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LIMIT BEHAVIORS FOR A β-MIXING SEQUENCE IN THE ST. PETERSBURG GAME

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ABSTRACT. We consider a sequence of non-negative β-mixing random variables $\{X, X_n : n \geq 1\}$ from the classical St. Petersburg game. The accumulated gains $S_n = X_1 + X_2 + \cdots + X_n$ in the St. Petersburg game are studied, and the large deviations and the weak law of large numbers of S_n are obtained.

1. Introduction

The classical St. Petersburg game is defined as follows: A player tosses a fair coin repeatedly. If the coin comes up heads the first time on the kth trial, he will wins 2^k dollars. Let the random variable X denote the gain of the player at a game; then we have

$$\mathbb{P}\left(X = 2^k\right) = 2^{-k} \quad \text{and} \quad \mathbb{P}\left(X > c\right) = 2^{-[\log_2 c]} \tag{1.1}$$

for any $k = 1, 2, \dots$ and $c \geq 1$, where $\log_2$ is the logarithm to the base 2 and $[x]$ is defined as the largest integer not exceeding x. It is easy to see that X has infinite expectation.

Let $\{X, X_n : n \geq 1\}$ be independent identically distributed (i.i.d.) random variables as above, and let $S_n = X_1 + \cdots + X_n$ denote the total winnings in n games. Feller [5] proved that

$$\frac{S_n}{n \log_2 n} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty.$$

Chow and Robbins [3] showed that

$$\frac{S_n}{n \log_2 n} \overset{a.s.}{\nrightarrow} 1 \quad \text{as } n \to \infty,$$

and the set of limit points of $S_n/n \log_2 n$ is the interval $[1, \infty)$. Csörgő and Simons [4] proved that, with probability one, S_n is asymptotic to $n \log_2 n$ if the m

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largest gains are ignored, i.e., for every $m \geq 1$,

$$\frac{S_n(m)}{n \log_2 n} \xrightarrow{a.s.} 1 \quad \text{as } n \to \infty,$$

where $S_n(m)$ denotes the same sum but with the m largest summands excised, which is defined as

$$S_n(m) := X_{n,1} + X_{n,2} + \cdots + X_{n,n-m}.$$

Here $X_{n,1} \leq X_{n,2} \leq \cdots \leq X_{n,n}$ denote the order statistics for $X_1, X_2, \ldots, X_n$, so that $S_n(0) = S_n$. Vardi [11] studied that this asymptotic equality is very rarely interrupted by a large gain that puts the player ahead for a relatively short period of time. Stoica [10] obtained the following logarithm tail asymptotic results for the accumulated gains of geometric size in the St. Petersburg game: for $\varepsilon > 0$ and $b > 1$, we have

$$\lim_{n\to\infty} \frac{\log_2 \mathbb{P}(S_n > \varepsilon b^n)}{n} = -\log_2 b$$

and

$$\lim_{n\to\infty} \frac{\log_2 \mathbb{P}(M_n > \varepsilon b^n)}{n} = -\log_2 b,$$

where $M_n = \max\{X_1, X_2, \ldots, X_n\}$ denotes the maximal gain in n St. Petersburg games. Some related results about heavy tailed sequences from generalized St. Petersburg games are studied (see [6, 7, 8]).

Based on the above works, we want to consider a β-mixing sequence with the distribution (1.1) in the St. Petersburg game. Let us recall the definition of the β-mixing coefficient, and for the definitions of other mixing coefficients as well as for the relations between them, we refer the reader to Bradley [2]. Let X and Z be two random variables. We denote the distribution of (X, Z) by $\mu_{(X,Z)}$ and the distributions of X and Z by μ_X and μ_Z. The β-mixing coefficient of X and Z is defined as

$$\beta(X, Z) = \frac{1}{2} \|\mu_{(X,Z)} - \mu_X \otimes \mu_Z\|,$$

where $\|\mu - \nu\|$ denotes the (total) variation norm of the signed measure $\mu - \nu$. Now for a sequence of random variables $\{Y_n : n \geq 1\}$, define

$$\beta(n) = \sup_{k\in\mathbb{N}} \beta((Y_1, Y_2, \ldots, Y_k), (Y_{k+n}, Y_{k+n+1}, \ldots)).$$

The sequence is called *absolutely regular* if $\beta(n) \to 0$ for $n \to \infty$.

