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THE w-CORE–EP INVERSE IN RINGS WITH INVOLUTION

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Abstract. The main goal of this paper is to present two new classes of generalized inverses in order to extend the concepts of the (dual) core–EP inverse and the (dual) w-core inverse. Precisely, we introduce the w-core–EP inverse and its dual for elements of a ring with involution. We characterize the (dual) w-core–EP invertible elements and develop several representations of the w-core–EP inverse and its dual in terms of different well-known generalized inverses. Using these results, we get new characterizations and expressions for the core–EP inverse and its dual. We apply the dual w-core–EP inverse to solve certain operator equations and give their general solution forms.

1. Introduction

Let $\mathcal{R}$ be an associative ring with unit 1. For $a \in \mathcal{R}$, we define the kernel ideals $a^\circ = \{x \in \mathcal{R} : ax = 0\}$ and ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$, and the image ideals $a\mathcal{R} = \{ax : x \in \mathcal{R}\}$ and $\mathcal{R}a = \{xa : x \in \mathcal{R}\}$.

An element $a \in \mathcal{R}$ is Drazin invertible if there exists $x \in \mathcal{R}$ such that

$$xax = x, \quad ax = xa \quad \text{and} \quad a^k = a^{k+1}x \tag{1.1}$$

for some nonnegative integer k. The Drazin inverse x of a is unique (if it exists) and denoted by a^D (see [6]). It is known that the Drazin inverse was defined in a semigroup [6] and in a semigroup without the identity we have $k > 0$, while for a semigroup with identity we have $k \geq 0$ and for $k = 0$ we define $a^0 = 1$. The smallest above mentioned k is called the Drazin index of a and denoted by $\mathrm{ind}(a)$. Recall that a^D double commutes with a, that is, $ay = ya$ implies $a^D y = ya^D$. For $\mathrm{ind}(a) = 1$, a is group invertible and its group inverse is denoted by $a^\#$. Notice that $a^\#$ satisfies $a^\# a a^\# = a^\#$, $a^\# a = aa^\#$ and $aa^\# a = a$. It is well known that $a^\#$ exists if and only if $a \in a^2\mathcal{R} \cap \mathcal{R}a^2$ if and only if $a\mathcal{R} = a^2\mathcal{R}$ and $\mathcal{R}a = \mathcal{R}a^2$ [6, 25]. The sets $\mathcal{R}^D$ and $\mathcal{R}^\#$ involve all Drazin invertible and all group invertible elements of $\mathcal{R}$, respectively.

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An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, i.e. $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{R}$. An element $p \in \mathcal{R}$ is an orthogonal projector if $p^2 = p = p^*$. Significant results related to orthogonal projectors can be seen in [16]. An element $a \in \mathcal{R}$ is Moore–Penrose invertible if there exists $x \in \mathcal{R}$ satisfying the so-called Penrose equations [26]:

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa.$$

The Moore–Penrose inverse x of a is uniquely determined (if it exists) and denoted by $x = a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

An element $x \in R$ is a $\{1\}$-inverse of $a \in R$ if $axa = a$ and, in this case, we say that a is regular. An element $x \in R$ is a $\{1,3\}$-inverse (or $\{1,4\}$-inverse) of a if $axa = a$ and $(ax)^* = ax$ ($axa = a$ and $(xa)^* = xa$). The symbol $a\{1,3\}$ (or $a\{1,4\}$) stands for the set of all $\{1,3\}$-inverses ($\{1,4\}$-inverses) of a. The set of all $\{1,3\}$-invertible ($\{1,4\}$-invertible) elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{\{1,3\}}$ ($\mathcal{R}^{\{1,4\}}$). An interesting class of $\{1\}$-inverses was studied in [4].

The notion of inverse along one element introduced by Mary [19] is important because a number of well-known generalized inverses, such as group inverse, Drazin inverse and Moore–Penrose inverse, are special cases of this inverse. For $d \in \mathcal{R}$, an element $a \in \mathcal{R}$ is invertible along d if there exists $x \in \mathcal{R}$ satisfying

$$xad = d = dax \quad \text{and} \quad x \in d\mathcal{R} \cap \mathcal{R}d.$$

The inverse x of a along d is unique (if it exists) and denoted by $a^{\|d}$ [19]. According to [19, 21], $a \in \mathcal{R}^\#$ if and only if $a^{\|a}$ exists if and only if $1^{\|a}$ exists. In addition, $a^\# = a^{\|a}$ and $1^{\|a} = aa^\#$. Also, $a \in \mathcal{R}^D$ if and only if $a^{\|a^k}$ exists for some positive integer k; and $a \in \mathcal{R}^\dagger$ if and only if $a^{\|a^*}$ exists. Furthermore, $a^D = a^{\|a^k}$ and $a^\dagger = a^{\|a^*}$. More results about the inverse along one element can be found in [2, 3, 20, 38].

The core–EP inverse was introduced in [27] for a square matrix over an arbitrary field, as an extension of the core inverse given in [1]. The core–EP inverse for elements of a ring was defined in [10] in the following way. Let $a \in \mathcal{R}$. Then a is core–EP (or pseudo core) invertible if there exists an element $x \in \mathcal{R}$ such that

$$ax^2 = x, \quad xa^{k+1} = a^k \quad \text{and} \quad (ax)^* = ax$$

for some positive integer k. The core–EP inverse of a is unique (if it exists) and denoted by $a^{\textcircled{D}}$. The smallest positive integer k in the definition of the core–EP inverse is called the pseudo core index of a and denoted by $I(a)$, either equals the Drazin index $\operatorname{ind}(a)$ of a if $\operatorname{ind}(a) > 0$, or is 1 if $\operatorname{ind}(a) = 0$ (see [10, Theorem 2.3] and observe that Gao et al. defined the Drazin index of a as the smallest positive integer k that satisfies (1.1)). Notice that a is core–EP invertible if and only if there exist a^D and $(a^k)^{(1,3)} \in a^k\{1,3\}$ for $k \geq \operatorname{ind}(a)$ [10, Theorem 2.3]. In addition, $a^{\textcircled{D}} = a^D a^k (a^k)^{(1,3)}$. The dual core–EP inverse $a_{\textcircled{D}}$ of a was introduced as the unique solution of equations $x^2a = x$, $a^{k+1}x = a^k$ and $(xa)^* = xa$ for some positive integer k. In a special case that $\operatorname{ind}(a) = 1$, the core–EP inverse of a

becomes the core inverse $a^{\textcircled{\#}} = a^{\#}aa^{\dagger}$ [1], and the dual core–EP inverse coincides with the dual core inverse $a_{\textcircled{\#}} = a^{\dagger}aa^{\#}$.

Recently, the core inverse and core–EP inverse were studied in numerous papers [5, 11, 13, 14, 15, 17, 30, 32, 39]. For instance, different properties and representations of the core–EP inverse were proved in [8, 9, 17, 18, 23, 31]; limit representations for the core-EP inverse were given in [32]; continuity of core-EP inverse was investigated in [12]; an iterative method for computing core-EP inverse was proved in [28, 29]. The core–EP inverse was extended for operators on Hilbert spaces in [22, 24] and for tensors in [30].

Two new classes of generalized inverses were recently presented in [36]. Precisely, the w-core inverse and its dual for elements of a ring with involution were introduced in [36] as generalizations of the core inverse and dual core inverse, respectively. We now state the definition of the w-core inverse. Let $a, w \in \mathcal{R}$; we say that a is w-core invertible if there exists an element $x \in \mathcal{R}$ such that

$$awx^2 = x, \quad xawa = a \quad \text{and} \quad (awx)^* = awx.$$

If such x exists, it is the uniquely determined w-core inverse of a [36] and denoted by $a_w^{\textcircled{\#}}$. Note that the 1-core inverse of a coincides with the core inverse of a, i.e. $a_1^{\textcircled{\#}} = a^{\textcircled{\#}}$. Some significant results about the w-core inverse can be found in [35].

Motivated by a number of researches and popularity of the core–EP inverse and a recent investigation about the w-core inverse, the aim of this paper is to introduce a new class of generalized inverses which includes the core–EP inverse and the w-core inverse. In particular, we present the w-core–EP inverse and its dual for elements of a ring with involution. In this way, we define two new wider classes of generalized inverses, extending the notions of the core–EP inverse, the w-core inverse and their duals. Various characterizations for the existence of the w-core–EP inverse and its dual are established as well as corresponding representations involving the inverse of w along a corresponding element, group inverse, Drazin inverse, $\{1, 3\}$-inverse and $\{1, 4\}$-inverse of adequate elements. Using these results, we obtain new characterizations and representations of the core–EP inverse and its dual. Applying the dual w-core–EP inverse, we solve several operator equations and give the forms of their general solutions.

