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EVOLUTION OF FIRST EIGENVALUES OF SOME GEOMETRIC OPERATORS UNDER THE RESCALED LIST'S EXTENDED RICCI FLOW

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Abstract. Let $(M, g(t), e^{-\phi}d\nu)$ be a compact weighted Riemannian manifold and let $(g(t), \phi(t))$ evolve by the rescaled List's extended Ricci flow. In this paper, we study the evolution equations for first eigenvalues of the geometric operators, $-\Delta_\phi + cS^a$, along the rescaled List's extended Ricci flow. Here $\Delta_\phi = \Delta - \nabla\phi.\nabla$ is a symmetric diffusion operator, $\phi \in C^\infty(M)$, $S = R - \alpha|\nabla\phi|^2$, R is the scalar curvature with respect to the metric $g(t)$ and a, c are some constants. As an application, we obtain some monotonicity results under the rescaled List's extended Ricci flow.

1. Introduction

It is well known that eigenvalues of geometric operators are vital to understanding geometric and topological natures of Riemannian manifolds. In recent years, there have been several results on evolution and monotonicity of eigenvalues of geometric operators under various geometric flows, most especially the Ricci flow and its variants such as Ricci-harmonic flow, Ricci–Bourguignon flow, rescaled Ricci flow and so on. Perelman's work [13] can be considered pioneering in this regard. In that work, Perelman established that the first eigenvalue of the operator $-4\Delta+R$ is monotonically nondecreasing along the Ricci flow and showed that the monotonicity of the first eigenvalue is due to the monotonicity of the energy functional $\mathcal{F}$ defined by $\mathcal{F} = \int_M (R + |\nabla f|^2)e^{-f}\, d\nu$. Perelman's ingenuity led to several groundbreaking results such as the no breather theorem, the non-collapsing theorem, and the solution of Poincaré's problem. Later in 2008, Cao [6] in an attempt to extend Perelman's work studied the first eigenvalue of the operator $-\Delta + cR$ with $c \geq \frac{1}{4}$ under the Ricci flow and obtained similar monotonicity results. The second author (Abolarinwa) considered in [1] the eigenvalues of the Witten Laplacian, $-\Delta_\phi$, under the extended Ricci flow and obtained some monotonic quantities involving the eigenvalue under the flow. The first author (Azami) investigated in [3] the first eigenvalue of geometric operator $-\Delta_\phi + cR$ under the Ricci–Bourguignon flow and in [4] studied the monotonicity of the first eigenvalue of the operator $-\Delta_\phi + cS$

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under the rescaled List's extended Ricci flow. He got several monotonicity results in both cases. Other interesting works include (to mention but a few) Fang et al. [8] on the evolution and monotonicity of eigenvalues of geometric operator $-\Delta_\phi + cR$ under the Ricci flow, Huang and Li [10] on the eigenvalues of geometric operators $-\Delta + bS$ along the rescaled List's extended Ricci flow, and Yang and Zhang [14] on general geometric operators under the Ricci flow. Yang and Zhang [14] considered the operator $-\Delta + cR^a$ for some constants c, a and derived evolution equations for the first eigenvalues of these operators under the Ricci flow. As an application of their evolution equations they prove some monotonicity results. We refer to [9, 11, 7] and the references therein for some other related works on monotonicity formulas of eigenvalues of geometric operators under some geometric flows.

Inspired by the above works, we consider the first nonzero eigenvalues of the geometric operator $-\Delta_\phi + cS^a$ (S is the generalized scalar curvature, defined below) under the rescaled List's extended Ricci flow for some constants a, c. Precisely, we prove the evolution equation for the first eigenvalue for constants $0 < a \leq 1$, $c > \frac{1}{4}$ under some suitable condition on the generalized scalar curvature. We also obtain some monotonic quantities depending on the first eigenvalue of $-\Delta_\phi + cS^a$ along the rescaled List's extended Ricci flow on closed Riemannian manifolds under some assumptions.

Before the main results are stated, we present the flow and some important facts around it so as to fix some notations. Let (M^n, g_0) be a smooth closed Riemannian manifold and $\phi_0 : M \to \mathbb{R}$ be a smooth function. The couple $(g(t), \phi(t))$, $t \in [0, T)$, is called the *extended Ricci flow* if it satisfies the following system of quasi-linear parabolic partial differential equations:

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Ric} + 2\alpha \nabla\phi \otimes \nabla\phi, & g(0) = g_0, \\ \frac{\partial}{\partial t}\phi = \Delta\phi, & \phi(0) = \phi_0. \end{cases} \tag{1.1}$$

Here, Ric is the Ricci tensor of M, ∇ is the gradient operator, $\alpha > 0$ is a coupling constant depending on n, and Δ denotes the Laplace–Beltrami operator with respect to the metric $g(t)$. In local coordinates $\{x^1, \ldots, x^n\}$ the Laplace–Beltrami operator is defined by

$$\Delta = g^{ij}\left(\frac{\partial^2}{\partial x^j \partial x^i} - \Gamma_{ij}^k \frac{\partial}{\partial x^k}\right),$$

where g^{ij} is the inverse of metric components g_{ij} and Γ_{ij}^k are the Christoffel symbols. For the special cases $\alpha = 2$ and $\alpha = \frac{n-1}{n-2}$, $n \geq 3$, the couple flow (1.1) has been studied by List in [12] and he has established its short time existence and uniqueness on any compact Riemannian manifold. Denote the generalized Ricci curvature, its components and generalized scalar curvature, respectively by

$$\mathcal{S} = \mathrm{Ric} - \alpha\nabla\phi \otimes \nabla\phi, \quad S_{ij} = R_{ij} - \alpha\phi_i\phi_j, \quad \text{and} \quad S = g^{ij}S_{ij} = R - \alpha|\nabla\phi|^2.$$

