REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 67, No. 1, 2024, Pages 245–256
Published online: May 19, 2024
https://doi.org/10.33044/revuma.3486

THE PRINCIPAL SMALL INTERSECTION GRAPH OF A COMMUTATIVE RING

SOHEILA KHOJASTEH

ABSTRACT. Let R be a commutative ring with non-zero identity. The small intersection graph of R, denoted by $G(R)$, is a graph with the vertex set $V(G(R))$, where $V(G(R))$ is the set of all proper non-small ideals of R and two distinct vertices I and J are adjacent if and only if $I \cap J$ is not small in R. In this paper, we introduce a certain subgraph $PG(R)$ of $G(R)$, called the principal small intersection graph of R. It is the subgraph of $G(R)$ induced by the set of all proper principal non-small ideals of R. We study the diameter, the girth, the clique number, the independence number and the domination number of $PG(R)$. Moreover, we present some results on the complement of the principal small intersection graph.

1. Introduction

There are many papers on assigning a graph to a ring R, see, for instance, [1, 3, 4]. Also, the intersection graph of some algebraic structures such as poset, group, ring and module have been studied by several authors, see [2, 7, 8, 9, 10] and [11]. Let R be a commutative ring, and let $I(R)^*$ be the set of all non-zero proper ideals of R. In [5], the *small intersection graph*, $G(R)$ of R was introduced and studied. The vertex set of $G(R)$, $V(G(R))$, is the set of all proper non-small ideals of R and two distinct vertices I and J in $V(G(R))$ are adjacent if and only if $I \cap J$ is not small in R. In this paper, we continue the study of $G(R)$ and introduce $PG(R)$, the induced subgraph of $G(R)$ on the set of all proper principal non-small ideals of R.

We first summarize the notations and concepts. Throughout the paper, all rings are commutative with non-zero identity and all modules are unitary. Let M be an R-module. A submodule N of M is called *small* in M (denoted by $N \ll M$) in case for every submodule L of M, $N + L = M$ implies that $L = M$. A module M is said to be a *hollow module* if every proper submodule of M is a small submodule. A *cyclic* module is a module that is generated by one element. We denote by $J(R)$ and $\mathrm{Max}(R)$ the Jacobson radical of R and the set of all maximal ideals of R, respectively. If R has a unique maximal ideal, then R is said to be a *local ring*.

2020 *Mathematics Subject Classification.* 05C12, 05C25, 05C40, 05C69, 13A15.

Key words and phrases. Small ideal, small intersection graph, clique number, independence number, domination number, chromatic number.

Also, an ideal I of R is *small* (denoted by $I \ll R$) if $I + K = R$ for some ideal K of R implies $K = R$, or equivalently, $I \subseteq J(R)$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the set of integers and the set of integers modulo n, respectively.

Let G be a graph with vertex set $V(G)$. If a is adjacent to b, then we write $a — b$. If $|V(G)| \geq 2$, then a *path* from a to b is a series of adjacent vertices $a — x_1 — x_2 — \cdots — x_n — b$. A graph G is *connected* if for every pair of distinct vertices $a, b \in V(G)$, there exists a path between a and b. For $a, b \in V(G)$ with $a \neq b$, $d(a, b)$ denotes the length of a shortest path from a to b. If there is no such path, then we will make the convention $d(a, b) = \infty$. The *diameter* of G is defined as $\operatorname{diam}(G) = \sup\{d(a, b) \mid a \text{ and } b \text{ are vertices of } G\}$. For any $a \in V(G)$, the *degree* of a, $d(a)$, is the number of edges incident with a. A *regular graph* is a graph where each vertex has the same degree. The *complement* of G, denoted by $\overline{G}$, is a graph on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. A graph G is *complete* if each pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices. Note that a graph whose edge-set is empty is *totally disconnected*. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. We use C_n to denote the cycle with n vertices, where $n \geq 3$. If a graph G has a cycle, then the *girth* of G (denoted by $\operatorname{gr}(G)$) is defined as the length of a shortest cycle of G; otherwise $\operatorname{gr}(G) = \infty$. A *forest* is a graph with no cycle. Also, a *unicyclic graph* is a connected graph with a unique cycle. Suppose that H is a non-empty subset of $V(G)$. The subgraph of a graph G whose vertex set is H and whose edge set is the set of those edges of G with both ends in H is called *the subgraph of G induced by H* and is denoted by $\langle H \rangle$. A graph G may be expressed uniquely as a disjoint union of connected graphs. These graphs are called the *connected components*, or simply the *components*, of G. For a connected graph G, x is a *cut vertex* of G if $\langle V(G) \setminus \{x\} \rangle$ is not connected. For every positive integer r, an *r-partite* graph is one whose vertex set can be partitioned into r subsets, or parts, in such a way that no edge has both ends in the same part. An r-partite graph is *complete r-partite* if any two vertices in different parts are adjacent. We denote the *complete bipartite graph* with part sizes m and n by $K_{m,n}$.

A *clique* of a graph is a complete subgraph and the number of vertices in a largest clique of a graph G, denoted by $\omega(G)$, is called the *clique number* of G. An *independent set* is a subset of the vertices of a graph such that no vertices are adjacent. The number of vertices in a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$. A *dominating set* is a subset S of $V(G)$ such that every vertex of $V(G) \setminus S$ is adjacent to at least one vertex in S. The number of vertices in a smallest dominating set, denoted by $\gamma(G)$, is called the *domination number* of G. By $\chi(G)$ we denote the *chromatic number* of G, i.e., the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A graph is *weakly perfect* if $\chi(G) = \omega(G)$.

