N_T is a holomorphic submanifold and $N_\perp$ is a totally real submanifold, in a Kaehler manifold. Furthermore, the author examined CR-warped product submanifolds of the type $N_T \times_\lambda N_\perp$ in Kaehler manifolds. Thereafter, many other studies explored the existence or non-existence of warped product submanifolds in various ambient space settings. In terms of their physical

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utility, warped product manifolds provide an excellent setting to model space-time near black holes or bodies with large gravitational force. For instance, the best relativistic model of Schwarzschild space-time that is used to depict the outer surface around the black hole or massive star is given as a warped product. In addition, many exact solutions like Robertson Walker's model and Schwarzschild's solution of the Einstein field equations are warped product structures.

On the other hand, the concept of slant submanifolds was given by Chen as a generalization of slant immersions (for details, see [4, 5]). Later, some more generalized classes of slant submanifolds came into existence and the subject matter was dealt with in detail by many others (see [3, 16, 19]). In [18], Sahin proved that there do not exist warped product semi-slant submanifolds of the type $N_T \times_\lambda N_\theta$ and $N_\theta \times_\lambda N_T$, where N_θ is a proper slant submanifold of a Kaehler manifold. Furthermore, Chen [7] studied pointwise CR-slant warped product submanifolds in Kaehler manifolds which are of the type $B \times_\lambda N_\theta$, where $B = N_T \times N_\perp$ is a CR-product and N_θ is a proper pointwise slant submanifold. The literature suggests that the majority of studies on warped products with a slant factor is confined only to manifolds with positive definite metric which may not be applicable to areas of mathematical physics involving indefinite metrics. However, in recent years, a few papers have appeared on warped product lightlike submanifolds (for details, see [8, 11, 12, 13, 14, 17]) but the concept of warped products considering the slant case still remains untouched for manifolds with indefinite metrics.

To this end, we consider warped product semi-slant lightlike submanifolds of indefinite Kaehler manifolds. Firstly, we obtain the non-existence of warped product semi-slant lightlike submanifolds of the type $N_T \times_\lambda N_\theta$, $N_\theta \times_\lambda N_T$, $N_\perp \times_\lambda N_\theta$ and $N_\theta \times_\lambda N_\perp$. Then, we derive several results in terms of the second fundamental form for warped product semi-slant lightlike submanifolds of the type $B \times_\lambda N_\theta$, where $B = N_T \times N_\perp$ in indefinite Kaehler manifolds. Furthermore, we provide an interesting characterization for the second fundamental form in terms of the Hessian of the warping function λ for such warped product lightlike submanifolds in an indefinite complex space form. Finally, we establish a geometric inequality for the second fundamental form in terms of the warping function of warped product lightlike submanifolds of the type $B \times_\lambda N_\theta$ and also present one non-trivial example.

2. Preliminaries

2.1. **Geometry of lightlike submanifolds.** Let (N^n, g) be an isometrically immersed submanifold of a semi-Riemannian manifold $(\tilde{N}^{m+n}, \tilde{g})$ of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$. The metric g is the induced metric of $\tilde{g}$ on N. If the metric $\tilde{g}$ becomes degenerate on the tangent bundle TN of N, then N is known as a lightlike submanifold of $\tilde{N}$. There exists locally a lightlike vector field $\xi \in \Gamma(TN)$, $\xi \neq \{0\}$, such that $g(\xi, Y) = 0$ for any $Y \in \Gamma(TN)$. Then, for each tangent space T_yN, we have that

$$T_yN^\perp = \bigcup \{u \in T_y\tilde{N} : \tilde{g}(u, v) = 0 \; \forall \, v \in T_yN, \, y \in N\}$$

is a degenerate n-dimensional subspace of $T_y\tilde{N}$. Thus, both the subspaces T_yN and $T_yN^{\perp}$ are degenerate orthogonal but no longer complementary. In this case, there exists a subspace $\operatorname{Rad}(T_yN) = T_yN \cap T_yN^{\perp}$, which is called the radical subspace and defined as

$$\operatorname{Rad}(T_yN) = \{\xi_y \in T_yN : g(\xi_y, Y) = 0 \;\forall\, Y \in T_yN\}.$$

If the mapping

$$\operatorname{Rad}(TN) : y \in N \longrightarrow \operatorname{Rad}(T_yN)$$

defines a smooth distribution on N of rank $r > 0$, then $\operatorname{Rad}(TN)$ is called the radical distribution on N, and N is said to be an r-lightlike submanifold of $\tilde{N}$. The following are four sub-cases of a lightlike submanifold $(N, g, S(TN), S(TN^{\perp}))$:

Case 1: r-lightlike if $r < \min\{m, n\}$.
Case 2: Co-isotropic if $r = m < n$, i.e., $S(TN^{\perp}) = \{0\}$.
Case 3: Isotropic if $r = n < m$, i.e., $S(TN) = \{0\}$.
Case 4: Totally lightlike if $r = m = n$, i.e., $S(TN) = \{0\} = S(TN^{\perp})$.

For Case 1, there exists a non-degenerate screen distribution $S(TN)$ which is a complementary orthogonal vector subbundle to $\operatorname{Rad}(TN)$ in TN. Thus, we can write

$$TN = \operatorname{Rad}(TN) \perp S(TN). \tag{2.1}$$

Although $S(TN)$ is not unique, it is canonically isomorphic to the vector bundle $TN/\operatorname{Rad}(TN)$. Let us denote an r-lightlike submanifold by $(N, g, S(TN), S(TN^{\perp}))$, where $S(TN^{\perp})$ is a complementary vector subbundle to $\operatorname{Rad}(TN)$ in $TN^{\perp}$. Therefore, we have the following essential result.

Theorem 2.1 ([9]). *For an r-lightlike submanifold $(N, g, S(TN), S(TN^{\perp}))$ of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$, there exists a complementary vector bundle $\operatorname{ltr}(TN)$ of $\operatorname{Rad}(TN)$ in $S(TN^{\perp})^{\perp}$ and a basis of $\Gamma(\operatorname{ltr}(TN)|_u)$ consisting of smooth sections $\{N_i\}$ of $S(TN^{\perp})^{\perp}|_u$, where u is a coordinate neighborhood of N such that*

$$\tilde{g}(N_i, N_j) = 0, \quad \tilde{g}(N_i, \xi_j) = \delta_{ij} \qquad \text{for } i, j \in \{1, 2, \ldots, r\}, \tag{2.2}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\operatorname{Rad}(TN))$.

It follows that there exists a lightlike transversal vector bundle $\operatorname{ltr}(TN)$ locally spanned by $\{N_i\}$.

Let $\operatorname{tr}(TN)$ and $\operatorname{ltr}(TN)$ be the vector bundles complementary (but not orthogonal) to TN in $T\tilde{N}|_N$ and to $\operatorname{Rad}(TN)$ in $S(TN^{\perp})$, respectively. Then we have

$$\begin{aligned} \operatorname{tr}(TN) &= \operatorname{ltr}(TN) \perp S(TN^{\perp}), \\ T\tilde{N}|_N &= TN \oplus \operatorname{tr}(TN) \\ &= (\operatorname{Rad}(TN) \oplus \operatorname{ltr}(TN)) \perp S(TN) \perp S(TN^{\perp}). \end{aligned} \tag{2.3}$$

Let $(N, g, S(TN), S(TN^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$. Let $\tilde{\nabla}$ be the Levi-Civita connection defined on $\tilde{N}$. Then, from the decomposition (2.3), the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.4}$$

$$\tilde{\nabla}_X N' = -A_{N'}X + \nabla^l_X N' + D^s(X,N'), \tag{2.5}$$

$$\tilde{\nabla}_X W = -A_W X + D^l(X,W) + \nabla^s_X W, \tag{2.6}$$

where $X,Y \in \Gamma(TN)$, $N' \in \Gamma(\mathrm{ltr}(TN))$ and $W \in \Gamma(S(TN^\perp))$. Furthermore, employing Eqs. (2.4)–(2.6), we derive

$$g(A_W X,Y) = \tilde{g}(h^s(X,Y),W) + \tilde{g}(Y,D^l(X,W)), \tag{2.7}$$

$$\tilde{g}(D^s(X,N'),W) = \tilde{g}(A_W X,N'). \tag{2.8}$$

Let P be the projection morphism of TN on $S(TN)$. Then using Eq. (2.1), we can induce some new geometric objects in the screen distribution $S(TN)$ on N as

$$\nabla_X PY = \nabla^*_X PY + h^*(X,Y), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi \tag{2.9}$$

for any $X,Y \in \Gamma(TN)$ and $\xi \in \Gamma(\mathrm{Rad}(TN))$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X,Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TN))$ and $\Gamma(\mathrm{Rad}(TN))$, respectively. Using Eqs. (2.4), (2.5) and (2.9), we obtain

$$\tilde{g}(h^l(X,PY),\xi) = g(A^*_\xi X,PY), \quad \tilde{g}(h^*(X,PY),N') = g(A_{N'}X,PY)$$

for any $X,Y \in \Gamma(TN)$, $\xi \in \Gamma(\mathrm{Rad}(TN))$ and $N' \in \Gamma(\mathrm{ltr}(TN))$.

Next, consider $\tilde{\nabla}$ is a metric connection, and using Eqs. (2.4)–(2.6) for $X,Y,Z \in \Gamma(TN)$ and $V,V' \in \Gamma(\mathrm{tr}(TN))$, we obtain

$$(\nabla_X \tilde{g})(Y,Z) = \tilde{g}(h^l(X,Y),Z) + \tilde{g}(h^l(X,Z),Y)$$

and

$$(\nabla^t_X \tilde{g})(V,V') = -\tilde{g}(A_V X,V') - \tilde{g}(A_{V'}X,V),$$

which implies that the induced linear connection ∇ on N and the transversal linear connection ∇^t on $\mathrm{tr}(TN)$ are generally not metric connections.