In this paper, we consider the *rescaled List's extended Ricci flow* as follows:

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathcal{S} + \frac{2r}{n} g, & g(0) = g_0, \\ \frac{\partial}{\partial t} \phi = \Delta \phi, & \phi(0) = \phi_0, \end{cases} \tag{1.2}$$

where $r(t)$ is a smooth function depending only on the time variable t. The rescaled List's extended Ricci flow is a generalization of the extended Ricci flow. When $r = 0$, then the couple flow (1.2) reduces to the extended Ricci flow (1.2). When one chooses $r = \frac{\int_M S\, d\mu}{\int_M d\mu}$ with $d\mu = e^{-\phi}\, d\nu$ being the volume form of metric $g(t)$, that is, r is the average of the generalized scalar curvature S, then the couple flow (1.2) is the normalized extended Ricci flow and the volume of $(M^n, g(t))$ is a constant. Let $g(t)$ be a solution to the extended Ricci flow (1.1). For any given function $r(t)$, assume

$$\gamma(t) = \frac{1}{1 - \frac{2}{n}\int_0^t r(\tau)\, d\tau}$$

and $\bar{t} = \int_0^t \gamma(\tau)\, d\tau$; then $\bar{g}(\bar{t}) = \gamma(t) g(t)$ solves the rescaled List's extended Ricci flow system

$$\begin{cases} \frac{\partial \bar{g}_{ij}}{\partial t} = -2(\bar{S}_{ij} - \frac{r}{n}\bar{g}_{ij}), \\ \frac{\partial \bar{\phi}}{\partial t} = \Delta \bar{\phi}. \end{cases}$$

Thus, there is a one-to-one relation between the extended Ricci flow (1.1) and the rescaled List's extended Ricci flow (1.2).

We state below the main results of this paper.

Theorem 1.1. *Let $(M^n, g(t), \phi(t), d\mu = e^{-\phi} d\nu)$, $t \in [0, T)$, be a solution to the rescaled List's extended Ricci flow (1.2) on a closed Riemannian manifold M. Suppose that λ is the first eigenvalue of the geometric operator $-\Delta_\phi + cS^a$, and the tensors $\mathcal{S}$ and $\nabla\phi \otimes \nabla\phi$ satisfy*

$$\left| \mathcal{S} - \frac{1}{4c-1}\nabla\nabla\phi \right| \geq \frac{2\sqrt{c}}{4c-1}|\nabla\nabla\phi|, \quad \nabla_i\phi \otimes \nabla_j\phi \geq (\Delta\phi) g_{ij} \quad \text{for all } t \in [0, T). \tag{1.3}$$

Then the quantity $\lambda(t) e^{\frac{2}{n}\int_0^t r(\tau)\, d\tau}$ is nondecreasing along the flow (1.2) for $0 < a \leq 1$ and $c > \frac{1}{4}$ if $0 < S < a^{\frac{1}{1-a}}$. Moreover, if $r(t) \leq 0$ for all t, then the eigenvalue $\lambda(t)$ is nondecreasing under the flow (1.2) for $0 < a \leq 1$ and $c > \frac{1}{4}$ if $0 < S < a^{\frac{1}{1-a}}$.

Corollary 1.2. *Let $(M^n, g(t), \phi(t), d\mu = e^{-\phi} d\nu)$, $t \in [0, T)$, be a solution to the flow*

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Ric} + \frac{2r(t)}{n} g, & g(0) = g_0, \\ \frac{\partial}{\partial t} \phi = \Delta \phi, & \phi(0) = \phi_0 \end{cases} \tag{1.4}$$

on a closed Riemannian manifold M. Suppose that λ is the first eigenvalue of the geometric operator $-\Delta_\phi + cR^a$ and the Ricci tensor satisfies

$$\left|\mathrm{Ric} - \frac{1}{4c-1}\nabla\nabla\phi\right| \geq \frac{2\sqrt{c}}{4c-1}|\nabla\nabla\phi| \quad \textit{for all } t \in [0,T).$$

Then the quantity $\lambda(t)e^{\frac{2}{n}\int_0^t r(\tau)\,d\tau}$ is nondecreasing along the flow (1.4) *for $0 < a \leq 1$ and $c > \frac{1}{4}$ if $0 < R < a^{\frac{1}{1-a}}$. Moreover, if $r(t) \leq 0$ for all t, then the eigenvalue $\lambda(t)$ is nondecreasing under the flow* (1.4) *for $0 < a \leq 1$ and $c > \frac{1}{4}$ if $0 < R < a^{\frac{1}{1-a}}$.*

Corollary 1.3. *Let $(M^2, g(t), \phi(t), d\mu = e^{-\phi}d\nu)$, $t \in [0,T)$, be a solution to the flow*

$$\begin{cases} \frac{\partial}{\partial t}g = -2\mathrm{Ric} + r(t)g, & g(0) = g_0, \\ \frac{\partial}{\partial t}\phi = \Delta\phi, & \phi(0) = \phi_0 \end{cases} \tag{1.5}$$

on a closed two-dimensional Riemannian manifold M. Suppose that λ is the first eigenvalue of the geometric operator $-\Delta_\phi + cR^a$ and the scalar curvature satisfies

$$R \geq \frac{\sqrt{2}}{2\sqrt{c}-1}|\nabla\nabla\phi| \quad \textit{for all } t \in [0,T). \tag{1.6}$$

Then the quantity $\lambda(t)e^{\int_0^t r(\tau)\,d\tau}$ is nondecreasing along the flow (1.5) *for $0 < a \leq 1$ and $c > \frac{1}{4}$ if $0 < R < a^{\frac{1}{1-a}}$. Moreover, if $r(t) \leq 0$ for all t, then the eigenvalue $\lambda(t)$ is nondecreasing under the flow* (1.5) *for $0 < a \leq 1$ and $c > \frac{1}{4}$ if $0 < R < a^{\frac{1}{1-a}}$.*

Remark 1.4. Note that the first equation in (1.6) can be read as

$$\frac{\partial}{\partial t}g = -(R - r(t))g$$

since in $\mathrm{Ric} = \frac{1}{2}Rg$ in dimension 2. We know that the Ricci flow and the Yamabe flow are equivalent in dimension 2 but differ in higher dimensions. This is clearly a coupled rescaled Yamabe flow. There is hope that this result can be extended to the Yamabe flow. Some new results have been obtained in this direction [2].