Here is a brief summary of the paper. We introduce the principal small intersection graph of a commutative ring R, denoted by $PG(R)$. In Section 2, we prove that $\mathrm{diam}(PG(R)) \in \{1, 2, \infty\}$ and $\mathrm{gr}(PG(R)) \in \{3, \infty\}$. Also, it is shown that $PG(R)$ is a forest if and only if $PG(R) \in \{\overline{K_2}, K_2 \cup K_2\}$. Moreover, it is proved that if R is a commutative ring with finitely many maximal ideals, then $\gamma(PG(R)) = 2$ and $\alpha(PG(R)) = |\operatorname{Max}(R)|$. In Section 3, we study the complement of the principal small intersection graph. It is proved that if $\operatorname{Max}(R)$ is finite, then $\mathrm{diam}(\overline{PG(R)}) \in \{1, 2, 3\}$ and $\mathrm{gr}(\overline{PG(R)}) \in \{3, 4, \infty\}$. Among other results, we prove that $\chi(\overline{PG(R)}) = |\operatorname{Max}(R)|$, where $\operatorname{Max}(R)$ is finite.

2. Basic properties of $PG(R)$

We begin with the following definition.

Definition. Let R be a ring. The *principal small intersection graph* $PG(R)$ is the graph with the vertex set $V(PG(R))$, where $V(PG(R))$ is the set of all proper principal non-small ideals of R, and two distinct vertices Rx and Ry are adjacent if and only if $Rx \cap Ry$ is not small in R.

Remark 2.1. Clearly, $PG(R)$ is an induced subgraph of the intersection graph of ideals of R. This is an important result of the definition.

To prove the next results, we use the prime avoidance theorem (see [12, p. 56]). If $\{M_i\}_{i=1}^{n} \subseteq \operatorname{Max}(R)$, then $M_i \nsubseteq \bigcup_{j\neq i} M_j$ and $\bigcap_{j\neq i} M_j \nsubseteq M_i$ for every i, $1 \le i \le n$.

Theorem 2.2. *Let R be a ring. Then $V(PG(R)) = \varnothing$ if and only if R is a local ring.*

Proof. First, suppose that $V(PG(R)) = \varnothing$. Assume to the contrary that R is a non-local ring and $M_1, M_2 \in \operatorname{Max}(R)$. Since $M_1 + M_2 = R$, we have $Rx_1 + Rx_2 = R$ for some $x_1 \in M_1 \setminus M_2$ and $x_2 \in M_2 \setminus M_1$. Therefore, $Rx_1, Rx_2 \in V(PG(R))$, a contradiction. Hence R is a local ring. Conversely, assume that R is a local ring. Then Rx is a small ideal of R for every non-unit element $x \in R$. Therefore, $V(PG(R)) = \varnothing$ and the proof is complete. □

Next, we study the case where $PG(R)$ is totally disconnected.

Theorem 2.3. *Let R be a ring. Then $PG(R)$ is totally disconnected if and only if $R \cong F_1 \times F_2$, where F_1, F_2 are fields.*

Proof. Assume that $PG(R)$ is totally disconnected. By the previous theorem, we have $|\operatorname{Max}(R)| \ge 2$. First, suppose that $|\operatorname{Max}(R)| \ge 3$. Let $M_1, M_2, M_3 \in \operatorname{Max}(R)$, $x \in M_1 \setminus (M_2 \cup M_3)$, and let $y \in M_2 \setminus (M_1 \cup M_3)$. Then $Rx \cap Ry \nsubseteq M_3$ and so $Rx \cap Ry$ is not small in R. Hence Rx and Ry are adjacent, a contradiction. Therefore, $|\operatorname{Max}(R)| = 2$. Let $\operatorname{Max}(R) = \{M_1, M_2\}$.

We claim that $M_1 = Rx_1$ and $M_2 = Rx_2$, where $x_1 \in M_1 \setminus M_2$ and $x_2 \in M_2 \setminus M_1$. If $x_1' \in M_1 \setminus M_2$ and $Rx_1 \neq Rx_1'$, then Rx_1 and Rx_1' are adjacent, which is impossible. Therefore, $M_1 = J(R) \cup Rx_1$. Similarly, $M_2 = J(R) \cup Rx_2$. Now, we show that $J(R) \subseteq Rx_1 \cap Rx_2$. Let $a \in J(R)$. Clearly, $a + x_i \in M_i \setminus J(R)$

for $i = 1, 2$. Therefore, $a + x_i \in Rx_i$ for $i = 1, 2$. Hence $a \in Rx_1 \cap Rx_2$. So $J(R) \subseteq Rx_1 \cap Rx_2$. This yields $M_1 = Rx_1$ and $M_2 = Rx_2$, and the claim is proved. Clearly, $M_2 = R(1 - x_1)$.

Now, we prove that $M_1M_2 = 0$. Since $x_1^2 \in M_1 \setminus M_2$, we have $Rx_1 = Rx_1^2$. Hence $x_1 = rx_1^2$ for some $r \in R$. This implies that $x_1(1 - rx_1) = 0 \in J(R)$ and so $1 - rx_1 \in M_2$. We note that $1 - rx_1 \notin M_1$. If not, $1 - rx_1, rx_1 \in M_1 = Rx_1$ which is impossible. Since $1 - rx_1 \in M_2 \setminus M_1$, we have $M_2 = R(1 - x_1) = R(1 - rx_1)$. On the other hand, we find that $Rx_1R(1 - rx_1) = M_1M_2 = 0$.

Next, we prove that $J(R) = 0$. Let $0 \neq a \in J(R)$. Then $a + x_1 \in M_1 \setminus M_2$ and so $a + x_1 = sx_1$ for some $s \in R$. This yields $a = (s - 1)x_1 \in M_1 \cap M_2$, which implies that $s - 1 \in M_2$. We have $a = (s - 1)x_1 \in M_1M_2$. Therefore, $J(R) = M_1M_2 = 0$. Now, by the Chinese remainder theorem [6, p. 7], $R \cong F_1 \times F_2$, where $F_1 = R/M_1$ and $F_2 = R/M_2$ are fields.