In the rest of the paper, let $\{X, X_n : n \geq 1\}$ be a sequence of non-negative β-mixing random variables, and we shall study the large deviations and the law of large numbers for the accumulated gains S_n with geometric size in the St. Petersburg game.

2. Main results

2.1. **The main results.** Let $\{X, X_n : n \geq 1\}$ be a sequence of nonnegative stationary β-mixing random variables with the distribution (1.1). It is not difficult to check that

$$\sup\{d > 0 : \mathbb{E}(X^d) < \infty\} = 1. \tag{2.1}$$

In particular, $\mathbb{E}(X^d)$ is finite for any $d < 1$.

Theorem 2.1. *Assume that the mixing coefficient $\beta(n)$ satisfies*

$$\frac{\log_2 \beta(n)}{n \log_2 \log_2 n} \to -\infty \quad \text{as } n \to \infty. \tag{2.2}$$

Then, for any $\varepsilon > 0$ and $b > 1$, we have

$$\lim_{n\to\infty} \frac{\log_2 \mathbb{P}\left(S_n > \varepsilon b^n\right)}{n} = -\log_2 b. \tag{2.3}$$

Remark 2.2. From the proof of Theorem 2.1, it is not difficult to check that the condition (2.2) can be weakened by the following condition. Assume that there exists a positive sequence $g(n)$ satisfying $g(n) \to \infty$ as $n \to \infty$ and let $k(n)$ be a subsequence from $\{1, 2, \ldots, n\}$ such that $k(n) \to \infty$ as $n \to \infty$. Suppose that there exists a positive constant C such that, for all n large enough,

$$\frac{g(k(n))}{g(n)} \geq C.$$

If the mixing coefficient $\beta(n)$ satisfies

$$\frac{\log_2 \beta(n)}{n g(n)} \to -\infty \quad \text{as } n \to \infty,$$

then (2.3) holds. Hence $g(n)$ can be chosen as $\log_2 \log_2 \log_2 n$.

Corollary 2.3. *Under the conditions in Theorem 2.1, we have*

$$\lim_{n\to\infty} \frac{\log_2 \mathbb{P}\left(M_n > \varepsilon b^n\right)}{n} = -\log_2 b,$$

where $M_n = \max\{X_1, X_2, \ldots, X_n\}$ denotes the maximal gain in n St. Petersburg games.

Theorem 2.4. *Assume that the mixing coefficient $\beta(n)$ satisfies*

$$\frac{\log_2 \beta(n)}{n \log_2 n} \to -\infty \quad \text{as } n \to \infty. \tag{2.4}$$

Then we have

$$\frac{S_n - n\mathbb{E}\left(X 1_{\{X < n \log_2 n\}}\right)}{n \log_2 n} \xrightarrow{\mathbb{P}} 0.$$

Remark 2.5. From (2.13), we know that

$$\mathbb{E}\left(X 1_{\{X < n \log_2 n\}}\right) = [\log_2(n \log_2 n)],$$

which implies that

$$\frac{S_n}{n \log_2 n} \xrightarrow{\mathbb{P}} 1.$$

Remark 2.6. Based on the proof of Theorem 2.4 (see the inequality (2.15)), the condition (2.4) cannot be weakened by the condition (2.2).

2.2. **Lemmas.** Before giving the proofs of the main results, we need to mention some lemmas. The following decoupling lemma is obtained by Berbee [1] and Schwarz [9]. It will be used to decouple X_i and X_j when $|i-j|$ is big enough.