We shortly describe the content of this paper. In Section 2, we define the w-core–EP inverse and investigate necessary and sufficient conditions for the existence of the w-core–EP inverse and its representations. New characterizations and expressions of the core–EP inverse are also given. The dual w-core–EP inverse is studied in Section 3 as well as the dual core–EP inverse. Section 4 contains applications of the dual w-core–EP inverse in solving some operator matrix equations.

2. The w-core–EP inverse

In order to extend the notions of the core–EP inverse and the w-core inverse, we define the w-core–EP inverse in a ring $\mathcal{R}$ with involution.

Definition 2.1. Let $a, w \in \mathcal{R}$. Then a is called w-core–EP invertible if there exists an element $x \in \mathcal{R}$ such that

$$awx^2 = x, \quad x(aw)^{k+1}a = (aw)^k a \quad \text{and} \quad (awx)^* = awx$$

for some nonnegative integer k. In this case, x is a w-core–EP inverse of a.

Observe that, for $k = 0$ in the above definition, the w-core–EP inverse becomes the w-core inverse. Notice that the 1-core-EP inverse is equal to the core–EP inverse. Thus, core–EP invertible and w-core invertible elements are w-core-EP invertible. The smallest nonnegative integer k in the definition of the w-core–EP inverse is called the w-core–EP index of a and denoted by $i_w(a)$.

Theorem 2.2. *Let $a, w \in \mathcal{R}$. Then a has at most one w-core-EP inverse.*

Proof. If x is the w-core–EP inverse of a, then $awx^2 = x$, $x(aw)^{k+1}a = (aw)^k a$ and $(awx)^* = awx$ for some nonnegative integer k. We have $x(aw)^{k+2} = (aw)^{k+1}$ and thus $x = (aw)^{Ⓓ}$. □

Since the w-core–EP inverse of a is unique, if it exists, by Theorem 2.2, we use the symbol $a_w^{Ⓓ}$ to denote the w-core–EP inverse of a.

Although core–EP invertible elements are w-core-EP invertible, the converse is not true in general. In the next example, we give a w-core-EP invertible element which is not core-EP invertible.

Example 2.3. Let $\mathcal{R} = \mathbb{Z}$ be the ring of all integers. For $a = 2$ and $w = 0$, we conclude that a is w-core–EP invertible with $a_w^{Ⓓ} = 0$. However, a is not Drazin invertible in $\mathbb{Z}$ and so it is not core–EP invertible.

Several necessary and sufficient conditions for the existence of the w-core–EP inverse are established now.

Theorem 2.4. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible;*

(ii) *there exists an element $x \in \mathcal{R}$ such that*

$$awx^2 = x, \quad x(aw)^{k+1}a = (aw)^k a, \quad xawx = x,$$

$$awx(aw)^k a = (aw)^k a \quad \text{and} \quad (awx)^* = awx$$

for some nonnegative integer k;

(iii) *there exists an element $x \in \mathcal{R}$ such that*

$$awx(aw)^k a = (aw)^k a, \quad (aw)^k a\mathcal{R} = x\mathcal{R} \quad \text{and} \quad \mathcal{R}x = \mathcal{R}\left((aw)^k a\right)^*$$

for some nonnegative integer k;

(iv) *there exists an element $x \in \mathcal{R}$ such that*

$$awx(aw)^k a = (aw)^k a \quad \text{and} \quad (aw)^k a\mathcal{R} = x\mathcal{R} = x^*\mathcal{R}$$

for some nonnegative integer k;

(v) *there exists an element* $x \in \mathcal{R}$ *such that*

$$awx(aw)^k a = (aw)^k a \quad \text{and} \quad (aw)^k a\mathcal{R} = x\mathcal{R} \supseteq x^*\mathcal{R}$$

for some nonnegative integer k*;*

(vi) *there exists an element* $x \in \mathcal{R}$ *such that*

$$awx(aw)^k a = (aw)^k a, \quad {}^\circ\left((aw)^k a\right) = {}^\circ x \quad \text{and} \quad x^\circ = \left(((aw)^k a)^*\right)^\circ$$

for some nonnegative integer k*;*

(vii) *there exists an element* $x \in \mathcal{R}$ *such that*

$$awx(aw)^k a = (aw)^k a, \quad {}^\circ\left((aw)^k a\right) = {}^\circ x \quad \text{and} \quad x^\circ \supseteq \left(((aw)^k a)^*\right)^\circ$$

for some nonnegative integer k*;*

(viii) *there exists an element* $x \in \mathcal{R}$ *such that*

$$awx^2 = x, \quad x(aw)^{k+1}a = (aw)^k a, \quad awx = (aw)^n x^n \quad \text{and} \quad (awx)^* = awx$$

for some nonnegative integer k *and all/some positive integer* n*.*

Proof. (i) ⇒ (ii): Assume that x is the w-core–EP inverse of a. Thus, for some nonnegative integer k, $awx^2 = x$, $x(aw)^{k+1}a = (aw)^k a$ and $(awx)^* = awx$. Then

$$\begin{aligned} x = awx^2 = (aw)^2x^3 = \cdots = (aw)^{k+1}x^{k+2} &= ((aw)^k a)wx^{k+2} \\ = x(aw)^{k+1}awx^{k+2} &= xawx \end{aligned}$$

and

$$(aw)^k a = x(aw)^{k+1}a = awx^2(aw)^{k+1}a = awx(aw)^k a.$$

(ii) ⇒ (iii): Using $x(aw)^{k+1}a = (aw)^k a$ and $awx^2 = x$, we have

$$(aw)^k a\mathcal{R} = x(aw)^{k+1}a\mathcal{R} \subseteq x\mathcal{R} = awx^2\mathcal{R} = (aw)^k awx^{k+2}\mathcal{R} \subseteq (aw)^k a\mathcal{R}.$$

Thus, $(aw)^k a\mathcal{R} = x\mathcal{R}$. The assumptions $awx(aw)^k a = (aw)^k a$ and $(awx)^* = awx$ imply

$$\mathcal{R}\left((aw)^k a\right)^* = \mathcal{R}\left(awx(aw)^k a\right)^* = \mathcal{R}\left((aw)^k a\right)^* awx \subseteq \mathcal{R}x.$$

Since $awx = (aw)^{k+1}x^{k+1}$, we have

$$\begin{aligned} x = xawx = x(awx)^* &= x\left((aw)^{k+1}x^{k+1}\right)^* \\ &= x\left((aw)^k awx^{k+1}\right)^* = x\left(wx^{k+1}\right)^*\left((aw)^k a\right)^*, \end{aligned}$$

which gives $\mathcal{R}x \subseteq \mathcal{R}\left((aw)^k a\right)^*$. Hence, $\mathcal{R}x = \mathcal{R}\left((aw)^k a\right)^*$.

(iii) ⇒ (iv) ⇒ (v): It is evident.

(v) ⇒ (i): From $awx(aw)^k a = (aw)^k a$ and $(aw)^k a\mathcal{R} = x\mathcal{R}$, we get, for some $u \in \mathcal{R}$,

$$x = (aw)^k au = awx\left((aw)^k au\right) = awx^2.$$

The hypothesis $(aw)^k a\mathcal{R} \supseteq x^*\mathcal{R}$ yields, for some $y \in \mathcal{R}$,

$$x = y\left((aw)^k a\right)^* = y\left(awx(aw)^k a\right)^* = y\left((aw)^k a\right)^*(awx)^* = x(awx)^*.$$

Further, $awx = awx(awx)^*$ implies that $(awx)^* = awx$ and so $x = xawx$. Because $(aw)^k a = xv$ for some $v \in \mathcal{R}$, we have $(aw)^k a = xv = xaw(xv) = x(aw)^{k+1}a$. Therefore, $x = a_w^{\text{Ⓓ}}$.

(iii) $\Rightarrow$ (vi) $\Rightarrow$ (vii): These implications are obvious.

(vii) $\Rightarrow$ (i): The condition $awx(aw)^k a = (aw)^k a$ gives $1 - awx \in {}^\circ\left((aw)^k a\right) = {}^\circ x$. Thus, $(1 - awx)x = 0$, i.e. $x = awx^2$. Since $((aw)^k a)^*(awx)^* = ((aw)^k a)^*$, we have $1 - (awx)^* \in \left(((aw)^k a)^*\right)^\circ \subseteq x^\circ$. Hence, $x = x(awx)^*$ yields $awx = awx(awx)^* = (awx)^*$ and $x = xawx$. Now, $1 - xaw \in {}^\circ x = {}^\circ\left((aw)^k a\right)$ implies $(aw)^k a = x(aw)^{k+1} a$. So, x is the w-core–EP inverse of a.

