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PROPERTIES OF THE CONVOLUTION OPERATION IN THE COMPLEXITY SPACE AND ITS DUAL

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Abstract. We give the basic properties of discrete convolution in the space of complexity functions and its dual space. Two inequalities are identified, and defined in the general context of an arbitrary binary operation in any weighted quasi-metric space. In that setting, some quasi-metric and convergence consequences of those inequalities are proven. Using convolution, we show a method for building improver functionals in the complexity space. We also consider convolution in three topologies within the dual space, obtaining two topological monoids.

1. Introduction

The main purpose of this article is to study the quasi-metric and topological properties of convolution in the complexity space $\mathcal{C}$ and in its dual $\mathcal{C}^*$. We are also interested in identifying those properties that can be extended to the general case of a binary operation in a weighted quasi-metric space. These include properties about continuity, quasi-uniform continuity, and several types of sequence convergence.

Discrete convolution has applications in a wide variety of fields. In mathematical analysis it is well known that convolution is associated with the product of power series [19]. In probability theory, convolution gives the distribution for the sum of two independent random variables with non-negative integer values. Convolution helps in solving many combinatorial problems [10]. It is also used in digital signal processing.

We use the term *quasi-metric* to denote the asymmetric concept of distance, as presented in [3], [5] and [13]. This non-symmetric interpretation of the term *quasi-metric* differs from the meaning given to it by authors in other fields, for example, in quasi-metric measure spaces [1], [11]. Matthews [14] introduced the concept of a weighted quasi-metric, as part of a topological, non-Hausdorff approach to the semantics of data-flow networks. The (weightable) quasi-metric space of

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complexity functions $\mathcal{C}$ was defined by Schellekens [20] as a topological foundation for the complexity analysis of algorithms. Later, Romaguera and Schellekens [18] introduced the dual complexity space $\mathcal{C}^*$ in order to obtain quasi-metric properties of the complexity space. They noted that, although $\mathcal{C}$ and $\mathcal{C}^*$ are isometric, the dual space is more appealing from a mathematical perspective since it admits the structure of a semilinear quasi-normed space [17].

The article is organized as follows. Section 2 covers the notation and concepts relevant to the results that will be proven later. Section 3 begins with the closure under convolution for the complexity spaces. Next we show some inequalities relating convolution to the weighted quasi-metric and the order structures in $\mathcal{C}$ and $\mathcal{C}^*$. In Section 4 we generalize the background context for two of those inequalities, and draw some of the consequences they have regarding the quasi-metric properties of any binary operation that satisfies them in a given weighted quasi-metric space. Section 5 gives the properties of a convolution-based functional in $\mathcal{C}^*$, as well as a way to use it in order to obtain a wide family of improver functionals in $\mathcal{C}$. We also examine convolution in relation to some equivalence classes of $d^s_{\mathcal{C}^*}$-Cauchy sequences. In Section 6 we examine two topologies for $\mathcal{C}^*$, other than the quasi-metric one, in relation to the question of whether or not convolution, along with each one of those topologies, yields a topological monoid.

2. Preliminaries

Throughout this article, the symbols $\mathbb{N}$, ω, $\mathbb{R}$, $\mathbb{R}^+$ and $(0,\infty]$ denote the positive integers, the non-negative integers, the real numbers, the non-negative real numbers, and the extended positive real numbers, respectively. If $a, b \in \mathbb{R}$, the notations $a \vee b$ and $a \wedge b$ indicate $\max\{a,b\}$ and $\min\{a,b\}$, respectively. We use the notations $\mathcal{O}$ ("big-oh") and Ω ("big-omega") with the standard meaning they have in computer science.

A *quasi-uniformity* $\mathcal{U}$ on a set X is a filter on $X \times X$ such that (i) every $U \in \mathcal{U}$ is a reflexive relation on X, and (ii) $\forall U \in \mathcal{U}$, $\exists V \in \mathcal{U} : V \circ V \subseteq U$. When $\mathcal{U}$ has the additional property (iii) $\forall U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$, then it is called a *uniformity* on X. The pair $(X,\mathcal{U})$ is a *quasi-uniform space* (or a *uniform space*, when $\mathcal{U}$ is a uniformity). For a presentation of the theory of quasi-uniform spaces, we recommend the book [6] by Fletcher and Lindgren. The elements of $\mathcal{U}$ are called *entourages.* The family $\mathcal{U}^{-1} = \left\{U^{-1} \mid U \in \mathcal{U}\right\}$ is also a quasi-uniformity called the *conjugate quasi-uniformity.* $\mathcal{U}$ is a uniformity if and only if $\mathcal{U}^{-1} = \mathcal{U}$. A function $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between quasi-uniform spaces is *quasi-uniformly continuous* when $\forall V \in \mathcal{V}$, $\exists U \in \mathcal{U} : (x,y) \in U \Rightarrow (f(x), f(y)) \in V$ for all $x, y \in X$.

A subset $\mathcal{B} \subseteq \mathcal{U}$ of a quasi-uniformity is a *base* of $\mathcal{U}$ when $\forall U \in \mathcal{U}$, $\exists B \in \mathcal{B}$: $B \subseteq U$. Then we say that $\mathcal{B}$ *generates* $\mathcal{U}$. Given a family $\mathcal{B} = \{U_i\}_{i\in I}$ of subsets of $X \times X$, there exists some quasi-uniformity $\mathcal{U}$ on X generated by $\mathcal{B}$ if and only if (i) $\mathcal{B}$ is a filterbase in $X \times X$, (ii) U_i is a reflexive relation on X for every $i \in I$, and (iii) $\forall i \in I$, $\exists j \in I : U_j \circ U_j \subseteq U_i$. If $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on X, with $\mathcal{B}_1$ being a base for $\mathcal{U}$, and $\mathcal{B}_2$ a base for $\mathcal{V}$, we say that $\mathcal{B}_1$ is *finer* than $\mathcal{B}_2$ (and $\mathcal{B}_2$ is *coarser* than $\mathcal{B}_1$) if each member of $\mathcal{B}_2$ contains a member of $\mathcal{B}_1$.

So, $\mathcal{U}$ is finer than $\mathcal{V}$ provided $\mathcal{V} \subseteq \mathcal{U}$. If $\mathcal{U}$ is a quasi-uniformity, then the family $\mathcal{B} = \{U \cap U^{-1} \mid U \in \mathcal{U}\}$ is a base for a uniformity that we denote by $\mathcal{U}^s$.

The topology $\tau(\mathcal{U})$ *induced* by a quasi-uniformity $\mathcal{U}$ on X is the only topology where, for each $x \in X$, the neighborhood filter of x is $\mathcal{N}_x = \{U(x) \mid U \in \mathcal{U}\}$. We call $\tau(\mathcal{U})$ the *quasi-uniform topology.* If τ is a topology on X, then $\mathcal{U}$ is said to be *compatible* with τ provided $\tau = \tau(\mathcal{U})$. In that case we say that (X, τ) *admits* $\mathcal{U}$.

In any quasi-uniform space $(X, \mathcal{U})$, the relation $\bigcap \mathcal{U}$ on X is a preorder. We call it the *quasi-uniform preorder* or the *preorder associated* with $\mathcal{U}$, and we denote it by $\leq_{\mathcal{U}}$. The topology $\tau(\mathcal{U})$ is T_0 when $\leq_{\mathcal{U}}$ is a partial order, and it is T_1 when $\leq_{\mathcal{U}}$ is the identity relation on X.

According to [5], the relationship of *strong inclusion* (denoted by $\ll$) between subsets $A, B \subseteq X$ in a quasi-uniform space $(X, \mathcal{U})$ is the following: $A \ll B$ when there exists an entourage $U \in \mathcal{U}$ such that $U \cap (A \times (X \setminus B)) = \varnothing$. This means $y \in B$ whenever $(x, y) \in U$ and $x \in A$ for all $x, y \in X$.

A *quasi-pseudo-metric* on a set X is a function $d : X \times X \to \mathbb{R}^+$ that satisfies, for every $x, y, z \in X$, the following conditions: (i) $d(x, x) = 0$, and (ii) $d(x, z) \leq d(x, y) + d(y, z)$. If, in addition, d satisfies $(d(x, y) = d(y, x) = 0) \Rightarrow x = y$, then it is called a *quasi-metric.* A quasi-metric is called a *metric* when it also satisfies the symmetry condition $d(y, x) = d(x, y)$. As references on the subject of quasi-metric spaces, we recommend the books [9] and [3]. The *conjugate quasi-pseudo-metric* is the map $d^{-1}(x, y) = d(y, x)$. The *induced pseudo-metric* is defined by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$. When d is a quasi-metric, d^s is a metric. A function $f : (X, d_X) \to (Y, d_Y)$ between quasi-pseudo-metric spaces is quasi-uniformly continuous provided for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$ for all $x, y \in X$.