2. Preliminaries

Let $\phi = \phi(t) : M \to \mathbb{R}$ be a family of smooth functions, $t \in [0,T)$, and let the closed Riemannian manifold $(M, g(t), \phi(t), d\mu = e^{-\phi}\,d\nu)$ be a solution to (1.2). Suppose that $\lambda(t)$ is the first nonzero eigenvalue of the geometric operator $-\Delta_\phi + cS^a$ for some constants a, c, where Δ is the Laplace–Beltrami operator and $d\mu = e^{-\phi}\,d\nu$ is the weighted volume measure on (M,g). Then the Witten–Laplace operator $\Delta_\phi = \Delta - \nabla\phi.\nabla$ is a symmetric operator on $L^2(M, d\mu)$ and satisfies the following integration by parts formula:

$$\int_M \nabla u.\nabla v\,d\mu = -\int_M (\Delta_\phi u)v\,d\mu = -\int_M (\Delta_\phi v)u\,d\mu \quad \text{for all } u, v \in C^\infty(M).$$

When ϕ is a constant function, the Witten Laplacian is just the Laplace–Beltrami operator. We say that Λ is an eigenvalue of $-\Delta_\phi + cS^a$ with associated eigenfunction f whenever

$$-\Delta_\phi f + cS^a f = \Lambda f. \tag{2.1}$$

Now, multiplying both sides of (2.1) by f and using integration by parts, we obtain

$$\Lambda \int_M f^2\, d\mu = \int_M (|\nabla f|^2 + cS^a f^2)\, d\mu.$$

The first nonzero eigenvalue of the operator $-\Delta_\phi + cS^a$ is defined by

$$\lambda(t) = \inf_{f\neq 0} \left\{ \int_M (|\nabla f|^2 + cS^a f^2)\, d\mu : f \in C^\infty(M),\ \int_M f^2\, d\mu = 1 \right\}.$$

The function f is called a normalized eigenfunction corresponding to the eigenvalue λ whenever

$$\lambda(t) = \int_M (|\nabla f|^2 + cS^a f^2)\, d\mu, \qquad \int_M f^2\, d\mu = 1.$$

We do not know whether the eigenvalue $\lambda(t)$ and its corresponding eigenfunction $f(t)$ are C^1-differentiable with respect to t or not along the rescaled List's extended Ricci flow (1.2). Then by a similar method as in [5], we give a general smooth function along the couple flow (1.2) in the following. At time $t_0 \in [0, T)$, we first assume that $f(t_0)$ is the normalized eigenfunction corresponding to the eigenvalue $\lambda(t_0)$ of $-\Delta_\phi + cS^a$. We consider the smooth function

$$u(t) = f(t_0) \left[\frac{\det(g_{ij}(t))}{\det(g_{ij}(t_0))} \right]^{\frac{1}{2}}$$

with respect to time t along the flow (1.2). Let a smooth function

$$f(t) = \frac{u(t)}{\sqrt{\int_M u^2\, d\mu}}$$

satisfy the normalization condition $\int_M f^2\, d\mu = 1$ under the flow (1.2), and at time t_0, f is an eigenfunction for λ of $-\Delta_\phi + cS^a$. Now, we define a smooth eigenfunction

$$\lambda(f(t), t) = \int_M \left(-f\Delta_\phi f + cSf^2 \right)\, d\mu = \int_M (|\nabla f|^2 + cS^a f^2)\, d\mu,$$

where $\lambda(f(t_0), t_0) = \lambda(t_0)$, f is a smooth function and satisfies $\int_M f^2\, d\mu = 1$.

3. The proof of the results

We first derive the evolution equation of $\lambda(f(t), t)$ under the general geometric flow $\frac{\partial g_{ij}}{\partial t} = h_{ij}$, where h_{ij} is a symmetric $(0, 2)$-tensor.

Proposition 3.1. *Let λ be an eigenvalue of the geometric operator $-\Delta_\phi + cS^a$, let f be the eigenfunction of $\lambda(t)$ at time t_0, and assume the metric $g(t)$ evolves by $\frac{\partial}{\partial t}g_{ij} = h_{ij}$, where h_{ij} is a symmetric $(0,2)$-tensor. Then, we get*

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} &= \int_M \left(h_{ij}f_{ij} - h_{ij}\phi_i f_j + caS^{a-1}\frac{\partial S}{\partial t}f + (\phi_t)_i f_i\right) f\, d\mu \\ &\quad + \int_M \left(h_{ij,i} - \frac{1}{2}H_j\right) f_j f\, d\mu. \end{aligned} \tag{3.1}$$

Here $H = \operatorname{tr}(h)$, $f_{ij} = \nabla_i\nabla_j f$, $f_i = \nabla_i f$, and $\phi_t = \frac{\partial \phi}{\partial t}$.

