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COMPLETE PRESENTATION AND HILBERT SERIES OF THE MIXED BRAID MONOID $MB_{1,3}$

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Abstract. The Hilbert series is the simplest way of finding dimension and degree of an algebraic variety defined explicitly by polynomial equations. The mixed braid groups were introduced by Sofia Lambropoulou in 2000. In this paper we compute the complete presentation and the Hilbert series of the canonical words of the mixed braid monoid $MB_{1,3}$.

1. Introduction

The braid group B_{n+1} for the Euclidean space consisting on $n+1$ strands is given by the following Artin presentation [3]:

$$B_{n+1} = \left\langle z_1, z_2, \ldots, z_n \;\middle|\; \begin{array}{l} z_i z_j = z_j\, z_i \text{ if } \mid i-j \mid \geq 2 \\ z_{i+1}\, z_i\, z_{i+1} = z_i\, z_{i+1}\, z_i \text{ if } 1 \leq i \leq n-1 \end{array} \right\rangle.$$

Elements of B_{n+1} are expressed in the generators $z_1, z_2, \ldots, z_n$ and their inverses. The presentation of the braid monoid MB_{n+1} is similar to the presentation of B_{n+1}. In [12] Lambropoulou gave the presentation of the mixed braid monoid $B_{m,n}$. Before this presentation she gave the presentation of $B_{1,n}$ in [11]. In this paper we compute the Hilbert series of $B_{1,3}$.

Definition 1.1 ([12])**.** The *mixed braid group* $B_{m,n}$ of $m+n$ strands is defined as

$$B_{m,n} = \left\langle \begin{array}{c} \alpha_1, \ldots, \alpha_m, \\ \beta_1, \ldots, \beta_{n-1} \end{array} \;\middle|\; \begin{array}{l} \beta_r \beta_s = \beta_s\, \beta_r \text{ if } |r-s| \geq 2 \\ \beta_{r+1}\, \beta_r\, \beta_{r+1} = \beta_r\, \beta_{r+1}\, \beta_r \text{ if } 1 \leq r \leq n-1 \\ \alpha_p\, \beta_s = \beta_s\, \alpha_p \text{ if } s \geq 2,\ 1 \leq p \leq m \\ \alpha_p\, (\beta_1\, \alpha_q\, \beta_1^{-1}) = (\beta_1\, \alpha_q\, \beta_1^{-1})\, \alpha_p \text{ if } q < p \end{array} \right\rangle.$$

In the mixed braid group $B_{m,n}$, the first index m denotes the strings which make the identity braid of m strings, and the next n strings show the braiding by itself and with m strings. The mixed braid group $B_{m,n}$ is a subgroup of the Artin braid group B_{m+n}. The associated Dynkin diagram for $B_{m,n}$ is given in [12]:

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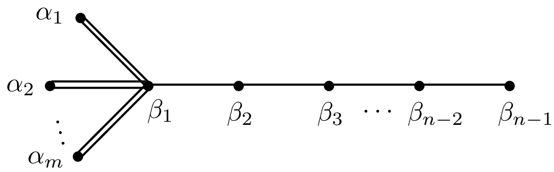

Figure 1.

In the above diagram, the double lines represent the relation of length 4, while the relation of length 3 is represented by the single line. However, if the there is no line among the generators, then they commute. Hence the Dynkin diagram for $MB_{1,2}$ reduces to

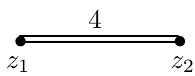

Figure 2.

Therefore we have

$$MB_{1,2} = \langle z_1, z_2 \mid z_2\, z_1\, z_2\, z_1 = z_1\, z_2\, z_1\, z_2 \rangle\,.$$

The complete structure and Hilbert series for $MB_{1,2}$ are computed in [2]. This motivated us to compute the Hilbert series of $MB_{1,3}$, where the Dynkin diagram for $MB_{1,3}$ is as follows:

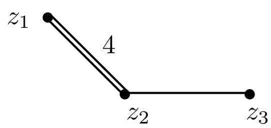

Figure 3.

Therefore we have the following presentation of $MB_{1,3}$:

$$MB_{1,3} = \big\langle z_1, z_2, z_3 \mid z_3\, z_2\, z_3 = z_2\, z_3\, z_2,\ z_2\, z_1\, z_2\, z_1 = z_1\, z_2\, z_1\, z_2,\ z_3\, z_1 = z_1\, z_3 \big\rangle.$$

In this case we have three Artin relations, namely, $R_0 : z_3\, z_1 = z_1\, z_3$, $R_1 : z_2\, z_1\, z_2\, z_1 = z_1\, z_2\, z_1\, z_2$, and $R_2 : z_3\, z_2\, z_3 = z_2\, z_3\, z_2$. The following is an example of a braid in $B_{1,3}$.

In [6], Zafar et al. constructed a linear system for the braid monoid MB_{n+1} and computed the Hilbert series for the braid monoids MB_3 and MB_4. The growth series of binomial edge ideals was computed by Kumar and Sarkar in [10]. In [6], growth series of the graded algebra of real regular functions on the symplectic quotient associated to an SU2-module has been given.

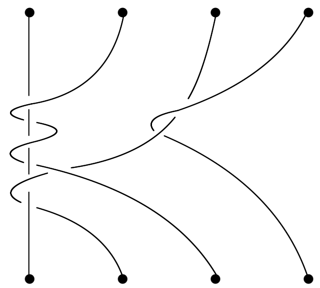

Figure 4.

In [9], the authors computed the Hilbert series of the braid monoid MB_4 in band generators. In [8], the authors constructed a linear system of canonical words of finite dimensional generalized Hecke algebras $H(Q_m, 3)$, where $Q_m = x^m - 1, m \in \{3, 4, 5\}$ and computed its Hilbert series. In [14] Saito computed the growth series of Artin monoids. In [13] Mairesse and Mathéus gave the growth series of Artin groups of dihedral type. In [1] we computed the Hilbert series of the Artin monoids $M(I_2(p))$, where $M(I_2(4))$ is isomorphic to $MB_{1,2}$ and $MB_{1,2}$ is isomorphic to the Artin monoid of type B_2. In this paper we construct a similar kind of linear system to compute the Hilbert series of $MB_{1,3}$ which is isomorphic to the Artin monoid of type B_3.