If (X, d) is a quasi-pseudo-metric space, then the family $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$, where $U_n = \{(x, y) \in X \times X \mid d(x, y) < 2^{-n}\}$ for each $n \in \mathbb{N}$, is a base for a quasi-uniformity $\mathcal{U}_d$ on X called the *quasi-uniformity generated by* d. The conjugate quasi-pseudo-metric d^{-1} generates the conjugate quasi-uniformity $\mathcal{U}^{-1}$, and the induced pseudo-metric d^s generates $\mathcal{U}^s$. If d is a pseudo-metric, then $d = d^{-1}$ and $\mathcal{U}_d$ is a uniformity.

A quasi-pseudo-metric d on a set X induces a topology $\tau(d)$ generated by the family of open balls $\mathcal{B} = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$. The topology $\tau(d)$ is called the *quasi-pseudo-metric topology*, or the topology *induced* by d. We always have $\tau(d) = \tau(\mathcal{U}_d)$. If d is a quasi-metric, then the topology $\tau(d)$ is T_0.

Given subsets $A, B \subseteq X$ in a quasi-pseudo-metric space (X, d), the condition stated above for A to be strongly contained in B in the quasi-uniformity $\mathcal{U}_d$ is equivalent to the existence of a real $\varepsilon > 0$ such that $y \in B$ when $d(x, y) < \varepsilon$ and $x \in A$ for all $x, y \in X$. A map $f : X \to X$ in a quasi-pseudo-metric space (X, d) is called a *d-contraction map* [20] provided there exists a value $c < 1$ such that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$.

A sequence $(x_n)_n$ in a quasi-metric space (X, d) is called a *left K-Cauchy sequence* [7] provided that $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N} : d(x_n, x_m) < \varepsilon$ whenever $m \geq n \geq n_0$. The sequence $(x_n)_n$ *converges* to $x \in X$, with respect to $\tau(d)$ [3] (we will just say "with respect to d"), provided $\lim_{n\to\infty} d(x, x_n) = 0$. This convergence is also referred to as *left d-convergence*. Statistical convergence is defined in terms of asymptotic density [12]. The *asymptotic density* $\varrho(A)$ of a subset $A \subseteq \mathbb{N}$ is $\varrho(A) = \lim_{n\to\infty} \frac{1}{n} |A \cap \{1, \dots, n\}|$, provided this limit exists. Here the symbol $|A|$ indicates the cardinality of a set A. The sequence $(x_n)_n$ is *forward statistically convergent* to $x \in X$ when the asymptotic density $\varrho(\{k \in \mathbb{N} \mid d(x, x_k) \geq \varepsilon\})$ equals zero for every $\varepsilon > 0$.

A quasi-metric space (X, d) is said to be *weighted*, or *weightable* (see [13], [20]) if there exists a function $w : X \to \mathbb{R}^+$ (called a *weighting function*) such that, for every $x, y \in X$, we have $d(x, y) + w(x) = d(y, x) + w(y)$. Consequently, in a weightable space, $w(x) \leq d(y, x) + w(y)$ holds for all $x, y \in X$.

A quasi-pseudo-metric d on a set X induces a preorder $\leq_d$ called the *preorder associated* with d, or the *quasi-metric preorder*, defined by $x \leq_d y \Leftrightarrow d(x, y) = 0$ for $x, y \in X$. In a preordered set $(X, \leq)$, given a subset $A \subseteq X$, its *increasing* (resp., *decreasing*) *hull*, also called the *upward* (resp., *downward*) *closure* of A, is denoted by $\uparrow A$ (resp., $\downarrow A$), and it is defined as follows (see [16]): $\uparrow A = \{x \in X \mid \exists a \in A : a \leq x\}$ and $\downarrow A = \{x \in X \mid \exists a \in A : x \leq a\}$. A is called *upward* (resp., *downward*) *closed* when $\uparrow A = A$ (resp., $\downarrow A = A$).

A *topological monoid* (X, m, τ) is a topological space (X, τ) together with a continuous and associative binary operation $m : X \times X \to X$ that has a unit [2]. An *Alexandroff space* is a topological space where arbitrary intersections of open sets are open. An equivalent condition is that every point has a minimum open neighborhood [8].

The quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$ of complexity functions has its origins in computer science as well as in topology. Scott and Strachey [22], [23] initiated the study of denotational semantics and domain theory in order to provide rigorous definitions of programming languages, and to develop mathematical models for them. Completion of partial orders plays an important role in domain theory. Some authors, notably Nachbin [15], have studied the connections between order theory and topology. Smyth [24] introduced the concept of a *topological quasi-uniform space*, a category extending that of the quasi-uniform spaces, and he developed a completion for quasi-uniform spaces, based also on the syntopological spaces of Császár [4]. According to Schellekens [21], "the Smyth completion allows one to develop denotational semantics completions as true topological completions." Sünderhauf [25] presented a construction of the Smyth completion that uses topological quasi-uniform spaces but makes no use of syntopological spaces. Künzi [13] characterized the property of Smyth-completability in terms of left K-Cauchy filters. He also proved that, if (X, d, w) is a weightable quasi-metric space, then the topological quasi-uniform space $(X, \mathcal{U}_d, \tau(\mathcal{U}_d))$ is Smyth-completable. In this context, Schellekens [20] defined his complexity distance (a weightable quasi-metric) as a way to measure the relative progress made by syntactic transformations in lowering the complexity of programs. Later, Schellekens and Romaguera [18] proved that $(\mathcal{C}^*, d_{\mathcal{C}^*})$, the dual complexity space, is Smyth-complete. We have the following definitions, where we adopt the convention that $1/\infty = 0$:

$$\mathcal{C} = \left\{ f : \omega \longrightarrow (0, \infty] \;\middle|\; \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\},$$

$$d_{\mathcal{C}}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left[\left(\frac{1}{g(n)} - \frac{1}{f(n)} \right) \vee 0 \right],$$

$$w_{\mathcal{C}}(f) = \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)}.$$

The dual complexity space $(\mathcal{C}^*, d_{\mathcal{C}^*})$, introduced by Romaguera and Schellekens in [18], has its own weighting function $w_{\mathcal{C}^*}$. We have

$$\mathcal{C}^* = \left\{ f : \omega \longrightarrow \mathbb{R}^+ \;\middle|\; \sum_{n=0}^{\infty} 2^{-n} f(n) < \infty \right\},$$

$$d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(g(n) - f(n)) \vee 0],$$

$$w_{\mathcal{C}^*}(f) = \sum_{n=0}^{\infty} 2^{-n} f(n).$$

We use the symbol $\mathcal{C}_0^*$ to denote the subset of $\mathcal{C}^*$ formed by all strictly positive functions. In the complexity space $\mathcal{C}$, the quasi-metric preorder $\leq_{d_{\mathcal{C}}}$ coincides with the pointwise order. On the other hand, in the dual complexity space $\mathcal{C}^*$ the associated preorder $\leq_{d_{\mathcal{C}^*}}$ is the opposite of the pointwise order. In both spaces we indicate the pointwise order by $\leq$. A functional $\xi : \mathcal{C} \to \mathcal{C}$ is *monotone* if $\xi f \leq \xi g$ whenever $f \leq g$ for all $f, g \in \mathcal{C}$. Monotone functionals in $\mathcal{C}^*$ are defined in the same way. Given a function $g \in \mathcal{C}$, the functional $\xi : \mathcal{C} \to \mathcal{C}$ is called an *improver with respect to* g when it satisfies $\forall n \in \omega$, $\xi^{n+1} g \leq \xi^n g$. If the functional ξ is monotone, in order for it to be an improver with respect to g, it is sufficient to verify that $\xi g \leq g$.

The *inversion map* $\Psi : (\mathcal{C}^*, d_{\mathcal{C}^*}) \to (\mathcal{C}, d_{\mathcal{C}})$ is specified by $\Psi(f) = 1/f$, where the sequence $1/f$ is defined pointwise by adopting the convention that $1/0 = \infty$. The map Ψ is an isometry, since it is a bijection satisfying $d_{\mathcal{C}}(\Psi(f), \Psi(g)) = d_{\mathcal{C}^*}(f, g)$. This implies the complexity space is Smyth-complete, also. Besides addition, the binary operations $\vee$ and $\wedge$ on sequences are defined pointwise in both spaces, $\mathcal{C}$ and $\mathcal{C}^*$. Given any value $c \in \mathbb{R}^+ \cup \{\infty\}$, the notation $\overline{c}$ will indicate the function with constant value c. Additionally, if $n \in \mathbb{N}$, we use the notation $\widehat{n}$ to represent the sequence with only n initial 1's, followed by the sequence $\overline{0}$.