Let us denote by $\tilde{R}$ the curvature tensors of $\tilde{\nabla}$. Then the equation of Codazzi is given by

$$\begin{aligned}(\tilde{R}(X,Y)Z)^\perp = {} & (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) + D^l(X,h^s(Y,Z)) \\ & - D^l(Y,h^s(X,Z)) + (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) \\ & + D^s(X,h^l(Y,Z)) - D^s(Y,h^l(X,Z)),\end{aligned} \tag{2.10}$$

where

$$(\nabla_X h^s)(Y,Z) = \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y,Z) - h^s(Y,\nabla_X Z), \tag{2.11}$$

$$(\nabla_X h^l)(Y,Z) = \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y,Z) - h^l(Y,\nabla_X Z) \tag{2.12}$$

for $X,Y,Z \in \Gamma(TN)$.

Definition 2.2. An indefinite almost Hermitian manifold $\tilde{N}$ with an indefinite Hermitian metric $\tilde{g}$ and an almost complex structure $\tilde{J}$ is known as an *indefinite Kaehler manifold* (cf. [1]) if

$$\tilde{J}^2 = -I, \quad \tilde{g}(\tilde{J}X, \quad \tilde{J}Y) = \tilde{g}(X,Y), \quad (\tilde{\nabla}_X \tilde{J})Y = 0 \qquad \text{for all } X, Y \in \Gamma(T\tilde{N}). \tag{2.13}$$

Let $\tilde{N}(c)$ be an indefinite complex space form of constant holomorphic curvature c. Then the curvature tensor $\tilde{R}$ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z = \frac{c}{4}\{&\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y + \tilde{g}(\tilde{J}Y,Z)\tilde{J}X - \tilde{g}(\tilde{J}X,Z)\tilde{J}Y \\ &+ 2\tilde{g}(X,\tilde{J}Y)\tilde{J}Z\}\end{aligned} \tag{2.14}$$

for X, Y, Z vector fields on $\tilde{N}$.

Next, we recall the definition of semi-slant lightlike submanifolds of an indefinite Kaehler manifold following [20].

Definition 2.3. A q-lightlike submanifold N of an indefinite Kaehler manifold $\tilde{N}$ of index $2q$ (provided that $2q < \dim N$) is known as a *semi-slant lightlike submanifold* of $\tilde{N}$ if

(a) $\tilde{J}\,\mathrm{Rad}(TN)$ is a distribution on N such that $\mathrm{Rad}(TN) \cap \tilde{J}\,\mathrm{Rad}(TN) = 0$;
(b) there exist non-degenerate orthogonal distributions D_0 and D_θ on N satisfying $S(TN) = \big(\tilde{J}\,\mathrm{Rad}(TN) \oplus \tilde{J}\,\mathrm{ltr}(TN)\big) \perp D_0 \perp D_\theta$;
(c) the distribution D_0 is an invariant distribution, i.e., $\tilde{J}D_0 = D_0$;
(d) the distribution D_θ is slant with angle $\theta(\neq 0)$, i.e., for each $y \in N$ and each non-zero vector $Y \in (D_\theta)_y$, the angle θ between the vector subspace $(D_\theta)_y$ and $\tilde{J}Y$ is a non-zero constant, which is independent of the choice of $y \in N$ and $Y \in (D_\theta)_y$.

In view of the above definition, the decomposition of $S(TN)$ becomes

$$S(TN) = (\tilde{J}\,\mathrm{Rad}(TN) \oplus \tilde{J}\,\mathrm{ltr}(TN)) \perp D_0 \perp D_\theta, \quad \tilde{J}D_0 = D_0.$$

Thus

$$TN = D_T \oplus D_\perp \oplus D_\theta, \quad TN^\perp = \tilde{J}D_\perp \oplus \mathrm{tr}(\tilde{J}D_\theta) \oplus \mu,$$

where

$$D_T = \tilde{J}\,\mathrm{Rad}(TN) \oplus \mathrm{Rad}(TN) \oplus D_0, \quad D_\perp = \tilde{J}\,\mathrm{ltr}(TN).$$

Now, for a vector field $U \in \Gamma(D_\theta)$, we write

$$\tilde{J}U = TU + FU, \tag{2.15}$$

where $TU \in \Gamma(TN)$ and $FU \in \Gamma(\mathrm{tr}(TN))$. Similarly,

$$\tilde{J}V = BV + CV$$

for $V \in \Gamma(\mathrm{tr}(TN))$, where BV and CV represent, respectively, sections of TN and $\mathrm{tr}(TN)$.

Applying $\tilde{J}$ to Eq. (2.15) and taking the tangential component, we get

$$-U = T^2U + BFU \quad \text{for all } U \in \Gamma(D_\theta).$$

From [20], for a semi-slant lightlike submanifold N of an almost Hermitian manifold $\tilde{N}$, one has

$$T^2U = -\cos^2\theta\, U, \tag{2.16}$$

$$g(TU, TV) = \cos^2\theta\, g(U, V), \tag{2.17}$$

and

$$\tilde{g}(FU, FV) = \sin^2\theta\, g(U, V)$$

for any $U, V \in \Gamma(TN)$. The next relation for semi-slant submanifolds of an almost Hermitian manifold follows easily from Eqs. (2.13) and (2.16) as

$$BFU = -\sin^2\theta\, U, \quad CFU = -FTU$$

for $U \in \Gamma(TN)$.

Lemma 2.4. *Consider a semi-slant lightlike submanifold N of an indefinite Kaehler manifold $\tilde{N}$. Then $FY \in \Gamma(S(TN^\perp))$ for any $Y \in \Gamma(D_\theta)$.*

Proof. Take any vector $Y \in \Gamma(D_\theta)$. Then $FY \in \Gamma(S(TN^\perp))$ if and only if $\tilde{g}(FY, \xi) = 0$ for $\xi \in \Gamma(\mathrm{Rad}(TN))$. Now, consider $\tilde{g}(FY, \xi) = \tilde{g}(\tilde{J}Y - TY, \xi) = \tilde{g}(\tilde{J}Y, \xi) = -g(Y, \tilde{J}\xi) = 0$, which implies that FY has no component in $\mathrm{ltr}(TN)$. Thus, the result holds. □

Note. From Lemma 2.4, we have $FD_\theta \subset S(TN^\perp)$, which implies that there exists $\mu \subset S(TN^\perp)$ such that

$$S(TN) = (\tilde{J}\,\mathrm{Rad}(TN) \oplus \tilde{J}\,\mathrm{ltr}(TN)) \perp \tilde{J}D_0 \perp TD_\theta, \quad S(TN^\perp) = FD_\theta \perp \mu,$$

and

$$T\tilde{N}|_N = \{\mathrm{Rad}(TN) \oplus \mathrm{ltr}(TN)\} \perp S(TN) \perp \{FD_\theta \perp \mu\}.$$

3. Warped product lightlike submanifolds with a slant factor

In this section, to investigate warped product semi-slant lightlike submanifolds of indefinite Kaehler manifolds, firstly, we present a basic lemma contributed by Bishop and O'Neill [2] on warped product manifolds as follows.

Lemma 3.1. *For a warped product manifold $N = N_1 \times_\lambda N_2$, one has*

$$\nabla_X V = \nabla_V X = \left(\frac{X\lambda}{\lambda}\right) V$$

for $X \in \Gamma(TN_1)$ and $V \in \Gamma(TN_2)$.

First, we consider warped product semi-slant lightlike submanifolds of the type $N_T \times_\lambda N_\theta$, $N_\theta \times_\lambda N_T$, $N_\perp \times_\lambda N_\theta$ and $N_\theta \times_\lambda N_\perp$ of an indefinite Kaehler manifold $\tilde{N}$ such that N_T denotes the invariant distribution, and $N_\perp$ and N_θ represent the anti-invariant distribution and proper slant distribution, respectively, of semi-slant lightlike submanifolds.

Theorem 3.2. *For a semi-slant lightlike submanifold N of an indefinite Kaehler manifold $\tilde{N}$, there does not exist any proper semi-slant lightlike warped product submanifold N of the type $N_T \times_\lambda N_\theta$ in $\tilde{N}$.*

Proof. Assume that N is a semi-slant lightlike warped product submanifold of the type $N = N_T \times_\lambda N_\theta$ in $\tilde{N}$. Then, using Lemma 3.1, we have $g(\nabla_{TZ}Y, Z) = (Y \ln \lambda) g(TZ, Z) = 0$ for $Y \in \Gamma(D_T)$ and $Z \in \Gamma(D_\theta)$, and employing Eq. (2.4), we obtain $0 = g(\nabla_{TZ}Y, Z) = \tilde{g}(\tilde{\nabla}_{TZ}Y, Z)$. Then, using Eqs. (2.4), (2.6), (2.7), (2.13), (2.15) and (2.16) and Lemma 3.1, we obtain

$$\begin{aligned} 0 &= \tilde{g}(\tilde{\nabla}_{TZ}Z, Y) \\ &= \tilde{g}(\tilde{\nabla}_{TZ}TZ, \tilde{J}Y) + \tilde{g}(\tilde{\nabla}_{TZ}FZ, \tilde{J}Y) \\ &= -g(TZ, \nabla_{TZ}\tilde{J}Y) - \tilde{g}(A_{FZ}TZ, \tilde{J}Y) + \tilde{g}(D^l(TZ, FZ), \tilde{J}Y) \\ &= -(\tilde{J}Y \ln \lambda) \cos^2\theta \, g(Z, Z) - \tilde{g}(h^s(TZ, \tilde{J}Y), FZ). \end{aligned}$$