Lemma 2.7 (Berbee [1, Lemma 2.1])**.** *Let $\{X, X_n : n \geq 1\}$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and, for every $1 \leq k \leq n$, define*

$$\beta_k := \beta((X_1, X_2, \dots, X_k), (X_{k+1}, X_{k+2}, \dots, X_n)).$$

Then there exists a sequence of independent random variables $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$ on the same probability space such that $\tilde{X}_i$ and X_i have the same distribution and

$$\|\mu_{(X_1,X_2,\dots,X_n)} - \mu_{(\tilde{X}_1,\tilde{X}_2,\dots,\tilde{X}_n)}\| \leq \beta_1 + \beta_2 + \cdots + \beta_n. \tag{2.5}$$

Lemma 2.8 (Hu and Nyrhinen [6, Lemma 3.2])**.** *Assume that $\{X, X_n : n \geq 1\}$ is a sequence of i.i.d. non-negative random variables with $\mathbb{E}(X^\alpha) < \infty$ for $\alpha \in (0, \infty)$. Denote $\theta = \min\{\alpha, 1\}$ and $\mu = \mathbb{E}(X^\theta)$. Then, for any $\nu > 0$, $t > 0$ and $n \in \mathbb{N}$, we have*

$$\mathbb{P}(S_n > t^{1/\theta}) \leq n\mathbb{P}\left(X > \left(\frac{t}{\nu}\right)^{1/\theta}\right) + e^\nu \left(\frac{\mu n}{t}\right)^{\nu^{1/\theta}}.$$

Lemma 2.9. *Let X obey the distribution* (1.1)*. Then, for every $c \geq 1$, we have*

$$\frac{1}{c} \leq \mathbb{P}(X > c) < \frac{2}{c}.$$

Proof. By using (1.1), the lemma can be obtained easily. □

2.3. **Proof of the main results.** In this subsection, we give the proofs of the main results.

Proof of Theorem 2.1. Decompose the set $\{1, 2, \dots, n\}$ into $l(n)$ blocks of length $k(n)$ and a block of length less than $k(n)$, where $k(n)$, $l(n)$ are integers such that

$$\frac{k(n)}{n/\log_2\log_2 n} \to 1 \quad \text{and} \quad \frac{l(n)}{\log_2\log_2 n} \to 1 \qquad \text{as } n \to \infty.$$

Hence for any $0 < \delta < 1/2$, if n is large enough, it follows that

$$(1-\delta)\log_2\log_2 n \leq l(n) \leq (1+\delta)\log_2\log_2 n. \tag{2.6}$$

From Lemma 2.7, we know that there exists a sequence of independent random variables $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$ such that for every $1 \leq i \leq n$, $\tilde{X}_i$ and X_i have the same distribution.

Step 1. We shall prove the lower bound of the limit (2.3),

$$\liminf_{n\to\infty} \frac{\log_2 \mathbb{P}\left(S_n > \varepsilon b^n\right)}{n} \geq -\log_2 b. \tag{2.7}$$