(i) $\Leftrightarrow$ (viii): This equivalence is clear. □

In the case that $k = 0$, notice that Theorem 2.4 recovers [36, Theorem 2.6] related to w-core invertible elements.

Remark 2.5. Let $a, b, c \in \mathcal{R}$. An element $x \in \mathcal{R}$ is a (b, c)-inverse of a if $xax = x$, $x\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}x = \mathcal{R}c$. The (b, c)-inverse of a is unique, if it exists, and denoted by $a^{\|(b,c)}$ [7]. By Theorem 2.4(iii), for $a, w \in \mathcal{R}$, we have that a is w-core–EP invertible if and only if aw is $((aw)^k a, ((aw)^k a)^*)$-invertible for some nonnegative integer k. In this case, $a_w^{\text{Ⓓ}} = (aw)^{\|((aw)^k a, ((aw)^k a)^*)}$.

Applying Theorem 2.4 for $w = 1$, we get new characterizations for core–EP invertible elements.

Remark 2.6. Let $a \in \mathcal{R}$. Then a is core–EP invertible if and only if there exists an element $x \in \mathcal{R}$ such that, for some nonnegative integer k and all/some positive integer n, one of the following equivalent statements holds:

- (i) $ax^2 = x$, $xa^{k+2} = a^{k+1}$, $xax = x$, $axa^{k+1} = a^{k+1}$ and $(ax)^* = ax$;
- (ii) $axa^{k+1} = a^{k+1}$, $a^{k+1}\mathcal{R} = x\mathcal{R}$ and $\mathcal{R}x = \mathcal{R}(a^{k+1})^*$;
- (iii) $axa^{k+1} = a^{k+1}$ and $a^{k+1}\mathcal{R} = x\mathcal{R} = x^*\mathcal{R}$;
- (iv) $axa^{k+1} = a^{k+1}$ and $a^{k+1}\mathcal{R} = x\mathcal{R} \supseteq x^*\mathcal{R}$;
- (v) $axa^{k+1} = a^{k+1}$, ${}^\circ(a^{k+1}) = {}^\circ x$ and $x^\circ = ((a^{k+1})^*)^\circ$;
- (vi) $axa^{k+1} = a^{k+1}$, ${}^\circ(a^{k+1}) = {}^\circ x$ and $x^\circ \supseteq ((a^{k+1})^*)^\circ$;
- (vii) $ax^2 = x$, $xa^{k+2} = a^{k+1}$ and $(ax)^* = ax = a^n x^n$.

By [36, Theorem 2.11], a is w-core invertible if and only if there exist $w^{\|a}$ and $a^{(1,3)} \in a\{1, 3\}$. In this case, $a_w^{\text{\#}} = w^{\|a} a^{(1,3)}$. We can develop a representation of the w-core–EP inverse in terms of the inverse along a corresponding element and $\{1, 3\}$-inverse, generalizing [36, Theorem 2.11] for the w-core inverse.

Theorem 2.7. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

- (i) *a is w-core–EP invertible;*
- (ii) *there exist $w^{\|(aw)^k a}$ and $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1, 3\}$ for some nonnegative integer k;*
- (iii) *there exist $w^{\|(aw)^k a}$ and $\left((aw)^{k+1}\right)^{(1,3)} \in \left((aw)^{k+1}\right)\{1, 3\}$ for some nonnegative integer k;*
- (iv) *there exist $w^{\|(aw)^k a}$ and $\left((aw)^{k+1} a\right)^{(1,3)} \in \left((aw)^{k+1} a\right)\{1, 3\}$ for some nonnegative integer k.*

In addition, if any of statements (i)–(iv) *holds, then, for some nonnegative integer* k *and* $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$,

$$a_w^{\textcircled{\scriptsize D}} = (aw)^k w^{\|(aw)^k a} \left((aw)^k a\right)^{(1,3)}.$$

Proof. (i) $\Rightarrow$ (ii): Let x be the w-core–EP inverse of a. Then $awx^2 = x$, $x(aw)^{k+1}a = (aw)^k a$ and $(awx)^* = awx$ for some nonnegative integer k. Because

$$\begin{aligned}(aw)^k a &= x(aw)^{k+1}a = x\left((aw)^k a\right)wa = x^2(aw)^{k+1}awa = x^2(aw)^k awawa = \dots \\ &= x^{k+1}(aw)^k aw(aw)^k a \in \mathcal{R}\left((aw)^k a\right) w\left((aw)^k a\right)\end{aligned}$$

and

$$\begin{aligned}(aw)^k a &= x(aw)^{k+1}a = awx^2(aw)^{k+1}a = \dots = (aw)^{2k+2}x^{2k+3}(aw)^{k+1}a \\ &= (aw)^k aw(aw)^k awx^{2k+3}(aw)^{k+1}a \in \left((aw)^k a\right) w \left((aw)^k a\right)\mathcal{R},\end{aligned}$$

by [21, Theorem 2.2], we deduce that $w \in \mathcal{R}^{\|(aw)^k a}$. Furthermore, from the relations

$$\begin{aligned}(aw)^k a &= (aw)^k aw(aw)^k awx^{2k+3}(aw)^{k+1}a \\ &= \left((aw)^k a\right) w(aw)^k awx^{2k+3}aw\left((aw)^k a\right)\end{aligned} \tag{2.1}$$

and

$$(aw)^k awx^{k+1} = awx = (awx)^* = \left((aw)^k awx^{k+1}\right)^*,$$

we observe that $(aw)^k a \in \mathcal{R}^{(1,3)}$.

(ii) $\Rightarrow$ (i): Suppose that $x = (aw)^k w^{\|(aw)^k a}\left((aw)^k a\right)^{(1,3)}$ for some nonnegative integer k and $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$. Notice that

$$(aw)^k a = (aw)^k aww^{\|(aw)^k a} = (aw)^{k+1}w^{\|(aw)^k a}$$

and $(aw)^k a = w^{\|(aw)^k a}w(aw)^k a = w^{\|(aw)^k a}(wa)^{k+1}$. Since $w^{\|(aw)^k a} = (aw)^k au = v(aw)^k a$ for some $u, v \in \mathcal{R}$, we get $w^{\|(aw)^k a} = (aw)^k a\left((aw)^k a\right)^{(1,3)} w^{\|(aw)^k a}$ and $w^{\|(aw)^k a} = w^{\|(aw)^k a}\left((aw)^k a\right)^{(1,3)}(aw)^k a$. Therefore,

$$awx = (aw)^{k+1}w^{\|(aw)^k a}\left((aw)^k a\right)^{(1,3)} = (aw)^k a\left((aw)^k a\right)^{(1,3)}$$

gives $(awx)^* = awx$ and

$$\begin{aligned}awx^2 &= (aw)^k a\left((aw)^k a\right)^{(1,3)}(aw)^k w^{\|(aw)^k a}\left((aw)^k a\right)^{(1,3)} \\ &= \left[(aw)^k a\left((aw)^k a\right)^{(1,3)}(aw)^k a\right](wa)^k u\left((aw)^k a\right)^{(1,3)} \\ &= (aw)^k[a(wa)^k u]\left((aw)^k a\right)^{(1,3)} = (aw)^k w^{\|(aw)^k a}\left((aw)^k a\right)^{(1,3)} \\ &= x.\end{aligned}$$

According to [21, Theorem 2.1], we have

$$w^{\|(aw)^k a} = ((aw)^{k+1})^{\#}(aw)^k a = (aw)^k((aw)^{k+1})^{\#}a.$$

So,

$$\begin{aligned} x(aw)^{k+1}a &= (aw)^k w^{\|(aw)^k a} \left((aw)^k a\right)^{(1,3)} (aw)^{k+1}a \\ &= (aw)^k \left[w^{\|(aw)^k a} \left((aw)^k a\right)^{(1,3)} (aw)^k a\right] wa = (aw)^k w^{\|(aw)^k a} wa \\ &= (aw)^k ((aw)^{k+1})^{\#} (aw)^k awa = [(aw)^k ((aw)^{k+1})^{\#} a](wa)^{k+1} \\ &= w^{\|(aw)^k a} (wa)^{k+1} = (aw)^k a \end{aligned}$$

and $x = a_w^{\textcircled{\tiny D}}$.