Proof. By direct computation, we obtain

$$\frac{\partial}{\partial t}\Delta_\phi f = \Delta_\phi f_t - h_{ij}f_{ij} - \frac{1}{2}g^{ij}\left(2(\operatorname{div} h)_i - H_i\right) f_j + h_{ij}\phi_i f_j - (\phi_t)_i f_i.$$

Therefore, we get

$$\begin{aligned} &\frac{d}{dt}\lambda(f,t)|_{t=t_0} \\ &= \int_M \left(-\Delta_\phi f + cS^a f\right) f\, d\mu \\ &= \int_M \left(h_{ij}f_{ij} + \frac{1}{2}g^{ij}\left(2(\operatorname{div} h)_i - H_i\right) f_j - h_{ij}\phi_i f_j + caS^{a-1}\frac{\partial S}{\partial t}f + (\phi_t)_i f_i\right) f\, d\mu \\ &\quad + \int_M \left(-\Delta_\phi f_t + cS^a f_t\right) f\, d\mu + \int_M \left(-\Delta_\phi f + cS^a f\right)\frac{d}{dt}(f\, d\mu). \end{aligned} \tag{3.2}$$

Integration by parts yields

$$\int_M \left(-\Delta_\phi f_t + cS^a f_t\right) f\, d\mu = \int_M \left(-\Delta_\phi f + cS^a f\right) f_t\, d\mu.$$

From the normalization condition $\int_M f^2\, d\mu = 1$, we infer

$$0 = \frac{d}{dt}\left(\int_M f^2\, d\mu\right) = \int_M f\big(f_t\, d\mu + (f\, d\mu)_t\big).$$

Hence, we conclude that

$$\begin{aligned} \int_M \left(-\Delta_\phi f_t + cS^a f_t\right) f\, d\mu + \int_M \left(-\Delta_\phi f + cS^a f\right)\frac{d}{dt}(f\, d\mu) \\ = \int_M \left(-\Delta_\phi f + cS^a f\right)\big(f_t\, d\mu + (f\, d\mu)_t\big) \\ = \lambda(t_0)\int_M f\big(f_t\, d\mu + (f\, d\mu)_t\big) = 0. \end{aligned} \tag{3.3}$$

Plugging (3.3) into (3.2), we arrive at (3.1). □

Now, we obtain the evolution equation of the first eigenvalue of $-\Delta_\phi + cS^a$ under the rescaled List's extended Ricci flow.

Proposition 3.2. *Suppose that* $(M^n, g(t), \phi(t), d\mu = e^{-\phi}\, d\nu)$, $t \in [0, T)$ *is a solution of the rescaled List's extended Ricci flow* (1.2) *on a closed Riemannian manifold* M. *Let* λ *be an eigenvalue of the geometric operator* $-\Delta_\phi + cS^a$ *and* f *be the eigenfunction of* $\lambda(t)$ *at time* t_0. *Then under the rescaled List's extended Ricci flow, we get*

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} = &-\frac{2r(t_0)}{n}\lambda(t_0) - 2\int_M S_{ij} f_{ij} f\, d\mu + 2\int_M S_{ij} f_j \phi_i f\, d\mu \\ &+ 2ac\int_M S^{a-1}|S_{ij}|^2 f^2\, d\mu + \int_M f_i(\phi_t)_i f\, d\mu \\ &+ 2a\alpha c\int_M S^{a-1}(\Delta\phi)^2 f^2\, d\mu + 2\alpha\int_M (\Delta\phi)\phi_i f_i f\, d\mu \\ &- ca(a-1)\int_M S^{a-2}|\nabla S|^2 f^2\, d\mu + c\int_M S^a \Delta(f^2 e^{-\phi})\, d\nu \\ &+ \frac{2r(t_0)}{n}c(1-a)\int_M S^a f^2\, d\mu. \end{aligned} \tag{3.4}$$

Proof. Under the rescaled List's extended Ricci flow, we have

$$h_{ij} = -2S_{ij} \tag{3.5}$$

and

$$\frac{\partial S}{\partial t} = \Delta S + 2|S_{ij}|^2 - \frac{2r}{n}S + 2\alpha(\Delta\phi)^2. \tag{3.6}$$

Applying (3.5) and (3.6) into the evolution formula (3.1), we deduce

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} = &\int_M \left[-2S_{ij} f_{ij} f + 2S_{ij}\phi_i f_j f + \frac{2r}{n} f\Delta_\phi f + (\phi_t)_i f_i f\right] d\mu \\ &+ ca\int_M S^{a-1}\left[\Delta S + 2|S_{ij}|^2 - \frac{2r}{n}S + 2\alpha(\Delta\phi)^2\right] f^2\, d\mu \\ &+ 2\alpha\int_M (\Delta\phi)\phi_i f_i f\, d\mu. \end{aligned} \tag{3.7}$$

Using integration by parts, we get

$$\int_M f^2 S^{a-1}\Delta S\, d\mu = -(a-1)\int_M f^2 S^{a-2}|\nabla S|^2\, d\mu - \int_M S^{a-1}\nabla S, \nabla(f^2 e^{-\phi})\rangle\, d\nu.$$

Note that

$$-\int_M S^{a-1}\nabla S, \nabla(f^2 e^{-\phi})\rangle\, d\nu = -\frac{1}{a}\int_M \nabla S^a, \nabla(f^2 e^{-\phi})\rangle\, d\nu = \frac{1}{a}\int_M S^a \Delta(f^2 e^{-\phi})\, d\nu.$$

Thus, we have

$$ca\int_M f^2 S^{a-1}\Delta S\, d\mu = -ca(a-1)\int_M S^{a-2}|\nabla S|^2 f^2\, d\mu + c\int_M S^a \Delta(f^2 e^{-\phi})\, d\nu. \tag{3.8}$$