Then, we get

$$(\tilde{J}Y \ln \lambda) \cos^2\theta \, g(Z, Z) = -\tilde{g}(h^s(TZ, \tilde{J}Y), FZ). \tag{3.1}$$

Upon considering $\tilde{J}Y$ in the place of Y in Eq. (3.1), we obtain

$$(Y \ln \lambda) \cos^2\theta \, g(Z, Z) = -\tilde{g}(h^s(TZ, Y), FZ). \tag{3.2}$$

Then, replacing Z by TZ in Eq. (3.2), we arrive at

$$\begin{aligned} (Y \ln \lambda) \cos^2\theta \, g(TZ, TZ) &= -\tilde{g}(h^s(T^2Z, Y), FTZ) \\ (Y \ln \lambda) \cos^2\theta \, g(Z, Z) &= \tilde{g}(h^s(Z, Y), FTZ). \end{aligned} \tag{3.3}$$

Furthermore, using Eq. (2.4), we have $\tilde{g}(h^s(TZ, Y), FW) = \tilde{g}(\tilde{\nabla}_Y TZ, FW) = -\tilde{g}(TZ, \tilde{\nabla}_Y FW)$ for $Y \in \Gamma(D_T)$ and $Z, W \in \Gamma(D_\theta)$. Thus, employing Eqs. (2.4), (2.15) and (2.16) and Lemma 3.1, we get

$$\begin{aligned} \tilde{g}(h^s(TZ, Y), FW) &= -\tilde{g}(TZ, \tilde{\nabla}_Y \tilde{J}W) + \tilde{g}(TZ, \tilde{\nabla}_Y TW) \\ &= \tilde{g}(\tilde{J}TZ, \tilde{\nabla}_Y W) + \tilde{g}(TZ, \tilde{\nabla}_Y TW) \\ &= g(T^2Z, \nabla_Y W) + \tilde{g}(FTZ, h^s(Y, W)) + g(TZ, \nabla_Y TW) \\ &= -(Y \ln \lambda) \cos^2\theta \, g(Z, W) + \tilde{g}(FTZ, h^s(Y, W)) \\ &\quad + (Y \ln \lambda) \cos^2\theta \, g(Z, W) \\ &= \tilde{g}(FTZ, h^s(Y, W)). \end{aligned} \tag{3.4}$$

Considering $Z = W$ in Eq. (3.4), we obtain

$$\tilde{g}(h^s(TZ, Y), FZ) = \tilde{g}(FTZ, h^s(Y, Z)) \tag{3.5}$$

for $Y \in \Gamma(D_T)$ and $Z \in \Gamma(D_\theta)$. Thus from Eqs. (3.2), (3.3) and (3.5), we obtain

$$(Y \ln \lambda) \cos^2\theta \, g(Z, Z) = 0.$$

As D_θ is a non-degenerate proper slant distribution we have $Y \ln \lambda = 0$. Hence, λ becomes constant on N_T, which implies that N is a lightlike product, and the result holds. □

Theorem 3.3. *For a semi-slant lightlike submanifold N of an indefinite Kaehler manifold $\tilde{N}$, there does not exist any proper warped product semi-slant lightlike submanifold N of the type $N_\theta \times_\lambda N_T$ in $\tilde{N}$.*

Proof. Assume that N is a warped product semi-slant lightlike submanifold of the type $N_\theta \times_\lambda N_T$ in $\tilde{N}$. Then employing Eq. (2.4) and Lemma 3.1, for $Y \in \Gamma(D_T)$ and $Z \in \Gamma(D_\theta)$, we obtain

$$\begin{aligned}(Z \ln \lambda)\|Y\|^2 &= g(\nabla_Z Y, Y)\\ &= \tilde{g}(\tilde{\nabla}_Z Y, Y) - \tilde{g}(h^l(Y, Z), Y)\\ &= -\tilde{g}(Y, \tilde{\nabla}_Z Y) - \tilde{g}(h^l(Y, Z), Y)\\ &= -(Z \ln \lambda)\|Y\|^2 - 2\tilde{g}(h^l(Y, Z), Y),\end{aligned}$$

which implies that

$$(Z \ln \lambda)\|Y\|^2 = -\tilde{g}(h^l(Y, Z), Y). \tag{3.6}$$

Therefore we have $\tilde{g}(\tilde{\nabla}_Z Y, Y) = 0$, which upon using Eq. (2.4) and Lemma 3.1, gives $\tilde{g}(\tilde{\nabla}_Y Y, Z) = 0$. Furthermore, employing Eqs. (2.4), (2.6), (2.13), (2.15), and Lemma 3.1, we derive

$$\begin{aligned}0 = \tilde{g}(\tilde{\nabla}_Y Y, Z) &= -\tilde{g}(\tilde{J}Y, \tilde{\nabla}_Y TZ) - \tilde{g}(\tilde{J}Y, \tilde{\nabla}_Y FZ)\\ &= -(TZ \ln \lambda) g(Y, \tilde{J}Y) - \tilde{g}(\tilde{J}Y, h^l(Y, TZ)) + \tilde{g}(\tilde{J}Y, A_{FZ}Y) - \tilde{g}(\tilde{J}Y, D^l(Y, FZ))\\ &= -\tilde{g}(\tilde{J}Y, h^l(Y, TZ)) + \tilde{g}(h^s(Y, \tilde{J}Y), FZ),\end{aligned}$$

which gives

$$\tilde{g}(h^s(Y, \tilde{J}Y), FZ) = \tilde{g}(\tilde{J}Y, h^l(Y, TZ)). \tag{3.7}$$

Interchanging Y and $\tilde{J}Y$ in Eq. (3.7), we obtain

$$\tilde{g}(h^s(Y, \tilde{J}Y), FZ) = \tilde{g}(Y, h^l(\tilde{J}Y, TZ)). \tag{3.8}$$

Furthermore, from Eqs. (3.7) and (3.8), we obtain

$$\tilde{g}(\tilde{J}Y, h^l(Y, TZ)) = \tilde{g}(Y, h^l(\tilde{J}Y, TZ)). \tag{3.9}$$

On the other hand, from Eqs. (2.4) and (2.13) and Lemma 3.1, we get

$$\begin{aligned}\tilde{g}(h^l(Y, Z), Y) &= \tilde{g}(\tilde{\nabla}_Z Y, Y) - (Z \ln \lambda)\|Y\|^2\\ &= g(\nabla_Z \tilde{J}Y, \tilde{J}Y) + \tilde{g}(h^l(Z, \tilde{J}Y), \tilde{J}Y) - (Z \ln \lambda)\|Y\|^2\\ &= \tilde{g}(h^l(Z, \tilde{J}Y), \tilde{J}Y).\end{aligned} \tag{3.10}$$

From Eqs. (2.13), (3.9) and (3.10), we obtain

$$\tilde{g}(h^l(Y, Z), Y) = \tilde{g}(h^l(Z, \tilde{J}Y), \tilde{J}Y) = \tilde{g}(h^l(Z, \tilde{J}^2 Y), Y) = -\tilde{g}(h^l(Y, Z), Y),$$

which yields $2\tilde{g}(h^l(Y, Z), Y) = 0$. Then, using Eq. (3.6), we get

$$(Z \ln \lambda)\|Y\|^2 = 0.$$

In particular, for $Y \in \Gamma(D_0 \perp \tilde{J}\,\mathrm{Rad}(TN))$, we have

$$Z \ln \lambda = 0.$$

Therefore, λ becomes constant on N_θ, which implies that N is a lightlike product. Hence, the result holds. □

Theorem 3.4. *Assume that N is a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\tilde{N}$. Then, there does not exist any proper semi-slant lightlike warped product submanifold N of the type $N_\perp \times_\lambda N_\theta$ in $\tilde{N}$.*

Proof. Let N be a semi-slant lightlike warped product submanifold of the type $N = N_\perp \times_\lambda N_\theta$ in $\tilde{N}$. Then, for $Y \in \Gamma(D_\perp)$ and $Z \in \Gamma(D_\theta)$, using Eq. (2.6), we have $g(A_{\tilde{J}Y}TZ, Z) = -\tilde{g}(\tilde{\nabla}_{TZ}\tilde{J}Y, Z) = \tilde{g}(\tilde{\nabla}_{TZ}Y, \tilde{J}Z)$. Furthermore, employing Eqs. (2.4), (2.15), (2.16) and Lemma 3.1, we obtain

$$\begin{aligned} g(A_{\tilde{J}Y}TZ, Z) &= \tilde{g}(\tilde{\nabla}_{TZ}Y, TZ) + \tilde{g}(\tilde{\nabla}_{TZ}Y, FZ) \\ &= g(\nabla_{TZ}Y, TZ) + \tilde{g}(h^s(TZ, Y), FZ) \\ &= (Y \ln \lambda) \cos^2\theta\, g(Z, Z) + \tilde{g}(h^s(TZ, Y), FZ). \end{aligned} \tag{3.11}$$

Upon replacing Z by TZ in Eq. (3.11) and employing Eqs. (2.16) and (2.17), we get

$$\begin{aligned} g(A_{\tilde{J}Y}T^2Z, TZ) &= (Y \ln \lambda) \cos^2\theta\, g(TZ, TZ) + \tilde{g}(h^s(T^2Z, Y), FTZ) \\ g(A_{\tilde{J}Y}Z, TZ) &= -(Y \ln \lambda) \cos^2\theta\, g(Z, Z) + \tilde{g}(h^s(Z, Y), FTZ). \end{aligned} \tag{3.12}$$