(ii) $\Rightarrow$ (iii): By [37, Lemma 2.2], recall that $u \in \mathcal{R}^{\{1,3\}}$ if and only if $u \in \mathcal{R}u^* u$. Since $(aw)^k a \in \mathcal{R}^{\{1,3\}}$ for some nonnegative integer k, we have $(aw)^k a \in \mathcal{R}\left((aw)^k a\right)^* (aw)^k a$, which yields $(aw)^{k+1} \in \mathcal{R}\left((aw)^k a\right)^* (aw)^{k+1}$. Notice that, by the equivalence (i) $\Leftrightarrow$ (ii), (2.1) holds. Hence, $(aw)^k a \in (aw)^{k+1}\mathcal{R}$, which gives $\left((aw)^k a\right)^* \in \mathcal{R}\left((aw)^{k+1}\right)^*$. Now $(aw)^{k+1} \in \mathcal{R}\left((aw)^{k+1}\right)^* (aw)^{k+1}$ implies that $(aw)^{k+1} \in \mathcal{R}^{\{1,3\}}$.

(iii) $\Rightarrow$ (iv): Because $(aw)^{k+1} \in \mathcal{R}^{\{1,3\}}$ gives $(aw)^{k+1} \in \mathcal{R}\left((aw)^{k+1}\right)^* (aw)^{k+1}$, then $(aw)^{k+1}a \in \mathcal{R}\left((aw)^{k+1}\right)^* (aw)^{k+1}a$. Also, $(aw)^k a = (aw)^{k+1} w^{\|(aw)^k a}$ and $w^{\|(aw)^k a} = (aw)^k au$ for some $u \in \mathcal{R}$ imply

$$(aw)^{k+1} = (aw)^k aw = (aw)^{k+1} w^{\|(aw)^k a} w = (aw)^{k+1} a(wa)^k uw.$$

Therefore, $\left((aw)^{k+1}\right)^* \in \mathcal{R}\left((aw)^{k+1}a\right)^*$ yields $(aw)^{k+1}a \in \mathcal{R}\left((aw)^{k+1}a\right)^* (aw)^{k+1}a$ and so $(aw)^{k+1}a \in \mathcal{R}^{\{1,3\}}$.

(iv) $\Rightarrow$ (ii): Since $w^{\|(aw)^k a}$ exists, by [21, p. 1132], $(aw)^k a$ is regular. Using $\left((aw)^{k+1}a\right)^{(1,3)} \in \left((aw)^{k+1}a\right)\{1,3\}$, for some nonnegative integer k, we have $(aw)^k awa\left((aw)^{k+1}a\right)^{(1,3)} = (aw)^{k+1}a\left((aw)^{k+1}a\right)^{(1,3)}$ and thus $\left((aw)^k a\right)\{1,3\} \neq \emptyset$. □

Remark 2.8. It is clear that the representation of the w-core–EP inverse given in Theorem 2.7 does not depend on the choice of $\{1,3\}$-inverse. Indeed, for $x, y \in \left((aw)^k a\right)\{1,3\}$, we have that $(aw)^k ax = (aw)^k ay$ and $w^{\|(aw)^k a} = w^{\|(aw)^k a} y(aw)^k a$, which imply $w^{\|(aw)^k a} x = w^{\|(aw)^k a} y(aw)^k ax = (w^{\|(aw)^k a} y(aw)^k a)y = w^{\|(aw)^k a} y$ and $(aw)^k w^{\|(aw)^k a} x = (aw)^k w^{\|(aw)^k a} y$.

As a consequence of Theorem 2.7, we obtain the following characterization of a core–EP invertible element and its expression based on the inverse along an element and the $\{1,3\}$-inverse.

Corollary 2.9. *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is core–EP invertible;*
(ii) *there exist $1^{\|a^{k+1}}$ and $(a^{k+1})^{(1,3)} \in (a^{k+1})\{1,3\}$ for some nonnegative integer k;*
(iii) *there exist $1^{\|a^{k+1}}$ and $\left(a^{k+2}\right)^{(1,3)} \in \left(a^{k+2}\right)\{1,3\}$ for some nonnegative integer k.*

In addition, if any of statements (i)–(ii) *holds, then, for some nonnegative integer* k *and* $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$,

$$a^{\textcircled{D}} = a^k 1^{\| a^{k+1}} (a^{k+1})^{(1,3)}.$$

By Theorem 2.7 and some properties of inverse along an element proved in [21], we can provide more characterizations of w-core–EP invertible elements.

Theorem 2.10. *Let* $a, w \in \mathcal{R}$. *Then the following statements are equivalent:*

(i) *a is w-core–EP invertible;*

(ii) $(aw)^k a \in (aw)^{k+1}\mathcal{R}$ *and there exist* $\left((aw)^{k+1}\right)^{\#}$ *and* $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$ *for some nonnegative integer k;*

(iii) $(aw)^k a \in \mathcal{R}(wa)^{k+1}$ *and there exist* $\left((wa)^{k+1}\right)^{\#}$ *and* $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$ *for some nonnegative integer k;*

(iv) $(aw)^k a \in (aw)^{2k+1}a\mathcal{R} \cap \mathcal{R}(aw)^{2k+1}a$ *and there exists* $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$ *for some nonnegative integer k.*

In addition, if any of statements (i)–(iv) *holds, then, for some nonnegative integer* k *and* $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$,

$$a_w^{\textcircled{D}} = \left((aw)^{k+1}\right)^{\#} (aw)^{2k} a \left((aw)^k a\right)^{(1,3)} = (aw)^{2k} a \left((wa)^{k+1}\right)^{\#} \left((aw)^k a\right)^{(1,3)}.$$

Proof. This result is evident by Theorem 2.7 and [21, Theorems 2.1 and 2.2]. $\square$

For $w = 1$ in Theorem 2.10, we get the next result.

Corollary 2.11. *Let* $a \in \mathcal{R}$. *Then the following statements are equivalent:*

(i) *a is core–EP invertible;*

(ii) $a^{k+1} \in \mathcal{R}^{\#} \cap \mathcal{R}^{(1,3)}$ *for some nonnegative integer k.*

In addition, if any of statements (i)–(ii) *holds, then, for some nonnegative integer* k *and* $(a^{k+1})^{(1,3)} \in a^{k+1}\{1,3\}$,

$$a^{\textcircled{D}} = (a^{k+1})^{\#} a^{2k+1} (a^{k+1})^{(1,3)} = a^k (a^{k+1})^{\textcircled{\#}}.$$

We can show that a is a core–EP invertible element if and only if a is a-core–EP invertible.

Theorem 2.12. *Let* $a \in \mathcal{R}$. *Then the following statements are equivalent:*

(i) *a is core–EP invertible;*

(ii) *a is a-core–EP invertible.*

Proof. Since a is core–EP invertible, by Corollary 2.11, $a^{k+1} \in \mathcal{R}^{\#} \cap \mathcal{R}^{(1,3)}$ for some nonnegative integer k. Then $a^{2k+2} = (a^{k+1})^2 \in \mathcal{R}^{\#}$ and $a^{2k+1} = a^k a^{k+1} = a^k (a^{k+1})^2 (a^{k+1})^{\#} \in a^{2k+2}\mathcal{R}$. For $y \in a^{k+1}\{1,3\}$, the equalities $a^{k+1} y a^{k+1} = a^{k+1}$ and $a^{k+1} y = (a^{k+1} y)^*$ imply $a^{2k+1} a (a^{k+1})^{\#} y a^{2k+1} = a^{2k+1}$ and

$$a^{2k+1} a (a^{k+1})^{\#} y = a^{k+1} y = (a^{k+1} y)^* = (a^{2k+1} a (a^{k+1})^{\#} y)^*,$$

i.e. $a(a^{k+1})^{\#} y \in a^{2k+1}\{1,3\}$. Using Theorem 2.10, we deduce that a is a-core–EP invertible.

If a is a-core EP-invertible, then, by Theorem 2.7, $a^{\|a^{2k+1}}$ exists and $a^{2k+1} \in \mathcal{R}^{(1,3)}$ for some nonnegative integer k. Since $a^{\|a^{2k+1}}$ exists, we have that $a \in R^D$ with $\text{ind}(a) \leq 2k+1$. So, by [10, Theorem 2.3], a is core-EP invertible. □

Consequently, when $w = a$ in Theorem 2.4, Theorem 2.7 and Theorem 2.10, we present a list of characterizations for core–EP invertible element using Theorem 2.12.