From the inequality (2.5), we have

$$\begin{aligned}
\mathbb{P}(S_n > \varepsilon b^n) &\geq \mathbb{P}\left(\sum_{j=1}^{l(n)} X_{(j-1)k(n)+1} > \varepsilon b^n\right) \\
&\geq \mathbb{P}\left(\max_{1\leq j\leq l(n)} X_{(j-1)k(n)+1} > \varepsilon b^n\right) \\
&\geq \mathbb{P}\left(\max_{1\leq j\leq n} X_{(j-1)k(n)+1} > \varepsilon b^n, \bigcap_{j=1}^{l(n)}\{X_j = \tilde{X}_j\}\right) \\
&\geq \mathbb{P}\left(\max_{1\leq j\leq n} \tilde{X}_{(j-1)k(n)+1} > \varepsilon b^n\right) - \mathbb{P}\left(\bigcup_{j=1}^{l(n)}\{X_j \neq \tilde{X}_j\}\right) \\
&\geq \mathbb{P}\left(\max_{1\leq j\leq n} \tilde{X}_{(j-1)k(n)+1} > \varepsilon b^n\right) - \big(\beta_1 + \beta_2 + \cdots + \beta_{l(n)}\big) \\
&\geq \mathbb{P}\left(\max_{1\leq j\leq l(n)} \tilde{X}_{(j-1)k(n)+1} > \varepsilon b^n\right) - l(n)\beta(k(n)) \\
&= 1 - (1-\mathbb{P}(X > \varepsilon b^n))^{l(n)} - l(n)\beta(k(n)) \\
&\geq 1 - (1-\mathbb{P}(X > \varepsilon b^n))^{(1-\delta)\log_2\log_2 n} - (1+\delta)(\log_2\log_2 n)\beta(k(n)).
\end{aligned} \tag{2.8}$$

Furthermore, by using Lemma 2.9, for $\varepsilon > 0$ and $b > 1$, we have

$$\mathbb{P}(X > \varepsilon b^n) > \frac{1}{\varepsilon b^n},$$

which, together with (2.8), implies that

$$\mathbb{P}(S_n > \varepsilon b^n) \geq 1 - \big(1 - (\varepsilon b^n)^{-1}\big)^{(1-\delta)\log_2\log_2 n} - (1+\delta)(\log_2\log_2 n)\beta(k(n)). \tag{2.9}$$

For $b > 1$, we have

$$(\varepsilon b^n)^{-1}\log_2\log_2 n \to 0 \quad \text{as } n \to \infty.$$

Therefore, by using the elementary inequalities

$$1 - y \leq e^{-y} \quad \text{for } y \geq 0$$

and

$$e^y \leq 1 + y + y^2 \quad \text{for } |y| \leq 1,$$

we get

$$\begin{aligned}
&\big(1 - (\varepsilon b^n)^{-1}\big)^{(1-\delta)\log_2\log_2 n} \\
&\qquad \leq e^{-(\varepsilon b^n)^{-1}(1-\delta)\log_2\log_2 n} \\
&\qquad \leq 1 - (\varepsilon b^n)^{-1}(1-\delta)\log_2\log_2 n + \big((\varepsilon b^n)^{-1}(1-\delta)\log_2\log_2 n\big)^2 \\
&\qquad \leq 1 - (1-2\delta)\,(\varepsilon b^n)^{-1}\log_2\log_2 n
\end{aligned} \tag{2.10}$$

for all n large enough. From (2.9) and (2.10), for all n large enough, we obtain

$$\begin{aligned}\mathbb{P}(S_n > \varepsilon b^n) &\geq (1-2\delta)\left(\varepsilon b^n\right)^{-1}\log_2\log_2 n - (1+\delta)(\log_2\log_2 n)\beta(k(n)) \\ &= (1-2\delta)\left(\varepsilon b^n\right)^{-1}\log_2\log_2 n\left(1 - \frac{(1+\delta)}{1-2\delta}\beta(k(n))(\varepsilon b^n)\right).\end{aligned}$$

From the condition $(\log_2 \beta(n))/(n\log_2\log_2 n) \to -\infty$, it follows that for any $M > 1$ and for all n large enough, we have

$$\beta(k(n)) \leq 2^{-2Mk(n)\log_2\log_2 k(n)} \leq 2^{-Mn}, \tag{2.11}$$

which implies

$$\beta(k(n))(\varepsilon b^n) \to 0 \quad \text{as } n \to \infty.$$

Hence we have

$$\liminf_{n\to\infty} \frac{\log_2 \mathbb{P}(S_n > \varepsilon b^n)}{n} \geq -\log_2 b.$$

Step 2. We shall prove the upper bound of the limit (2.3),

$$\limsup_{n\to\infty} \frac{\log_2 \mathbb{P}\left(S_n > \varepsilon b^n\right)}{n} \leq -\log_2 b. \tag{2.12}$$