Furthermore, using Eq. (2.6) and Lemma 3.1, we obtain

$$\begin{aligned} g(A_{\tilde{J}Y}TZ, Z) &= -\tilde{g}(\tilde{\nabla}_{TZ}\tilde{J}Y, Z) \\ &= \tilde{g}(\tilde{\nabla}_{TZ}Y, \tilde{J}Z) = \tilde{g}(\tilde{\nabla}_Y TZ, \tilde{J}Z) \\ &= \tilde{g}(\tilde{J}TZ, \tilde{\nabla}_Z Y) = \tilde{g}(TZ, A_{\tilde{J}Y}Z). \end{aligned} \tag{3.13}$$

Moreover, employing Eqs. (2.4), (2.6), (2.15), (2.16), (2.17) and Lemma 3.1, we obtain

$$\begin{aligned} g(A_{FZ}Y, TZ) &= -\tilde{g}(\tilde{\nabla}_Y FZ, TZ) \\ &= \tilde{g}(\tilde{\nabla}_Y Z, \tilde{J}TZ) + \tilde{g}(\tilde{\nabla}_Y TZ, TZ) \\ &= g(\nabla_Y Z, T^2Z) + \tilde{g}(h^s(Y, Z), FTZ) + g(\nabla_Y TZ, TZ) \\ &= -(Y \ln \lambda) \cos^2\theta\, g(Z, Z) + \tilde{g}(h^s(Y, Z), FTZ) \\ &\qquad + (Y \ln \lambda) \cos^2\theta\, g(Z, Z) \\ &= \tilde{g}(h^s(Y, Z), FTZ). \end{aligned}$$

Furthermore, using Eq. (2.7), we obtain

$$\tilde{g}(h^s(Y, TZ), FZ) = \tilde{g}(h^s(Y, Z), FTZ). \tag{3.14}$$

Thus, from Eqs. (3.11)–(3.14), we derive

$$(Y \ln \lambda) \cos^2\theta\, \|Z\|^2 = 0.$$

As D_θ is a proper non-degenerate distribution we get

$$Y \ln \lambda = 0.$$

Therefore, it is clear that λ is constant on $N_\perp$, and N becomes a trivial lightlike product. Hence, the result holds. □

Theorem 3.5. *For a semi-slant lightlike submanifold N of an indefinite Kaehler manifold $\tilde{N}$, there does not exist any proper semi-slant lightlike warped product submanifold N of the type $N_\theta \times_\lambda N_\perp$ in $\tilde{N}$.*

Proof. Consider a semi-slant lightlike warped product submanifold N of the type $N = N_\theta \times_\lambda N_\perp$ in $\tilde{N}$. Then using Eq. (2.4) and Lemma 3.1 for $Y \in \Gamma(D_\perp)$ and $Z \in \Gamma(D_\theta)$, we obtain

$$\begin{aligned}(Z \ln \lambda)\|Y\|^2 &= g(\nabla_Z Y, Y) = \tilde{g}(\tilde{\nabla}_Z Y, Y)\\ &= -\tilde{g}(Y, \tilde{\nabla}_Z Y) = -g(Y, \nabla_Z Y)\\ &= -(Z \ln \lambda)\|Y\|^2.\end{aligned}$$

Then, we get

$$(Z \ln \lambda)\|Y\|^2 = 0.$$

Furthermore, as $D_\perp$ is non-degenerate, we get

$$Z \ln \lambda = 0.$$

Hence, λ becomes constant on N_θ, which implies that N is a lightlike product, and the result holds. □

From Theorems 3.2 and 3.5, we conclude that indefinite Kaehler manifolds do not admit semi-slant warped product lightlike submanifolds of the type $N_T \times_\lambda N_\theta$, $N_\theta \times_\lambda N_T$, $N_\theta \times_\lambda N_\perp$ and $N_\perp \times_\lambda N_\theta$. Next, we ask an obvious question:

Question. *Does there exist any warped product lightlike submanifold in an indefinite Kaehler manifold with a slant factor?*

To answer this question affirmatively, we investigate a new type of warped products of semi-slant lightlike submanifold of an indefinite Kaehler manifold $\tilde{N}$, namely, $B \times_\lambda N_\theta$, where $B = N_T \times N_\perp$ is a lightlike product submanifold and N_θ is a proper slant lightlike submanifold of $\tilde{N}$. To start with, we give one non-trivial example for this class of warped products as follows.

Example 3.6. Consider N a 7-dimensional submanifold of $(R_2^{14}, \tilde{g})$ given by

$$\begin{aligned}
&x^1 = \sqrt{2}u^1, && x^2 = \sqrt{2}u^2, && x^3 = u^1 \sin u^3, && x^4 = u^4 \sin u^3,\\
&x^5 = u^1 \cos u^3, && x^6 = u^4 \cos u^3, && x^7 = u^1 \sin u^7, && x^8 = u^4 \sin u^7,\\
&x^9 = u^1 \cos u^7, && x^{10} = u^4 \cos u^7, && x^{11} = \sqrt{2}u^5, && x^{12} = \sqrt{2}u^6,\\
&x^{13} = u^3, && x^{14} = u^7,
\end{aligned}$$

where $u^3, u^7 \in R - \{\frac{n\pi}{2}, n \in \mathbb{Z}\}$. Then TN is spanned by $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7$ such that

$$\begin{aligned}
Z_1 &= \sqrt{2}\partial x_1 + \sin u^3 \partial x_3 + \cos u^3 \partial x_5 + \sin u^7 \partial x_7 + \cos u^7 \partial x_9,\\
Z_2 &= \sqrt{2}\partial x_2,
\end{aligned}$$

$$
\begin{aligned}
Z_3 &= u^1 \cos u^3 \partial x_3 + u^4 \cos u^3 \partial x_4 - u^1 \sin u^3 \partial x_5 - u^4 \sin u^3 \partial x_6 + \partial x_{13},\\
Z_4 &= \sin u^3 \partial x_4 + \cos u^3 \partial x_6 + \sin u^7 \partial x_8 + \cos u^7 \partial x_{10},\\
Z_5 &= \sqrt{2} \partial x_{11},\\
Z_6 &= \sqrt{2} \partial x_{12},\\
Z_7 &= u^1 \cos u^7 \partial x_7 + u^4 \cos u^7 \partial x_8 - u^1 \sin u^7 \partial x_9 - u^4 \sin u^7 \partial x_{10} + \partial x_{14}.
\end{aligned}
$$

Clearly, N is a 1-lightlike submanifold with $\operatorname{Rad}(TN) = \operatorname{Span}\{Z_1\}$, and $\operatorname{ltr}(TN)$ is spanned by

$$N_1 = \frac{1}{4}\big\{ -\sqrt{2}\partial x_1 + \sin u^3 \partial x_3 + \cos u^3 \partial x_5 + \sin u^7 \partial x_7 + \cos u^7 \partial x_9 \big\}.$$

It follows that $\tilde{J} Z_1 = Z_2 + Z_4$ and $\tilde{J} N_1 = -\frac{1}{2}(Z_2 - Z_4)$, which implies that $\tilde{J} \operatorname{Rad}(TN)$ and $\tilde{J} \operatorname{ltr}(TN)$ are distributions on N. As $\tilde{J} Z_5 = Z_6$ we have $D_0 = \operatorname{Span}\{Z_5, Z_6\}$. Furthermore,

$$
\begin{aligned}
S(TN^{\perp}) = \operatorname{Span}\{W = &- u^4 \cos u^3 \partial x_3 + u^1 \cos u^3 \partial x_4 + u^4 \sin u^3 \partial x_5\\
&- u^1 \sin u^3 \partial x_6 - u^4 \cos u^7 \partial x_7 + u^1 \cos u^7 \partial x_8\\
&+ u^4 \sin u^7 \partial x_9 - u^1 \sin u^7 \partial x_{10}\}.
\end{aligned}
$$

Hence, $D_\theta = \operatorname{Span}\{Z_3, Z_7\}$ is a slant distribution with slant angle

$$\cos^{-1} \frac{1}{(u^1)^2 + (u^4)^2 + 1}.$$

Therefore, N is a proper semi-slant lightlike submanifold of R^{14}_2. It is easy to check that D_θ is integrable. Now, if we denote the leaves of $D_T \oplus D_\perp$ and D_θ by B and N_θ, respectively, where $B = N_T \times N_\perp$, then the induced metric tensor of $N = B \times_\lambda N_\theta$ is given by

$$ds^2 = 2(du_2^2 + du_4^2 + du_5^2 + du_6^2) + ((u^1)^2 + (u^4)^2 + 1)(du_3^2 + du_7^2).$$

Thus, N is a proper semi-slant lightlike warped product submanifold of the type $B \times_\lambda N_\theta$ in R^{14}_2, with $\lambda = \sqrt{(u^1)^2 + (u^4)^2 + 1}$.

Next, we prove the following basic results.

Lemma 3.7. *For a semi-slant lightlike warped product submanifold $N = B \times_\lambda N_\theta$ of an indefinite Kaehler manifold $\tilde{N}$ with $B = N_T \times N_\perp$, we have*

(i) $\tilde{g}(h^s(X_1, Y_1), FX_3) = \tilde{g}(h^l(X_1, X_3), \tilde{J} Y_1) + \tilde{g}(Y_1, h^l(X_1, TX_3))$,

(ii) $\tilde{g}(h^s(X_1, X_3), FY_3) = -(\tilde{J} X_1 \ln \lambda) g(X_3, Y_3) - (X_1 \ln \lambda) g(X_3, TY_3)$

for any $X_1, Y_1 \in \Gamma(D_T)$ and $X_3, Y_3 \in \Gamma(D_\theta)$.