Corollary 2.13. *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is core–EP invertible;*

(ii) *there exist $a^{\|a^{2k+1}}$ and $(a^{2k+1})^{(1,3)} \in a^{2k+1}\{1,3\}$ for some nonnegative integer k;*

(iii) *there exists an element $x \in \mathcal{R}$ such that*

$$a^2x^2 = x, \quad xa^{2k+3} = a^{2k+1} \quad \text{and} \quad (a^2x)^* = a^2x$$

for some nonnegative integer k;

(iv) *there exists an element $x \in \mathcal{R}$ such that*

$$a^2x^2 = x, \quad xa^{2k+3} = a^{2k+1}, \quad xa^2x = x, \quad a^2xa^{2k+1} = a^{2k+1} \quad \text{and} \quad (a^2x)^* = a^2x$$

for some nonnegative integer k;

(v) *there exists an element $x \in \mathcal{R}$ such that*

$$a^2xa^{2k+1} = a^{2k+1}, \quad a^{2k+1}\mathcal{R} = x\mathcal{R} \quad \text{and} \quad \mathcal{R}x = \mathcal{R}(a^{2k+1})^*$$

for some nonnegative integer k;

(vi) *there exists an element $x \in \mathcal{R}$ such that*

$$a^2xa^{2k+1} = a^{2k+1} \quad \text{and} \quad a^{2k+1}a\mathcal{R} = x\mathcal{R} = x^*\mathcal{R}$$

for some nonnegative integer k;

(vii) *there exists an element $x \in \mathcal{R}$ such that*

$$a^2xa^{2k+1} = a^{2k+1} \quad \text{and} \quad a^{2k+1}\mathcal{R} = x\mathcal{R} \supseteq x^*\mathcal{R}$$

for some nonnegative integer k;

(viii) *there exists an element $x \in \mathcal{R}$ such that*

$$a^2xa^{2k+1} = a^{2k+1}, \quad {}^\circ(a^{2k+1}) = {}^\circ x \quad \text{and} \quad x^\circ = \left((a^{2k+1})^*\right)^\circ$$

for some nonnegative integer k;

(ix) *there exists an element $x \in \mathcal{R}$ such that*

$$a^2xa^{2k+1} = a^{2k+1}, \quad {}^\circ(a^{2k+1}) = {}^\circ x \quad \text{and} \quad x^\circ \supseteq \left((a^{2k+1})^*\right)^\circ$$

for some nonnegative integer k;

(x) *$a^{2k+1} \in a^{2k+2}\mathcal{R}$ and there exist $(a^{2k+2})^\#$ and $(a^{2k+1})^{(1,3)} \in a^{2k+1}\{1,3\}$ for some nonnegative integer k;*

(xi) *$a^{2k+1} \in \mathcal{R}a^{2k+2}$ and there exist $(a^{2k+2})^\#$ and $(a^{2k+1})^{(1,3)} \in a^{2k+1}\{1,3\}$ for some nonnegative integer k;*

(xii) *$a^{2k+1} \in a^{4k+3}\mathcal{R} \cap \mathcal{R}a^{4k+3}$ and there exists $(a^{2k+1})^{(1,3)} \in a^{2k+1}\{1,3\}$ for some nonnegative integer k.*

In addition, if any of statements (i)–(xii) *holds, then, for some nonnegative integer k and $(a^{2k+1})^{(1,3)} \in (a^{2k+1})\{1,3\}$,*

$$a_a^{\circledD} = a^{2k}a^{\|a^{2k+1}}(a^{2k+1})^{(1,3)} = (a^{2k+1})^{\#}a^{4k+1}(a^{2k+1})^{(1,3)}.$$

It is interesting to observe that a being w-core–EP invertible is equivalent to aw being core–EP invertible.

Theorem 2.14. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible;*
(ii) *aw is core–EP invertible;*
(iii) *there exist $(aw)^D$ and $\left((aw)^k\right)^{(1,3)} \in (aw)^k\{1,3\}$ for $k \geq \operatorname{ind}(aw)$;*
(iv) *there exist $(aw)^D$ and the unique orthogonal projector $p \in \mathcal{R}$ such that $p\mathcal{R} = (aw)^k a\mathcal{R}$ for $k \geq \operatorname{ind}(aw)$.*

In addition, if any of statements (i)–(ii) *holds, then $i_w(a) \leq I(aw) \leq i_w(a) + 1$ and, for $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$,*

$$a_w^{\circledD} = (aw)^{\circledD} = (aw)^D p = (aw)^D (aw)^k a \left((aw)^k a\right)^{(1,3)}.$$

Proof. (i) $\Rightarrow$ (ii): It is clear by Theorem 2.2.

(ii) $\Rightarrow$ (i): If x is the core–EP inverse of aw, then $awx^2 = x$, $x(aw)^{k+1} = (aw)^k$ and $(awx)^* = awx$ for some positive integer k. Because $x(aw)^{k+1}a = (aw)^k a$, we conclude that x is the w-core–EP inverse of a.

(ii) $\Leftrightarrow$ (iii): This equivalence follows by [10, Theorem 2.3].

(iii) $\Rightarrow$ (iv): For $k \geq \operatorname{ind}(aw)$ and $\left((aw)^k\right)^{(1,3)} \in (aw)^k\{1,3\}$, we observe that $y = w(aw)^D \left((aw)^k\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$ by

$$(aw)^k ay = (aw)^k aw(aw)^D \left((aw)^k\right)^{(1,3)} = (aw)^k \left((aw)^k\right)^{(1,3)}$$

and

$$(aw)^k ay(aw)^k a = (aw)^k \left((aw)^k\right)^{(1,3)} (aw)^k a = (aw)^k a.$$

Set $p = (aw)^k ay$. Hence, $p = p^* = p^2$ and $p\mathcal{R} = (aw)^k ay\mathcal{R} = (aw)^k a\mathcal{R}$.

To prove the uniqueness of p, let two orthogonal projectors p and p_1 satisfy $p\mathcal{R} = (aw)^k a\mathcal{R} = p_1\mathcal{R}$. Then $p = p_1 p$ and $p_1 = pp_1$ gives $p = p^* = (p_1 p)^* = pp_1 = p_1$.

(iv) $\Rightarrow$ (i): Because there exist $(aw)^D$ and the unique orthogonal projector $p \in \mathcal{R}$ such that $p\mathcal{R} = (aw)^k a\mathcal{R}$ for $k \geq \operatorname{ind}(aw)$, we have $p = (aw)^k au$ for some $u \in \mathcal{R}$, and $(aw)^k a = p(aw)^k a$. Therefore, $(aw)^k a = (aw)^k au(aw)^k a$ and $((aw)^k au)^* = p = (aw)^k au$, that is, $(aw)^k a \in \mathcal{R}^{(1,3)}$. We now observe that $p = (aw)^k au = (aw)^k a \left((aw)^k a\right)^{(1,3)} (aw)^k au = (aw)^k a \left((aw)^k a\right)^{(1,3)} p$, where $\left((aw)^k a\right)^{(1,3)} \in (aw)^k\{1,3\}$. So,

$$p = p^* = p(aw)^k a \left((aw)^k a\right)^{(1,3)} = (aw)^k a \left((aw)^k a\right)^{(1,3)}.$$

Denote by $x = (aw)^D p = (aw)^D (aw)^k a \left((aw)^k a\right)^{(1,3)}$. From the relations

$$awx = (aw(aw)^D (aw)^k)a \left((aw)^k a\right)^{(1,3)} = (aw)^k a \left((aw)^k a\right)^{(1,3)} = p,$$

$$awx^2 = px = \left[(aw)^k a \left((aw)^k a\right)^{(1,3)} (aw)^k a\right] w((aw)^D)^2 a \left((aw)^k a\right)^{(1,3)}$$
$$= (aw)^k (aw)^D a \left((aw)^k a\right)^{(1,3)} = x$$

and

$$x(aw)^{k+1}a = (aw)^D p(aw)^{k+1}a = (aw)^D (aw)^{k+1}a = (aw)^k a,$$

we deduce that x is the w-core–EP inverse of a. □

As a consequence of Theorem 2.14 and [34, Theorem 4.4], we develop one more representation for the w-core–EP inverse.

Corollary 2.15. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible;*
(ii) $\mathcal{R} = \mathcal{R}(aw)^k \oplus {}^\circ((aw)^k) = \mathcal{R}((aw)^k)^* \oplus {}^\circ((aw)^k)$ *for some positive integer k;*
(iii) $\mathcal{R} = (aw)^k\mathcal{R} \oplus ((aw)^k)^\circ = \mathcal{R}((aw)^k)^* \oplus {}^\circ((aw)^k)$ *for some positive integer k.*

In addition, if any of statements (i)–(iii) *holds, then $a_w^{\textcircled{D}} = (aw)^{2k-1}b^2a^ks^*$, where $b, s \in \mathcal{R}$, $c \in ((aw)^k)^\circ$ and $t \in {}^\circ((aw)^k)$ such that $(aw)^kb + c = s((aw)^k)^* + t = 1$.*

Proof. (i) ⇔ (ii) ⇔ (iii): These equivalences follow by Theorem 2.14 and [34, Theorem 4.4]. □

Under the assumption $(aw)^ka \in \mathcal{R}^\dagger$, we prove that the w-core–EP inverse of a is equal to the inverse of aw along $(aw)^ka((aw)^ka)^*$.