2. Complete presentation of $MB_{1,3}$

To obtain a canonical form of a word in an algebra, the diamond lemma by G. Bergman [4] is extremely useful. To understand the notions of ambiguities and canonical words, we start with his terminology.

Definition 2.1 ([4])**.** Let $\alpha_1 = ut$ and $\alpha_2 = tv$ be two words consisting of the left-hand sides of two relations R_i and R_j in $MB_{1,3}$. The word of the form utv is said to be an *ambiguity* and we denote it by $R_i - R_j$.

A word containing a sub-word of the left-hand side of any relation of a braid monoid is called a *reducible word*, and a word that does not contain any sub-word of the left-hand side of any relation is called an *irreducible* (or *canonical*) *word*.

Definition 2.2 ([5])**.** Let G be a finitely generated group and S be a finite set of generators of G. The *word length* $l_S(g)$ of an element $g \in G$ is the smallest integer n for which there exist $s_1, \dots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$.

The diamond lemma says that a set of relations is complete if all the ambiguities are solved. We call a complete set of relations in $MB_{1,3}$ a *complete presentation* of $MB_{1,3}$. The other names for the complete presentation are being used as Gröbner bases, presentation with solvable ambiguities and rewriting system, etc. We find the system of linear equations of the canonical words of $MB_{1,3}$ and solve this system, which consequently leads to the Hilbert series of $MB_{1,3}$.

In a relation in $MB_{1,3}$ we place the equivalent words on the left-hand side which are greater in length-lexicographic ordering [7] (we choose a natural total order $z_1 < z_2 < \cdots < z_n$ between the generators). For example, the words $z_2z_1z_2z_1$ and $z_1z_2z_1z_2$ are equivalent in the mixed braid monoid $MB_{1,3}$. Hence we write $z_2z_1z_2z_1 = z_1z_2z_1z_2$ as the basic braid relation. We use the notation $R_j^{(4)}$ to express j^{th} generalized relation in $MB_{1,3}$. The words $Xz_2 \times_2 z_2Y$ and $Xz_2z_1 \times_{21} z_2z_1Y$ denote the products Xz_2Y and Xz_2z_1Y, respectively.

The ambiguity utv has two resolutions, namely $(ut)v$ and $u(tv)$. Let $w = utv$. Then by $L(w)$ we mean the canonical form of $(ut)v$ and by $R(w)$ we mean the canonical form of $u(tv)$. If $L(w)$ and $R(w)$ are identical, then the ambiguity is solvable. If $L(w)$ and $R(w)$ differ by lexicographic order, then we get a new relation in $MB_{1,3}$.

Theorem 2.3. *The complete presentation of $MB_{1,3}$ is given by*

$$\langle z_1, z_2, z_3 \mid z_3\, z_1 = z_1\, z_3, z_3\, z_2\, z_3 = z_2\, z_3\, z_2, z_2\, z_1\, z_2\, z_1 = z_1\, z_2\, z_1\, z_2, R_1^{(4)}, \dots, R_{11}^{(4)} \rangle,$$

where

(1) $R_1^{(4)}$: $z_2z_1^{n+1}z_2z_1z_2 = z_1z_2z_1z_2^2z_1^n$
(2) $R_2^{(4)}$: $z_3z_2z_1^nz_3 = z_2z_3z_2z_1^n$
(3) $R_3^{(4)}$: $z_3z_2^nz_3z_2 = z_2z_3z_2^2z_3^n$
(4) $R_4^{(4)}$: $z_3z_2z_1^nz_2^{n_1}z_3z_2 = z_2z_3z_2z_1^nz_2z_3^{n_1}$
(5) $R_5^{(4)}$: $z_3z_2z_1z_2^nz_3z_2 = z_2z_3z_2z_1z_2z_3^n$
(6) $R_6^{(4)}$: $z_3z_2^nz_1^{n_1}z_3z_2z_1z_2 = z_2z_3z_2^2z_1z_3^nz_2z_1^{n_1}$
(7) $R_7^{(4)}$: $z_3z_2z_1^nz_2^{n_1}z_1^{n_2}z_3z_2z_1z_2 = z_2z_3z_2z_1^nz_2z_1z_3^{n_1}z_2z_1^{n_2}$
(8) $R_8^{(4)}$: $z_3z_2z_1z_2^nz_1^{n_1}z_3z_2z_1z_2 = z_2z_1z_3z_2z_1z_2z_3^nz_2z_1^{n_1}$
(9) $R_9^{(4)}$: $z_3(z_2^nz_1^{n_1}z_2^{n_2}z_1^{n_3}\cdots)z_3z_2z_1z_2z_3 = z_2z_3z_2^2z_1z_3^nz_2z_1^{n_1}z_3(z_2^{n_2}z_1^{n_3}\cdots)$
(10) $R_{10}^{(4)}$: $z_3z_2(z_1^nz_2^{n_1}z_1^{n_2}z_2^{n_3}z_1^{n_4}\cdots)z_3z_2z_1z_2z_3$
$= z_2z_3z_2z_1^nz_2z_1z_3^{n_1}z_2z_1^{n_2}z_3(z_2^{n_3}z_1^{n_4}\cdots)$
(11) $R_{11}^{(4)}$: $z_3z_2z_1(z_2^nz_1^{n_1}z_2^{n_2}\cdots)z_3z_2z_1z_2z_3$
$= z_2z_1z_3z_2z_1z_2z_3^nz_2z_1^{n_1}z_3(z_2^{n_2}z_1^{n_3}\cdots)$,

with $n, n_1, n_2, n_3, \ldots \in \mathbb{N}$.

Proof. In this proof we use the inductive argument. We compute the relations by solving the ambiguities involving the relations R_0, R_1, and R_2 and the new relations.