We use the term *convolution* in its discrete sense, as defined in [10] for real-valued, ω-indexed sequences. That is, if $f, g \in \mathbb{R}^\omega$, then their convolution $f \otimes g$ is given by

$$(f \otimes g)(n) = \sum_{k=0}^{n} f(k) g(n-k) \quad \text{for all } n \in \omega.$$

Notice that, in case $f, g \in \mathcal{C}^*$, the same formula works as far as producing output values that belong to $\mathbb{R}^+$. Similarly, when $f, g \in \mathcal{C}$, the formula outputs values belonging to $(0, \infty]$. It is well known that convolution is associative, commutative, and it distributes over addition. Also, when $f, g, h \in \mathcal{C}^*$ are such that $f \otimes g = f \otimes h$, if $f(0) > 0$, then $g = h$. Convolution can be used to express some recurrences. For example, if f represents the Fibonacci sequence, then $f(0) = 0$, $f(1) = 1$ and $f(n) = \left(f \otimes \widehat{2}\right)(n-1)$ for $n \geq 2$. We use exponents to indicate repeated convolution. That is, $f^0 = \widehat{1}$ and $f^n = f \otimes f^{n-1}$ for all $n > 0$.

3. Convolution in $\mathcal{C}$ and $\mathcal{C}^*$: closure and inequalities

In this section we show that $\mathcal{C}$ and $\mathcal{C}^*$ are closed under convolution. Also, we give some inequalities that relate convolution to the following: the inversion map, the pointwise order in each space, the asymptotic order of growth in $\mathcal{C}^*$, and the quasi-metric distance and weight function in $\mathcal{C}^*$.

Proposition 3.1. *The complexity space $\mathcal{C}$ is closed under convolution. Also, for any two sequences in $\mathcal{C}$, the weight of their convolution is no greater than the product of their weights.*

Proof. If $f, g \in \mathcal{C}$,

$$\begin{aligned}
\sum_{s=0}^{\infty} 2^{-s} \frac{1}{(f \otimes g)(s)} &= \sum_{s=0}^{\infty} 2^{-s} \frac{1}{\sum_{n=0}^{s} f(n) g(s-n)} \\
&\leq \sum_{s=0}^{\infty} \sum_{n=0}^{s} \frac{2^{-s}}{f(n) g(s-n)} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{2^{-n-k}}{f(n) g(k)} \\
&= \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} \sum_{k=0}^{\infty} 2^{-k} \frac{1}{g(k)}.
\end{aligned}$$

□

Proposition 3.2. *The dual complexity space $\mathcal{C}^*$ is closed under convolution. Furthermore, given any two sequences $f, g \in \mathcal{C}^*$, the weight of their convolution equals the product of their weights.*

Proof. If $f, g \in \mathcal{C}^*$,

$$\begin{aligned}
\sum_{s=0}^{\infty} 2^{-s} (f \otimes g)(s) &= \sum_{s=0}^{\infty} 2^{-s} \sum_{n=0}^{s} f(n) g(s-n) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} 2^{-n-k} f(n) g(k) \\
&= \sum_{n=0}^{\infty} 2^{-n} f(n) \sum_{k=0}^{\infty} 2^{-k} g(k).
\end{aligned}$$

□

In view of the previous result, if $A \subseteq \mathbb{R}^+$ is closed under multiplication, then the subspace of $\mathcal{C}^*$ defined by $W_A = \{f \in \mathcal{C}^* \mid w_{\mathcal{C}^*}(f) \in A\}$ is closed under convolution. Also, if a sequence f belongs to $\mathcal{C}$, then the sequence of partial sums of its series $\sum f$ belongs to $\mathcal{C}$ as well, since $\sum_{k=0}^{n} f(k) = \left(f \otimes \overline{1}\right)(n)$. The same applies to $\mathcal{C}^*$. The function $\widehat{1} \in \mathcal{C}^*$ defined by $\widehat{1}(0) = 1$ and $\widehat{1}(n) = 0$ if $n > 0$ is the neutral element for convolution. This makes the triple $\left(\mathcal{C}^*, \otimes, \widehat{1}\right)$ an abelian monoid.

Lemma 3.3. *In both spaces $\mathcal{C}$ and $\mathcal{C}^*$, convolution is consistent with the pointwise order. That is, given f, g, h such that $g \leq h$, then $f \otimes g \leq f \otimes h$.*

It follows that, in both spaces $\mathcal{C}$ and $\mathcal{C}^*$, for any functions f, g, h there, we have

$$f \otimes (g \wedge h) \leq (f \otimes g) \wedge (f \otimes h) \leq (f \otimes g) \vee (f \otimes h) \leq f \otimes (g \vee h).$$

Proposition 3.4. *Given any $f, g \in \mathcal{C}^*$, we have $\Psi(f \otimes g) \leq \Psi f \otimes \Psi g$ in $\mathcal{C}$. Also, if $f, g \in \mathcal{C}$, then $\Psi^{-1}(f \otimes g) \leq \Psi^{-1} f \otimes \Psi^{-1} g$ in $\mathcal{C}^*$.*

Proof. For every $s \in \omega$, we have the inequality

$$\Psi(f \otimes g)(s) = \frac{1}{\sum_{n=0}^{s} f(n) g(s-n)} \leq \sum_{n=0}^{s} \frac{1}{f(n)} \cdot \frac{1}{g(s-n)} = (\Psi f \otimes \Psi g)(s).$$

The proof for the second part is similar. □

Proposition 3.5. *If $f, g, h \in \mathcal{C}_0^*$ and $f \in \mathcal{O}(g)$, then $f \otimes h \in \mathcal{O}(g \otimes h)$.*

Proof. Since $f \in \mathcal{O}(g)$, there are $n_0 \in \mathbb{N}$ and $c > 0$ such that $n \geq n_0 \Rightarrow f(n) \leq c g(n)$. Define $c_n = f(n)/g(n)$ for each $0 \leq n < n_0$ and take $k = \max\{c, c_0, \ldots, c_{n_0-1}\}$. So $k > 0$ and $f(n) = c_n g(n) \leq k g(n)$ for $0 \leq n < n_0$. Now if $s \geq n_0$, we have

$$(f \otimes h)(s) = \sum_{n=0}^{s} f(n) h(s-n) \leq \sum_{n=0}^{s} k g(n) h(s-n) = k(g \otimes h)(s). \qquad \square$$

In general, for arbitrary functions in $\mathcal{C}^*$, the above result is not necessarily true, as shown in the next two examples.

Example 3.6. $\widehat{1} \in \mathcal{O}\left(\overline{0}\right)$; however, $\widehat{1} \otimes (n)_n = (n)_n \notin \mathcal{O}\left(\overline{0}\right) = \mathcal{O}\left(\overline{0} \otimes (n)_n\right)$.

Example 3.7. Let $f = \widehat{2}$ and the function h be given by $h(n) = (n/2+1)^2$ if n is even and $h(n) = (n+1)/2$ if n is odd. Then every single term of $f \otimes h$ is of quadratic order. So we have $f \in \mathcal{O}\left(\widehat{1}\right)$ but $f \otimes h \notin \mathcal{O}(h) = \mathcal{O}\left(\widehat{1} \otimes h\right)$.