Proof. For $X_1, Y_1 \in \Gamma(D_T)$ and $X_3 \in \Gamma(D_\theta)$, using Eqs. (2.4), (2.15) and Lemma 3.1, we derive

$$
\begin{aligned}
\tilde{g}(h^s(X_1, Y_1), FX_3) &= \tilde{g}(\tilde{\nabla}_{X_1} Y_1, FX_3)\\
&= \tilde{g}(\tilde{J} Y_1, \tilde{\nabla}_{X_1} X_3) - \tilde{g}(\tilde{\nabla}_{X_1} Y_1, TX_3)\\
&= g(\tilde{J} Y_1, \nabla_{X_1} X_3) + \tilde{g}(\tilde{J} Y_1, h^l(X_1, X_3)) + \tilde{g}(Y_1, \tilde{\nabla}_{X_1} TX_3)
\end{aligned}
$$

$$\begin{aligned}
&= (X_1 \ln\lambda) g(\tilde{J}Y_1, X_3) + \tilde{g}(h^l(X_1, X_3), \tilde{J}Y_1) + g(Y_1, \nabla_{X_1} TX_3) \\
&\quad + \tilde{g}(Y_1, h^l(X_1, TX_3)) \\
&= \tilde{g}(h^l(X_1, X_3), \tilde{J}Y_1) + \tilde{g}(Y_1, h^l(X_1, TX_3)).
\end{aligned}$$

Similarly, using Eqs. (2.4) and (2.15) and Lemma 3.1, we get

$$\begin{aligned}
\tilde{g}(h^s(X_1, X_3), FY_3) &= \tilde{g}(\tilde{\nabla}_{X_3} X_1, FY_3) \\
&= -\tilde{g}(\tilde{\nabla}_{X_3} \tilde{J}X_1, Y_3) - \tilde{g}(\tilde{\nabla}_{X_3} X_1, TY_3) \\
&= -g(\nabla_{X_3} \tilde{J}X_1, Y_3) - g(\nabla_{X_3} X_1, TY_3) \\
&= -(\tilde{J}X_1 \ln\lambda) g(X_3, Y_3) - (X_1 \ln\lambda) g(X_3, TY_3)
\end{aligned}$$

for $X_1 \in \Gamma(D_T)$ and $X_3, Y_3 \in \Gamma(D_\theta)$. □

Lemma 3.8. *Consider a semi-slant lightlike warped product submanifold $N = B \times_\lambda N_\theta$ of an indefinite Kaehler manifold $\tilde{N}$ with $B = N_T \times N_\perp$. Then we have*

(i) $\tilde{g}(h^s(X_1, X_2), FX_3) = 0$,
(ii) $\tilde{g}(h^s(X_2, Y_2), FX_3) = 0$

for any $X_1 \in \Gamma(D_T)$, $X_2, Y_2 \in \Gamma(D_\perp)$ and $X_3 \in \Gamma(D_\theta)$.

Proof. For any $X_1 \in \Gamma(D_T)$, $X_2 \in \Gamma(D_\perp)$ and $X_3 \in \Gamma(D_\theta)$, employing Eqs. (2.4) and (2.15) and Lemma 3.1, we obtain

$$\begin{aligned}
\tilde{g}(h^s(X_1, X_2), FX_3) &= \tilde{g}(\tilde{\nabla}_{X_1} X_2, FX_3) \\
&= \tilde{g}(\tilde{J}X_2, \tilde{\nabla}_{X_1} X_3) - \tilde{g}(\tilde{\nabla}_{X_1} X_2, TX_3) \\
&= (X_1 \ln\lambda) \tilde{g}(\tilde{J}X_2, X_3) + (X_1 \ln\lambda) g(X_2, TX_3) \\
&= 0.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\tilde{g}(h^s(X_2, Y_2), FX_3) &= \tilde{g}(\tilde{\nabla}_{X_2} Y_2, FX_3) \\
&= \tilde{g}(\tilde{J}Y_2, \tilde{\nabla}_{X_2} X_3) + \tilde{g}(Y_2, \tilde{\nabla}_{X_2} TX_3) \\
&= (X_2 \ln\lambda) \tilde{g}(\tilde{J}Y_2, X_3) + (X_2 \ln\lambda) g(Y_2, TX_3) \\
&= 0.
\end{aligned}$$ □

Definition 3.9. A warped product $(N_T \times N_\perp) \times_\lambda N_\theta$ is called $D_1 \oplus D_2$*-mixed totally geodesic* if the second fundamental form h satisfies

$$h(D_1, D_2) = 0,$$

where D_1 and D_2 are distributions from $\{D_T, D_\perp, D_\theta\}$.

Proposition 3.10. *Consider a semi-slant lightlike warped product submanifold $N = B \times_\lambda N_\theta$ of an indefinite Kaehler manifold $\tilde{N}$ with $B = N_T \times N_\perp$. If N is $D_T \oplus D_\theta$-mixed totally geodesic, then the warping function λ depends only on $N_\perp$.*

Proof. According to Lemma 3.7, we obtain

$$\tilde{g}(h^s(X_1, X_3), FY_3) = -(\tilde{J}X_1 \ln\lambda)g(X_3, Y_3) - (X_1 \ln\lambda)g(X_3, TY_3), \tag{3.15}$$

where $X_1 \in \Gamma(D_T)$ and $X_3, Y_3 \in \Gamma(D_\theta)$. On interchanging the role of X_1 and $\tilde{J}X_1$, the above equation takes the form

$$\tilde{g}(h^s(\tilde{J}X_1, X_3), FY_3) = (X_1 \ln\lambda)g(X_3, Y_3) - (\tilde{J}X_1 \ln\lambda)g(X_3, TY_3),$$

which further gives

$$\tilde{g}(h^s(\tilde{J}X_1, X_3), FY_3) = (X_1 \ln\lambda)g(X_3, Y_3) + (\tilde{J}X_1 \ln\lambda)g(TX_3, Y_3).$$

Now, on interchanging the role of X_3 and TX_3, the above equation becomes

$$\tilde{g}(h^s(\tilde{J}X_1, TX_3), FY_3) = (X_1 \ln\lambda)g(TX_3, Y_3) - (\tilde{J}X_1 \ln\lambda)\cos^2\theta\, g(X_3, Y_3). \tag{3.16}$$

Then, subtracting Eq. (3.16) from Eq. (3.15), we obtain

$$\tilde{g}(h^s(X_1, X_3), FY_3) - \tilde{g}(h^s(\tilde{J}X_1, TX_3), FY_3) = -(\tilde{J}X_1 \ln\lambda)\sin^2\theta\, g(X_3, Y_3).$$

Employing the hypothesis along with the non-degeneracy of the slant distribution D_θ in the above equation, we obtain $\tilde{J}X_1 \ln\lambda = 0$, which yields that λ is constant on N_T, proving the result. □

Lemma 3.11. *For a semi-slant lightlike warped product submanifold $N = B \times_\lambda N_\theta$ of an indefinite Kaehler manifold $\tilde{N}$ with $B = N_T \times N_\perp$, we have*

$$\tilde{g}(h^s(X_1, X_3), FX_3) = -(\tilde{J}X_1 \ln\lambda)\|X_3\|^2, \tag{3.17}$$

$$\tilde{g}(h^s(\tilde{J}X_1, X_3), FX_3) = (X_1 \ln\lambda)\|X_3\|^2 \tag{3.18}$$

for any $X_1 \in \Gamma(D_T)$ and $X_3 \in \Gamma(D_\theta)$.

Proof. For any $X_1 \in \Gamma(D_T)$ and $X_3 \in \Gamma(D_\theta)$, employing Eqs. (2.4), (2.15) and Lemma 3.1, we obtain

$$\begin{aligned}\tilde{g}(h^s(X_1, X_3), FX_3) &= \tilde{g}(\tilde{\nabla}_{X_3}X_1, FX_3)\\ &= -\tilde{g}(\tilde{\nabla}_{X_3}\tilde{J}X_1, X_3) - g(\nabla_{X_3}X_1, TX_3)\\ &= -(\tilde{J}X_1 \ln\lambda)\|X_3\|^2 - (X_1 \ln\lambda)g(X_3, TX_3)\\ &= -(\tilde{J}X_1 \ln\lambda)\|X_3\|^2.\end{aligned}$$

On the other hand, using Eqs. (2.4), (2.13) and (2.15) and Lemma 3.1, we derive

$$\begin{aligned}\tilde{g}(h^s(\tilde{J}X_1, X_3), FX_3) &= \tilde{g}(\tilde{\nabla}_{X_3}\tilde{J}X_1, FX_3)\\ &= \tilde{g}(\tilde{\nabla}_{X_3}\tilde{J}X_1, \tilde{J}X_3) - \tilde{g}(\tilde{\nabla}_{X_3}\tilde{J}X_1, TX_3)\\ &= (X_1 \ln\lambda)\|X_3\|^2 - (\tilde{J}X_1 \ln\lambda)g(X_3, TX_3)\\ &= (X_1 \ln\lambda)\|X_3\|^2.\end{aligned}$$

Hence, the result holds. □

Corollary 3.12. *For a semi-slant lightlike warped product submanifold* $N = B \times_\lambda N_\theta$ *of an indefinite Kaehler manifold* $\tilde{N}$ *with* $B = N_T \times N_\perp$*, one has*

(i) $\tilde{g}(h^s(\tilde{J}X_1, \nabla_{X_1}X_3), \tilde{J}X_3) = (X_1 \ln\lambda)^2 \|X_3\|^2$,

(ii) $\tilde{g}(h^s(X_1, \nabla_{\tilde{J}X_1}X_3), \tilde{J}X_3) = -(\tilde{J}X_1 \ln\lambda)^2 \|X_3\|^2$

for $X_1 \in \Gamma(D_T)$ *and* $X_3 \in \Gamma(D_\theta)$.

Next, we establish a geometric estimate for semi-slant lightlike warped product submanifolds of an indefinite Kaehler manifold $\tilde{N}$ in terms of the second fundamental form.