Conversely, if $R \cong F_1 \times F_2$, then $\mathrm{Max}(R) = \{F_1 \times 0, 0 \times F_2\} = V(PG(R))$ and $PG(R) \cong \overline{K_2}$. This completes the proof. □

Now, we have an immediate corollary.

Corollary 2.4. *Let R be a ring. Then $PG(R)$ is totally disconnected if and only if $G(R)$ is totally disconnected. Moreover, $PG(R)$ is totally disconnected if and only if $PG(R) = G(R) \cong \overline{K_2}$.*

Proof. If $PG(R)$ is totally disconnected, then by the above theorem $R \cong F_1 \times F_2$, where F_1, F_2 are fields. Hence $\mathrm{Max}(R) = \{F_1 \times 0, 0 \times F_2\}$ and $F_1 \times 0, 0 \times F_2$ are distinct cyclic hollow R-modules (see [13, p. 352]). Then by [5, Theorem 2.4], $G(R)$ is totally disconnected. The proof of the converse is clear. □

Also, we have the following result for the case where $G(R)$ is totally disconnected.

Corollary 2.5. *Let R be a ring. Then $G(R)$ is totally disconnected if and only if $R \cong F_1 \times F_2$, where F_1, F_2 are fields.*

Theorem 2.6. *Let R be a ring. Then the following statements are equivalent:*

(i) *$PG(R)$ is disconnected;*
(ii) *$|\mathrm{Max}(R)| = 2$;*
(iii) *$PG(R) = G_1 \cup G_2$, where G_1, G_2 are two disjoint complete subgraphs of $PG(R)$.*

Proof. (i) ⇒ (ii) Assume that $PG(R)$ is disconnected, G_1 and G_2 are two components of $PG(R)$ and Rx, Ry are two vertices such that $Rx \in G_1$ and $Ry \in G_2$. Let $\mathrm{Max}(R) = \{M_i\}_{i \in I}$ and let $A = \{i \in I \mid Rx \nsubseteq M_i\}$, $B = \{i \in I \mid Ry \nsubseteq M_i\}$. Since $Rx \cap Ry \ll R$, we have $Rx \cap Ry \subseteq J(R)$. This implies that $A \cap B = \varnothing$. Let $a \in A$ and $b \in B$. If $|\mathrm{Max}(R)| \geq 3$, then $\mathrm{Max}(R) \setminus \{M_a, M_b\} \neq \varnothing$. Suppose that $M_c \in \mathrm{Max}(R) \setminus \{M_a, M_b\}$ and $z \in M_c \setminus (M_a \cup M_b)$. Clearly, $Rx \cap Rz \nsubseteq M_a$ and $Ry \cap Rz \nsubseteq M_b$. Hence we have a path $Rx — Rz — Ry$, a contradiction. Therefore, $|\mathrm{Max}(R)| \leq 2$. If $|\mathrm{Max}(R)| = 1$, then by Theorem 2.2, we conclude that $V(PG(R)) = \varnothing$, a contradiction. Therefore, $|\mathrm{Max}(R)| = 2$.

(ii) $\Rightarrow$ (iii) Let $\operatorname{Max}(R) = \{M_1, M_2\}$ and let $G_i = \{0 \neq Rx \mid Rx \subseteq M_i, Rx$ is not small in $R\}$ for $i = 1, 2$. If $Rx, Ry \in G_1$ and Rx and Ry are not adjacent then $Rx \cap Ry \ll R$, which implies $Rx \cap Ry \subseteq M_2$. Hence $Rx \subseteq M_2$ or $Ry \subseteq M_2$, which gives $Rx \ll R$ or $Ry \ll R$, a contradiction. So G_1 is a complete subgraph of $PG(R)$. Similarly, G_2 is a complete subgraph of $PG(R)$. Clearly, there is no path between G_1 and G_2. Therefore, $PG(R) = G_1 \cup G_2$, where G_1 and G_2 are disjoint complete subgraphs.

(iii) $\Rightarrow$ (i) It is clear. □

From the above theorem and [5, Theorem 2.6], we can deduce the next result.

Corollary 2.7. *Let R be a ring. Then $PG(R)$ is connected if and only if $G(R)$ is connected.*

Now, we study the diameter of $PG(R)$.

Theorem 2.8. *Let R be a ring. If $PG(R)$ is connected, then* $\operatorname{diam}(PG(R)) \leq 2$.

Proof. Let Rx and Ry be two non-adjacent vertices of $PG(R)$. So $Rx \cap Ry \ll R$. Let $\operatorname{Max}(R) = \{M_i\}_{i \in I}$, $A = \{i \in I \mid Rx \nsubseteq M_i\}$ and $B = \{i \in I \mid Ry \nsubseteq M_i\}$. Since $Rx \cap Ry \ll R$, we have $Rx \cap Ry \subseteq J(R)$. This implies that $A \cap B = \varnothing$. Assume that $a \in A$ and $b \in B$. By Theorem 2.6, $|\operatorname{Max}(R)| \geq 3$ which implies that $\operatorname{Max}(R) \setminus \{M_a, M_b\} \neq \varnothing$. Suppose that $M_c \in \operatorname{Max}(R) \setminus \{M_a, M_b\}$ and $z \in M_c \setminus (M_a \cup M_b)$. Clearly, $Rx \cap Rz \nsubseteq M_a$ and $Ry \cap Rz \nsubseteq M_b$. Hence $Rx — Rz — Ry$ is a path in $PG(R)$. Therefore, $\operatorname{diam}(PG(R)) \leq 2$. □

In [5, Theorem 2.8], it was proved that if $G(R)$ is connected, then $\operatorname{diam}(G(R)) \leq 2$. In the above theorem, we deduce the same result for $PG(R)$. The following theorem shows that the girth of $PG(R)$ has two possible values.

Theorem 2.9. *Let R be a ring. Then* $\operatorname{gr}(PG(R)) \in \{3, \infty\}$.

