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BOUNDEDNESS OF GEOMETRIC INVARIANTS NEAR A SINGULARITY WHICH IS A SUSPENSION OF A SINGULAR CURVE

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Dedicated to Professors Masaaki Umehara and Kotaro Yamada on the occasion of their sixtieth birthdays

Abstract. Near a singular point of a surface or a curve, geometric invariants diverge in general, and the orders of this divergence, in particular the boundedness about these invariants, represent the geometry of the surface and the curve. In this paper, we study the boundedness and orders of several geometric invariants near a singular point of a surface which is a suspension of a singular curve in the plane, and those of the curves passing through the singular point. We evaluate the orders of the Gaussian and mean curvatures, as well as those of the geodesic and normal curvatures, and the geodesic torsion for the curve.

1. Introduction

In this paper, we study the boundedness of several geometric invariants near a singular point of a surface which is a suspension of a singular curve in the plane. More precisely, let σ be an $\mathcal{A}$-equivalence class of singular plane curve-germs. A *σ-edge* is a map-germ $f:(\mathbf{R}^2,0)\to(\mathbf{R}^3,0)$ such that it is $\mathcal{A}$-equivalent to $(u,v)\mapsto(u,c_1(v),c_2(v))$, where $c=(c_1,c_2)$ is a representative of σ, namely, a one-dimensional *suspension* of σ. Here, two map-germs $h_1,h_2:(\mathbf{R}^m,0)\to(\mathbf{R}^n,0)$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphisms $\Phi_s:(\mathbf{R}^m,0)\to(\mathbf{R}^m,0)$ and $\Phi_t:(\mathbf{R}^n,0)\to(\mathbf{R}^n,0)$ such that $h_2=\Phi_t\circ h_1\circ\Phi_s^{-1}$. A cuspidal edge ($\mathcal{A}$-equivalent to the germ $(u,v)\mapsto(u,v^2,v^3)$ at the origin) and a 5/2-cuspidal edge ($\mathcal{A}$-equivalent to the germ $(u,v)\mapsto(u,v^2,v^5)$) are examples of σ-edges, and σ are a 3/2-cusp and 5/2-cusp, respectively. If σ is of finite multiplicity, then the σ-edge is a frontal. A frontal is a class of surfaces with singular points, and it is well known that surfaces with constant curvature are frequently in this class. In these decades, there are several studies of frontals from the viewpoint of differential geometry and various geometric invariants at singular points are introduced (for instance,

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[3, 5, 6, 8, 10, 12, 13]). If a surface is invariant under a group action on $\mathbf{R}^3$, then σ-edges will appear naturally. Singularities appearing on surfaces of revolution, and a helicoidal surface are examples of such surfaces [11, 15]. Moreover, such singularities appear on the dual surface at cone-like singular points of a constant mean curvature surface in the de Sitter 3-space (see [7]).

In this paper, we study geometry of σ-edges. For this, we consider two classes of singular map-germs, which we shall call *m-type* and *(m, n)-type edges*, the first including (m, n)-type edges and also σ-edges when σ has finite multiplicity (see Section 2). One observes that m-type edges are frontals. In order to proceed with our study, we find a normal form for each one of these map-germs preserving the geometry of the initial map, since we only use isometries in the target (Proposition 2.9). In [10, 13] the authors define singular, normal and cuspidal curvatures, as well as cuspidal torsion for frontals. In an analogous way, we define similar geometric invariants for m-type edges, using the same names, except for the cuspidal curvature, which we call *$(m, m+i)$-cuspidal curvature*. These invariants are related with the coefficients of the normal form given in Proposition 2.9. It is worth mentioning that these cuspidal curvatures are similar. In fact, we know that a frontal-germ is a front if and only if the cuspidal curvature is not zero. We conclude from Proposition 2.11 that an m-type edge is a front if and only if the $(m, m+1)$-cuspidal curvature is non-zero at 0. In particular, we study orders of geometric invariants and geometric invariants of curves passing through the singular point. We evaluate the orders of Gaussian and mean curvatures (Theorem 2.17) and the minimum orders of geodesic, normal curvatures and geodesic torsion for a singular curve passing through the singular point (Theorem 3.5). These minimum orders are written in terms of singular, cuspidal and normal curvatures and the cuspidal torsion. As a corollary, we give the boundedness of these curvatures under certain generic conditions (Corollary 3.6).

2. Geometry of σ-edges

We give several classes similar to σ-edges. They include σ-edges, and these classes will be useful to treat. We recall that a map-germ $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ is a *frontal* if there exists a unit vector field ν along f such that $\langle df_p(X_p), \nu(p)\rangle = 0$ holds at any $p \in (\mathbf{R}^2, 0)$ and any $X_p \in T_p\mathbf{R}^2$, where $\langle \cdot, \cdot \rangle$ is the canonical inner product of $\mathbf{R}^3$. The vector field ν is called a *unit normal vector field* of f. A map-germ $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ is an *m-type edge* if it is $\mathcal{A}$-equivalent to $(u, v^m, v^{m+1}a(u, v))$ for a function $a(u, v)$. A map-germ $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ is a *(m, n)-type edge* $(m < n)$ if it is $\mathcal{A}$-equivalent to $(u, v^m, v^n h(u, v))$, where $h(0, 0) = 1$. This is equivalent to being $\mathcal{A}^n$-equivalent to (u, v^m, v^n). Two map-germs are $\mathcal{A}^n$-equivalent if their n-jets at the origin are $\mathcal{A}$-equivalent. We establish the following lemma.