Proposition 3.8. *Given any $f, f_1, g, g_1 \in \mathcal{C}^*$,*

$$d_{\mathcal{C}^*}\left(f \otimes g, f_1 \otimes g_1\right) \leq w_{\mathcal{C}^*}(g) \cdot d_{\mathcal{C}^*}\left(f, f_1\right) + w_{\mathcal{C}^*}\left(f_1\right) \cdot d_{\mathcal{C}^*}\left(g, g_1\right).$$

Proof.

$$\begin{aligned}
&d_{\mathcal{C}^*}\left(f \otimes g, f_1 \otimes g_1\right) \\
&\quad = \sum_{s=0}^{\infty} 2^{-s}\left[\left(\left(f_1 \otimes g_1\right)(s) - (f \otimes g)(s)\right) \vee 0\right] \\
&\quad \leq \sum_{s=0}^{\infty}\left[\sum_{n=0}^{s}\left(2^{-s}\left[f_1(n) g_1(s-n) - f(n) g(s-n)\right] \vee 0\right)\right] \\
&\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty}\left(2^{-(n+k)}\left[f_1(n) g_1(k) - f(n) g(k)\right] \vee 0\right) \\
&\quad \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} 2^{-(n+k)}\Big(\left[f_1(n)\left(g_1(k) - g(k)\right) \vee 0\right] + \left[g(k)\left(f_1(n) - f(n)\right) \vee 0\right]\Big) \\
&\quad = \sum_{n=0}^{\infty} 2^{-n}\left(f_1(n) \cdot d_{\mathcal{C}^*}\left(g, g_1\right) + \left[\left(f_1(n) - f(n)\right) \vee 0\right] \cdot w_{\mathcal{C}^*}(g)\right) \\
&\quad = d_{\mathcal{C}^*}\left(g, g_1\right) \cdot w_{\mathcal{C}^*}\left(f_1\right) + w_{\mathcal{C}^*}(g) \cdot d_{\mathcal{C}^*}\left(f, f_1\right).
\end{aligned}$$

□

Corollary 3.9. *Given $f, f_1, g, g_1 \in \mathcal{C}^*$, and $\varepsilon, \delta > 0$, if $d_{\mathcal{C}^*}(f, f_1) < \varepsilon$ and $d_{\mathcal{C}^*}(g, g_1) < \delta$, then $d_{\mathcal{C}^*}(f \otimes g, f_1 \otimes g_1) < (\varepsilon \vee \delta) \cdot w_{\mathcal{C}^*}(f + g) + \varepsilon\delta$.*

The inequality in Proposition 3.8 has important consequences that can be generalized to other weighted quasi-metric spaces. We do that in the following section.

4. Steady operations in a weightable quasi-metric space

The inequalities in Propositions 3.1 and 3.8 may not hold for any binary operation in a weighted quasi-metric space. Here we give a few examples of spaces and operations that satisfy those inequalities. We also prove some of the continuity, quasi-uniform continuity, and sequence convergence properties that follow as a consequence of a given operation satisfying those inequalities.

Definition 4.1 (Steady, and sub-multiplicative operations)**.** Let (X, d, w) be a weightable quasi-metric space with weight function w. Given a binary operation $* : X \times X \to X$, we call it

(i) *steady* with respect to d and w if and only if, for all $x, y, u, v \in X$, it satisfies

$$d(x * y, u * v) \leq w(y) d(x, u) + w(u) d(y, v).$$

(ii) *sub-multiplicative* with respect to w, provided $w(x * y) \leq w(x) w(y)$ for all $x, y \in X$.

A constant binary operation is trivially steady with respect to any weighted quasi-metric and its weighting function(s).

Lemma 4.2. *Let (X, d, w) be a weightable quasi-metric space with weight function w, and let $*$ be a binary operation on X. If $*$ is steady with respect to d and w, then, for all $a, x, y \in X$,*

$$\max\{d(a * x, a * y),\ d(x * a, y * a)\} \leq w(a)d(x, y).$$

Lemma 4.3. *If (X, d, w) is a weighted quasi-metric space, and $*$ is an operation on X that is steady with respect to d and w, then $*$ is steady with respect to d^{-1} and w if and only if d is a metric.*

Proof. If $*$ is steady with respect to d^{-1} and w, then w is a weighting function for d^{-1}. Since w is already a weighting function for d, it follows that $d = d^{-1}$. □

Example 4.4. In the dual complexity space $(\mathcal{C}^*, d_{\mathcal{C}^*}, w_{\mathcal{C}^*})$, the convolution of sequences $\otimes$ is both steady with respect to $d_{\mathcal{C}^*}$ and $w_{\mathcal{C}^*}$, and also sub-multiplicative with respect to $w_{\mathcal{C}^*}$.

Example 4.5. Let X be the interval $(0, 1)$, and define $d(x, x) = 0$, $d(x, y) = y$ when $x \neq y$, and $w(x) = x$ for all $x, y \in X$. Then (X, d, w) is a weightable quasi-metric space where multiplication is steady with respect to d and w, and sub-multiplicative with respect to w. The function $r(x) = 1 - x$ is a weighting function for d^{-1}, the conjugate quasi-metric. However, multiplication is not sub-multiplicative with respect to r, and it is not steady with respect to d^{-1} and r.

Example 4.6. As stated in [13], the space $(\mathbb{R}^+, u, w)$, with $u(x, y) = (y - x) \vee 0$ and the weight function given by $w(x) = x$, is a weightable quasi-metric space. Here, multiplication is steady with respect to u and w, and it is also sub-multiplicative with respect to w. In this space, w is not a weighting function for u^{-1}, so $*$ is not steady with respect to u^{-1} and w, even though the inequality $u^{-1}(a * b, x * y) \leq w(b)u^{-1}(a, x) + w(x)u^{-1}(b, y)$ is valid for all $a, b, x, y \in \mathbb{R}^+$.

Example 4.7. A metric space (X, d) is weightable by any constant function $w(x) = c \in \mathbb{R}^+$. In particular, let $X \subseteq \mathbb{R}$ be closed under addition. If we take d to be the Euclidean metric, and set $w(x) = 1$ for all $x \in X$, then the operation of addition is steady with respect to d and w, and it is also sub-multiplicative with respect to w.

Example 4.8. If F is a non-empty finite set, make $X = \mathcal{P}(F)$, and define, for all $A, B \subseteq F$, $d(A, B) = |B \setminus A|$, and $w(A) = |A|$. In this case (X, d, w) is a weightable quasi-metric space. Here, the binary operation of intersection between subsets of F is both steady with respect to d and w and sub-multiplicative with respect to w.

Example 4.9. This example is very similar to the dual complexity space, only extending its basic concepts to the continuous case. Consider the space X, consisting of the weighted integrable functions from $\mathbb{R}^+$ to $\mathbb{R}^+$, and equipped with the following quasi-metric distance d, weighting function w, and the binary operation of continuous convolution:

$$X = \left\{ f : \mathbb{R}^+ \to \mathbb{R}^+ \;\middle|\; \int_0^\infty e^{-x} f(x)\, dx < \infty \right\},$$

$$d(f,g) = \int_0^\infty e^{-x} \left[(g(x) - f(x)) \vee 0 \right] dx,$$

$$(f * g)(s) = \int_0^s f(t) g(s-t)\, dt, \quad \text{and} \quad w(f) = \int_0^\infty e^{-x} f(x)\, dx.$$

(X, d, w) is a weightable quasi-metric space with weighting function w.

If $f, g, h, l \in X$, we have

$$\begin{aligned} \int_0^\infty e^{-s} (f * g)(s)\, ds &= \int_0^\infty \int_0^s e^{-s} f(t) g(s-t)\, dt ds \\ &= \int_0^\infty \int_0^\infty e^{-x-y} f(x) g(y)\, dy dx = w(f) w(g) < \infty. \end{aligned}$$

Thus $*$ is closed and sub-multiplicative on X. The proof that $*$ is steady with respect to d and w is quite similar to the one given in the discrete case (Proposition 3.8).

Lemma 4.10. *Let $*$ be a binary operation in the weightable quasi-metric space (X, d, w). For any $x \in X$, define its right powers, relative to $*$, by $x^1 = x$ and $x^{n+1} = x^n * x$ for all $n \geq 1$. Suppose that $*$ is steady with respect to d and w. If, additionally, $*$ is sub-multiplicative with respect to w, then the following inequality holds for all $n \in \mathbb{N}$:*

$$d(x^n, y^n) \leq d(x,y) \sum_{k=1}^{n} w(x)^{n-k} w(y)^{k-1}.$$

Proof. The base case for induction, with $n = 1$, is trivially true. For the induction step we have:

$$\begin{aligned} d\left(x^{n+1}, y^{n+1}\right) &= d\left(x^n * x, y^n * y\right) \\ &\leq w(x) d\left(x^n, y^n\right) + w\left(y^n\right) d(x,y) \\ &\leq w(x) d(x,y) \sum_{k=1}^{n} w(x)^{n-k} w(y)^{k-1} + w\left(y^n\right) d(x,y) \\ &= d(x,y) \left(\sum_{k=1}^{n} \left[w(x)^{n-k+1} w(y)^{k-1} \right] + w\left(y^n\right) \right) \\ &\leq d(x,y) \sum_{k=1}^{n+1} w(x)^{n+1-k} w(y)^{k-1}. \end{aligned}$$