Theorem 3.13. *Consider a semi-slant lightlike warped product submanifold* $N = B \times_\lambda N_\theta$ *of an indefinite Kaehler manifold* $\tilde{N}$ *with* $B = N_T \times N_\perp$*. Then we have*

$$\|h^s(\tilde{J}X_1, X_3)\|^2 + \|h^s(X_1, X_3)\|^2 = (X_1 \ln\lambda)^2\|X_3\|^2 + (\tilde{J}X_1 \ln\lambda)^2\|X_3\|^2 + 2\tilde{g}(\tilde{J}h^s(X_1, X_3), h^s(\tilde{J}X_1, X_3))$$

for $X_1 \in \Gamma(D_T)$ *and* $X_3 \in \Gamma(D_\theta)$.

Proof. For $X_1 \in \Gamma(D_T)$ and $X_3 \in \Gamma(D_\theta)$, using Eqs. (2.4), (2.15), (3.18) and Lemma 3.1, we obtain

$$\begin{aligned}\|h^s(\tilde{J}X_1, X_3)\|^2 &= \tilde{g}(h^s(\tilde{J}X_1, X_3), h^s(\tilde{J}X_1, X_3)) \\ &= \tilde{g}(\tilde{J}\tilde{\nabla}_{X_3}X_1, h^s(\tilde{J}X_1, X_3)) \\ &= (X_1 \ln\lambda)\tilde{g}(\tilde{J}X_3, h^s(\tilde{J}X_1, X_3)) + \tilde{g}(\tilde{J}h^s(X_1, X_3), h^s(\tilde{J}X_1, X_3)) \\ &= (X_1 \ln\lambda)\tilde{g}(FX_3, h^s(\tilde{J}X_1, X_3)) + \tilde{g}(\tilde{J}h^s(X_1, X_3), h^s(\tilde{J}X_1, X_3)) \\ &= (X_1 \ln\lambda)^2\|X_3\|^2 + \tilde{g}(\tilde{J}h^s(X_1, X_3), h^s(\tilde{J}X_1, X_3)). \end{aligned} \tag{3.19}$$

Similarly, from Eqs. (2.4), (2.13) and (3.17) and Lemma 3.1, we get

$$\begin{aligned}\|h^s(X_1, X_3)\|^2 &= \tilde{g}(h^s(X_1, X_3), h^s(X_1, X_3)) \\ &= \tilde{g}(\tilde{\nabla}_{X_3}X_1, h^s(X_1, X_3)) = \tilde{g}(\tilde{\nabla}_{X_3}\tilde{J}X_1, \tilde{J}h^s(X_1, X_3)) \\ &= (\tilde{J}X_1 \ln\lambda)\tilde{g}(X_3, \tilde{J}h^s(X_1, X_3)) + \tilde{g}(\tilde{J}h^s(X_1, X_3), h^s(\tilde{J}X_1, X_3)) \\ &= (\tilde{J}X_1 \ln\lambda)^2\|X_3\|^2 + \tilde{g}(\tilde{J}h^s(X_1, X_3), h^s(\tilde{J}X_1, X_3)). \end{aligned} \tag{3.20}$$

Hence, the assertion follows from Eqs. (3.19) and (3.20). □

Consider $(\tilde{N}, \tilde{g})$ an $(m+n)$-dimensional semi-Riemannian manifold and λ a smooth function on $\tilde{N}$. Then, the Hessian of λ is defined as

$$H^\lambda(U, V) = UV\lambda - (\nabla_U V)\lambda \tag{3.21}$$

for $U, V \in \Gamma(T\tilde{N})$.

Next, we present an important geometric result on semi-slant lightlike warped products in an indefinite complex space form $\tilde{N}(c)$ involving the squared norm of the second fundamental form and the Hessian of the warping function λ.

Theorem 3.14. *Consider a semi-slant lightlike warped product submanifold $N = B \times_\lambda N_\theta$ of indefinite complex space form $\tilde{N}(c)$ with $B = N_T \times N_\perp$. Then we have*

$$\begin{aligned}\|h^s(\tilde{J}X_1, X_3)\|^2 + \|h^s(X_1, X_3)\|^2 = &\left\{H^{\ln\lambda}(X_1, X_1) + H^{\ln\lambda}(\tilde{J}X_1, \tilde{J}X_1)\right\}\|X_3\|^2 \\ &+ \frac{c}{2}\|X_1\|^2\|X_3\|^2 + (X_1 \ln\lambda)^2\|X_3\|^2 \\ &+ (\tilde{J}X_1 \ln\lambda)^2\|X_3\|^2 + \tilde{g}(A_{h^s(\tilde{J}X_1, X_3)}X_1, \tilde{J}X_3) \\ &- \tilde{g}(A_{h^s(X_1, X_3)}\tilde{J}X_1, \tilde{J}X_3)\end{aligned}$$

for $X_1 \in \Gamma(D_T)$ and $X_3 \in \Gamma(D_\theta)$.

Proof. For $X_1 \in \Gamma(D_T)$ and $X_3 \in \Gamma(D_\theta)$, taking into account Eq. (2.14), we derive

$$\tilde{R}(X_1, \tilde{J}X_1, X_3, \tilde{J}X_3) = -\frac{c}{2}\|X_1\|^2\|X_3\|^2. \tag{3.22}$$

Taking the inner product of Eq. (2.10) with $\tilde{J}X_3$ and using Eqs. (2.11) and (2.12), for $X_1 \in \Gamma(D_T)$ and $X_3 \in \Gamma(D_\theta)$, we obtain

$$\begin{aligned}\tilde{R}(X_1, \tilde{J}X_1, X_3, \tilde{J}X_3) = &\tilde{g}(\nabla^s_{X_1} h^s(\tilde{J}X_1, X_3), FX_3) - \tilde{g}(h^s(\nabla_{X_1}\tilde{J}X_1, X_3), FX_3) \\ &- \tilde{g}(h^s(\tilde{J}X_1, \nabla_{X_1}X_3), FX_3) - \tilde{g}(\nabla^s_{\tilde{J}X_1} h^s(X_1, X_3), FX_3) \\ &+ \tilde{g}(h^s(\nabla_{\tilde{J}X_1}X_1, X_3), FX_3) + \tilde{g}(h^s(X_1, \nabla_{\tilde{J}X_1}X_3), FX_3) \\ &+ \tilde{g}(D^s(X_1, h^l(\tilde{J}X_1, X_3)), FX_3) \\ &- \tilde{g}(D^s(\tilde{J}X_1, h^l(X_1, X_3)), FX_3).\end{aligned} \tag{3.23}$$

Next, considering Eq. (2.6), we get

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_{X_1} h^s(\tilde{J}X_1, X_3), FX_3) = &-\tilde{g}(A_{h^s(\tilde{J}X_1, X_3)}X_1, FX_3) + \tilde{g}(\nabla^s_{X_1} h^s(\tilde{J}X_1, X_3), FX_3) \\ &+ \tilde{g}(D^l(h^s(\tilde{J}X_1, X_3), X_1), FX_3),\end{aligned}$$

which further yields

$$\tilde{g}(\tilde{\nabla}_{X_1} h^s(\tilde{J}X_1, X_3), FX_3) = \tilde{g}(\nabla^s_{X_1} h^s(\tilde{J}X_1, X_3), FX_3). \tag{3.24}$$

As $\tilde{\nabla}$ is a metric connection, for $X_1 \in \Gamma(D_T)$ and $X_3 \in \Gamma(D_\theta)$, we obtain

$$\tilde{g}(\tilde{\nabla}_{X_1} h^s(\tilde{J}X_1, X_3), FX_3) = X_1\tilde{g}(h^s(\tilde{J}X_1, X_3), FX_3) - \tilde{g}(h^s(\tilde{J}X_1, X_3), \tilde{\nabla}_{X_1}FX_3). \tag{3.25}$$

Furthermore, from Eqs. (3.24) and (3.25), we derive

$$\tilde{g}(\nabla^s_{X_1} h^s(\tilde{J}X_1, X_3), FX_3) = X_1\tilde{g}(h^s(\tilde{J}X_1, X_3), FX_3) - \tilde{g}(h^s(\tilde{J}X_1, X_3), \tilde{\nabla}_{X_1}FX_3). \tag{3.26}$$

Then employing Eqs. (2.4), (3.18) and (3.19) and Lemma 3.1 in Eq. (3.26), we obtain

$$\begin{aligned}\tilde{g}(\nabla^s_{X_1} h^s(\tilde{J}X_1, X_3), FX_3) &= X_1\{(X_1 \ln\lambda)\|X_3\|^2\} - \tilde{g}(h^s(\tilde{J}X_1, X_3), \tilde{\nabla}_{X_1}FX_3) \\ &= X_1\{(X_1 \ln\lambda)\|X_3\|^2\} - \tilde{g}(h^s(\tilde{J}X_1, X_3), \tilde{\nabla}_{X_1}\tilde{J}X_3) \\ &\quad + \tilde{g}(h^s(\tilde{J}X_1, X_3), \tilde{\nabla}_{X_1}TX_3) \\ &= X_1\{(X_1 \ln\lambda)\|X_3\|^2\} - \tilde{g}(h^s(\tilde{J}X_1, X_3), \tilde{J}\tilde{\nabla}_{X_1}X_3)\end{aligned}$$

$$
\begin{aligned}
&+ \tilde{g}(h^s(\tilde{J}X_1, X_3), h^s(X_1, TX_3))\\
&= X_1(X_1 \ln\lambda)\|X_3\|^2 - \tilde{g}(h^s(\tilde{J}X_1, X_3), \tilde{J}\nabla_{X_1}X_3)\\
&\quad + 2(X_1\ln\lambda)^2\|X_3\|^2 - \tilde{g}(h^s(\tilde{J}X_1, X_3), \tilde{J}h^s(X_1, X_3))\\
&\quad + \tilde{g}(h^s(\tilde{J}X_1, X_3), h^s(X_1, TX_3))\\
&= X_1(X_1 \ln\lambda)\|X_3\|^2 - (X_1\ln\lambda)\tilde{g}(h^s(\tilde{J}X_1, X_3), \tilde{J}X_3)\\
&\quad + 2(X_1\ln\lambda)^2\|X_3\|^2 - \|h^s(\tilde{J}X_1, X_3)\|^2\\
&\quad + (X_1\ln\lambda)^2\|X_3\|^2 + \tilde{g}(h^s(\tilde{J}X_1, X_3), h^s(X_1, TX_3))\\
&= X_1(X_1 \ln\lambda)\|X_3\|^2 + 2(X_1\ln\lambda)^2\|X_3\|^2\\
&\quad - \|h^s(\tilde{J}X_1, X_3)\|^2 + \tilde{g}(h^s(\tilde{J}X_1, X_3), h^s(X_1, TX_3)).
\end{aligned}
\tag{3.27}
$$