(1) In [1] we computed the first relation (for $p = 4$) $R_1^{(4)}$, which is given by

$$R_1^{(4)}:\ z_2z_1^{n+1}z_2z_1z_2 = z_1z_2z_1z_2^2z_1^n.$$

(2) For an ambiguity $R_2 - R_0 = z_3z_2z_3z_1 = w_1$ (say), we have

$$R(w_1) = z_3z_2\underline{z_3z_1} = z_3z_2z_1z_3, \qquad L(w_1) = \underline{z_3z_2z_3}z_1 = z_2z_3z_2z_1.$$

Hence we have a relation $R_{w_1} : z_3z_2z_1z_3 = z_2z_3z_2z_1$. Again by solving a new ambiguity $R_{w_1} - R_0 = z_3z_2z_1z_3z_1 = w_2$ we have

$$R(w_2) = z_3z_2z_1\underline{z_3z_1} = z_3z_2z_1^2z_3, \qquad L(w_2) = \underline{z_3z_2z_1z_3}z_1 = z_2z_3z_2z_1z_2,$$

which gives another relation $R_{w_2} : z_3z_2z_1^2z_3 = z_2z_3z_2z_1z_2$. By continuing the same process we have the general relation

$$R_2^{(4)} :\ z_3z_2z_1^nz_3 = z_2z_3z_2z_1^n.$$

(3) In the ambiguity $R_2 - R_2 = z_3z_2z_3z_2z_3 = w_3$, we have

$$R(w_3) = z_3z_2\underline{z_3z_2z_3} = z_3z_2^2z_3z_2, \qquad L(w_3) = \underline{z_3z_2z_3}z_2z_3 = z_2z_3z_2^2z_3.$$

Hence we have a relation $R_{w_3} : z_3z_2^2z_3z_2 = z_2z_3z_2^2z_3$. Therefore in general we have

$$R_3^{(4)} :\ z_3z_2^nz_3z_2 = z_2z_3z_2^2z_3^n.$$

(4) Successive ambiguities of $R_2^{(4)}$ and R_2 lead to the relation

$$R_4^{(4)} :\ z_3z_2z_1^nz_2^{n_1}z_3z_2 = z_2z_3z_2z_1^nz_2z_3^{n_1}.$$

(5) By solving $R_{w_1} - R_2 = z_3z_2z_1z_3z_2z_3$ and generalizing, we have

$$R_5^{(4)} :\ z_3z_2z_1z_2^nz_3z_2 = z_2z_3z_2z_1z_2z_3^n.$$

(6) $R_6^{(4)} :\ z_3z_2^nz_1^{n_1}z_3z_2z_1z_2 = z_2z_3z_2^2z_1z_3^nz_2z_1^{n_1}$ is obtained by solving the ambiguity of the relations $R_3^{(4)}$ and R_1.

(7) Solving the ambiguities formed by $R_4^{(4)}$ and R_1, we get

$$R_7^{(4)} :\ z_3z_2z_1^nz_2^{n_1}z_1^{n_2}z_3z_2z_1z_2 = z_2z_3z_2z_1^nz_2z_1z_3^{n_1}z_2z_1^{n_2}.$$

(8) Successive ambiguities of $R_5^{(4)}$ and R_1 lead to the relation

$$R_8^{(4)} :\ z_3z_2z_1z_2^nz_1^{n_1}z_3z_2z_1z_2 = z_2z_1z_3z_2z_1z_2z_3^nz_2z_1^{n_1}.$$

(9) Now, solving the ambiguity formed by $R_6^{(4)}$, $R_4^{(4)}$, and R_0, we have

$$R_9^{(4)} :\ z_3(z_2^nz_1^{n_1}z_2^{n_2}z_1^{n_3}\cdots)z_3z_2z_1z_2z_3 = z_2z_3z_2^2z_1z_3^nz_2z_1^{n_1}z_3(z_2^{n_2}z_1^{n_3}\cdots).$$

(10) The relation

$$R_{10}^{(4)} :\ z_3z_2(z_1^nz_2^{n_1}z_1^{n_2}z_2^{n_3}z_1^{n_4}\cdots)z_3z_2z_1z_2z_3 = z_2z_3z_2z_1^nz_2z_1z_3^{n_1}z_2z_1^{n_2}z_3(z_2^{n_3}z_1^{n_4}\cdots)$$

is obtained by solving the ambiguities formed by $R_7^{(4)}$, $R_4^{(4)}$, and R_0.

(11) Successively solving the ambiguities formed by $R_8^{(4)}$, $R_4^{(4)}$, and R_0, we get

$$R_{11}^{(4)} :\ z_3z_2z_1(z_2^nz_1^{n_1}z_2^{n_2}z_1^{n_3}\cdots)z_3z_2z_1z_2z_3 = z_2z_1z_3z_2z_1z_2z_3^nz_2z_1^{n_1}z_3(z_2^{n_2}z_1^{n_3}\cdots).$$

All other ambiguities are solvable. Hence we have the complete set of relations. □

3. Hilbert series of $MB_{1,3}$

Definition 3.1 ([5])**.** Let M be a group or a monoid and a_n be the number of elements of M of word length n. The *Hilbert series* of M for arbitrary variable t is denoted by $H_M(t)$ and is defined by $H_M(t) = \sum_{n=0}^{\infty} a_n t^n$.