Proof. If $|\operatorname{Max}(R)| = 2$, then $PG(R)$ is a union of two disjoint complete graphs by Theorem 2.6. Hence $\operatorname{gr}(PG(R)) \in \{3, \infty\}$. If $|\operatorname{Max}(R)| \geq 3$, then suppose that $M_1, M_2, M_3 \in \operatorname{Max}(R)$. Let $x \in M_1 \setminus (M_2 \cup M_3)$, $y \in M_2 \setminus (M_1 \cup M_3)$ and $z \in M_3 \setminus (M_1 \cup M_2)$. Clearly, $Rx — Ry — Rz — Rx$ is a cycle in $PG(R)$. Therefore, $\operatorname{gr}(G(R)) = 3$. □

Theorem 2.10. *Let R be a ring such that* $\operatorname{Max}(R)$ *is finite. Then the following hold:*

(i) *there is no vertex in $PG(R)$ that is adjacent to every other vertex;*
(ii) *$PG(R)$ can not be a complete graph.*

Proof. (i) Suppose, to the contrary, that Rx is a vertex of $PG(R)$ adjacent to every other vertex. Let $\operatorname{Max}(R) = \{M_1, M_2, \ldots, M_n\}$. By Theorem 2.6, we know that $n \geq 3$. Since Rx is a vertex of $PG(R)$, we have $x \in M_i$ for some $M_i \in \operatorname{Max}(R)$. Let $y \in \bigcap_{j \neq i} M_j \setminus M_i$. We note that Rx and Ry are distinct vertices of $PG(R)$. But Rx is not adjacent to Ry, a contradiction.

(ii) If the edge-set is empty, then $PG(R)$ is totally disconnected with one vertex. Corollary 2.4, shows that $PG(R) \cong \overline{K_2}$ and $PG(R)$ has two vertices, a contradiction. Hence $PG(R)$ has at least one edge, which is a contradiction by (i). Thus $PG(R)$ can not be a complete graph. □

Theorem 2.11. *If R is a ring, then $PG(R)$ contains a pendant vertex if and only if $|\operatorname{Max}(R)| = 2$ and $PG(R) \cong K_2 \cup K_2$.*

Proof. Let $\operatorname{Max}(R) = \{M_i\}_{i \in I}$. First, suppose that there exists $Rx \in V(PG(R))$ such that $d(Rx) = 1$. Since $Rx \in V(PG(R))$, we have $x \notin M_j$ for some $M_j \in \operatorname{Max}(R)$. Suppose, for contradiction, that $|\operatorname{Max}(R)| \geq 3$. Let $M_1, M_2 \in \operatorname{Max}(R) \setminus \{M_j\}$. It is not hard to see that Rx is adjacent to both Ry and Rz for every $y \in M_1 \setminus (M_j \cup M_2)$ and $z \in M_2 \setminus (M_j \cup M_1)$, a contradiction. Therefore, $|\operatorname{Max}(R)| = 2$. Also, by Theorem 2.6, we conclude that $PG(R) \cong K_2 \cup K_2$. The proof of the converse is obvious. □

In the following result, we determine that all forests can occur as the principal small intersection graph of a commutative ring.

Corollary 2.12. *Let R be a ring. Then $PG(R)$ is a forest if and only if $PG(R) \in \{\overline{K_2}, K_2 \cup K_2\}$.*

Example 2.13. There are some rings R for which $PG(R) \cong K_2 \cup K_2$. For instance, suppose that $R = \mathbb{Z}_{p^2q^2}$ for some distinct prime numbers p, q. Then $\operatorname{Max}(R) = \{p\mathbb{Z}_{p^2q^2}, q\mathbb{Z}_{p^2q^2}\}$ and $V(PG(R)) = \{p\mathbb{Z}_{p^2q^2}, q\mathbb{Z}_{p^2q^2}, p^2\mathbb{Z}_{p^2q^2}, q^2\mathbb{Z}_{p^2q^2}\}$. Also, $p\mathbb{Z}_{p^2q^2} — p^2\mathbb{Z}_{p^2q^2}$ and $q\mathbb{Z}_{p^2q^2} — q^2\mathbb{Z}_{p^2q^2}$ are two paths. Hence $PG(R) \cong K_2 \cup K_2$.

Corollary 2.14. *Let R be a ring. Then $PG(R)$ is not a unicyclic graph.*

Proof. Suppose, for contradiction, that $PG(R)$ is a unicyclic graph. Since $PG(R)$ is a connected graph, $|\operatorname{Max}(R)| \geq 3$. Then by Theorem 2.11, $PG(R)$ does not have a pendant vertex. Hence by Theorem 2.9, $PG(R)$ is a 3-cycle. On the other hand, Theorem 2.10 shows that $PG(R)$ can not be a complete graph. In particular, $PG(R)$ can not be a 3-cycle, a contradiction. This completes the proof. □

Now, we provide a lower bound for the clique number of $PG(R)$.

Theorem 2.15. *Let R be a ring. Then the following hold:*

(i) $\omega(PG(R)) = 1$ *if and only if* $R \cong F_1 \times F_2$, *where* F_1, F_2 *are fields;*
(ii) *if* $\omega(PG(R)) \geq 2$, *then* $|\operatorname{Max}(R)| \leq \omega(PG(R))$;
(iii) *if* $\omega(PG(R)) < \infty$, *then* $|\operatorname{Max}(R)| < \infty$;
(iv) *if* $\operatorname{Max}(R)$ *is finite, then* $\omega(PG(R)) \geq 2^{|\operatorname{Max}(R)|-1} - 1$.

Proof. (i) It is clear by Theorem 2.6.