Similarly, one has

$$
\begin{aligned}
\tilde{g}(\nabla^s_{\tilde{J}X_1} h^s(X_1, X_3), FX_3) &= -\tilde{J}X_1(\tilde{J}X_1\ln\lambda)\|X_3\|^2 - 2(\tilde{J}X_1\ln\lambda)^2\|X_3\|^2\\
&\quad + \|h^s(X_1, X_3)\|^2 + \tilde{g}(h^s(X_1, X_3), h^s(\tilde{J}X_1, TX_3)).
\end{aligned}
\tag{3.28}
$$

Furthermore, from Eqs. (2.7) and (3.18), we have

$$
\begin{aligned}
g(A_{FX_3}X_3, \tilde{J}X_1) &= \tilde{g}(h^s(\tilde{J}X_1, X_3), FX_3) + \tilde{g}(D^l(X_3, FX_3), \tilde{J}X_1)\\
&= (X_1\ln\lambda)\|X_3\|^2 + \tilde{g}(D^l(X_3, FX_3), \tilde{J}X_1).
\end{aligned}
$$

As $N_T \times N_\perp$ is totally geodesic, for $X_1 \in N_T$, we have $\nabla_{X_1}X_1 \in N_T \times N_\perp$; then replacing X_1 by $\nabla_{X_1}X_1$, the above equation reduces to

$$
g(A_{FX_3}X_3, \tilde{J}\nabla_{X_1}X_1) = (\nabla_{X_1}X_1\ln\lambda)\|X_3\|^2 + \tilde{g}(D^l(X_3, FX_3), \tilde{J}\nabla_{X_1}X_1). \tag{3.29}
$$

Then, for $X_1 \in \Gamma(D_T)$ and $X_3 \in \Gamma(D_\theta)$, using Eqs. (2.4) and (2.7), we derive

$$
\begin{aligned}
g(A_{FX_3}X_3, \tilde{J}\nabla_{X_1}X_1) &= g(A_{FX_3}X_3, \nabla_{X_1}\tilde{J}X_1)\\
&= \tilde{g}(h^s(X_3, \nabla_{X_1}\tilde{J}X_1), FX_3) + \tilde{g}(D^l(X_3, FX_3), \nabla_{X_1}\tilde{J}X_1),
\end{aligned}
$$

which further gives

$$
\tilde{g}(h^s(X_3, \nabla_{X_1}\tilde{J}X_1), FX_3) = g(A_{FX_3}X_3, \tilde{J}\nabla_{X_1}X_1) - \tilde{g}(D^l(X_3, FX_3), \nabla_{X_1}\tilde{J}X_1). \tag{3.30}
$$

Then using Eqs. (2.4), (3.29) and (3.30), we obtain

$$
\tilde{g}(h^s(X_3, \nabla_{X_1}\tilde{J}X_1), FX_3) = (\nabla_{X_1}X_1\ln\lambda)\|X_3\|^2. \tag{3.31}
$$

Next, replacing X_1 by $\tilde{J}X_1$ in Eq. (3.31), we obtain

$$
\tilde{g}(h^s(X_3, \nabla_{\tilde{J}X_1}X_1), FX_3) = -(\nabla_{\tilde{J}X_1}\tilde{J}X_1\ln\lambda)\|X_3\|^2. \tag{3.32}
$$

On the other hand, from Eqs. (2.4), (2.6) and (2.8) and Lemma 3.1, we derive

$$
\begin{aligned}
\tilde{g}(D^s(X_1, h^l(\tilde{J}X_1, X_3)), FX_3) &= \tilde{g}(A_{FX_3}X_1, h^l(\tilde{J}X_1, X_3))\\
&= -\tilde{g}(\tilde{\nabla}_{X_1}FX_3, h^l(\tilde{J}X_1, X_3))
\end{aligned}
$$

$$\begin{aligned}
&= -\tilde{g}(\tilde{J}\tilde{\nabla}_{X_1}X_3, h^l(\tilde{J}X_1, X_3)) \\
&\quad + \tilde{g}(\nabla_{X_1}TX_3, h^l(\tilde{J}X_1, X_3)) \\
&= -\tilde{g}(\tilde{J}\nabla_{X_1}X_3, h^l(\tilde{J}X_1, X_3)) \\
&\quad + (X_1 \ln\lambda)\tilde{g}(TX_3, h^l(\tilde{J}X_1, X_3)) \\
&= 0. \qquad (3.33)
\end{aligned}$$

Similarly, it follows that

$$\tilde{g}(D^s(\tilde{J}X_1, h^l(X_1, X_3)), FX_3) = 0. \qquad (3.34)$$

Now employing Eqs. (3.27), (3.28), (3.31), (3.32), (3.33) and (3.34) and Corollary 3.12 in Eq. (3.23), we derive

$$\begin{aligned}
\tilde{R}(X_1, \tilde{J}X_1, X_3, \tilde{J}X_3) &= \{X_1(X_1 \ln\lambda) - \nabla_{X_1}X_1 \ln\lambda\}\|X_3\|^2 \\
&\quad + \{\tilde{J}X_1(\tilde{J}X_1 \ln\lambda) - \nabla_{\tilde{J}X_1}\tilde{J}X_1 \ln\lambda\}\|X_3\|^2 \\
&\quad + \{(X_1 \ln\lambda)^2 + (\tilde{J}X_1 \ln\lambda)^2\}\|X_3\|^2 - \|h^s(\tilde{J}X_1, X_3)\|^2 \\
&\quad - \|h^s(X_1, X_3)\|^2 + \tilde{g}(h^s(\tilde{J}X_1, X_3), h^s(X_1, TX_3)) \\
&\quad - \tilde{g}(h^s(X_1, X_3), h^s(\tilde{J}X_1, TX_3)). \qquad (3.35)
\end{aligned}$$

Next, using Eq. (3.21) in Eq. (3.35), we obtain

$$\begin{aligned}
\tilde{R}(X_1, \tilde{J}X_1, X_3, \tilde{J}X_3) &= \{H^{\ln\lambda}(X_1, X_1) + H^{\ln\lambda}(\tilde{J}X_1, \tilde{J}X_1)\}\|X_3\|^2 \\
&\quad + (X_1 \ln\lambda)^2\|X_3\|^2 + (\tilde{J}X_1 \ln\lambda)^2\|X_3\|^2 - \|h^s(\tilde{J}X_1, X_3)\|^2 \\
&\quad - \|h^s(X_1, X_3)\|^2 + \tilde{g}(h^s(\tilde{J}X_1, X_3), h^s(X_1, TX_3)) \\
&\quad - \tilde{g}(h^s(X_1, X_3), h^s(\tilde{J}X_1, TX_3)). \qquad (3.36)
\end{aligned}$$

From Eqs. (3.22) and (3.36), we derive

$$\begin{aligned}
-\frac{c}{2}\|X_1\|^2\|X_3\|^2 &= \{H^{\ln\lambda}(X_1, X_1) + H^{\ln\lambda}(\tilde{J}X_1, \tilde{J}X_1)\}\|X_3\|^2 + (X_1 \ln\lambda)^2\|X_3\|^2 \\
&\quad + (\tilde{J}X_1 \ln\lambda)^2\|X_3\|^2 - \|h^s(\tilde{J}X_1, X_3)\|^2 - \|h^s(X_1, X_3)\|^2 \\
&\quad + \tilde{g}(h^s(\tilde{J}X_1, X_3), h^s(X_1, TX_3)) - \tilde{g}(h^s(X_1, X_3), h^s(\tilde{J}X_1, TX_3)),
\end{aligned}$$

which can be written as

$$\begin{aligned}
\|h^s(\tilde{J}X_1, X_3)\|^2 + \|h^s(X_1, X_3)\|^2 &= \{H^{\ln\lambda}(X_1, X_1) + H^{\ln\lambda}(\tilde{J}X_1, \tilde{J}X_1)\}\|X_3\|^2 \\
&\quad + \frac{c}{2}\|X_1\|^2\|X_3\|^2 + (X_1 \ln\lambda)^2\|X_3\|^2 \\
&\quad + (\tilde{J}X_1 \ln\lambda)^2\|X_3\|^2 \\
&\quad + \tilde{g}(h^s(\tilde{J}X_1, X_3), h^s(X_1, TX_3)) \\
&\quad - \tilde{g}(h^s(X_1, X_3), h^s(\tilde{J}X_1, TX_3)).
\end{aligned}$$