We use the complete presentation of $MB_{1,3}$ to compute the Hilbert series. Let $A^{(m+n)}$ and $B^{(m+n)}$ denote the set of all canonical and reducible words in $MB_{m,n}$, respectively. In particular assume that $A_{\mu}^{(m+n)}$ and $B_{\mu,\nu}^{(m+n)}$ denote the set of all canonical and reducible words in $MB_{m,n}$, respectively, where μ is related to the prefix of a word while ν is the suffix of the word. For example, $A_{j(j-1)\ldots k}^{(n+m)}$ denotes the collection of all canonical words in $MB_{m,n}$ that start with $z_j z_{j-1} \ldots z_k$ and $B_{j,v}^{(m+n)}$, $B_{j(j+1),v}^{(m+n)}$ denote the collection of all reducible words that start with $z_{(m+n)-1} z_{(m+n)-2} \ldots z_j$ and $z_{(m+n)-1} z_{(m+n)-2} \cdots z_1 z_2 \cdots z_j$, respectively, and v is a word in the generators $z_1, \ldots, z_n$. The set $B_{*,v}^{(m+n)}$ denotes all the reducible words starting with any word and ending in the generators $z_1, \ldots, z_n$. Hence in $MB_{1,3}$ we have the following set of reducible words:

$$B_{1,2}^{(4)} = \{z_2 z_1 z_2 z_1\}, \quad B_{1,212}^{(4)} = \{z_2 z_1^{n+1} z_2 z_1 z_2\}, \quad B_{2,3}^{(4)} = \{z_3 z_2 z_3\},$$
$$B_{1,3}^{(4)} = \{z_3 z_2 z_1^n z_3\}, \quad B_{2,32}^{(4)} = \{z_3 z_2^n z_3 z_2\}, \quad B_{1,32}^{(4)} = \{z_3 z_2 z_1^n z_2^{n_1} z_3 z_2\},$$
$$B_{12,32}^{(4)} = \{z_3 z_2 z_1 z_2^n z_3 z_2\},$$
$$B_{2,3212}^{(4)} = \{z_3 z_2^n z_1^{n_1} z_3 z_2 z_1 z_2\}, \quad B_{1,3212}^{(4)} = \{z_3 z_2 z_1^n z_2^{n_1} z_1^{n_2} z_3 z_2 z_1 z_2\},$$
$$B_{12,3212}^{(4)} = \{z_3 z_2 z_1 z_2^n z_1^{n_1} z_3 z_2 z_1 z_2\}, \quad B_{2,32123}^{(4)} = \{z_3 (z_2^n z_1^{n_1} z_2^{n_2} z_1^{n_3} \cdots) z_3 z_2 z_1 z_2 z_3\},$$
$$B_{1,32123}^{(4)} = \{z_3 z_2 (z_1^n z_2^{n_1} z_1^{n_2} z_2^{n_3} z_1^{n_4} \cdots) z_3 z_2 z_1 z_2 z_3\},$$
$$B_{12,32123}^{(4)} = \{z_3 z_2 z_1 (z_2^n z_1^{n_1} z_2^{n_2} \cdots) z_3 z_2 z_1 z_2 z_3\}.$$

Assume that $Q_{\mu,\nu}^{(m+n)}(t)$ denotes the Hilbert series of $B_{\mu,\nu}^{(m+n)}$ and $P_{\mu}^{(m+n)}(t)$ denotes the Hilbert series of $A_{\mu}^{(m+n)}$. If $A_{*}^{(m+n)}$ denotes a set of canonical words in $MB_{m,n}$, then $\Sigma A_{*}^{(m+n)}$ denotes the same set of canonical words with each index increased by 1. For example, for $A_1^{(2)} = \{z_1, z_1^2, z_1^3, \ldots\}$, we have $\Sigma A_1^{(2)} = \{z_2, z_2^2, z_2^3, \ldots\}$. Therefore

$$P_1^{(2)} = t + t^2 + t^3 + \cdots = \frac{t}{1-t}.$$

Lemma 3.2 ([2])**.** *The following equations hold for the canonical words in $MB_{1,2}$:*

(1) $P_1^{(3)}(t) = \frac{t}{(1-t)(1-t-t^2-t^3)}$,
(2) $P_2^{(3)}(t) = \frac{t(1+t+t^2)}{(1-t-t^2-t^3)}$,
(3) $P_{21}^{(3)}(t) = \frac{t^2(1+t)}{(1-t-t^2-t^3)}$,
(4) $P_{212}^{(3)}(t) = \frac{t^3}{(1-t-t^2-t^3)}$.

Corollary 3.3 ([2]). *The Hilbert series for the canonical words in $MB_{1,2}$ is*

$$H_M^{(3)}(t) = \frac{1}{(1-t)(1-t-t^2-t^3)}.$$

Now, we have to find $P_1^{(4)}(t)$, $P_2^{(4)}(t)$, and $P_3^{(4)}(t)$ for the computation of the Hilbert series $H_M^{(4)}(t)$ of $MB_{1,3}$.

Lemma 3.4. *The following equations hold for the reducible words in $MB_{1,3}$:*

(1) $Q_{1,2}^{(4)} = t^4$

(2) $Q_{1,212}^{(4)} = \dfrac{t^6}{1-t}$

(3) $Q_{2,3}^{(4)} = t^3$

(4) $Q_{1,3}^{(4)} = \dfrac{t^4}{1-t}$

(5) $Q_{\mu,3}^{(4)} = \dfrac{t^3}{1-t}$

(6) $Q_{2,32}^{(4)} = \dfrac{t^5}{1-t}$

(7) $Q_{1,32}^{(4)} = \dfrac{t^6}{(1-t)^2}$

(8) $Q_{12,32}^{(4)} = \dfrac{t^6}{1-t}$

(9) $Q_{*,32}^{(4)} = \dfrac{t^5}{(1-t)^2}$

(10) $Q_{2,3212}^{(4)} = \dfrac{t^8}{(1-t)^2}$

(11) $Q_{1,3212}^{(4)} = \dfrac{t^{10}}{(1-t)^3}$

(12) $Q_{12,3212}^{(4)} = \dfrac{t^{10}}{(1-t)^2}$

(13) $Q_{*,3212}^{(4)} = \dfrac{t^8(1-t+2t^2-t^3)}{(1-t)^3}$

(14) $Q_{2,32123}^{(4)} = \dfrac{t^{10}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2}$

(15) $Q_{1,32123}^{(4)} = \dfrac{2t^{13}}{(1-t-t^2-t^3)(1-t)^3}$

(16) $Q_{12,32123}^{(4)} = \dfrac{t^{12}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2}$

(17) $Q_{*,32123}^{(4)} = \dfrac{t^{10}(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3}$.