Theorem 2.16. *Let $a, w \in \mathcal{R}$ such that $(aw)^ka \in \mathcal{R}^\dagger$ for some nonnegative integer k. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible with $i_w(a) = k$;*
(ii) *aw is invertible along $(aw)^ka((aw)^ka)^*$.*

In addition, if any of statements (i)–(ii) *holds, then $a_w^{\textcircled{D}} = (aw)^{\|(aw)^ka((aw)^ka)^*}$.*

Proof. (i) ⇒ (ii): For $d = (aw)^ka((aw)^ka)^*$ and $x = a_w^{\textcircled{D}}$, we have

$$xawd = x(aw)^{k+1}a((aw)^ka)^* = (aw)^ka((aw)^ka)^* = d$$

and

$$dawx = (awxd)^* = (awx(aw)^ka((aw)^ka)^*)^* = ((aw)^ka((aw)^ka)^*)^* = d^* = d.$$

Applying Theorem 2.4 and the hypothesis $(aw)^ka \in \mathcal{R}^\dagger$, it is clear that $x \in (aw)^ka\mathcal{R} \cap \mathcal{R}\left((aw)^ka\right)^* = (aw)^ka\left((aw)^ka\right)^*\mathcal{R} \cap \mathcal{R}(aw)^ka\left((aw)^ka\right)^* = d\mathcal{R} \cap \mathcal{R}d$. So, we deduce that $x = (aw)^{\|(aw)^ka((aw)^ka)^*}$.

(ii) ⇒ (i): Let $x = (aw)^{\|(aw)^ka((aw)^ka)^*}$ and $d = (aw)^ka((aw)^ka)^*$. Then $x\mathcal{R} = d\mathcal{R} = (aw)^ka\mathcal{R}$ and $\mathcal{R}x = \mathcal{R}d = \mathcal{R}((aw)^ka)^*$. We observe that $(aw)^ka((aw)^ka)^* =$

$d = dawx = (aw)^k a((aw)^k a)^* awx$ and $x = du = (aw)^k a((aw)^k a)^* u$ for some $u \in \mathcal{R}$, which imply

$$\begin{aligned} awx &= (aw)^{k+1} a((aw)^k a)^* u = (aw)^k a((aw)^k a)^\dagger ((aw)^{k+1} a((aw)^k a)^* u) \\ &= (aw)^k a((aw)^k a)^\dagger awx = [((aw)^k a)^\dagger]^* ((aw)^k a)^\dagger ((aw)^k a((aw)^k a)^* awx) \\ &= [((aw)^k a)^\dagger]^* ((aw)^k a)^\dagger (aw)^k a((aw)^k a)^* = (aw)^k a((aw)^k a)^\dagger. \end{aligned}$$

Thus, $(awx)^* = awx$. Since

$$\begin{aligned} ((aw)^k a)^* &= ((aw)^k a)^\dagger ((aw)^k a((aw)^k a)^*) = ((aw)^k a)^\dagger (aw)^k a((aw)^k a)^* awx \\ &= ((aw)^k a)^* awx, \end{aligned}$$

we get $(aw)^k a = awx(aw)^k a$. By Theorem 2.4, we conclude that $x = a_w^{\textcircled{\tiny D}}$. ☐

We also verify that a being w-core–EP invertible implies that $awa_w^{\textcircled{\tiny D}}a$ is w-core invertible.

Theorem 2.17. *Let $a, w \in \mathcal{R}$. If a is w-core–EP invertible, then $awa_w^{\textcircled{\tiny D}}a$ is w-core invertible and*

$$(awa_w^{\textcircled{\tiny D}}a)_w^{\textcircled{\#}} = a_w^{\textcircled{\tiny D}}.$$

Proof. Suppose that a is w-core–EP invertible and $a' = awa_w^{\textcircled{\tiny D}}a$. Then $aw(a_w^{\textcircled{\tiny D}})^2 = a_w^{\textcircled{\tiny D}}$, $a_w^{\textcircled{\tiny D}}(aw)^{k+1}a = (aw)^k a$ and $(awa_w^{\textcircled{\tiny D}})^* = awa_w^{\textcircled{\tiny D}}$ for some nonnegative integer k. Now, $a'wa_w^{\textcircled{\tiny D}} = aw(a_w^{\textcircled{\tiny D}}awa_w^{\textcircled{\tiny D}}) = awa_w^{\textcircled{\tiny D}}$, which yields $(a'wa_w^{\textcircled{\tiny D}})^* = (awa_w^{\textcircled{\tiny D}})^* = awa_w^{\textcircled{\tiny D}} = a'wa_w^{\textcircled{\tiny D}}$ and $a'w(a_w^{\textcircled{\tiny D}})^2 = aw(a_w^{\textcircled{\tiny D}})^2 = a_w^{\textcircled{\tiny D}}$. Furthermore, since

$$\begin{aligned} a_w^{\textcircled{\tiny D}}a'wa' &= (a_w^{\textcircled{\tiny D}}awa_w^{\textcircled{\tiny D}})aw(awa_w^{\textcircled{\tiny D}})a = a_w^{\textcircled{\tiny D}}aw(aw)^{k+1}(a_w^{\textcircled{\tiny D}})^{k+1}a \\ &= (a_w^{\textcircled{\tiny D}}(aw)^{k+1}a)w(a_w^{\textcircled{\tiny D}})^{k+1}a = (aw)^k aw(a_w^{\textcircled{\tiny D}})^{k+1}a \\ &= awa_w^{\textcircled{\tiny D}}a = a', \end{aligned}$$

we deduce that $(awa_w^{\textcircled{\tiny D}}a)_w^{\textcircled{\#}} = a_w^{\textcircled{\tiny D}}$. ☐

3. The dual w-core–EP inverse

This section is dedicated to investigating the dual w-core–EP inverse.

Definition 3.1. Let $a, w \in \mathcal{R}$. Then a is called dual w-core–EP invertible if there exists an element $x \in \mathcal{R}$ such that

$$x^2wa = x, \quad (aw)^{k+1}ax = (aw)^k a \quad \text{and} \quad (xwa)^* = xwa$$

for some nonnegative integer k. In this case, x is a dual w-core–EP inverse of a.

When $k = 0$ in the above definition, the dual w-core–EP inverse coincides with the dual w-core inverse. Also, the dual 1-core-EP inverse is the dual core–EP inverse, i.e. dual core–EP invertible elements are w-core-EP invertible. The smallest nonnegative integer k in the definition of the dual w-core–EP inverse is called the dual w-core–EP index of a and denoted by $i'_w(a)$.

As in Theorem 2.2, we can check the following result.

Theorem 3.2. *Let $a, w \in \mathcal{R}$. Then a has at most one dual w-core-EP inverse.*

Thus, if the dual w-core-EP inverse of a exists, it is unique and denoted by $a_{Ⓓ,w}$.

Lemma 3.3. *Let $a, w \in \mathcal{R}$. Then a is dual w-core–EP invertible if and only if a^* is w^*-core–EP invertible. In addition, $(a_{Ⓓ,w})^* = (a^*)^{Ⓓ}_{w^*}$ and $i'_w(a) = i_{w^*}(a^*)$.*

Proof. Note that x is the dual w-core-EP inverse of a if and only if $x^2wa = x$, $(aw)^{k+1}ax = (aw)^ka$ and $(xwa)^* = xwa$ for some nonnegative integer k, which is equivalent to $a^*w^*(x^*)^2 = x^*$, $x^*(a^*w^*)^{k+1}a^* = (a^*w^*)^ka^*$ and $(a^*w^*x^*)^* = a^*w^*x^*$ for some nonnegative integer k, that is, x^* is the w^*-core–EP inverse of a^*. □

Note that, for $w = 1$, Lemma 3.3 recovers the well-known fact that a is dual core–EP invertible if and only if a^* is core–EP invertible [10]. In this case, $(a_{Ⓓ})^* = (a^*)^{Ⓓ}$.

Using Theorem 2.4 and Lemma 3.3, we can present the next characterizations of dual w-core–EP invertible elements.