Hence, inserting (3.8) into (3.7) we obtain (3.4). □

Proposition 3.3. *Let $(M^n, g(t), \phi(t), d\mu = e^{-\phi(x)}\, d\nu)$, $t \in [0,T)$ be a solution of the rescaled List's extended Ricci flow* (1.2) *on a closed Riemannian manifold M. Suppose that λ is an eigenvalue of the geometric operator $-\Delta_\phi + cS^a$ and f is the corresponding eigenfunction of $\lambda(t)$ at time t_0. Then along the rescaled List's extended Ricci flow, we have*

$$
\begin{aligned}
\frac{d}{dt}\lambda(f,t)|_{t=t_0} = & -\frac{2r(t_0)}{n}\lambda(t_0) + \frac{1}{2}\int_M |S_{ij} + \psi_{ij} + \phi_{ij}|^2 e^{-\psi}\, d\mu \\
& + \frac{4c-1}{2}\int_M \left(\left| S_{ij} - \frac{1}{4c-1}\phi_{ij}\right|^2 - \frac{4c}{(4c-1)^2}|\phi_{ij}|^2 \right) e^{-\psi}\, d\mu \\
& + \alpha \int_M (\nabla_i\phi \otimes \nabla_j\phi)(\phi_i + \frac{1}{2}\psi_i)\psi_j e^{-\psi}\, d\mu \\
& - \alpha \int_M (\Delta\phi)\phi_i\psi_i e^{-\psi}\, d\mu + 2a\alpha c\int_M S^{a-1}(\Delta\phi)^2 e^{-\psi}\, d\mu \\
& + \int_M (2acS^{a-1} - 2c)|S_{ij}|^2 e^{-\psi}\, d\mu - ca(a-1)\int_M S^{a-2}|\nabla S|^2 e^{-\psi}\, d\mu \\
& + \frac{2r(t_0)}{n}c(1-a)\int_M S^a e^{-\psi}\, d\mu
\end{aligned}
\tag{3.9}
$$

for $c \neq \frac{1}{4}$, where ψ satisfies $e^{-\psi} = f^2$.

Proof. Let ψ be a smooth function satisfying $e^{-\psi} = f^2$; replacing it into (3.4), we obtain

$$
\begin{aligned}
\frac{d}{dt}\lambda(f,t)|_{t=t_0} = & -\frac{2r(t_0)}{n}\lambda(t_0) + 2ac\int_M S^{a-1}|S_{ij}|^2 e^{-\psi}\, d\mu \\
& + 2a\alpha c\int_M S^{a-1}(\Delta\phi)^2 e^{-\psi}\, d\mu - ca(a-1)\int_M S^{a-2}|\nabla S|^2 e^{-\psi}\, d\mu \\
& - \alpha\int_M (\Delta\phi)\phi_i\psi_i e^{-\psi}\, d\mu + \int_M S_{ij}\psi_{ij} e^{-\psi}\, d\mu - \frac{1}{2}\int_M S_{ij}\psi_i\psi_j e^{-\psi}\, d\mu \\
& - \int_M S_{ij}\phi_i\psi_j e^{-\psi}\, d\mu - \frac{1}{2}\int_M \psi_j(\Delta\phi)_i e^{-\psi}\, d\mu + c\int_M S^a \Delta e^{-\psi-\phi}\, d\mu \\
& + \frac{2r(t_0)}{n}c(1-a)\int_M S^a e^{-\psi}\, d\mu.
\end{aligned}
\tag{3.10}
$$

By the definition of S_{ij} and the contracted second Bianchi identity, we arrive at

$$
S_{ij,j} = \frac{1}{2}S_{,i} - \alpha(\Delta\phi)\phi_i. \tag{3.11}
$$

Then, integration by parts yields

$$\begin{aligned}\int_M S_{ij}\psi_{ij}e^{-\psi}\,d\mu &= \int_M S_{ij}\psi_i\psi_j e^{-\psi}\,d\mu + \int_M S_{ij}\psi_i\phi_j e^{-\psi}\,d\mu \\ &\quad - \int_M \left(\frac{1}{2}S_{,i} - \alpha(\Delta\phi)\phi_i\right)\psi_i e^{-\psi}\,d\mu \\ &= \int_M S_{ij}\psi_i\psi_j e^{-\psi}\,d\mu + \int_M S_{ij}\psi_i\phi_j e^{-\psi}\,d\mu \\ &\quad + \frac{1}{2}\int_M S(\psi_i e^{-\psi-\phi})_i\,d\nu + \alpha\int_M (\Delta\phi)\phi_i\psi_i e^{-\psi}\,d\mu. \end{aligned} \tag{3.12}$$

Since

$$(\psi_i e^{-\psi-\phi})_i = -\Delta e^{-\psi-\phi} + e^{-\psi}\Delta e^{-\phi} + \psi_i\phi_i e^{-\psi-\phi}, \tag{3.13}$$

we can rewrite (3.12) as follows:

$$\begin{aligned}\int_M S_{ij}\psi_{ij}e^{-\psi}\,d\mu &= \int_M S_{ij}\psi_i\psi_j e^{-\psi}\,d\mu + \int_M S_{ij}\psi_i\phi_j e^{-\psi}\,d\mu \\ &\quad - \frac{1}{2}\int_M S\Delta e^{-\psi-\phi}\,d\nu + \frac{1}{2}\int_M S e^{-\psi}\Delta e^{-\phi}\,d\nu \\ &\quad + \frac{1}{2}\int_M S\psi_i\phi_i e^{-\psi}\,d\mu + \alpha\int_M (\Delta\phi)\phi_i\psi_i e^{-\psi}\,d\mu. \end{aligned} \tag{3.14}$$

On a closed Riemannian manifold M^n, for any smooth function ψ, we have the Bochner–Weizenböck formula as follows:

$$|\nabla\nabla\psi|^2 = \frac{1}{2}\Delta|\nabla\psi|^2 - \nabla\psi.\nabla(\Delta\psi) - \mathrm{Ric}(\nabla\psi,\nabla\psi). \tag{3.15}$$