Proof. We proceed with the proof by considering tail-wise reducible words. Here for all the reducible words we use the decompositions.

(1) We have only one word that starts and ends with z_2z_1, i.e., $B_{1,2}^{(4)} = \{z_2z_1z_2z_1\}$. Hence we have $Q_{1,2}^{(4)} = t^4$.

(2) Since $B_{1,212}^{(4)} = \{z_2z_1^{n+1}z_2z_1z_2\} = \{z_2z_1\} \times A_1^{(2)} \times \{z_2z_1z_2\}$, we have $Q_{1,212}^{(4)} = \frac{t^6}{1-t}$.

(3) For $B_{2,3}^{(4)} = \{z_3z_2z_3\}$, we have $Q_{2,3}^{(4)} = t^3$.

(4) Similarly, $B_{1,3}^{(4)} = \{z_3z_2z_1^nz_3\} = \{z_3z_2z_1\} \times_1 A_1^{(2)} \times \{z_3\}$. Therefore $Q_{1,3}^{(4)} = \frac{t^4}{1-t}$.

(5) As there are two types of reducible words whose tail is z_3, that is, $B_{*,3}^{(4)} = B_{2,3}^{(4)} \sqcup B_{1,3}^{(4)}$, we have

$$Q_{*,3}^{(4)} = t^3 + \frac{t^4}{1-t} = \frac{t^3}{1-t}.$$

(6) The decomposition $B_{2,32}^{(4)} = \{z_3z_2^nz_3z_2\} = \{z_3z_2\} \times \Sigma A_1^{(2)} \times \{z_3z_2\}$ gives $Q_{2,32}^{(4)} = \frac{t^5}{1-t}$.

(7) The set $B^{(4)}_{1,32} = \{z_3z_2z_1^nz_2^{n_1}z_3z_2\} = \{z_3z_2z_1\} \times_1 A^{(2)}_1 \times \Sigma A^{(2)}_1 \times \{z_3z_2\}$ gives the relation $Q^{(4)}_{1,32} = \frac{t^6}{(1-t)^2}$.

(8) Similarly, $B^{(4)}_{12,32} = \{z_3z_2z_1z_2^nz_3z_2\} = \{z_3z_2z_1z_2\} \times_2 \Sigma A^{(2)}_1 \times \{z_3z_2\}$. Therefore $Q^{(4)}_{12,32} = \frac{t^6}{1-t}$.

(9) Using complete presentation, we have two different types of reducible words ending with z_3z_2 (as $B^{(4)}_{12,32}$ is a subword of $B^{(4)}_{1,32}$ for $n = 1$), i.e., $B^{(4)}_{*,32} = B^{(4)}_{2,32} \sqcup B^{(4)}_{1,32}$. Hence, we have

$$Q^{(4)}_{*,32} = Q^{(4)}_{2,32} + Q^{(4)}_{1,32} = \frac{t^5}{1-t} + \frac{t^6}{(1-t)^2} = \frac{t^5}{(1-t)^2}.$$

(10) As $B^{(4)}_{2,3212} = \{z_3z_2^nz_1^{n_1}z_3z_2z_1z_2\} = \{z_3z_2\} \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\}$, we have $Q^{(4)}_{2,3212} = \frac{t^8}{(1-t)^2}$.

(11) $B^{(4)}_{1,3212} = \{z_3z_2z_1^nz_2^{n_1}z_1^{n_2}z_3z_2z_1z_2\} = \{z_3z_2z_1\} \times A^{(2)}_1 \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\}$ gives $Q^{(4)}_{1,3212} = \frac{t^{10}}{(1-t)^3}$.

(12) $B^{(4)}_{12,3212} = \{z_3z_2z_1z_2^nz_1^{n_1}z_3z_2z_1z_2\} = \{z_3z_2z_1\} \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\}$ gives the relation $Q^{(4)}_{12,3212} = \frac{t^{10}}{(1-t)^2}$.

(13) Using reduced complete presentation, we have three types of reducible words ending with $z_3z_2z_1z_2$, i.e., $B^{(4)}_{*,3212} = B^{(4)}_{2,3212} \sqcup B^{(4)}_{1,3212} \sqcup B^{(4)}_{12,3212}$. Hence we get

$$\begin{aligned} Q^{(4)}_{*,3212} &= Q^{(4)}_{2,3212} + Q^{(4)}_{1,3212} + Q^{(4)}_{12,3212} \\ &= \frac{t^8}{(1-t)^2} + \frac{t^{10}}{(1-t)^3} + \frac{t^{10}}{(1-t)^2} \\ &= \frac{t^8(1-t+2t^2-t^3)}{(1-t)^3}. \end{aligned}$$

(14) The word $B^{(4)}_{2,32123} = \{z_3(z_2^nz_1^{n_1}z_2^{n_2}z_1^{n_3}\cdots)z_3z_2z_1z_2z_3\} = \{z_3z_2\} \times A^{(3)}_2 \times \{z_3z_2z_1z_2z_3\}$ can be written as $\{z_3z_2\} \times \Sigma A^{(2)}_1 \times \{z_3z_2\} \times \{z_1z_2z_3\}$ as well as $\{z_3z_2\} \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\} \times \{z_3\}$. In this case we have reducible subwords, which will be subtracted. Hence we have

$$B^{(4)}_{2,32123} = \{z_3z_2\} \times A^{(3)}_2 \times \{z_3z_2z_1z_2z_3\} \backslash \left((B^{(4)}_{2,32} \times \{z_1z_2z_3\}) \sqcup (B^{(4)}_{2,3212} \times \{z_3\}) \right),$$

for which we have

$$Q^{(4)}_{2,32123} = t^7P^3_2 - \frac{t^8}{1-t} - \frac{t^9}{(1-t)^2}.$$

Using Lemma 3.2 we have

$$\begin{aligned} Q^{(4)}_{2,32123} &= \frac{t^8(1+t+t^2)}{1-t-t^2-t^3} - \frac{t^8}{1-t} - \frac{t^9}{(1-t)^2} \\ &= \frac{t^{10}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2}. \end{aligned}$$