Theorem 3.4. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is dual w-core–EP invertible;*

(ii) *there exists an element $x \in \mathcal{R}$ such that*

$$x^2wa = x, \quad (aw)^{k+1}ax = (aw)^ka, \quad xwax = x,$$
$$(aw)^kaxwa = (aw)^ka \quad \text{and} \quad (xwa)^* = xwa$$

for some nonnegative integer k;

(iii) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^kaxwa = (aw)^ka, \quad \mathcal{R}(aw)^ka = \mathcal{R}x \quad \text{and} \quad x\mathcal{R} = \left((aw)^ka\right)^* \mathcal{R}$$

for some nonnegative integer k;

(iv) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^kaxwa = (aw)^ka \quad \text{and} \quad \left((aw)^ka\right)^* \mathcal{R} = x\mathcal{R} = x^*\mathcal{R}$$

for some nonnegative integer k;

(v) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^kaxwa = (aw)^ka \quad \text{and} \quad \left((aw)^ka\right)^* \mathcal{R} = x^*\mathcal{R} \supseteq x\mathcal{R}$$

for some nonnegative integer k;

(vi) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^kaxwa = (aw)^ka, \quad \left((aw)^ka\right)^\circ = x^\circ \quad \text{and} \quad {}^\circ x = {}^\circ\left(((aw)^ka)^*\right)$$

for some nonnegative integer k;

(vii) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^kaxwa = (aw)^ka, \quad \left((aw)^ka\right)^\circ = x^\circ \quad \text{and} \quad {}^\circ x \supseteq {}^\circ\left(((aw)^ka)^*\right)$$

for some nonnegative integer k;

(viii) *there exists an element* $x \in \mathcal{R}$ *such that*

$$x^2wa = x, \quad (aw)^{k+1}ax = (aw)^k a, \quad xwa = x^n(wa)^n \quad \text{and} \quad (xwa)^* = xwa$$

for some nonnegative integer k *and all/some positive integer* n.

Consequently, we have the following result concerning dual core–EP invertible elements.

Corollary 3.5. *Let* $a \in \mathcal{R}$. *Then the following statements are equivalent:*

(i) a *is dual core–EP invertible;*

(ii) *there exists an element* $x \in \mathcal{R}$ *such that*

$$x^2a = x, \quad a^{k+2}x = a^{k+1}, \quad xax = x, \quad a^{k+1}xa = a^{k+1} \quad \text{and} \quad (xa)^* = xa$$

for some nonnegative integer k*;*

(iii) *there exists an element* $x \in \mathcal{R}$ *such that*

$$a^{k+1}xa = a^{k+1}, \quad \mathcal{R}a^{k+1} = \mathcal{R}x \quad \text{and} \quad x\mathcal{R} = (a^{k+1})^*\mathcal{R}$$

for some nonnegative integer k*;*

(iv) *there exists an element* $x \in \mathcal{R}$ *such that*

$$a^{k+1}xa = a^{k+1} \quad \text{and} \quad (a^{k+1})^*\mathcal{R} = x\mathcal{R} = x^*\mathcal{R}$$

for some nonnegative integer k*;*

(v) *there exists an element* $x \in \mathcal{R}$ *such that*

$$a^{k+1}xa = a^{k+1} \quad \text{and} \quad (a^{k+1})^*\mathcal{R} = x^*\mathcal{R} \supseteq x\mathcal{R}$$

for some nonnegative integer k*;*

(vi) *there exists an element* $x \in \mathcal{R}$ *such that*

$$a^{k+1}xa = a^{k+1}, \quad (a^{k+1})^\circ = x^\circ \quad \text{and} \quad {}^\circ x = {}^\circ\left((a^{k+1})^*\right)$$

for some nonnegative integer k*;*

(vii) *there exists an element* $x \in \mathcal{R}$ *such that*

$$a^{k+1}xa = a^{k+1}, \quad (a^{k+1})^\circ = x^\circ \quad \text{and} \quad {}^\circ x \supseteq {}^\circ\left((a^{k+1})^*\right)$$

for some nonnegative integer k*;*

(viii) *there exists an element* $x \in \mathcal{R}$ *such that*

$$x^2a = x, \quad a^{k+2}x = a^{k+1} \quad \text{and} \quad (xa)^* = xa = x^na^n$$

for some nonnegative integer k *and all/some positive integer* n.

Based on $w^{\|(aw)^ka}$ and $\left((aw)^ka\right)^{(1,4)}$, we give an expression for the w-core–EP inverse of a.

Theorem 3.6. *Let* $a, w \in \mathcal{R}$. *Then the following statements are equivalent:*

(i) a *is dual* w*-core–EP invertible;*

(ii) *there exist* $w^{\|(aw)^ka}$ *and* $\left((aw)^ka\right)^{(1,4)} \in \left((aw)^ka\right)\{1,4\}$ *for some nonnegative integer* k*;*

(iii) *there exist* $w^{\|(aw)^ka}$ *and* $\left((aw)^{k+1}\right)^{(1,4)} \in \left((aw)^{k+1}\right)\{1,4\}$ *for some nonnegative integer* k*;*

(iv) *there exist* $w^{\|(aw)^k a}$ *and* $\left((aw)^{k+1}a\right)^{(1,4)} \in \left((aw)^{k+1}a\right)\{1,4\}$ *for some nonnegative integer* k*;*
(v) $(aw)^k a \in (aw)^{k+1}\mathcal{R}$ *and there exist* $\left((aw)^{k+1}\right)^{\#}$ *and* $\left((aw)^k a\right)^{(1,4)} \in \left((aw)^k a\right)\{1,4\}$ *for some nonnegative integer* k*;*
(vi) $(aw)^k a \in \mathcal{R}(wa)^{k+1}$ *and there exist* $\left((wa)^{k+1}\right)^{\#}$ *and* $\left((aw)^k a\right)^{(1,4)} \in \left((aw)^k a\right)\{1,4\}$ *for some nonnegative integer* k*;*
(vii) $(aw)^k a \in (aw)^{2k+1}a\mathcal{R} \cap \mathcal{R}(aw)^{2k+1}a$ *and there exists* $\left((aw)^k a\right)^{(1,4)} \in \left((aw)^k a\right)\{1,4\}$ *for some nonnegative integer* k*.*

In addition, if any of statements (i)–(ii) *holds, then, for some nonnegative integer* k *and* $\left((aw)^k a\right)^{(1,4)} \in \left((aw)^k a\right)\{1,4\}$,

$$a_{\textcircled{D},w} = \left((aw)^k a\right)^{(1,4)} w^{\|(aw)^k a}(wa)^k = \left((aw)^k a\right)^{(1,4)} (aw)^{2k}a\left((wa)^{k+1}\right)^{\#}$$
$$= \left((aw)^k a\right)^{(1,4)} \left((aw)^{k+1}\right)^{\#} (aw)^{2k}a.$$

Now, we get new representations for the dual core–EP inverse.

Corollary 3.7. *Let* $a \in \mathcal{R}$*. Then the following statements are equivalent:*

(i) *a is dual core–EP invertible;*
(ii) *there exist* $1^{\|a^{k+1}}$ *and* $(a^{k+1})^{(1,4)} \in (a^{k+1})\{1,4\}$ *for some nonnegative integer* k
(iii) *there exist* $1^{\|a^{k+1}}$ *and* $\left(a^{k+2}\right)^{(1,4)} \in \left(a^{k+2}\right)\{1,4\}$ *for some nonnegative integer* k*;*
(iv) $a^{k+1} \in \mathcal{R}^{\#} \cap \mathcal{R}^{(1,4)}$ *for some nonnegative integer* k*.*

In addition, if any of statements (i)–(ii) *holds, then, for some nonnegative integer* k *and* $(a^{k+1})^{(1,4)} \in (a^{k+1})\{1,4\}$,

$$a_{\textcircled{D}} = (a^{k+1})^{(1,4)}1^{\|a^{k+1}}a^k = (a^{k+1})^{(1,4)}a^{2k+1}(a^{k+1})^{\#} = (a^{k+1})_{\textcircled{\#}}a^k.$$

Theorem 2.7 and Theorem 3.6 imply the following result.

Corollary 3.8. *Let* $a, w \in \mathcal{R}$*. Then the following statements are equivalent:*

(i) *a is both w-core–EP invertible and dual w-core–EP invertible;*
(ii) *there exist* $w^{\|(aw)^k a}$ *and* $\left((aw)^k a\right)^{\dagger}$ *for some nonnegative integer* k*;*
(iii) *there exist* $w^{\|(aw)^k a}$ *and* $\left((aw)^{k+1}\right)^{\dagger}$ *for some nonnegative integer* k*;*
(iv) *there exist* $w^{\|(aw)^k a}$ *and* $\left((aw)^{k+1}a\right)^{\dagger}$ *for some nonnegative integer* k*.*

Clearly, we have the next relation between dual w-core–EP invertibility of a and core–EP invertibility of wa.

Theorem 3.9. *Let* $a, w \in \mathcal{R}$*. Then the following statements are equivalent:*

(i) *a is dual w-core–EP invertible;*
(ii) *wa is dual core–EP invertible;*
(iii) *there exist* $(wa)^D$ *and* $\left((aw)^k\right)^{(1,4)} \in (aw)^k\{1,4\}$ *for* $k \geq \operatorname{ind}(wa)$*;*

(iv) *there exist* $(wa)^D$ *and the unique orthogonal projector* $p \in \mathcal{R}$ *such that* $p\mathcal{R} = \left((aw)^k a\right)^* \mathcal{R}$ *for* $k \geq \operatorname{ind}(aw)$.