(ii) Suppose, for contradiction, that $\omega(PG(R)) = n \geq 2$ and $|\operatorname{Max}(R)| \geq n+1 \geq 3$. Let $M_1, \ldots, M_{n+1} \in \operatorname{Max}(R)$ and let $x_i \in M_i \setminus \bigcup_{j \neq i} M_j$ for $i = 1, \ldots, n+1$. It is not hard to see that $\{Rx_1, \ldots, Rx_{n+1}\}$ is a clique of $PG(R)$, a contradiction. Therefore, $|\operatorname{Max}(R)| \leq \omega(PG(R))$.

(iii) It is clear by (ii).

(iv) If $|\operatorname{Max}(R)| = 1$, then by Theorem 2.2, $V(PG(R)) = \varnothing$. So, consider $|\operatorname{Max}(R)| \geq 2$. Let $\operatorname{Max}(R) = \{M_1, \dots, M_n\}$, $A = \{M_2, \dots, M_n\}$ and let $P(A)$ be the power set of A. For each $\varnothing \neq X \in P(A)$, set $x_X \in \bigcap_{M_i \in X} M_i \setminus M_1$. It is not hard to see that if $\varnothing \neq X, Y \in P(A)$ and $X \neq Y$, then $Rx_X \neq Rx_Y$. Also, $Rx_X \cap Rx_Y \nsubseteq M_1$. This implies that the subgraph of $PG(R)$ with the vertex set $\{Rx_X \mid \varnothing \neq X \in P(A)\}$ is a clique of $PG(R)$. We note that $|P(A)\setminus\{\varnothing\}| = 2^{n-1}-1$, so $|\{Rx_X \mid \varnothing \neq X \in P(A)\}| = 2^{|\operatorname{Max}(R)|-1} - 1$. This completes the proof. □

Example 2.16. (i) The lower bound in part (iv) of the previous theorem is sharp. To see this, consider $R = F_1 \times F_2$, where F_1, F_2 are fields. Then we have $\omega(PG(R)) = 2^{|\operatorname{Max}(R)|-1} - 1 = 1$.

(ii) There are some rings R for which $\omega(PG(R)) > 2^{|\operatorname{Max}(R)|-1} - 1$. For instance, suppose that $R = \mathbb{Z}_{p^n q^m}$ for some distinct prime numbers p, q and positive integers n, m with $\max\{n, m\} \geq 2$. Then $\operatorname{Max}(R) = \{p\mathbb{Z}_{p^n q^m}, q\mathbb{Z}_{p^n q^m}\}$. It is not hard to see that $PG(R) \cong K_n \cup K_m$. We have $\omega(PG(R)) = \max\{n, m\}$ and $2^{|\operatorname{Max}(R)|-1} - 1 = 1$. Clearly, $\omega(PG(R)) > 2^{|\operatorname{Max}(R)|-1} - 1$.

To prove Theorem 2.18, we need the following simple lemma.

Lemma 2.17. *Let R be a ring. If $Rx, Ry \in V(PG(R))$ and $Rx \subset Ry$, then the following hold:*

(i) *$d(Rx) \leq d(Ry)$.*
(ii) *If Rz is adjacent to Rx, then Rz is adjacent to Ry.*

Proof. Apply the proof of [5, Theorem 2.15]. □

Theorem 2.18. *If R is a ring and $PG(R)$ is an r-regular graph, then $|\operatorname{Max}(R)| = 2$ and $PG(R) \cong K_{r+1} \cup K_{r+1}$.*

Proof. Let $PG(R)$ be an r-regular graph. By Theorem 2.15, $\operatorname{Max}(R)$ is finite. First, assume that $|\operatorname{Max}(R)| = n \geq 3$, $x \in M_1 \setminus \bigcup_{i=2}^n M_i$ and $y \in (M_1 \cap M_2) \setminus \bigcup_{i=3}^n M_i$. By Lemma 2.17, $d(Rxy) \leq d(Rx)$. We claim that $d(Rxy) < d(Rx)$. Let $z \in \bigcap_{i=3}^n M_i \setminus (M_1 \cup M_2)$. It is clear that Rz is adjacent to Rx, but Rz is not adjacent to Rxy. Therefore, $d(Rxy) < d(Rx)$ and the claim is proved. This is a contradiction because $PG(R)$ is a regular graph and $d(Rxy) = d(Rx)$. Hence $|\operatorname{Max}(R)| = 2$ and by Theorem 2.6, $PG(R) \cong K_{r+1} \cup K_{r+1}$. □

Now, we are in a position to state one of the main results of this section.

Theorem 2.19. *Let R be a ring. Then $PG(R)$ can not be a complete r-partite graph.*

Proof. Suppose, for contradiction, that $PG(R)$ is a complete r-partite graph with r parts $V_1, \dots, V_r$. Then by Theorem 2.15, $|\operatorname{Max}(R)| \leq r$. In view of the proof of Theorem 2.15, we find that $\{Rx_1, \dots, Rx_n\}$ is a clique of $PG(R)$, where $\operatorname{Max}(R) = \{M_1, \dots, M_n\}$ and $x_i \in M_i \setminus \bigcup_{j \neq i} M_j$ for $i = 1, \dots, n$. With no loss of generality, assume that $Rx_i \in V_i$ for $i = 1, \dots, n$. Suppose that $y_i \in \bigcap_{j \neq i} M_j \setminus M_i$ for every i, $1 \leq i \leq n$. Then Rx_i and Ry_i are not adjacent. This implies that $\{Rx_i, Ry_i\} \subseteq V_i$

for every i, $1 \leq i \leq n$. Let $Rx \in V(PG(R))$. Hence $Rx \nsubseteq M_t$ for some t, $1 \leq t \leq n$. Therefore, Rx and Ry_t are adjacent. Since $Rx_t \in V_t$, Rx and Rx_t are adjacent, a contradiction. □

Theorem 2.20. *Let R be a ring such that $PG(R)$ is connected. Then $PG(R)$ has no cut vertex.*