Hence, the proof is completed. □

Consider a semi-slant lightlike warped product submanifold $N = B \times_\lambda N_\theta$ in an indefinite Kaehler manifold $\tilde{N}$ with $\dim N = n$, $\dim \tilde{N} = m$ and $B = N_T \times N_\perp$. We choose a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ of TN such that

$$\begin{aligned}
D_\perp &= \mathrm{Span}\{e_1, e_2, \ldots, e_q\},\\
D_T &= \mathrm{Span}\{e_{q+1} = \tilde{e}_1, \ldots, e_{q+p} = \tilde{e}_p, \ldots, e_{q+p+1} = \tilde{e}_{p+1} = \tilde{J}\tilde{e}_1, \ldots,\\
&\qquad e_{q+2p} = \tilde{e}_{2p} = \tilde{J}\tilde{e}_p, e_{q+2p+1} = \tilde{e}_{2p+1}, \ldots, e_{2p+2q} = \tilde{e}_{2p+q},\\
&\qquad e_{2p+2q+1} = \tilde{e}_{2p+q+1} = \tilde{J}\tilde{e}_{2p+1}, \ldots, e_{2p+3q} = \tilde{e}_{2p+2q} = \tilde{J}\tilde{e}_{2p+q}\},\\
D_\theta &= \mathrm{Span}\{e_{2p+3q+1} = e_1^*, \ldots, e_{2p+3q+t} = e_t^*, e_{2p+3q+t+1} = \sec\theta\, Te_1^*, \ldots,\\
&\qquad e_{2p+3q+2t} = \sec\theta\, Te_t^*\},
\end{aligned}$$

with $q = \dim N_\perp$, $p + q = \frac{1}{2}\dim N_T$ and $t = \frac{1}{2}\dim N_\theta$. Also, we choose a local orthonormal frame $\{E_1, \ldots, E_{m-n}\}$ of the normal bundle $TN^\perp$ such that

$$\begin{aligned}
\tilde{J}D_\perp &= \mathrm{Span}\{E_1 = \tilde{J}e_1, \ldots, E_q = \tilde{J}e_q\},\\
F(D_\theta) &= \mathrm{Span}\{E_{q+1} = \csc\theta\, Fe_1^*, \ldots, E_{q+t} = \csc\theta\, Fe_t^*, E_{q+t+1} = \csc\theta\sec\theta\, FTe_1^*,\\
&\qquad \ldots, E_{q+2t} = \csc\theta\sec\theta\, FTe_t^*\},\\
\mu &= \mathrm{Span}\{E_{q+2t+1}, \ldots, E_{m-n}\},
\end{aligned}$$

where μ is a screen transversal subbundle of $TN^\perp$ such that $\tilde{J}\mu = \mu$.

The following theorem gives a sharp inequality involving the squared norm of the second fundamental form for semi-slant lightlike warped product submanifolds in an indefinite Kaehler manifold $\tilde{N}$.

Theorem 3.15. *Let $N = B \times_\lambda N_\theta$ be a semi-slant lightlike warped product submanifold in an indefinite Kaehler manifold $\tilde{N}$ with $B = N_T \times N_\perp$. If N is D_θ-totally geodesic, then the squared norm of the second fundamental form of N satisfies*

$$\|h\|^2 \geq 4t(\csc^2\theta + \cot^2\theta)\|\nabla^T(\ln\lambda)\|^2,$$

where $\nabla^T(\ln\lambda)$ and $\nabla^\perp(\ln\lambda)$ denote the gradient components of $\ln\lambda$ along N_T and $N_\perp$, respectively, and $t = \frac{1}{2}\dim N_\theta$.

Proof. From the definition of h, we have

$$\|h\|^2 = \sum_{i,j=1}^{n}\{\|h^s(e_i, e_j)\|^2 + \|h^l(e_i, e_j)\|^2\},$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of TN. From Eq. (2.2), the above relation takes the form

$$\|h\|^2 = \sum_{i,j=1}^{n}\|h^s(e_i, e_j)\|^2 = \sum_{r=n+1}^{2m-n}\sum_{i,j=1}^{n}\tilde{g}(h^s(e_i, e_j), E_r)^2$$

$$= \sum_{r=n+1}^{q} \sum_{i,j=1}^{n} (\tilde{g}(h^s(e_i, e_j), \tilde{J}e_r))^2 + \sum_{r=q+1}^{q+2t} \sum_{i,j=1}^{n} (\tilde{g}(h^s(e_i, e_j), E_r))^2$$
$$+ \sum_{r=q+2t+1}^{m-n} \sum_{i,j=1}^{n} (\tilde{g}(h^s(e_i, e_j), E_r))^2,$$

where $\{E_{n+1}, \ldots, E_{m-n}\}$ is a local orthonormal frame of the normal bundle. Leaving the last μ-components term in the equation above, we obtain

$$\|h\|^2 \geq \sum_{r=q+1}^{q+2t} \sum_{i,j=1}^{n} (\tilde{g}(h^s(e_i, e_j), E_r))^2$$
$$= \csc^2\theta \sum_{s=1}^{t} \sum_{i,j=1}^{q} \big[(\tilde{g}(h^s(e_i, e_j), Fe_s^*))^2 + \sec^2\theta(\tilde{g}(h^s(e_i, e_j), FTe_s^*))^2\big]$$
$$+ \csc^2\theta \sum_{s=1}^{t} \sum_{i,j=1}^{2p+2s} \big[(\tilde{g}(h^s(\tilde{e}_i, \tilde{e}_j), Fe_s^*))^2 + \sec^2\theta(\tilde{g}(h^s(\tilde{e}_i, \tilde{e}_j), FTe_s^*))^2\big]$$
$$+ \csc^2\theta \sum_{s=1}^{t} \sum_{i,j=1}^{2t} \big[(\tilde{g}(h^s(e_i^*, e_j^*), Fe_s^*))^2 + \sec^2\theta(\tilde{g}(h^s(e_i^*, e_j^*), FTe_s^*))^2\big]$$
$$+ \csc^2\theta \sum_{s=1}^{t} \sum_{i=1}^{q} \sum_{j=1}^{2p+2s} \big[(\tilde{g}(h^s(e_i, \tilde{e}_j), Fe_s^*))^2 + \sec^2\theta(\tilde{g}(h^s(e_i, \tilde{e}_j), FTe_s^*))^2\big]$$
$$+ \csc^2\theta \sum_{s=1}^{t} \sum_{i=1}^{q} \sum_{j=1}^{2t} \big[(\tilde{g}(h^s(e_i, e_j^*), Fe_s^*))^2 + \sec^2\theta(\tilde{g}(h^s(e_i, e_j^*), FTe_s^*))^2\big]$$
$$+ \csc^2\theta \sum_{s=1}^{t} \sum_{i=1}^{2p+2s} \sum_{j=1}^{2t} \big[(\tilde{g}(h^s(\tilde{e}_i, e_j^*), Fe_s^*))^2 + \sec^2\theta(\tilde{g}(h^s(\tilde{e}_i, e_j^*), FTe_s^*))^2\big],$$
$$\geq \csc^2\theta \sum_{s=1}^{t} \sum_{i=1}^{2p+2s} \sum_{j=1}^{2t} \big[(\tilde{g}(h^s(\tilde{e}_i, e_j^*), Fe_s^*))^2 + \sec^2\theta(\tilde{g}(h^s(\tilde{e}_i, e_j^*), FTe_s^*))^2\big]$$
$$= 2\csc^2\theta \sum_{j,s=1}^{t} \sum_{i=1}^{p+q} \big[(-\tilde{J}\tilde{e}_i \ln\lambda)^2 (g(e_j^*, e_s^*))^2\big] + (\tilde{e}_i \ln\lambda)^2 (g(e_j^*, Te_s^*))^2$$
$$+ 2\csc^2\theta \sum_{j,s=1}^{q} \sum_{i=1}^{p+q} \big[\sec^2\theta(\tilde{J}\tilde{e}_i \ln\lambda)^2 (g(e_j^*, Te_s^*))^2 + (\tilde{e}_i \ln\lambda)^2 (g(e_j^*, T^2 e_s^*))^2\big]$$
$$= 4t\csc^2\theta \sum_{i=1}^{p+q} \big[(-\tilde{J}\tilde{e}_i \ln\lambda)^2 + (\tilde{e}_i \ln\lambda)^2\big]$$
$$+ 4t\cot^2\theta \sum_{j,s=1}^{t} \sum_{i=1}^{p+q} [\sec^4\theta(\tilde{J}\tilde{e}_i \ln\lambda)^2 (g(e_j^*, Te_s^*))^2 + (\tilde{e}_i \ln\lambda)^2 (g(e_j^*, e_s^*))^2]$$

$$= 4t(\csc^2\theta + \cot^2\theta)\sum_{i=1}^{2p+2q}(\tilde{e}_i \ln\lambda)^2$$
$$= 4t(\csc^2\theta + \cot^2\theta)\|\nabla^T(\ln\lambda)\|^2.$$

Hence, the result holds. □

4. Concluding remark

The literature suggests that a vast majority of studies available on warped products are bounded to manifolds with positive definite metric. The geometry of warped products has potential applications in the study of black holes and cosmological models (see [10, 15]). In reference to the study under consideration, for the case of cosmological models, there do exist some points where the warping function becomes zero. These points are called *singular points*. Furthermore, at singular points, the metric of the product manifold becomes degenerate. Therefore, for effective analysis of such models, it is suggested that the theory of manifolds with indefinite metrics be used. Moreover, the present work focuses on the geometry of warped product lightlike submanifolds with a slant factor, which in itself provides new insights into the geometry of lightlike submanifolds and warped products with broader application perspectives. For the proposed class of warped products, a non-trivial example is also presented. On a similar note, the same idea of warped product lightlike submanifolds with a slant factor can be generalized to contact manifolds with indefinite metrics.

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