In addition, if any of statements (i)–(ii) *holds, then, for* $\left((aw)^k a\right)^{(1,4)} \in \left((aw)^k a\right)\{1,4\}$,

$$a_{\textcircled{\tiny D},w} = (wa)_{\textcircled{\tiny D}} = p(wa)^D = \left((aw)^k a\right)^{(1,4)} (aw)^k a(wa)^D.$$

Theorem 3.10. *Let* $a, w \in \mathcal{R}$ *be such that* $(aw)^k a \in \mathcal{R}^\dagger$ *for some nonnegative integer* k. *Then the following statements are equivalent:*

(i) *a is dual w-core–EP invertible with* $i'_w(a) = k$;
(ii) *wa is invertible along* $((aw)^k a)^*(aw)^k a$.

In addition, if any of statements (i)–(ii) *holds, then* $a_{\textcircled{\tiny D},w} = (wa)^{\|((aw)^k a)^*(aw)^k a}$.

Note that the dual w-core–EP invertibility of a gives dual w-core invertibility of an adequate element.

Theorem 3.11. *Let* $a, w \in \mathcal{R}$. *If a is dual w-core–EP invertible, then* $aa_{\textcircled{\tiny D},w}wa$ *is dual w-core invertible and*

$$(aa_{\textcircled{\tiny D},w}wa)_{\textcircled{\tiny \#},w} = a_{\textcircled{\tiny D},w}.$$

We also consider characterizations of dual a^*-core–EP invertibility. Recall that, by [33, Theorem 3.12], $a \in \mathcal{R}$ is Moore–Penrose invertible if and only if $a \in aa^*a\mathcal{R}$ if and only if $a \in \mathcal{R}aa^*a$.

Theorem 3.12. *Let* $a \in \mathcal{R}$. *Then the following statements are equivalent:*

(i) *a is dual a^*-core–EP invertible;*
(ii) $(aa^*)^k a$ *is Moore–Penrose invertible for some nonnegative integer k;*
(iii) *a is a^*-core–EP invertible.*

Proof. (i) ⇒ (ii): Since a is dual a^*-core–EP invertible, by Theorem 3.6, $(a^*)^{\|(aa^*)^k a}$ exists for some nonnegative integer k. So,

$$(aa^*)^k a \in (aa^*)^k aa^*(aa^*)^k a\mathcal{R} = (aa^*)^{2k+1} a\mathcal{R},$$

which gives $(aa^*)^k a \in (aa^*)^{k+1}(aa^*)^k a\mathcal{R} \subseteq (aa^*)^{k+1}(aa^*)^{2k+1}a\mathcal{R} = (aa^*)^{3k+2}a\mathcal{R}$. According to [33, Theorem 3.12], we deduce that $(aa^*)^k a$ is Moore–Penrose invertible.

(ii) ⇒ (iii): If $(aa^*)^k a$ is Moore–Penrose invertible, by [33, Theorem 3.12], $(aa^*)^k a \in (aa^*)^{3k+1}a\mathcal{R} \cap \mathcal{R}(aa^*)^{3k+1}a \subseteq (aa^*)^{2k+1}a\mathcal{R} \cap \mathcal{R}(aa^*)^{2k+1}a$. Thus, $(a^*)^{\|(aa^*)^k a}$ exists and, by Theorem 2.7, a is a^*-core–EP invertible.

(iii) ⇒ (i): The hypothesis a that is a^*-core–EP invertible and Theorem 2.7 imply that $(aa^*)^k a$ is Moore–Penrose invertible as in the implication (i) ⇒ (ii). Using Theorem 3.6, we conclude that a is dual a^*-core–EP invertible. □

4. Applications of the dual w-core–EP inverse

We can investigate solvability of some equations applying the dual w-core–EP inverse. Precisely, we solve some operator equations using the following notations in this section. Let $\mathcal{B}(X,Y)$ be the set of all bounded linear operators from X to Y, where X and Y are arbitrary Hilbert spaces. Especially, $\mathcal{B}(X,X)=\mathcal{B}(X)$. For $W\in\mathcal{B}(Y,X)$ and $A\in\mathcal{B}(X,Y)$, according to [22], observe that Drazin invertibility of WA (or, equivalently, W-weighted Drazin invertibility of A) implies the existence of $A_{\textcircled{D},W}\in\mathcal{B}(X)$. Notice that, for complex rectangular matrices A and W of appropriated sizes, $A_{\textcircled{D},W}$ always exists.

Theorem 4.1. *Let $W\in\mathcal{B}(Y,X)$ and $A\in\mathcal{B}(X,Y)$ be such that WA is Drazin invertible and $i'_W(A)=k$. For $b\in X$, the equation*

$$(AW)^{k+1}Ax=(AW)^kAb \tag{4.1}$$

is consistent and its general solution is

$$x=A_{\textcircled{D},W}b+(I-A_{\textcircled{D},W}WA)y \tag{4.2}$$

for arbitrary $y\in X$.

Proof. Assume that x has the form (4.2). Then

$$(AW)^{k+1}Ax=(AW)^{k+1}AA_{\textcircled{D},W}b+(AW)^{k+1}A(I-A_{\textcircled{D},W}WA)y=(AW)^kAb,$$

which shows that x is a solution to (4.1).

If x is a solution to (4.1), by the properties of the dual w-core–EP inverse $A_{\textcircled{D},W}$, we obtain

$$\begin{aligned}A_{\textcircled{D},W}b&=A^2_{\textcircled{D},W}WAb=A^{k+2}_{\textcircled{D},W}(WA)^{k+1}b=A^{k+2}_{\textcircled{D},W}W((AW)^kAb)\\&=A^{k+2}_{\textcircled{D},W}W(AW)^{k+1}Ax=A^{k+2}_{\textcircled{D},W}(WA)^{k+2}x\\&=A_{\textcircled{D},W}WAx.\end{aligned}$$

Therefore,

$$x=A_{\textcircled{D},W}b+x-A_{\textcircled{D},W}WAx=A_{\textcircled{D},W}b+(I-A_{\textcircled{D},W}WA)x,$$

i.e. x has the form (4.2). □

In the case that $A_{\textcircled{\#},W}$ exists, we obtain the next result as a particular case of Theorem 4.1 for $k=0$.

Corollary 4.2. *Let $W\in\mathcal{B}(Y,X)$ and $A\in\mathcal{B}(X,Y)$ be such that $A_{\textcircled{\#},W}$ exists. For $b\in X$, the equation*

$$AWAx=Ab$$

is consistent and its general solution is

$$x=A_{\textcircled{\#},W}b+(I-A_{\textcircled{\#},W}WA)y$$

for arbitrary $y\in X$.

When $X=Y$ and $W=I$ in Theorem 4.1 and Corollary 4.2, we get solvability of the following equations in terms of the dual core–EP inverse and dual core inverse.

Corollary 4.3. *Let $W \in \mathcal{B}(Y,X)$ and $A \in \mathcal{B}(X,Y)$ be such that WA is Drazin invertible and $i'_W(A) = k$, and let $b \in X$.*

(i) *The equation*
$$A^{k+2}x = A^{k+1}b$$
is consistent and its general solution is
$$x = A_{\textcircled{\scriptsize D}}b + (I - A_{\textcircled{\scriptsize D}}A)y$$
for arbitrary $y \in X$.

(ii) *If $A_{\textcircled{\#}}$ exists, the equation*
$$A^2x = Ab$$
is consistent and its general solution is
$$x = A_{\textcircled{\#}}b + (I - A_{\textcircled{\#}}A)y$$
for arbitrary $y \in X$.

For $W = A^*$ in Theorem 4.1 and Corollary 4.2, we can solve the equations $(AA^*)^{k+1}Ax = (AA^*)^kAb$ and $AA^*Ax = Ab$ as special cases.

Corollary 4.4. *Let $A \in \mathcal{B}(X,Y)$ be such that A^*A is Drazin invertible and $i'_{A^*}(A) = k$, and let $b \in X$.*

(i) *The equation*
$$(AA^*)^{k+1}Ax = (AA^*)^kAb$$
is consistent and its general solution is
$$x = A_{\textcircled{\scriptsize D},A^*}b + (I - A_{\textcircled{\scriptsize D},A^*}A^*A)y$$
for arbitrary $y \in X$.

(ii) *If $A_{\textcircled{\#},A^*}$ exists and $b \in X$, the equation*
$$AA^*Ax = Ab$$
is consistent and its general solution is
$$x = A_{\textcircled{\#},A^*}b + (I - A_{\textcircled{\#},A^*}A^*A)y$$
for arbitrary $y \in X$.

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