Proof. Suppose, for contradiction, that Rx is a cut vertex of $PG(R)$. Then the induced subgraph $\langle V(PG(R)) \setminus \{Rx\}\rangle$ is disconnected. Hence there exist vertices Ry and Rz such that Rx lies on every path from Ry to Rz. Theorem 2.6 shows that $|\operatorname{Max}(R)| \geq 3$. Let $M_1, M_2, M_3 \in \operatorname{Max}(R)$. Obviously, Ry and Rz are proper non-small ideals of R. With no loss of generality, we may assume that $Ry \nsubseteq M_1, Rz \nsubseteq M_2$, because $Ry \cap Rz \ll R$. Since $Ry \cap Rz \ll R$, we have $Ry \subseteq M_2$ and $Rz \subseteq M_1$. If there exists $w \in M_3 \setminus (M_1 \cup M_2)$ such that $Rw \neq Rx$, then we have a path between Ry and Rz in $PG(R)$, a contradiction. Therefore, $Rw = Rx$ for every $w \in M_3 \setminus (M_1 \cup M_2)$. If $|\operatorname{Max}(R)| \geq 4$, then by a similar argument as above, we conclude that $Rw = Rx$ for every $w \in M \setminus (M_1 \cup M_2)$ and for every $M \in \operatorname{Max}(R) \setminus \{M_1, M_2, M_3\}$, which is impossible. Therefore, $|\operatorname{Max}(R)| = 3$. Let $x_1 \in M_1 \setminus (M_2 \cup M_3)$ and $x_2 \in M_2 \setminus (M_1 \cup M_3)$. It is clear that $Ry — Rx_2 — Rx_1 — Rz$ is a path in $\langle V(PG(R)) \setminus \{Rx\}\rangle$, a contradiction. □

In the rest of this section, we study the domination number and the independence number of the principal small intersection graph of R.

Theorem 2.21. *Let R be a ring. If $\operatorname{Max}(R)$ is finite, then $\gamma(PG(R)) = 2$.*

Proof. Since $V(PG(R)) \neq \varnothing$, $|\operatorname{Max}(R)| \geq 2$. We divide the proof into two cases:

Case 1. $|\operatorname{Max}(R)| = 2$. Then by Theorem 2.6, we deduce that $\gamma(PG(R)) = 2$.

Case 2. $|\operatorname{Max}(R)| \geq 3$. Let $\operatorname{Max}(R) = \{M_1, \ldots, M_n\}$, $x_i \in M_i \setminus \bigcup_{j \neq i} M_j$ for $i = 1, 2$, and let $S = \{Rx_1, Rx_2\}$. If Rx is a vertex of $PG(R)$ and $Rx \notin S$, then Rx is adjacent to Rx_1 or Rx_2. Otherwise, $Rx \cap Rx_1 \subseteq J(R)$ and $Rx \cap Rx_2 \subseteq J(R)$. Hence $Rx \subseteq \bigcap_{j \neq 1} M_j$ and $Rx \subseteq \bigcap_{j \neq 2} M_j$. Therefore, $Rx \subseteq \bigcap_{j=1}^{n} M_j$, a contradiction. This implies that S is a dominating set of $PG(R)$ and so $\gamma(PG(R)) \leq 2$. Now, Theorem 2.10 shows that $\gamma(PG(R)) = 2$. □

In [5], it was proved that $\alpha(G(R)) = |\operatorname{Max}(R)|$, where $\operatorname{Max}(R)$ is finite. Next, we prove that if $\operatorname{Max}(R)$ is finite, then $\alpha(PG(R)) = \alpha(G(R))$.

Theorem 2.22. *Let R be a ring such that $\operatorname{Max}(R)$ is finite. Then $\alpha(PG(R)) = |\operatorname{Max}(R)|$.*

Proof. Let $\operatorname{Max}(R) = \{M_1, \ldots, M_n\}$ and let $S_1 = \{Rx_i \mid x \in \bigcap_{j \neq i} M_j \setminus M_i$ for $i = 1, \ldots, n\}$. Clearly, S_1 is an independent set for $PG(R)$. Therefore, $n \leq \alpha(PG(R))$. Suppose that $S_2 = \{Ry_1, \ldots, Ry_m\}$ is an independent set of $PG(R)$. If $m > n$, then by the pigeonhole principle, we find that there exist i, j, $1 \leq i < j \leq m$, and $M_t \in \operatorname{Max}(R)$ such that $Ry_i \nsubseteq M_t$ and $Ry_j \nsubseteq M_t$. This yields $Ry_i \cap Ry_j \nsubseteq M_t$. On the other hand, we have $Ry_i, Ry_j \in S_2$ and S_2 is an independent set of $PG(R)$. This shows that $Ry_i \cap Ry_j \ll R$, a contradiction. Therefore, $\alpha(PG(R)) = |\operatorname{Max}(R)|$. □

Corollary 2.23. *If R is an Artinian ring, then $\alpha(PG(R)) = |\operatorname{Max}(R)|$.*

Proof. By the structure theorem of Artinian rings [6, Theorem 8.7], there exists a positive integer n such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ and $(R_i, \mathfrak{m}_i)$ is a local ring for all $1 \leq i \leq n$. The above theorem shows that $\alpha(PG(R)) = |\operatorname{Max}(R)| = n$. □

The following example approves the equality $\alpha(PG(R)) = |\operatorname{Max}(R)|$.

Example 2.24. Let F_1, F_2, F_3 be fields and let $R = F_1 \times F_2 \times F_3$. In view of the proof of Corollary 2.23, we find that $\alpha(PG(R)) = 3$. We draw the graph $PG(R)$ in Fig. 1. One can easily see that $\{F_1 \times 0 \times 0, 0 \times F_2 \times 0, 0 \times 0 \times F_3\}$ is an independent set of $PG(R)$.