(15) As we have $B^{(4)}_{1,32123} = z_3z_2(z_1^n z_2^{n_1} z_1^{n_2} z_2^{n_3} z_1^{n_4} \cdots)z_3z_2z_1z_2z_3$, we can write $B^{(4)}_{1,32123} = \{z_3z_2z_1\} \times A^{(3)}_1 \times \{z_3z_2z_1z_2z_3\}$. Using the above argument we have

$$\begin{aligned} B^{(4)}_{1,32123} = \{z_3z_2z_1\} &\times A^{(3)}_1 \times \{z_3z_2z_1z_2z_3\} \\ &\setminus \Big((\{z_3z_2z_1\} \times A^{(2)}_1 \times \{z_2z_1z_2\} \times_{212} \times A^{(3)}_{212} \times \{z_3z_2z_1z_2z_3\}) \\ &\quad \sqcup (\{z_3z_2z_1\} \times A^{(2)}_1 \times \{z_3\} \times \{z_2z_1z_2z_3\}) \\ &\quad \sqcup (\{z_3z_2z_1\} \times A^{(2)}_1 \times \Sigma A^{(2)}_1 \times \{z_3z_2\} \times \{z_1z_2z_3\}) \\ &\quad \sqcup (\{z_3z_2z_1\} \times A^{(2)}_1 \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\} \times \{z_3\}) \Big). \end{aligned}$$

Hence

$$\begin{aligned} Q^{(4)}_{1,32123} &= t^8 P^{(3)}_1 - \frac{t^9}{1-t} P^{(3)}_{212} - \frac{t^9}{1-t} - \frac{t^{10}}{(1-t)^2} - \frac{t^{11}}{(1-t)^3} \\ &= \frac{2t^{13}}{(1-t-t^2-t^3)(1-t)^3}. \end{aligned}$$

(16) Similarly, as $B^{(4)}_{12,32123} = z_3z_2z_1(z_2^n z_1^{n_1} z_2^{n_2} \cdots)z_3z_2z_1z_2z_3$, we can write

$$\begin{aligned} B^{(4)}_{12,32123} = \{z_3z_2z_1z_2\} &\times A^{(3)}_2 \times \{z_3z_2z_1z_2z_3\} \\ &\setminus \Big((\{z_3z_2z_1z_2\} \times \Sigma A^{(2)}_1 \times \{z_3z_2\} \times \{z_1z_2z_3\}) \\ &\quad \sqcup (\{z_3z_2z_1z_2\} \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\} \times \{z_3\}) \Big). \end{aligned}$$

Hence, using Lemma 3.2, we have

$$\begin{aligned} Q^{(4)}_{12,32123} &= t^9 P^{(3)}_2 - \frac{t^{10}}{1-t} - \frac{t^{11}}{(1-t)^2} \\ &= \frac{t^{12}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2}. \end{aligned}$$

(17) We have three types of reducible words ending with $z_3z_2z_1z_2z_3$, i.e.,

$$B^{(4)}_{*,32123} = B^{(4)}_{2,32123} \sqcup B^{(4)}_{1,32123} \sqcup B^{(4)}_{12,32123}.$$

Therefore we get

$$\begin{aligned} Q^{(4)}_{*,32123} &= Q^{(4)}_{2,32123} + Q^{(4)}_{1,32123} + Q^{(4)}_{12,32123} \\ &= \frac{t^{10}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} + \frac{2t^{13}}{(1-t-t^2-t^3)(1-t)^3} \\ &\quad + \frac{t^{12}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} \\ &= \frac{t^{10}(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3}. \end{aligned}$$ □

For the computation of the Hilbert series of $MB_{1,3}$, we have the following linear system for the canonical words.

Lemma 3.5. *The following equations hold for the canonical words in $MB_{1,3}$:*

(1) $P_1^{(4)} = P_1^{(3)} + P_1^{(3)} P_3^{(4)}$
(2) $P_2^{(4)} = P_2^{(3)} + P_2^{(3)} P_3^{(4)}$
(3) $P_{21}^{(4)} = P_{21}^{(3)} + P_{21}^{(3)} P_3^{(4)}$
(4) $P_{212}^{(4)} = P_{212}^{(3)} + P_{212}^{(3)} P_3^{(4)}$
(5) $P_3^{(4)} = t + tP_3^{(4)} + P_{32}^{(4)}$
(6) $P_{32}^{(4)} = tP_2^{(4)} - \frac{t^2}{1-t} P_3^{(4)} - \frac{t^3(1+t-t^2)}{(1-t)^2} P_{32}^{(4)} - \frac{t^4(1-t+2t^2-t^3)}{(1-t)^3} P_{3212}^{(4)} - \frac{t^5(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)}$
(7) $P_{321}^{(4)} = tP_{21}^{(4)} - \frac{t^3}{1-t} P_3^{(4)} - \frac{t^4}{(1-t)^2} P_{32}^{(4)} - \frac{t^6(2-t)}{(1-t)^3} P_{3212}^{(4)} - \frac{t^7(1+t+t^2-t^3)}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)}$
(8) $P_{3212}^{(4)} = tP_{212}^{(4)} - \frac{t^4}{1-t} P_{32}^{(4)} - \frac{t^6}{(1-t)^2} P_{3212}^{(4)} - \frac{t^7(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} P_{32123}^{(4)}$
(9) $P_{32123}^{(4)} = t^4 P_3^{(4)} - t^4 P_{32}^{(4)}$.

Proof. The canonical words may start with z_1, z_2, z_2z_1, $z_2z_1z_2$, z_3, z_3z_2, $z_3z_2z_1$, $z_3z_2z_1z_2$ or $z_3z_2z_1z_2z_3$. By $\sqcup$ we mean the disjoint union of sets.