From Lemma 2.7 and the inequalities in (2.6), for all n large enough, we have

$$\begin{aligned}\mathbb{P}(S_n > \varepsilon b^n) &= \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \frac{\varepsilon b^n}{n}\right) \leq \mathbb{P}\left(\frac{1}{k(n)l(n)}\sum_{i=1}^{n} X_i > \frac{\varepsilon b^n}{n}\right) \\ &\leq \mathbb{P}\left(\frac{1}{k(n)}\sum_{j=1}^{k(n)}\frac{1}{l(n)}\sum_{i=1}^{l(n)+1} X_{(i-1)k(n)+j} > \frac{\varepsilon b^n}{n}\right) \\ &= \mathbb{P}\left(\sum_{j=1}^{k(n)}\frac{1}{l(n)}\sum_{i=1}^{l(n)+1} X_{(i-1)k(n)+j} > k(n)\frac{\varepsilon b^n}{n}\right) \\ &\leq k(n)\mathbb{P}\left(\frac{1}{l(n)}\sum_{i=1}^{l(n)+1} X_{(i-1)k(n)+1} > \frac{\varepsilon b^n}{n}\right) \\ &\leq k(n)\mathbb{P}\left(\sum_{i=1}^{l(n)+1} X_{(i-1)k(n)+1} > (1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n\right) \\ &\leq k(n)\mathbb{P}\left(\sum_{i=1}^{l(n)+1} \tilde{X}_{(i-1)k(n)+1} > (1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n\right) \\ &\quad + k(n)[l(n)+1]\beta(k(n)) \\ &\leq k(n)\mathbb{P}\left(\tilde{S}_{l(n)+1} > (1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n\right) + 2n\beta(k(n)),\end{aligned}$$

where

$$\tilde{S}_{l(n)+1} := \sum_{i=1}^{l(n)+1} \tilde{X}_{(i-1)k(n)+1}.$$

From (2.1), let us take $\theta < 1$,

$$t = \left((1-\delta)\frac{\log_2 \log_2 n}{n}\varepsilon b^n\right)^{\theta}$$

and $\mu = \mathbb{E}(X^\theta)$ in Lemma 2.8. Then, for any $\nu > 0$, we obtain

$$\begin{aligned}
k(n)\mathbb{P}&\left(\tilde{S}_{l(n)+1} > (1-\delta)\frac{\log_2 \log_2 n}{n}\varepsilon b^n\right) \\
&\leq k(n)(l(n)+1)\mathbb{P}\left(X > \frac{(1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n}{\nu^{\frac{1}{\theta}}}\right) \\
&\qquad + k(n)e^{\nu}\mu^{\nu^{\frac{1}{\theta}}}\left(\frac{l(n)+1}{\left((1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n\right)^{\theta}}\right)^{\nu^{\frac{1}{\theta}}}.
\end{aligned}$$

From Lemma 2.9, for all n large enough, we have

$$\mathbb{P}\left(X > \frac{(1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n}{\nu^{\frac{1}{\theta}}}\right) < \frac{2\nu^{\frac{1}{\theta}}}{(1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n},$$

which implies that

$$\begin{aligned}
\mathbb{P}(S_n > \varepsilon b^n) \leq{}& \frac{4n\nu^{\frac{1}{\theta}}}{(1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n} + k(n)e^{\nu}\mu^{\nu^{\frac{1}{\theta}}}\left(\frac{l(n)+1}{\left((1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n\right)^{\theta}}\right)^{\nu^{\frac{1}{\theta}}} \\
&+ 2n\beta(k(n)).
\end{aligned}$$