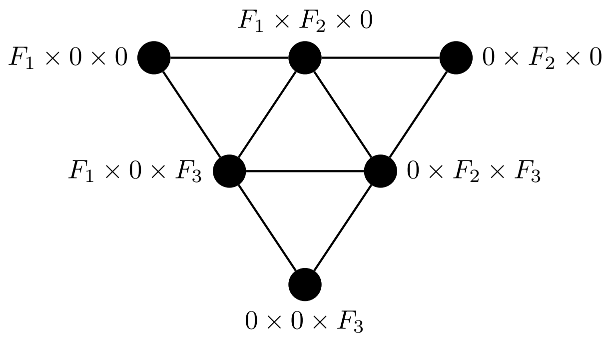

FIGURE 1. $PG(F_1 \times F_2 \times F_3) = G(F_1 \times F_2 \times F_3)$.

3. The complement of $PG(R)$

In this section, we determine the diameter, the girth and the chromatic number of the complement of the principal small intersection graph of R. As we mentioned in the introduction, the complement of the principal small intersection graph of R, $\overline{PG(R)}$, is the graph with the vertex set $V(\overline{PG(R)}) = V(PG(R))$, and two distinct vertices Rx and Ry are adjacent if and only if $Rx \cap Ry \ll R$.

First, we determine the diameter of $\overline{PG(R)}$.

Theorem 3.1. *Let R be a ring such that* $\operatorname{Max}(R)$ *is finite. Then $\overline{PG(R)}$ is connected and* $\operatorname{diam}(\overline{PG(R)}) \in \{1, 2, 3\}$.

Proof. If R is a local ring, then by Theorem 2.2 we have $V(\overline{PG(R)}) = \varnothing$. Also, if $|\operatorname{Max}(R)| = 2$, then by Theorem 2.6, $\overline{PG(R)}$ is a complete bipartite graph and so $\operatorname{diam}(\overline{PG(R)}) \in \{1, 2\}$. Now, suppose that $|\operatorname{Max}(R)| \geq 3$ and $Rx, Ry \in V(\overline{PG(R)})$. Let $\operatorname{Max}(R) = \{M_1, \ldots, M_n\}$, with $n \geq 3$. If Rx and Ry are not adjacent in $\overline{PG(R)}$, then assume that $A = \{M_i \mid 1 \leq i \leq n,\, Rx \subseteq M_i\}$, $B = \{M_i \mid 1 \leq i \leq n,\, Ry \subseteq M_i\}$, $\operatorname{Max}(R) \setminus A = A'$ and $\operatorname{Max}(R) \setminus B = B'$. We have the following two cases:

Case 1. $A \cap B \in \{A, B\}$. With no loss of generality, we may assume that $A \cap B = A$. Then $B' \subseteq A'$. Let $z \in (\bigcap_{M_i \in A'} M_i) \setminus J(R)$. It is clear that Rz is adjacent to both Rx and Ry. Therefore, $d(Rx, Ry) = 2$.

Case 2. $A \cap B \notin \{A, B\}$. Then $A' \cup B \neq \mathrm{Max}(R)$ and $B' \cup A \neq \mathrm{Max}(R)$. Let $z_1 \in (\bigcap_{M_i \in (A' \cup B)} M_i) \setminus J(R)$ and $z_2 \in (\bigcap_{M_i \in (B' \cup A)} M_i) \setminus J(R)$. Clearly, $Rx — Rz_1 — Rz_2 — Ry$ is a path between Rx and Ry in $\overline{PG(R)}$. Hence $d(Rx, Ry) \leq 3$. This completes the proof. □

As an immediate consequence of the previous theorem, we have the next result.

Corollary 3.2. *Let R be a ring such that* $\mathrm{Max}(R)$ *is finite. Then the following hold:*

(i) $\mathrm{diam}(\overline{PG(R)}) = 1$ *if and only if* $|\mathrm{Max}(R)| = 2$ *and* $\overline{PG(R)} \cong K_2$.
(ii) $\mathrm{diam}(\overline{PG(R)}) = 2$ *if and only if* $|\mathrm{Max}(R)| = 2$, $\overline{PG(R)}$ *is a complete bipartite graph and* $\overline{PG(R)} \ncong K_2$.
(iii) $\mathrm{diam}(\overline{PG(R)}) = 3$ *if and only if* $|\mathrm{Max}(R)| \geq 3$.

Proof. Parts (i) and (ii) are clear.

(iii) Let $\mathrm{Max}(R) = \{M_1, \ldots, M_n\}$, $x \in M_1 \setminus \bigcup_{i \neq 1} M_i$ and $y \in M_2 \setminus \bigcup_{i \neq 2} M_i$. Clearly, $Rx \cap Ry \nsubseteq M_3$. This implies that Rx and Ry are not adjacent. We claim that $d(Rx, Ry) = 3$. Otherwise, the previous theorem shows that there exists a vertex, say Rz, such that Rz is adjacent to both Rx and Ry. Since Rz is adjacent to Rx, $z \in \bigcap_{i \neq 1} M_i$. On the other hand, since Rz is adjacent to Ry, $z \in \bigcap_{i \neq 2} M_i$. This implies that $z \in \bigcap_{i=1}^{n} M_i$, which is impossible. Therefore, the claim is proved. Now, by Theorem 3.1, $\mathrm{diam}(\overline{PG(R)}) = 3$. □

Example 3.3. By Theorem 3.1, if R is a ring with finitely many maximal ideals, then $\overline{PG(R)}$ is connected. But there are some rings R with infinite maximal ideals whose $\overline{PG(R)}$ is not connected. Let $R = \mathbb{Z}$. It is clear that $\mathrm{Max}(\mathbb{Z})$ is infinite and the only small ideal of $\mathbb{Z}$ is 0. Also, $\mathrm{diam}(\overline{PG(\mathbb{Z})}) = \infty$ and $\overline{PG(\mathbb{Z})}$ is totally disconnected because $I \cap J \neq 0$ for every two non-zero ideals I and J.