□

Proposition 4.11. *Let (X, d, w) be a weightable quasi-metric space, equipped with a binary operation $* : X \times X \to X$. If this operation is steady with respect to d and w, then it is continuous with respect to the quasi-metric topology $\tau(d)$ in X, and the corresponding product topology in $X \times X$.*

Proof. Let $x, y \in X$ and $\varepsilon > 0$. Make $a = w(x) + w(y)$, and define the quantity δ as follows:

$$\delta = \frac{-a}{2} + \sqrt{\varepsilon + \frac{a^2}{4}}.$$

So, $\delta > 0$ and $\delta^2 + a\delta = \varepsilon$. Consider $u, v \in X$ such that $d(x,u), d(y,v) < \delta$. We have

$$\begin{aligned} d(x * y, u * v) &\leq w(y)d(x,u) + w(u)d(y,v) \\ &< \delta\,[w(y) + w(u)] \\ &\leq \delta\,[w(y) + d(x,u) + w(x)] \\ &< \delta(a + \delta) = \varepsilon. \end{aligned}$$

□

Corollary 4.12. *Under the hypothesis of Proposition* 4.11*, if the operation $*$ is associative and has a neutral element, then $(X, *, \tau(d))$ is a topological monoid.*

Proposition 4.13. *If (X, d, w) is a weightable quasi-metric space with weight function w, and there is a binary operation $* : X \times X \to X$, steady with respect to d and w, then, given a fixed element $a \in X$ with positive weight, the function $f_a : X \to X$ defined by $f_a(x) = a * x$ for all $x \in X$ is quasi-uniformly continuous.*

Proof. Let $\varepsilon > 0$. Define $\delta = \varepsilon / w(a)$, and take any $x, y \in X$ such that $d(x,y) < \delta$. Therefore

$$d\left(f_a(x), f_a(y)\right) = d(a * x, a * y) \leq w(a)d(x,y) < \varepsilon.$$

□

Proposition 4.14. *Let $(x_n)_n, (y_n)_n$ be left K-Cauchy sequences in a weightable quasi-metric space (X, d, w). Additionally, suppose $* : X \times X \to X$ is a steady binary operation with respect to d and w. Then $(x_n * y_n)_n$ is also a left K-Cauchy sequence.*

Proof. There are $n_1, n_2 \in \mathbb{N}$ such that $d\left(x_{n_1}, x_m\right) < 1$ if $m \geq n_1$, and $d\left(y_{n_2}, y_m\right) < 1$ whenever $m \geq n_2$. Therefore $w\left(x_m\right) \leq d\left(x_{n_1}, x_m\right) + w\left(x_{n_1}\right) < w\left(x_{n_1}\right) + 1$, and also $w\left(y_m\right) < w\left(y_{n_2}\right) + 1$. For $\varepsilon > 0$, with $a = w\left(x_{n_1}\right) + 1$, $b = w\left(y_{n_2}\right) + 1$, and $\delta = \varepsilon/(a+b) > 0$, there exist $n_3, n_4 \in \mathbb{N}$ such that $d\left(x_n, x_m\right) < \delta$ whenever $m \geq n \geq n_3$, and $d\left(y_n, y_m\right) < \delta$ whenever $m \geq n \geq n_4$. If $n_0 = \max\left\{n_1, n_2, n_3, n_4\right\}$, and $m \geq n \geq n_0$:

$$\begin{aligned} d\left(x_n * y_n, x_m * y_m\right) &\leq w\left(y_n\right) d\left(x_n, x_m\right) + w\left(x_m\right) d\left(y_n, y_m\right) \\ &< \delta\left(w\left(x_m\right) + w\left(y_n\right)\right) < \varepsilon. \end{aligned}$$

□

Proposition 4.15. *Assume (X, d, w) is a weightable quasi-metric space, and $* : X \times X \to X$ is a binary operation on X, steady with respect to d and w. Let $x, y \in X$, and suppose $(x_n)_n$ and $(y_n)_n$ are sequences in X. Then we have:*

(i) *If $(x_n)_n$ and $(y_n)_n$ converge to x and y, respectively, with respect to d, then the sequence $(x_n * y_n)_n$ converges to $x * y$ with respect to d.*
(ii) *If $(x_n)_n$ and $(y_n)_n$ converge to x and y, respectively, with respect to d^s, then the sequence $(x_n * y_n)_n$ converges to $x * y$ with respect to d^s.*

(iii) *If* $(x_n)_n$ *and* $(y_n)_n$ *converge to* x *and* y*, respectively, with respect to* d^{-1}*, and either* $(w(y_n))_n$ *is bounded, or the operation* $*$ *is commutative and* $(w(x_n))_n$ *is bounded, then* $(x_n * y_n)_n$ *converges to* $x*y$ *with respect to* d^{-1}*.*

(iv) *If* $(x_n)_n$ *and* $(y_n)_n$ *are forward statistically convergent to* x *and* y*, respectively, then the sequence* $(x_n * y_n)_n$ *is forward statistically convergent to* $x * y$*.*

Proof. For (i), given $\varepsilon > 0$, we make δ the same as in the proof of Proposition 4.11. There exists $n_0 \in \mathbb{N}$ such that $d(x, x_n), d(y, y_n) < \delta$ for $n \geq n_0$. Then

$$\begin{aligned}
d(x * y, x_n * y_n) &\leq w(x_n) \cdot d(y, y_n) + w(y) \cdot d(x, x_n) \\
&< \delta(w(x_n) + w(y)) \\
&\leq \delta(d(x, x_n) + w(x) + w(y)) \\
&< \delta(\delta + a) = \varepsilon.
\end{aligned}$$

To prove (ii), observe that, as $(x_n)_n$ and $(y_n)_n$ converge to x and y, respectively, with respect to d^s, they also converge with respect to d and d^{-1}. Part (i) implies $(x_n * y_n)_n$ converges to $x * y$ with respect to d. In order to show its convergence with respect to d^s, it will be sufficient to prove convergence with respect to d^{-1}. Take any $\varepsilon > 0$, and define a and δ the same way as in part (i). Then, there exists $n_0 \in \mathbb{N}$ such that $d^s(x, x_n) < \delta$ and $d^s(y, y_n) < \delta$ whenever $n \geq n_0$. For such $n \geq n_0$, we have

$$\begin{aligned}
d^{-1}(x * y, x_n * y_n) &= d(x_n * y_n, x * y) \\
&\leq w(x) \cdot d(y_n, y) + w(y_n) \cdot d(x_n, x) \\
&< \delta(w(x) + w(y_n)) \\
&< \delta(a + \delta) = \varepsilon.
\end{aligned}$$

For (iii), first consider the case where $(w(y_n))_n$ is bounded by $M \in \mathbb{R}^+$. Take any $\varepsilon > 0$ and define $\delta = \varepsilon / (w(x) + (M \vee 1))$. So, there exists $n_0 \in \mathbb{N}$ such that $d^{-1}(x, x_n) < \delta$ and $d^{-1}(y, y_n) < \delta$ whenever $n \geq n_0$. Then, for large enough n,

$$\begin{aligned}
d^{-1}(x * y, x_n * y_n) &\leq w(x) \cdot d^{-1}(y, y_n) + w(y_n) \cdot d^{-1}(x, x_n) \\
&< \delta(w(x) + (M \vee 1)) = \varepsilon.
\end{aligned}$$

For the second case, assume the operation $*$ is commutative, and that the sequence $(w(x_n))_n$ is bounded by $K \in \mathbb{R}^+$. Given $\varepsilon > 0$, let $\delta = \varepsilon / (w(y) + (K \vee 1))$. Then there exists $n_0 \in \mathbb{N}$ such that $d^{-1}(x, x_n) < \delta$ and $d^{-1}(y, y_n) < \delta$ whenever $n \geq n_0$. For such n,

$$\begin{aligned}
d^{-1}(x * y, x_n * y_n) &= d(x_n * y_n, x * y) \\
&= d(y_n * x_n, y * x) \\
&\leq w(x_n) \cdot d(y_n, y) + w(y) \cdot d(x_n, x) \\
&< \delta(w(y) + (K \vee 1)) = \varepsilon.
\end{aligned}$$

We turn now to part (iv). For $r > 0$, $n \in \mathbb{N}$, and $z \in \{x, y\}$, let us define the sets $A_r^z = \{k \in \mathbb{N} \mid d(z, z_k) \geq r\}$, $A_{r,n}^z = A_r^z \cap \{1, \ldots, n\}$. Similarly, make