Now it is not difficult to prove that

$$\lim_{n\to\infty}\frac{1}{n}\log_2\frac{4n\nu^{\frac{1}{\theta}}}{(1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n} = -\log_2 b$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log_2\left(k(n)e^{\nu}\mu^{\nu^{\frac{1}{\theta}}}\left(\frac{l(n)+1}{\left((1-\delta)\frac{\log_2\log_2 n}{n}\varepsilon b^n\right)^{\theta}}\right)^{\nu^{\frac{1}{\theta}}}\right) = -\theta\nu^{\frac{1}{\theta}}\log_2 b.$$

Moreover, from (2.11) and the arbitrariness of M, we have

$$\lim_{M\to\infty}\lim_{n\to\infty}\frac{1}{n}\log_2\left(2n\beta(k(n))\right) \leq \lim_{M\to\infty}(-M) = -\infty.$$

Therefore, by taking $\nu > 0$ such that $\theta\nu^{\frac{1}{\theta}} > 1$, we have

$$\limsup_{n\to\infty} \frac{\log_2 \mathbb{P}(S_n > \varepsilon b^n)}{n} \le -\log_2 b.$$

From (2.7) and (2.12), the desired result can be obtained. □

Proof of Corollary 2.3. From the proof of the lower bound and the inequality (2.8), we have

$$\liminf_{n\to\infty} \frac{\log_2 \mathbb{P}(M_n > \varepsilon b^n)}{n} \ge -\log_2 b.$$

From (2.12) and $M_n \le S_n$, we get

$$\limsup_{n\to\infty} \frac{\log_2 \mathbb{P}(M_n > \varepsilon b^n)}{n} \le -\log_2 b. \qquad \square$$

Proof of Theorem 2.4. For all n and $1 \le k \le n$, define

$$Y_{n,k} = X_k 1_{\{X_k \le n \log_2 n\}}, \quad S_n' = \sum_{k=1}^{n} Y_{n,k},$$

and, for any $r \ge 1$, it is easy to see that

$$\mathbb{E}Y_{n,k}^r = \sum_{k=1}^{[\log_2(n\log_2 n)]} 2^{rk}\frac{1}{2^k} = \begin{cases} [\log_2(n\log_2 n)], & r = 1, \\ \frac{2^{r-1}(1-2^{(r-1)[\log_2(n\log_2 n)]})}{1-2^{r-1}}, & r > 1. \end{cases} \tag{2.13}$$

For any $\varepsilon > 0$, we have

$$\begin{aligned} &\mathbb{P}\left(|S_n - n\mathbb{E}\left(X1_{\{X<n\log_2 n\}}\right)| > \varepsilon n \log_2 n\right) \\ &= \mathbb{P}\left(\left|S_n - n\mathbb{E}\left(X1_{\{X<n\log_2 n\}}\right)\right| > \varepsilon n\log_2 n, \bigcap_{k=1}^{n}\{X_k < n\log_2 n\}\right) \\ &\quad + \mathbb{P}\left(\left|S_n - n\mathbb{E}\left(X1_{\{X<n\log_2 n\}}\right)\right| > \varepsilon n\log_2 n, \bigcup_{k=1}^{n}\{X_k \ge n\log_2 n\}\right) \\ &\le \mathbb{P}\left(|S_n' - n\mathbb{E}\left(X1_{\{X<n\log_2 n\}}\right)| > \varepsilon n\log_2 n\right) + n\mathbb{P}\left(X \ge n\log_2 n\right). \end{aligned}$$

We split up S_n' in blocks as follows:

$$S_n' - n\mathbb{E}\left(X1_{\{X<n\log_2 n\}}\right) = S_n^{(o)} + S_n^{(e)} + V_n,$$

where

$$S_n^{(o)} = \sum_{j=1}^{[l(n)/2]} \sum_{i=(2j-2)k(n)+1}^{(2j-1)k(n)} (Y_{n,i} - \mathbb{E}Y_{n,i}),$$

$$S_n^{(e)} = \sum_{j=1}^{[l(n)/2]} \sum_{i=(2j-1)k(n)+1}^{2jk(n)} (Y_{n,i} - \mathbb{E}Y_{n,i}),$$