Multiplying both sides of (3.15) with $e^{-\psi}$ and integrating on M, we conclude that

$$\begin{aligned}\int_M |\psi_{ij}|^2 e^{-\psi}\,d\mu &= \frac{1}{2}\int_M \Delta|\nabla\psi|^2 e^{-\psi-\phi}\,d\nu - \int_M \nabla\psi.\nabla(\Delta\psi)e^{-\psi-\phi}\,d\nu \\ &\quad - \int_M R_{ij}\psi_i\psi_j e^{-\psi-\phi}\,d\nu \\ &= -\int_M S_{ij}\psi_i\psi_j e^{-\psi-\phi}\,d\nu - \alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi-\phi}\,d\nu \\ &\quad + \frac{1}{2}\int_M \Delta|\nabla\psi|^2 e^{-\psi-\phi}\,d\nu - \int_M \nabla\psi.\nabla(\Delta\psi)e^{-\psi-\phi}d\,\nu. \end{aligned}$$

Using integration by parts we get

$$\begin{aligned}\int_M |\psi_{ij}|^2 e^{-\psi}\,d\mu = &-\int_M S_{ij,j}\psi_i e^{-\psi-\phi}\,d\nu - \int_M S_{ij}\psi_{ij}e^{-\psi-\phi}\,d\nu\\ &+\int_M S_{ij}\psi_i\phi_j e^{-\psi-\phi}\,d\nu - \alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi-\phi}\,d\nu\\ &+\frac{1}{2}\int_M |\nabla\psi|^2\Delta e^{-\psi-\phi}\,d\nu\\ &-\int_M (\Delta\psi)(\Delta\psi-\psi_i\phi_i-|\nabla\psi|^2)e^{-\psi-\phi}\,d\nu. \qquad (3.16)\end{aligned}$$

Substituting (3.11) and (3.13) into (3.16), we infer

$$\begin{aligned}\int_M |\psi_{ij}|^2 e^{-\psi}\,d\mu = &-\frac{1}{2}\int_M S_{,i}\psi_i e^{-\psi-\phi}\,d\nu + \alpha\int_M (\Delta\phi)\psi_i\phi_i e^{-\psi-\phi}\,d\nu\\ &-\int_M S_{ij}\psi_{ij}e^{-\psi-\phi}\,d\nu + \int_M S_{ij}\psi_i\phi_j e^{-\psi-\phi}\,d\nu\\ &-\alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi-\phi}\,d\nu + \frac{1}{2}\int_M |\nabla\psi|^2\Delta e^{-\psi-\phi}\,d\nu\\ &-\int_M (\Delta\psi)(\Delta\psi-\psi_i\phi_i-|\nabla\psi|^2)e^{-\psi-\phi}\,d\nu\\ = &-\int_M S_{ij}\psi_{ij}e^{-\psi}\,d\mu - \int_M \Delta e^{-\psi-\phi}\left(\Delta\psi+\frac{1}{2}S-\frac{1}{2}|\nabla\psi|^2\right)d\nu\\ &+\int_M (\Delta\psi)(\Delta e^{-\phi})e^{-\psi}\,d\nu + \int_M (\Delta\psi)\psi_i\phi_i e^{-\psi}\,d\mu\\ &+\int_M S_{ij}\psi_i\phi_j e^{-\psi}\,d\mu + \frac{1}{2}\int_M S(\Delta e^{-\phi})e^{-\psi}\,d\nu\\ &+\frac{1}{2}\int_M S\psi_i\phi_i e^{-\psi}\,d\mu + \alpha\int_M (\Delta\phi)\psi_i\phi_i e^{-\psi}\,d\mu\\ &-\alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi}\,d\mu. \qquad (3.17)\end{aligned}$$

Since $2\lambda = \Delta_\phi\psi - \frac{1}{2}|\nabla\psi|^2 + 2cS^a$, we can write (3.17) as follows:

$$\begin{aligned}\int_M |\psi_{ij}|^2 e^{-\psi}\,d\mu = &-\int_M S_{ij}\psi_{ij}e^{-\psi}\,d\mu + \int_M (2cS^a-\frac{1}{2}S)\Delta e^{-\psi-\phi}\,d\nu\\ &-\int_M \psi_i\phi_i\Delta e^{-\psi-\phi}\,d\nu + \int_M (\Delta\psi)(\Delta e^{-\phi})e^{-\psi}\,d\nu\\ &+\int_M (\Delta\psi)\psi_i\phi_i e^{-\psi}\,d\mu + \int_M S_{ij}\psi_i\phi_j e^{-\psi}\,d\mu\\ &+\frac{1}{2}\int_M S(\Delta e^{-\phi})e^{-\psi}\,d\nu + \frac{1}{2}\int_M S\psi_i\phi_i e^{-\psi}\,d\mu \qquad (3.18)\\ &+\alpha\int_M (\Delta\phi)\psi_i\phi_i e^{-\psi}\,d\mu - \alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi}\,d\mu.\end{aligned}$$

Inserting (3.14) into (3.18), we get

$$\begin{aligned}\int_M |\psi_{ij}|^2 e^{-\psi}\, d\mu = &- \int_M S_{ij}\psi_i\psi_j e^{-\psi}\, d\mu + 2c\int_M S^a \Delta e^{-\psi-\phi}\, d\nu \\ &- \int_M \psi_i\phi_i \Delta e^{-\psi-\phi}\, d\nu + \int_M (\Delta\psi)(\Delta e^{-\phi}) e^{-\psi}\, d\nu \\ &+ \int_M (\Delta\psi)\psi_i\phi_i e^{-\psi}\, d\mu - \alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi}\, d\mu,\end{aligned}$$

and

$$\begin{aligned}2c\int_M S^a\Delta e^{-\psi-\phi}\, d\nu = &\int_M |\psi_{ij}|^2 e^{-\psi}\, d\mu + \int_M S_{ij}\psi_i\psi_j e^{-\psi}\, d\mu \\ &+ \int_M \psi_i\phi_i\Delta e^{-\psi-\phi}\, d\nu - \int_M (\Delta\psi)(\Delta e^{-\phi})e^{-\psi}\, d\nu \\ &- \int_M (\Delta\psi)\psi_i\phi_i e^{-\psi}\, d\mu + \alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi}\, d\mu.\end{aligned} \tag{3.19}$$