$H_r = \{k \in \mathbb{N} \mid d(x * y, x_k * y_k) \geq r\}$, and $H_{r,n} = H_r \cap \{1, \ldots, n\}$. By hypothesis, $\varrho(A^z_\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \left|A^z_{\varepsilon,n}\right| = 0$ for every $\varepsilon > 0$. Now, take $\varepsilon > 0$ and define δ the same way it was defined in part (i). As shown there, for any $k \in \mathbb{N}$,

$$(d(x, x_k) < \delta \quad \text{and} \quad d(y, y_k) < \delta) \Rightarrow d(x * y, x_k * y_k) < \varepsilon.$$

Therefore $\mathbb{N} \setminus (A^x_\delta \cup A^y_\delta) = (\mathbb{N} \setminus A^x_\delta) \cap (\mathbb{N} \setminus A^y_\delta) \subseteq \mathbb{N} \setminus H_\varepsilon$. Then, $H_\varepsilon \subseteq A^x_\delta \cup A^y_\delta$. So, for every $n \in \mathbb{N}$, $H_{\varepsilon,n} \subseteq A^x_{\delta,n} \cup A^y_{\delta,n}$, which implies $|H_{\varepsilon,n}| \leq \left|A^x_{\delta,n}\right| + \left|A^y_{\delta,n}\right|$, and we have

$$\varrho(H_\varepsilon) = \lim_{n\to\infty} \frac{1}{n} |H_{\varepsilon,n}| \leq \lim_{n\to\infty} \frac{1}{n} \left|A^x_{\delta,n}\right| + \lim_{n\to\infty} \frac{1}{n} \left|A^y_{\delta,n}\right| = 0. \qquad \square$$

5. Convolution functionals in $\mathcal{C}$ and $\mathcal{C}^*$

In the complexity analysis of divide-and-conquer algorithms presented in [20], a key concept is that of an improver functional in the complexity space $\mathcal{C}$. In this section we use convolution to build such a functional in $\mathcal{C}$ from each function $f \in \mathcal{C}^*$ such that $f(0) \geq 1$. Additionally, we show that convolution is consistent with a certain equivalence relation defined between $d^s_{\mathcal{C}^*}$-Cauchy sequences.

Definition 5.1 (Convolution functional)**.** Given $f \in \mathcal{C}^*$, we define the functional $\Phi_f : \mathcal{C}^* \to \mathcal{C}^*$ by $\Phi_f(g) = f \otimes g$ for all $g \in \mathcal{C}^*$.

Proposition 5.2. *If $f \in \mathcal{C}^*$, the functional Φ_f satisfies the following:*

(i) *Φ_f is a monotone functional.*
(ii) *When $w_{\mathcal{C}^*}(f) < 1$, the functional Φ_f is a $d_{\mathcal{C}^*}$-contraction map.*
(iii) *If $f(0) > 0$, then Φ_f is injective.*
(iv) *If $f(0) \geq 1$, then $\Phi_f(g) \geq g$ for every $g \in \mathcal{C}^*$.*
(v) *When $f(0) > 0$ and $A, B \subseteq \mathcal{C}^*$ with $\Phi_f A \ll \Phi_f B$, we have $A \ll B$.*
(vi) *$\Phi_f : (\mathcal{C}^*, d_{\mathcal{C}^*}) \to (\mathcal{C}^*, d_{\mathcal{C}^*})$ is quasi-uniformly continuous.*

Proof. Part (i) follows from Lemma 3.3. Part (ii) is a consequence of Lemma 4.2. Part (iii) is easy to verify by induction. To prove (iv), take any $s \in \omega$; then

$$(f \otimes g)(s) = \sum_{n=0}^{s} f(n)g(s-n) = f(0)g(s) + \sum_{n=1}^{s} f(n)g(s-n) \geq g(s).$$

To verify (v), notice there exists $\varepsilon > 0$ such that, if $x \in \Phi_f A$ and $d_{\mathcal{C}^*}(x, y) < \varepsilon$, then $y \in \Phi_f B$ for all $x, y \in \mathcal{C}^*$. Let $\delta = \varepsilon / w_{\mathcal{C}^*}(f)$, and take $a \in A$ and $z \in \mathcal{C}^*$ with $d_{\mathcal{C}^*}(a, z) < \delta$. Using Lemma 4.2, we see $d_{\mathcal{C}^*}(f \otimes a, f \otimes z) < w_{\mathcal{C}^*}(f) \cdot \delta = \varepsilon$. This implies $f \otimes z \in \Phi_f B$, so there exists $b \in B$ such that $f \otimes z = f \otimes b$, but Φ_f here is injective, so $z = b \in B$, and then $A \ll B$. Part (vi) follows from Proposition 4.13. $\square$

Corollary 5.3. (i) *If $f \in \mathcal{C}^*$ and $(g_n)_n$ is a left K-Cauchy sequence in $\mathcal{C}^*$, then the sequence $(\Phi_f(g_n))_n$ is also left K-Cauchy.*
(ii) *Given a filter $\mathcal{F}$ in $\mathcal{C}^*$ and $h \in \mathcal{C}^*$ such that $\mathcal{F}$ converges to h, the image filter $\mathcal{G}$ generated by the filterbase $\Gamma = \{\Phi_f(F) \mid F \in \mathcal{F}\}$ converges to $\Phi_f(h)$.*

Definition 5.4 (The ICI functional)**.** To any given function f in the dual complexity space $\mathcal{C}^*$ we associate a functional $\Upsilon_f : \mathcal{C} \to \mathcal{C}$ defined on the complexity space $\mathcal{C}$ by means of the composition $\Upsilon_f = \Psi \circ \Phi_f \circ \Psi^{-1}$. We call Υ_f the *inversion-convolution-inversion (ICI) functional* associated with f:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Upsilon_f"] \arrow[d, "\Psi^{-1}"'] & \mathcal{C} \\ \mathcal{C}^* \arrow[r, "\Phi_f"'] & \mathcal{C}^* \arrow[u, "\Psi"'] \end{tikzcd}$$

Proposition 5.5. *Let $f \in \mathcal{C}^*$. The ICI functional Υ_f associated with f satisfies the following:*

(i) Υ_f *is monotone.*
(ii) *If $f(0) \geq 1$, then Υ_f is an improver with respect to any function $g \in \mathcal{C}$.*

Proof. For (i), take $g, h \in \mathcal{C}$ such that $g \leq h$. Then $\Psi^{-1}g = 1/g \geq 1/h = \Psi^{-1}h$, and so $\Phi_f\Psi^{-1}g \geq \Phi_f\Psi^{-1}h$, since Φ_f is monotone. Now we have

$$\Upsilon_f(g) = \Psi \circ \Phi_f \circ \Psi^{-1}g = \frac{1}{\Phi_f\Psi^{-1}g} \leq \frac{1}{\Phi_f\Psi^{-1}h} = \Psi \circ \Phi_f \circ \Psi^{-1}h = \Upsilon_f(h).$$

To prove (ii), consider any $g \in \mathcal{C}$ and let $h = 1/g = \Psi^{-1}g \in \mathcal{C}^*$. Therefore $g = 1/h = \Psi h$. Also, because of part (iv) of Proposition 5.2, $h \leq \Phi_f(h) = \Phi_f\Psi^{-1}g$. Then,

$$g(n) = \frac{1}{h(n)} \geq \frac{1}{[\Phi_f(h)](n)} = \left[\Psi \circ \Phi_f \circ \Psi^{-1}g\right](n) = [\Upsilon_f(g)](n). \qquad \square$$

Schellekens [20], in his sequential completion of a quasi-uniform space $(X,\mathcal{U})$ with a countable base, defines an equivalence relation $\approx$ between $\mathcal{U}^s$-Cauchy sequences. In the particular case of the quasi-metric dual complexity space, convolution can be defined between equivalence classes using representatives. The equivalence relation here looks as follows.

Definition 5.6 ([20])**.** If $(x_n)_n, (y_n)_n$ are $d^s_{\mathcal{C}^*}$-Cauchy sequences, then $(x_n)_n \approx (y_n)_n$ if and only if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d^s_{\mathcal{C}^*}(x_m, y_n) < \varepsilon$ whenever $m, n \geq n_0$.

To prove that convolution is consistent with the equivalence relation defined above, we will use the corollary following the next lemma.