By Theorem 2.6, we have the next corollary.

Corollary 3.4. *Let R be a ring. Then the following statements are equivalent:*

(i) $|\mathrm{Max}(R)| = 2$;
(ii) $\overline{PG(R)}$ *is a complete bipartite graph.*

Theorem 3.5. *Let R be a ring such that* $\mathrm{Max}(R)$ *is finite. Then* $\mathrm{gr}(\overline{PG(R)}) \in \{3, 4, \infty\}$.

Proof. If $|\mathrm{Max}(R)| = 2$, then $\overline{PG(R)}$ is a complete bipartite graph by Corollary 3.4. Hence $\mathrm{gr}(\overline{PG(R)}) \in \{4, \infty\}$. If $|\mathrm{Max}(R)| \geq 3$, then suppose that $\mathrm{Max}(R) = \{M_1, \ldots, M_n\}$, with $n \geq 3$. Let $x_i \in \bigcap_{j \neq i} M_j \setminus M_i$ for $i = 1, 2, 3$. Clearly, $Rx_1 — Rx_2 — Rx_3 — Rx_1$ is a 3-cycle in $\overline{PG(R)}$. Therefore, $\mathrm{gr}(\overline{PG(R)}) = 3$. □

In view of the proof of Theorem 3.5 and by Corollary 3.4, we deduce the following result.

Corollary 3.6. *Let R be a ring. Then the following statements are equivalent:*
- (i) $|\operatorname{Max}(R)| = 2$*;*
- (ii) $\overline{PG(R)}$ *is a complete bipartite graph;*
- (iii) $\overline{PG(R)}$ *is a bipartite graph.*

Theorem 2.22 shows that if $\operatorname{Max}(R)$ is finite, then $\alpha(PG(R)) = \omega(\overline{PG(R)}) = |\operatorname{Max}(R)|$. We close this paper with the following main result, which implies that the complement of the principal small intersection graph is weakly perfect.

Theorem 3.7. *Let R be a ring such that* $\operatorname{Max}(R)$ *is finite. Then* $\chi(\overline{PG(R)}) = |\operatorname{Max}(R)| = \omega(\overline{PG(R)})$.

Proof. Let $\operatorname{Max}(R) = \{M_1, \dots, M_n\}$. We define the map $c : V(\overline{PG(R)}) \longrightarrow \{1, \dots, n\}$ by $c(Rx) = \min\{i \mid 1 \leq i \leq n, Rx \nsubseteq M_i\}$. It suffices to show that c is a proper vertex coloring of $\overline{PG(R)}$. If $c(Rx) = c(Ry) = t$ for some $Rx, Ry \in V(\overline{PG(R)})$ and for some $t \in \{1, \dots, n\}$, then we have $Rx \nsubseteq M_t$ and $Ry \nsubseteq M_t$. This implies that $Rx \cap Ry$ is non-small and so Rx and Ry are not adjacent in $\overline{PG(R)}$. Therefore, c is a proper vertex coloring. Thus $\chi(\overline{PG(R)}) \leq |\operatorname{Max}(R)|$. Now, the result follows from Theorem 2.22. □

Acknowledgments

I would like to thank the referee(s) for valuable and fruitful comments.

References

[1] S. Akbari and S. Khojasteh, Commutative rings whose cozero-divisor graphs are unicyclic or of bounded degree, *Comm. Algebra* **42** no. 4 (2014), 1594–1605. DOI MR Zbl

[2] S. Akbari, H. A. Tavallaee, and S. Khalashi Ghezelahmad, Intersection graph of submodules of a module, *J. Algebra Appl.* **11** no. 1 (2012), Paper no. 1250019, 8 pp. DOI MR Zbl

[3] D. F. Anderson and A. Badawi, The total graph of a commutative ring, *J. Algebra* **320** no. 7 (2008), 2706–2719. DOI MR Zbl

[4] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** no. 2 (1999), 434–447. DOI MR Zbl

[5] S. E. Atani, S. D. Pish Hesari, and M. Khoramdel, A graph associated to proper non-small ideals of a commutative ring, *Comment. Math. Univ. Carolin.* **58** no. 1 (2017), 1–12. DOI MR Zbl

[6] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass.-London-Don Mills, Ont., 1969. MR

[7] I. Chakrabarty, S. Ghosh, T. K. Mukherjee, and M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.* **309** no. 17 (2009), 5381–5392. DOI MR Zbl

[8] B. Csákány and G. Pollák, The graph of subgroups of a finite group (Russian), *Czechoslovak Math. J.* **19 (94)** no. 2 (1969), 241–247. DOI MR Zbl

[9] F. Heydari, The M-intersection graph of ideals of a commutative ring, *Discrete Math. Algorithms Appl.* **10** no. 3 (2018), Paper no. 1850038, 11 pp. DOI MR Zbl

[10] S. Khojasteh, The intersection graph of ideals of $\mathbb{Z}_m$, *Discrete Math. Algorithms Appl.* **11** no. 4 (2019), Paper no. 1950037, 12 pp. DOI MR Zbl

[11] S. Khojasteh, The complement of the intersection graph of ideals of a poset, *J. Algebra Appl.* **22** no. 11 (2023), Paper no. 2350236, 13 pp. DOI MR Zbl

[12] R. Y. Sharp, *Steps in commutative algebra*, London Mathematical Society Student Texts 19, Cambridge University Press, Cambridge, 1990. MR Zbl

[13] R. Wisbauer, *Foundations of module and ring theory*, Algebra, Logic and Applications 3, Gordon and Breach Science Publishers, Philadelphia, PA, 1991. MR Zbl

Soheila Khojasteh
Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran
s_khojasteh@liau.ac.ir, khojaste.soheila@gmail.com

Received: July 15, 2022
Accepted: March 17, 2023