Applying (3.19) into (3.10), we conclude that

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} = &-\frac{2r(t_0)}{n}\lambda(t_0) + 2ac\int_M S^{a-1}|S_{ij}|^2 e^{-\psi}\, d\mu \\ &+ 2a\alpha c\int_M S^{a-1}(\Delta\phi)^2 e^{-\psi}\, d\mu - ca(a-1)\int_M S^{a-2}|\nabla S|^2 e^{-\psi}\, d\mu \\ &- \alpha\int_M (\Delta\phi)\phi_i\psi_i e^{-\psi}\, d\mu + \int_M S_{ij}\psi_{ij}e^{-\psi}\, d\mu - \int_M S_{ij}\phi_i\psi_j e^{-\psi}\, d\mu \\ &- \frac{1}{2}\int_M \psi_j(\Delta\phi)_i e^{-\psi}\, d\mu + \frac{1}{2}\int_M |\psi_{ij}|^2 e^{-\psi}\, d\mu \\ &+ \frac{1}{2}\int_M \psi_i\phi_i\Delta e^{-\psi-\phi}\, d\nu - \frac{1}{2}\int_M (\Delta\psi)(\Delta e^{-\phi})e^{-\psi}\, d\nu \\ &- \frac{1}{2}\int_M (\Delta\psi)\psi_i\phi_i e^{-\psi}\, d\mu + \frac{1}{2}\alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi}\, d\mu \\ &+ \frac{2r(t_0)}{n}c(1-a)\int_M S^a e^{-\psi}\, d\mu.\end{aligned} \tag{3.20}$$

Notice that, integrating by parts again, we have the following formulas:

$$
\begin{aligned}
\int_M (-\Delta\phi + |\nabla\phi|^2 + \phi_i\psi_i)(\Delta\psi)e^{-\psi}\,d\mu &= \int_M \phi_i(\Delta\psi)_i e^{-\psi}\,d\mu \\
&= -\int_M \psi_{ij}\phi_{ij}e^{-\psi}\,d\mu - \int_M R_{ij}\phi_i\psi_j e^{-\psi}\,d\mu \\
&\quad + \int_M \phi_i\psi_{ij}\psi_j e^{-\psi}\,d\mu + \int_M \phi_i\psi_{ij}\phi_j e^{-\psi}\,d\mu, \\
\int_M (\Delta\phi - |\nabla\phi|^2 - \phi_i\psi_i)\psi_i\phi_i e^{-\psi}\,d\mu &= -\int_M \phi_{ij}\phi_i\psi_j e^{-\psi}\,d\mu - \int_M \psi_{ij}\phi_i\phi_j e^{-\psi}\,d\mu, \\
\int_M (\Delta\psi - |\nabla\psi|^2 - \psi_i\psi_i)\psi_i\phi_i e^{-\psi}\,d\mu &= -\int_M \phi_{ij}\psi_i\psi_j e^{-\psi}\,d\mu - \int_M \psi_{ij}\phi_i\psi_j e^{-\psi}\,d\mu,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_M \psi_i(\Delta\phi)_i e^{-\psi}\,d\mu = &-\int_M \psi_{ij}\phi_{ij}e^{-\psi}\,d\mu - \int_M R_{ij}\phi_i\psi_j e^{-\psi}\,d\mu \\
&+ \int_M \psi_i\phi_{ij}\phi_j e^{-\psi}\,d\mu + \int_M \psi_i\phi_{ij}\psi_j e^{-\psi}\,d\mu.
\end{aligned}
$$

Combining the last four formulas, we obtain

$$
\begin{aligned}
&-2\int_M \psi_{ij}\phi_{ij}e^{-\psi}\,d\mu - 2\int_M R_{ij}\psi_i\phi_j e^{-\psi}\,d\mu \\
&\qquad = \int_M \Delta\psi\left(\Delta e^{-\phi} + \phi_i\psi_i e^{-\phi}\right)e^{-\psi}\,d\mu - \int_M \phi_i\psi_i\Delta e^{-\psi-\phi}\,d\nu \\
&\qquad\quad + \int_M \psi_i(\Delta\phi)_i e^{-\psi}\,d\mu.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\int_M \psi_{ij}\phi_{ij}e^{-\psi}\,d\mu + \alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\phi_i\psi_j e^{-\psi}\,d\mu \\
&\qquad = -\int_M S_{ij}\psi_i\phi_j e^{-\psi}\,d\mu - \frac{1}{2}\int_M \Delta\psi\left(\Delta e^{-\phi} + \phi_i\psi_i e^{-\phi}\right)e^{-\psi}\,d\mu \\
&\qquad\quad + \frac{1}{2}\int_M \phi_i\psi_i\Delta e^{-\psi-\phi}\,d\nu - \frac{1}{2}\int_M \psi_i(\Delta\phi)_i e^{-\psi}\,d\mu. \qquad (3.21)
\end{aligned}
$$