(1) For the canonical words starting with z_1, we have the decomposition of the form $A_1^{(4)} = A_1^{(3)} \sqcup (A_1^{(3)} \times A_3^{(4)})$. The associated Hilbert series becomes

$$P_1^{(4)} = P_1^{(3)} + P_1^{(3)} P_3^{(4)}.$$

(2) The canonical words starting with z_2 have the form $A_2^{(4)} = A_2^{(3)} \sqcup (A_2^{(3)} \times A_3^{(4)})$. Hence

$$P_2^{(4)} = P_2^{(3)} + P_2^{(3)} P_3^{(4)}.$$

(3) The decomposition $A_{21}^{(4)} = A_{21}^{(3)} \sqcup (A_{21}^{(3)} \times A_3^{(4)})$ gives

$$P_{21}^{(4)} = P_{21}^{(3)} + P_{21}^{(3)} P_3^{(4)}.$$

(4) The decomposition $A_{212}^{(4)} = A_{212}^{(3)} \sqcup (A_{212}^{(3)} \times A_3^{(4)})$ gives

$$P_{212}^{(4)}(t) = P_{212}^{(3)} + P_{212}^{(3)} P_3^{(4)}.$$

(5) The canonical words starting with z_3 can be written as $A_3^{(4)} = \{z_3\} \sqcup (\{z_3\} \times A_3^{(4)}) \sqcup A_{32}^{(4)}$. Therefore the corresponding Hilbert series is

$$P_3^{(4)} = t + tP_3^{(4)} + P_{32}^{(4)}.$$

(6) By taking the product of z_3 on the left side of the set of canonical words starting with z_2, we may have reducible words of any one of the form $B_{\mu,\nu}^{(4)}$. In order to get canonical words starting with z_3z_2, we have to get rid of the above-mentioned

reducible words from the $\{z_3\} \times A_2^{(4)}$. Therefore,

$$A_{32}^{(4)} = \{z_3\} \times A_2^{(4)} \backslash \Big((B_{*,3}^{(4)} \times_3 A_3^{(4)}) \sqcup (B_{*,32}^{(4)} \times_{32} A_{32}^{(4)}) \sqcup (B_{*,3212}^{(4)} \times_{3212} A_{3212}^{(4)}) \\ \sqcup (B_{*,32123}^{(4)} \times_{32123} A_{32123}^{(4)}) \Big).$$

Hence we have

$$P_{32}^{(4)} = tP_2^{(4)} - \frac{t^2}{1-t} P_3^{(4)} - \frac{t^3}{(1-t)^2} P_{32}^{(4)} - \frac{t^4(1-t+2t^2-t^3)}{(1-t)^3} P_{3212}^{(4)} \\ - \frac{t^5(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)}.$$

Equivalently we have

$$tP_2^{(4)} - \frac{t^2}{1-t} P_3^{(4)} - (1 + \frac{t^3}{(1-t)^2}) P_{32}^{(4)} - \frac{t^4(1-t+2t^2-t^3)}{(1-t)^3} P_{3212}^{(4)} \\ - \frac{t^5(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)} = 0.$$

(7) Similarly we can write

$$A_{321}^{(4)} = \{z_3 z_2 z_1\} \times_{21} A_{21}^{(4)} \backslash \Big((B_{1,3}^{(4)} \times_3 A_3^{(4)}) \sqcup (B_{1,32}^{(4)} \times_{32} A_{32}^{(4)}) \\ \sqcup (B_{1,3212}^{(4)} \times_{3212} A_{3212}^{(4)}) \sqcup (B_{12,3212}^{(4)} \times_{3212} A_{3212}^{(4)}) \\ \sqcup (B_{1,32123}^{(4)} \times_{32123} A_{32123}^{(4)}) \sqcup (B_{12,32123}^{(4)} \times_{32123} A_{32123}^{(4)}) \Big).$$

Therefore we get

$$P_{321}^{(4)} = tP_{21}^{(4)} - \frac{t^3}{1-t} P_3^{(4)} - \frac{t^4}{(1-t)^2} P_{32}^{(4)} - \frac{t^6}{(1-t)^3} P_{3212}^{(4)} - \frac{t^6}{(1-t)^2} P_{3212}^{(4)} \\ - \frac{2t^8}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)} - \frac{t^7(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} P_{32123}^{(4)}$$

or

$$tP_{21}^{(4)} - \frac{t^3}{1-t} P_3^{(4)} - \frac{t^4}{(1-t)^2} P_{32}^{(4)} - P_{321}^{(4)} - \frac{t^6(2-t)}{(1-t)^3} P_{3212}^{(4)} \\ - \frac{t^7(1+t+t^2-t^3)}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)} = 0.$$

(8) Similarly we have

$$A_{3212}^{(4)} = \{z_3 z_2 z_1 z_2\} \times_{212} A_{212}^{(4)} \backslash \Big((B_{12,32}^{(4)} \times_{32} A_{32}^{(4)}) \sqcup (B_{12,3212}^{(4)} \times_{3212} A_{3212}^{(4)}) \\ \sqcup (B_{12,32123}^{(4)} \times_{32123} A_{32123}^{(4)}) \Big).$$

The corresponding Hilbert series becomes

$$P_{3212}^{(4)} = tP_{212}^{(4)} - \frac{t^4}{1-t} P_{32}^{(4)} - \frac{t^6}{(1-t)^2} P_{3212}^{(4)} - \frac{t^7(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} P_{32123}^{(4)}$$

or

$$tP^{(4)}_{212} - \frac{t^4}{1-t}P^{(4)}_{32} - \frac{t^6+t^2-2t+1}{(1-t)^2}P^{(4)}_{3212} - \frac{t^7(1+t^2)}{(1-t-t^2-t^3)(1-t)^2}P^{(4)}_{32123} = 0.$$

(9) The decomposition $A^{(4)}_{32123} = \{z_3z_2z_1z_2z_3\} \times_3 A^{(4)}_3 \setminus (\{z_3z_2z_1z_2z_3z_2\} \times_{32} A^{(4)}_{32})$ gives $P^{(4)}_{32123} = t^4P^{(4)}_3 - t^4P^{(4)}_{32}$ or $t^4P^{(4)}_3 - t^4P^{(4)}_{32} - P^{(4)}_{32123} = 0$. □

Finally we have our main result.