Lemma 5.7. *Suppose a sequence $(x_n)_n$ in $\mathbb{R}^+$ has the following property: for every $\varepsilon > 0$, there exists $z \in \mathbb{N}$ such that $x_m \leq x_n + \varepsilon$ whenever $m \geq n \geq z$. Then $(x_n)_n$ converges.*

Proof. Notice that $(x_n)_n$ must be bounded, so it has a convergent subsequence $(x_{n_k})_k$. Let $L = \lim_{k\to\infty} x_{n_k}$ and take $\varepsilon > 0$. For a $K \in \mathbb{N}$ such that $|L - x_{n_k}| < \varepsilon/2$ if $k \geq K$, and a $z \in \mathbb{N}$ such that $x_m \leq x_n + \varepsilon/2$ whenever $m \geq n \geq z$, define $M = K \vee z$, and take any $m \geq n_M$. Therefore $|L - x_{n_M}| < \varepsilon/2$ and $m \geq n_M \geq n_z \geq z$, which implies $x_m \leq x_{n_M} + \varepsilon/2 < L + \varepsilon$. Now by way of contradiction assume $x_m \leq L-\varepsilon$. Since $m \geq n_M \geq M \geq K$, we have $|L - x_{n_m}| < \varepsilon/2$. Besides, $n_m \geq m \geq z$, and from this we conclude the contradiction $x_{n_m} \leq x_m + \varepsilon/2 \leq (L-\varepsilon) + \varepsilon/2 = L - \varepsilon/2$. Therefore $L - \varepsilon < x_m$, and so $(x_n)_n$ converges to L. $\square$

Corollary 5.8. *Given a left K-Cauchy sequence $(f_n)_n$ in $\mathcal{C}^*$, the sequence of weights $(w_{\mathcal{C}^*}(f_n))_n$ converges in $\mathbb{R}^+$.*

Proof. Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ such that $d_{\mathcal{C}^*}(f_n, f_m) < \varepsilon$ whenever $m \geq n \geq n_0$. If that is the case, then $w_{\mathcal{C}^*}(f_m) = d_{\mathcal{C}^*}(\overline{0}, f_m) \leq d_{\mathcal{C}^*}(\overline{0}, f_n) + d_{\mathcal{C}^*}(f_n, f_m) < w_{\mathcal{C}^*}(f_n) + \varepsilon$ and this means the sequence $(w_{\mathcal{C}^*}(f_n))_n$ satisfies the hypothesis of Lemma 5.7. $\square$

Proposition 5.9. *Given $d^s_{\mathcal{C}^*}$-Cauchy sequences $(a_n)_n, (b_n)_n, (f_n)_n, (g_n)_n$ satisfying the relations $(a_n)_n \approx (f_n)_n$ and $(b_n)_n \approx (g_n)_n$, we have $(a_n \otimes b_n)_n \approx (f_n \otimes g_n)_n$.*

Proof. Since $(f_n)_n$ and $(b_n)_n$ are $d^s_{\mathcal{C}^*}$-Cauchy sequences, they are also left K-Cauchy sequences in $\mathcal{C}^*$. In view of Corollary 5.8, the sequences $(w_{\mathcal{C}^*}(f_n))_n$ and $(w_{\mathcal{C}^*}(b_n))_n$ converge to $L_f, L_b \in \mathbb{R}^+$, respectively. Now take any $\varepsilon > 0$ and define the quantity

$$\delta = -\frac{L_f + L_b}{4} + \sqrt{\frac{(L_f + L_b)^2}{16} + \frac{\varepsilon}{2}}.$$

Therefore $\delta > 0$ and also $\delta(2\delta + L_f + L_b) = \varepsilon$, so there are $n_i \in \mathbb{N}$, $i = 1, \ldots, 4$ such that $d^s_{\mathcal{C}^*}(a_m, f_n) < \delta$ when $m, n \geq n_1$, $d^s_{\mathcal{C}^*}(b_m, g_n) < \delta$ if $m, n \geq n_2$, $|L_f - w_{\mathcal{C}^*}(f_n)| < \delta$ provided $n \geq n_3$, and $|L_b - w_{\mathcal{C}^*}(b_n)| < \delta$ when $n \geq n_4$. If $n_0 = \max\{n_i\}$ and $m, n \geq n_0$ then, applying Proposition 3.8, we have

$$\begin{aligned} d_{\mathcal{C}^*}(a_m \otimes b_m, f_n \otimes g_n) &\leq w_{\mathcal{C}^*}(f_n) \cdot d_{\mathcal{C}^*}(b_m, g_n) + w_{\mathcal{C}^*}(b_m) \cdot d_{\mathcal{C}^*}(a_m, f_n) \\ &\leq w_{\mathcal{C}^*}(f_n) \cdot d^s_{\mathcal{C}^*}(b_m, g_n) + w_{\mathcal{C}^*}(b_m) \cdot d^s_{\mathcal{C}^*}(a_m, f_n) \\ &< \delta(2\delta + L_f + L_b) = \varepsilon. \end{aligned}$$

Since convolution is commutative, $d_{\mathcal{C}^*}(f_n \otimes g_n, a_m \otimes b_m) = d_{\mathcal{C}^*}(g_n \otimes f_n, b_m \otimes a_m)$ and we can take the same steps as above to verify that $d_{\mathcal{C}^*}(f_n \otimes g_n, a_m \otimes b_m) < \varepsilon$. Thus, $d^s_{\mathcal{C}^*}(f_n \otimes g_n, a_m \otimes b_m) < \varepsilon$. $\square$

6. Convolution and topological monoids in $\mathcal{C}^*$

As a consequence of Corollary 4.12, we know that $(\mathcal{C}^*, \otimes, \tau(d_{\mathcal{C}^*}))$ is a topological monoid. In this section we work with two topologies for $\mathcal{C}^*$, different from the quasi-metric one but closely related to it, and we show that convolution also yields a topological monoid with one of them but not with the other.

Proposition 6.1. *The smallest Alexandroff topology in $\mathcal{C}^*$ finer than $\tau(d_{\mathcal{C}^*})$ is the one generated by the base $\mathcal{B} = \{\downarrow\{f\} \mid f \in \mathcal{C}^*\}$.*

Proof. To show that $\mathcal{B}$ is a basis for a topology, take $A, B \in \mathcal{B}$, and $x \in A \cap B$. Then, there exist $f, g \in \mathcal{C}^*$ such that $A = \downarrow\{f\}$ and $B = \downarrow\{g\}$. It follows that $x \leq f$ and $x \leq g$. Therefore $x \in \downarrow\{x\} \subseteq A \cap B$. Let us call α the topology generated by $\mathcal{B}$. It is well known ([9, Section 4.2]) that the upward closed subsets in a preordered set form an Alexandroff topology. Given any $A \subseteq \mathcal{C}^*$ such that $\downarrow A = A$, it is clear that $A = \bigcup_{a \in A} \downarrow \{a\}$. It follows that α is the Alexandroff topology given by the upward closed sets in $(\mathcal{C}^*, \geq)$. Now let σ be an Alexandroff topology defined on $\mathcal{C}^*$ and finer than $\tau(d_{\mathcal{C}^*})$. Since arbitrary intersections of open sets are open in σ, given any $f \in \mathcal{C}^*$, we have the following:

$$\begin{aligned} \downarrow \{f\} &= \{g \in \mathcal{C}^* \mid g \leq f\} \\ &= \{g \in \mathcal{C}^* \mid d_{\mathcal{C}^*}(f,g) = 0\} \\ &= \bigcap_{\varepsilon > 0} \{g \in \mathcal{C}^* \mid d_{\mathcal{C}^*}(f,g) < \varepsilon\} \in \sigma. \end{aligned}$$

Therefore $\mathcal{B} \subseteq \sigma$, so σ is finer than α. All that is left to prove is that α is finer than $\tau(d_{\mathcal{C}^*})$. Take $f \in \mathcal{C}^*$ and $\varepsilon > 0$. We need to show that $B_{d_{\mathcal{C}^*}}(f,\varepsilon)$ is a union of elements of $\mathcal{B}$. If $g \in B_{d_{\mathcal{C}^*}}(f,\varepsilon)$, this means $d_{\mathcal{C}^*}(f,g) < \varepsilon$. Then if $h \leq g$, by the triangle inequality, $d_{\mathcal{C}^*}(f,h) \leq d_{\mathcal{C}^*}(f,g) + d_{\mathcal{C}^*}(g,h) < \varepsilon$. So $g \in \downarrow \{g\} \subseteq B_{d_{\mathcal{C}^*}}(f,\varepsilon)$. □

Proposition 6.2. *In the topological space $(\mathcal{C}^*, \alpha)$, with α being the Alexandroff topology generated by the base $\mathcal{B} = \{\downarrow \{f\} \mid f \in \mathcal{C}^*\}$, the binary operation of convolution, considered as a function $\otimes : (\mathcal{C}^*, \alpha)^2 \to (\mathcal{C}^*, \alpha)$, is continuous.*

Proof. Let $f, g, h \in \mathcal{C}^*$ and assume $f \otimes g \leq h$. Given $a, b \in \mathcal{C}^*$, if $a \leq f$ and $b \leq g$, then $a \otimes b \leq f \otimes g$. This means $\otimes(\downarrow \{f\} \times \downarrow \{g\}) \subseteq \downarrow \{h\}$. Therefore the inverse image under convolution of any basic set of α is the union of basic sets for the product topology in $(\mathcal{C}^*, \alpha)^2$. □

Corollary 6.3. *The triple $(\mathcal{C}^*, \otimes, \alpha)$ is a topological monoid.*

To finish, we exhibit another topology in $\mathcal{C}^*$, one that is strictly finer than $\tau(d_{\mathcal{C}^*})$, and strictly coarser than α, but with respect to which convolution is not continuous. Given $f, g \in \mathcal{C}^*$, we use the notation $f \prec g$ to indicate $f(n) < g(n)$ for all $n \in \omega$.