Inserting (3.21) into (3.20), we have

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} = &-\frac{2r(t_0)}{n}\lambda(t_0) + 2ac\int_M S^{a-1}|S_{ij}|^2 e^{-\psi}\,d\mu \\ &+ 2a\alpha c\int_M S^{a-1}(\Delta\phi)^2 e^{-\psi}\,d\mu - ca(a-1)\int_M S^{a-2}|\nabla S|^2 e^{-\psi}\,d\mu \\ &- \alpha\int_M (\Delta\phi)\phi_i\psi_i e^{-\psi}\,d\mu + \int_M S_{ij}\psi_{ij}e^{-\psi}\,d\mu + \int_M \psi_{ij}\phi_{ij}e^{-\psi}\,d\mu \\ &+ \alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\phi_i\psi_j e^{-\psi}\,d\mu + \frac{1}{2}\int_M |\psi_{ij}|^2 e^{-\psi}\,d\mu \\ &+ \frac{1}{2}\alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi}\,d\mu + \frac{2r(t_0)}{n}c(1-a)\int_M S^a e^{-\psi}\,d\mu.\end{aligned} \tag{3.22}$$

We can write (3.22) as follows:

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} = &-\frac{2r(t_0)}{n}\lambda(t_0) + \frac{1}{2}\int_M |S_{ij}+\psi_{ij}+\phi_{ij}|^2 e^{-\psi}\,d\mu \\ &+ \frac{4c-1}{2}\int_M |S_{ij}+\phi_{ij}|^2 e^{-\psi}\,d\mu - 2c\int_M |\phi_{ij}|^2 e^{-\psi}\,d\mu \\ &- 4c\int_M S_{ij}\phi_{ij}e^{-\psi}\,d\mu + \alpha\int_M (\nabla_i\phi\otimes\nabla_j\phi)\phi_i\psi_j e^{-\psi}\,d\mu \\ &+ \frac{\alpha}{2}\int_M (\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi}\,d\mu - \alpha\int_M (\Delta\phi)\phi_i\psi_i e^{-\psi}\,d\mu \\ &+ 2a\alpha c\int_M S^{a-1}(\Delta\phi)^2 e^{-\psi}\,d\mu + \int_M (2acS^{a-1}-2c)|S_{ij}|^2 e^{-\psi}\,d\mu \\ &- ca(a-1)\int_M S^{a-2}|\nabla S|^2 e^{-\psi}\,d\mu \\ &+ \frac{2r(t_0)}{n}c(1-a)\int_M S^a e^{-\psi}\,d\mu.\end{aligned} \tag{3.23}$$

Finally, it is easy to see that (3.9) follows from the formula (3.23). □

Now, using Propositions 3.2 and 3.3 we prove our main results.

Proof of Theorem 1.1. Since $\alpha\int_M(\nabla_i\phi\otimes\nabla_j\phi)\psi_i\psi_j e^{-\psi}\,d\mu \geq 0$, replacing (1.3) into (3.9) we conclude that

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} \geq &-\frac{2r(t_0)}{n}\lambda(t_0) + 2a\alpha c\int_M S^{a-1}(\Delta\phi)^2 e^{-\psi}\,d\mu \\ &+ \int_M (2acS^{a-1}-2c)|S_{ij}|^2 e^{-\psi}\,d\mu - ca(a-1)\int_M S^{a-2}|\nabla S|^2 e^{-\psi}\,d\mu \\ &+ \frac{2r(t_0)}{n}c(1-a)\int_M S^a e^{-\psi}\,d\mu.\end{aligned}$$

Using the conditions $c > \frac{1}{4}$, $0 < a \leq 1$, and $0 < S < a^{\frac{1}{1-a}}$, we obtain

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} \geq -\frac{2r(t_0)}{n}\lambda(t_0).$$

Since $\lambda(f,t)$ is smooth with respect to time t, in any sufficiently small neighborhood of t_0 we get

$$\frac{d}{dt}\lambda(f(t),t) \geq -\frac{2r(t)}{n}\lambda(f(t),t). \tag{3.24}$$

Since t_0 is arbitrary, for any $t \in [0,T)$ the inequality (3.24) holds and it implies

$$\frac{d}{dt}\left(\lambda(t)e^{\frac{2}{n}\int_0^t r(\tau)\,d\tau}\right) \geq 0.$$

Therefore the quantity $\lambda(t)e^{\frac{2}{n}\int_0^t r(\tau)\,d\tau}$ is nondecreasing along the flow (1.2). Moreover, if $r(t) \leq 0$ for all $t \in [0,T)$, then the function $e^{\frac{2}{n}\int_0^t r(\tau)\,d\tau}$ is decreasing along the flow (1.2). Therefore $\lambda(t)$ is increasing under the flow. □

Remark 3.4. In Theorem 1.1, if we consider $\phi = 0$ then our theorem reduces to [14, Theorems 1.1 and 1.2]. Also, in Theorem 1.1, if we assume that $a = 1$ then our theorem reduces to [4, Theorem 1.1].

Proof of Corollary 1.2. Similarly to the proof of Theorem 1.1, if in (3.9), we set $\alpha = 0$, then for any $t \in [0,T)$ we get

$$\frac{d}{dt}\lambda(f(t),t) \geq -\frac{2r(t)}{n}\lambda(f(t),t).$$

This implies that the quantity $\lambda(t)e^{\frac{2}{n}\int_0^t r(\tau)\,d\tau}$ is nondecreasing along the flow (1.4). □

Remark 3.5. In Corollary 1.2, if we let $r = 0$ and $a = 1$, our corollary reduces to [8, Theorem 1.1]. So our result is an extension version of [8].

Proof of Corollary 1.3. In a two-dimensional Riemannian manifold we have $R_{ij} = \frac{1}{2}Rg_{ij}$. Since $\sqrt{2}|\nabla\nabla\phi| \geq |\Delta\phi|$, the condition (1.6) implies that $|R_{ij} - \frac{1}{4c-1}\phi_{ij}| \geq \frac{2\sqrt{c}}{4c-1}|\phi_{ij}|$. Therefore the result of Corollary 1.3 follows from Corollary 1.2. □

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