Theorem 3.6. *The Hilbert series of $MB_{1,3}$ is*

$$H^{(4)}_M(t) = \frac{1}{(1-t)(1-2t-t^2+t^4+t^5+t^6+t^7+t^8)}.$$

Proof. Let $T_1 = 1-t-t^2-t^3$ and $T_2 = 1-2t-t^2+t^4+t^5+t^6+t^7+t^8$. Then solving the linear system given in Lemma 3.5 we have the augmented matrix of the system:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & \frac{-t}{(1-t)T_1} & 0 & 0 & 0 & 0 & \frac{t}{(1-t)T_1} \\
0 & 1 & 0 & 0 & \frac{-t(1+t+t^2)}{T_1} & 0 & 0 & 0 & 0 & \frac{t(1+t+t^2)}{T_1} \\
0 & 0 & 1 & 0 & \frac{-t^2(1+t)}{T_1} & 0 & 0 & 0 & 0 & \frac{t^2(1+t)}{T_1} \\
0 & 0 & 0 & 1 & \frac{-t^3}{T_1} & 0 & 0 & 0 & 0 & \frac{t^3}{T_1} \\
0 & 0 & 0 & 0 & 1-t & -1 & 0 & 0 & 0 & t \\
0 & t & 0 & 0 & \frac{-t^2}{1-t} & -\frac{(1-t)^2+t^3}{(1-t)^2} & 0 & \frac{-t^4(1-t+2t^2-t^3)}{(1-t)^3} & \frac{-t^5(1-t+2t^2+t^4-t^5)}{(1-t)^3T_1} & 0 \\
0 & 0 & t & 0 & \frac{-t^3}{1-t} & \frac{-t^4}{(1-t)^2} & -1 & \frac{-t^6(2-t)}{(1-t)^3} & \frac{-t^7(1+t+t^2-t^3)}{(1-t)^3T_1} & 0 \\
0 & 0 & 0 & t & 0 & \frac{-t^4}{1-t} & 0 & -\frac{(1-t)^2+t^6}{(1-t)^2} & \frac{-t^7(1+t^2)}{(1-t)^2T_1} & 0 \\
0 & 0 & 0 & 0 & t^4 & -t^4 & 0 & 0 & -1 & 0
\end{bmatrix}$$

The solution gives the following values:

$$\begin{aligned}
&P^{(4)}_1 = \frac{t}{(1-t)(T_2)}, \quad P^{(4)}_2 = \frac{t(1+t+t^2)}{T_2}, \\
&P^{(4)}_3 = \frac{t(1-t^2-t^3-t^4-t^5-t^6-t^7)}{T_2}, \quad P^{(4)}_{21} = \frac{t^2(1+t)}{T_2}, \\
&P^{(4)}_{212} = \frac{t^3}{T_2}, \quad P^{(4)}_{32} = \frac{t^2(1-t^3-t^4-t^5-t^6)}{T_2}, \\
&P^{(4)}_{321} = \frac{t^3(1-t^2-t^3-t^4-t^5)}{T_2}, \quad P^{(4)}_{3212} = \frac{t^4(1-t^2-t^3-t^4)}{T_2}, \\
&P^{(4)}_{32123} = \frac{t^5(1-t-t^2-t^3)}{T_2}.
\end{aligned}$$

All the canonical words in $MB_{1,3}$ are expressed as $A^{(4)} = \{e\} \sqcup A_1^{(4)} \sqcup A_2^{(4)} \sqcup A_3^{(4)}$. Hence the corresponding Hilbert series is given by

$$\begin{aligned} H_M^{(4)}(t) &= 1 + P_1^{(4)}(t) + P_2^{(4)}(t) + P_3^{(4)}(t) \\ &= 1 + \frac{t}{(1-t)(T_2)} + \frac{t(1+t+t^2)}{T_2} + \frac{t(1-t^2-t^3-t^4-t^5-t^6-t^7)}{T_2} \\ &= \frac{1}{(1-t)(1-2t-t^2+t^4+t^5+t^6+t^7+t^8)} \\ &= 1 + 3t + 8t^2 + 20t^3 + 48t^4 + 112t^5 + 263t^6 + \cdots + a_k^{(4)} t^k + \cdots, \end{aligned}$$

where $a_k^{(4)}$ is an arbitrary constant. □

Definition 3.7. Let $\{a_k\}_{k\geq 1}$ be a sequence of positive numbers and r be a positive real number. The *growth rate* r of the sequence $\{a_k\}_{k\geq 1}$ is defined as

$$r = \overline{\lim_k} \exp\left(\frac{\log a_k}{k}\right).$$

Corollary 3.8. *The growth rate of $MB_{1,3}$ is 2.29.*

Proof. The Hilbert series in rational form obtained in Theorem 3.6 can be resolved (approximately) into its partial fraction, using Maple, as follows:

$$\begin{aligned} &\frac{1}{(1-t)(1-2t-t^2+t^4+t^5+t^6+t^7+t^8)} \\ &\qquad = \frac{0.65564t + 0.51628}{t^2 + 0.98567t + 1.35852} + \frac{0.33333}{1-t} \\ &\qquad\quad + \frac{0.60593t - 0.39941}{t^2 - 0.98615t + 1.49520} + \frac{0.56106t + 0.60272}{t^2 + 2.21096t + 1.45727} \\ &\qquad\quad + \frac{0.80972}{0.4364 + t}. \end{aligned}$$

The first four terms have negligible contribution in the approximation of the series; however, the last term can be approximated as

$$\frac{0.809722}{0.43644 + t} = 1.8552\,\{1 + 2.29t + (2.29)^2 t^2 + (2.29)^3 t^3 + \cdots\}.$$

Therefore $a_k^{(4)} \approx 1.8552\,(2.29)^k$. Hence the growth rate of $MB_{1,3}$ is 2.29. □

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