Proposition 6.4. *Given $h \in \mathcal{C}_0^*$, let $U_h = \left\{(f,g) \in \mathcal{C}^* \times \mathcal{C}^* \mid (g - f) \vee \overline{0} \prec h\right\}$. Then, the family $\mathcal{B} = \{U_h \mid h \in \mathcal{C}_0^*\}$ is a base for a quasi-uniformity $\mathcal{U}$ in $\mathcal{C}^*$ with the following properties:*

- (i) *$\tau(\mathcal{U})$ is strictly finer than $\tau(d_{\mathcal{C}^*})$.*
- (ii) *α is strictly finer than $\tau(\mathcal{U})$.*
- (iii) *$\otimes : (\mathcal{C}^*, \tau(\mathcal{U}))^2 \to (\mathcal{C}^*, \tau(\mathcal{U}))$ is not continuous.*

Proof. Given any $h \in \mathcal{C}_0^*$, it is clear that $(f,f) \in U_h$ for all $f \in \mathcal{C}^*$. Then $U_h \neq \varnothing$ and $\varnothing \notin \mathcal{B}$. Consider $h_1, h_2 \in \mathcal{C}_0^*$, put $h_3 = h_1 \wedge h_2$, and take $(f,g) \in U_{h_3}$. Therefore $(g - f) \vee \overline{0} \prec h_1 \wedge h_2$. From this it follows that $(f,g) \in U_{h_1} \cap U_{h_2}$, so $\mathcal{B}$ is a filterbase in $\mathcal{C}^* \times \mathcal{C}^*$. To finish proving that $\mathcal{B}$ is a base for a quasi-uniformity, define, for $h \in \mathcal{C}_0^*$, $b = h/2$. Now, if $(x,y), (y,z) \in U_b$, then $(y - x) \vee \overline{0} \prec b$ and $(z - y) \vee \overline{0} \prec b$. Therefore $(z - x) \vee \overline{0} \prec 2b = h$, so we have $U_b \circ U_b \subseteq U_h$. Call $\mathcal{U}$ the quasi-uniformity generated by $\mathcal{B}$. To prove (i), take $f \in \mathcal{C}^*$, a real $\varepsilon > 0$, and a function $g \in B_{d_{\mathcal{C}^*}}(f, \varepsilon)$. If $a = d_{\mathcal{C}^*}(f, g)$, take $\delta = \varepsilon - a > 0$ and define the constant function $h = \overline{\delta/2} \in \mathcal{C}_0^*$. Given $u \in U_h(g)$, we have $(u - g) \vee \overline{0} \prec h$, so

$$\begin{aligned} d_{\mathcal{C}^*}(f, u) &\leq d_{\mathcal{C}^*}(f, g) + d_{\mathcal{C}^*}(g, u) \\ &= a + \sum_{n=0}^{\infty} 2^{-n} \left[(u(n) - g(n)) \vee 0\right] \\ &< a + \sum_{n=0}^{\infty} 2^{-n} \left[\frac{\delta}{2}\right] = \varepsilon. \end{aligned}$$

So we see that $U_h(g) \subseteq B_{d_{\mathcal{C}^*}}(f, \varepsilon)$. This implies that $\tau(\mathcal{U})$ is finer than $\tau\left(d_{\mathcal{C}^*}\right)$. To see that this refinement is strict, consider the function $h = (1/n)_n \in \mathcal{C}_0^*$. We claim there is no $\varepsilon > 0$ such that $B_{d_{\mathcal{C}^*}}(f, \varepsilon) \subseteq U_h(f)$. For $k \in \omega$, define the function $g_k \in \mathcal{C}^*$ by setting $g_k(n) = f(n)$ if $n \neq k$, and $g_k(k) = f(k) + \varepsilon \cdot 2^{k-1}$. Then

$$\begin{aligned} d_{\mathcal{C}^*}\left(f, g_k\right) &= \sum_{n=0}^{\infty} 2^{-n} \left[(g_k(n) - f(n)) \vee 0\right] \\ &= 2^{-k} \left(f(k) + \varepsilon \cdot 2^{k-1} - f(k)\right) = \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

So, $g_k \in B_{d_{\mathcal{C}^*}}(f, \varepsilon)$ for all $k \in \omega$. Now assume $g_k \in U_h(f)$. This implies that $(g_k - f) \vee \overline{0} \prec h$, and therefore $\forall n \in \omega$, $(g_k(n) - f(n)) \vee 0 < 1/n$. In particular, for $n = k$, we have $\varepsilon \cdot 2^{k-1} < 1/k$, and this is false for large k. For (ii), given $f, g \in \mathcal{C}^*$ and $h \in \mathcal{C}_0^*$ with $g \in U_h(f)$, we have $g \prec f + h$. If $b = g + (f + h - g)/2$, then $g \in \downarrow\{b\}$. Now, for any $a \in \downarrow\{b\}$ and each $n \in \omega$, we have

$$a(n) \leq (f(n) + h(n) + g(n))/2 < f(n) + h(n).$$

Then $(f, a) \in U_h$, and this shows that $\downarrow\{b\} \subseteq U_h(f)$. So α is finer than $\tau(\mathcal{U})$. To see that this refinement is strict, consider any $f \in \mathcal{C}^*$ that takes the value zero at some point $n_0 \in \omega$. No basic open set in $\tau(\mathcal{U})$ is contained in $\downarrow\{f\}$ since, given any $a \in \mathcal{C}^*$ and $h \in \mathcal{C}_0^*$, the function $g = a + h/2 \in U_h(a)$, but $g\left(n_0\right) = a\left(n_0\right) + h\left(n_0\right)/2 > 0 = f\left(n_0\right)$. Therefore g does not belong to $\downarrow\{f\}$. In order to prove part (iii), consider the basic open set $U_h\left(\overline{0}\right) \in \tau(\mathcal{U})$, where h is the sequence $\left((n-1)n(n+1)/6 + 2^{-n}\right)_n \in \mathcal{C}_0^*$. To show that $\otimes^{-1} U_h\left(\overline{0}\right)$ is not an open set in $\left(\mathcal{C}^*, \tau(\mathcal{U})\right)^2$, consider $a = b = (n)_n \in \mathcal{C}^*$, and notice that $a \otimes b = ((n-1)n(n+1)/6)_n \in U_h\left(\overline{0}\right)$. We claim that, for any $h_1, h_2 \in \mathcal{C}_0^*$, the image set $\otimes\left(U_{h_1}(a) \times U_{h_2}(b)\right)$ is not contained in $U_h\left(\overline{0}\right)$. For $u = a + \frac{1}{2}h_1$ and

$v = b + \frac{1}{2}h_2$, and for large enough values of $s \in \omega$, we have

$$\begin{aligned}
(u \otimes v)(s) &= \sum_{n=0}^{s} \left(n + \frac{1}{2}h_1(n)\right)\left(s - n + \frac{1}{2}h_2(s-n)\right) \\
&= \frac{(s-1)s(s+1)}{6} + \frac{1}{2}\sum_{n=0}^{s} n\left[h_1(s-n) + h_2(s-n)\right] + \frac{1}{4}\left(h_1 \otimes h_2\right)(s) \\
&\geq \frac{(s-1)s(s+1)}{6} + \frac{1}{2}\left[h_1(0) + h_2(0)\right]s \\
&> \frac{(s-1)s(s+1)}{6} + 2^{-s} = h(s).
\end{aligned}$$

Therefore $u \in U_{h_1}(a)$ and $v \in U_{h_2}(b)$, but $u \otimes v \notin U_h\left(\overline{0}\right)$. □

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