$$V_n = \sum_{i=2[l(n)/2]k(n)+1}^{n} (Y_{n,i} - \mathbb{E}Y_{n,i})$$

and

$$k(n) = \left[\frac{\log_2 n}{\log_2 \log_2 n}\right], \quad l(n) = [n/k(n)].$$

Firstly, we consider the term $S_n^{(o)}$. Let

$$\sigma_j = \sum_{i=(2j-2)k(n)+1}^{(2j-1)k(n)} (Y_{n,i} - \mathbb{E}Y_{n,i})$$

and observe that

$$S_n^{(o)} = \sum_{j=1}^{[l(n)/2]} \sigma_j.$$

By using Lemma 2.7, there are independent random variables $\tilde{\sigma}_j$, distributed as σ_j, $1 \le j \le [l(n)/2]$. Hence for any $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{P}\left(|S_n^{(o)}| > \varepsilon n \log_2 n\right) &= \mathbb{P}\left(\left|\sum_{j=1}^{[l(n)/2]} \sigma_j\right| > \varepsilon n \log_2 n\right) \\ &\le \mathbb{P}\left(\left|\sum_{j=1}^{[l(n)/2]} \tilde{\sigma}_j\right| > \varepsilon n \log_2 n\right) + l(n)\beta(k(n)).\end{aligned}$$

From (2.13), we have

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{j=1}^{[l(n)/2]} \tilde{\sigma}_j\right| > \varepsilon n \log_2 n\right) &\le \frac{[l(n)/2]}{\varepsilon^2 n^2 \log_2^2 n}\mathbb{E}|\tilde{\sigma}_1|^2 \\ &\le \frac{[l(n)/2]}{\varepsilon^2 n^2 \log_2^2 n} k^2(n)\mathbb{E}Y_{n,1}^2 \\ &\le \frac{C_1[l(n)/2]}{\varepsilon^2 n^2 \log_2^2 n} k^2(n) n \log_2 n \le C_2 \frac{k(n)}{\log_2 n} \to 0,\end{aligned} \tag{2.14}$$

where C_1 and C_2 are two positive constants. Furthermore, by using the condition $(\log_2 \beta(n))/(n \log_2 n) \to -\infty$, it follows that for any $M > 1$ and for all n large enough, we have

$$\beta(k(n)) \le 2^{-2Mk(n)\log_2 k(n)} \le 2^{-M \log_2 n} = \frac{1}{n^M}. \tag{2.15}$$

So we get

$$l(n)\beta(k(n)) = O\left(\frac{n \log_2 \log_2 n}{\log_2 n} \frac{1}{n^M}\right) \to 0 \quad \text{as } n \to \infty,$$

which, together with (2.14), implies that

$$\frac{S_n^{(o)}}{n \log_2 n} \xrightarrow{\mathbb{P}} 0.$$

With the same proof used for $S_n^{(o)}$, we have

$$\frac{S_n^{(e)}}{n\log_2 n} \xrightarrow{\mathbb{P}} 0.$$

Now we consider the term V_n. By Markov's inequality and (2.13), we have

$$\begin{aligned}\mathbb{P}\left(|V_n| > \varepsilon n \log_2 n\right) &\le \mathbb{P}\left(\sum_{i=1}^{2k(n)} |Y_{n,i} - \mathbb{E}Y_{n,i}| > \varepsilon n \log_2 n\right)\\ &\le \frac{4k(n)}{\varepsilon n \log_2 n}\mathbb{E}Y_{n,1} \to 0,\end{aligned}$$

which yields

$$\frac{V_n}{n\log_2 n} \xrightarrow{\mathbb{P}} 0.$$

At last, from Lemma 2.9, we get

$$n\mathbb{P}\left(X \ge n\log_2 n\right) \le \frac{2n}{n\log_2 n} \to 0.$$

Based on the above discussion, the desired result can be obtained. $\square$

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