Lemma 2.1. *Let f be a σ-edge (respectively, an m-type edge, an (m, n)-type edge). Then an intersection curve of f with a surface T which is transversal to $f(S(f))$ passing through $p \in S(f)$ near 0 is $\mathcal{A}$-equivalent to σ (respectively, $\mathcal{A}^m$-equivalent to $(t^m, 0)$, $\mathcal{A}^n$-equivalent to (t^m, t^n)).*

Proof. Since the assumption and the assertion do not depend on the choice of the coordinate systems, we can assume f is given by $(u, c_1(v), c_2(v))$, where $c = (c_1, c_2)$ is $\mathcal{A}$-equivalent to σ. Then T can be represented by the graph $\{(x, y, z) \mid x = h(y, z)\}$ in $(\mathbf{R}^3, 0)$ as the xyz-space, and the intersection curve is $(h(c_1(v), c_2(v)), c_1(v), c_2(v))$. Since T is transverse to the x-axis, the orthogonal projection of T onto the yz-plane is a diffeomorphism, and thus, we see the assertion. One can show the other claims in a similar way. □

2.1. **A sufficient condition.** We give a sufficient condition for a frontal-germ being an m- or (m, n)-type edge under the assumption $n < 2m$. We assume $n < 2m$ throughout this subsection. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be a frontal-germ satisfying $\operatorname{rank} df_0 = 1$. Then there exists a vector field η such that η_p generates $\ker df_p$ if $p \in S(f)$. We call $\eta|_{S(f)}$ a *null vector field*, and η an *extended null vector field*. An extended null vector field is also called a null vector field if it does not induce a confusion. We assume that the set of singular points $S(f)$ is a regular curve, and the tangent direction of $S(f)$ is not in $\ker df_0$. Let ξ be a vector field such that ξ_p is a non-zero tangent vector of $S(f)$ for $p \in S(f)$. We consider the following conditions for (ξ, η):

[2.1] $\eta^i f = 0$ $(1 \leq i \leq m-1)$ on $S(f)$.
[2.2] $\operatorname{rank}(\xi f, \eta^m f) = 2$ on $S(f)$.
[2.3] $\operatorname{rank}(\xi f, \eta^m f, \eta^i f) = 2$ $(m < i < n)$ on $S(f)$.
[2.4] $\operatorname{rank}(\xi f, \eta^m f, \eta^n f) = 3$ at p.

Here, for a vector field ζ and a map f, the symbol $\zeta^i f$ stands for the i-times directional derivative of f by ζ. Moreover, for a coordinate system (u, v) and a map f, the symbol f_{v^i} stands for $\partial^i f/\partial v^i$.

Proposition 2.2. *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be a frontal-germ satisfying $\operatorname{rank} df_0 = 1$. Assume that the set of singular points $S(f)$ is a regular curve, and the tangent direction of $S(f)$ generated by ξ is not in $\ker df_0$. If there exists a null vector field η satisfying* [2.1], *and (ξ, η) satisfies* [2.2], *then f is an m-type edge. Moreover, if (ξ, η) also satisfies* [2.3]–[2.4], *then f is an (m, n)-type edge.*

As we will see, the conditions [2.2]–[2.4] do not depend on the choice of null vector field η satisfying [2.1]. To show this fact, we show several lemmas which we shall need later. Firstly we show that the conditions do not depend on the choice of the diffeomorphism on the target. In what follows in this section, f is as in Proposition 2.2.

Lemma 2.3. *Let Φ be a diffeomorphism-germ on $(\mathbf{R}^3, 0)$, and set $\hat{f} = \Phi(f)$. If f and (ξ, η) satisfy the condition C, then $\hat{f}$ and (ξ, η) satisfy C, where $C = \{[2.1]\}$, $C = \{[2.1], [2.2]\}$, $C = \{[2.1]–[2.3]\}$ and $C = \{[2.1]–[2.4]\}$.*

Proof. Let us assume η satisfies [2.1]. By a direct calculation, we have $\eta \hat{f} = d\Phi(f)\eta f$, and

$$\eta^i \hat{f} = \sum_{j=0}^{i-1} c_{ij} \eta^j (d\Phi(f)) \eta^{i-j} f \quad (c_{ij} \in \mathbf{R} \setminus \{0\}). \tag{2.1}$$

By [2.1], $\eta^i\hat{f} = 0$ $(2 \leq i \leq m-1)$ and $\eta^m\hat{f} = d\Phi(f)\eta^m f$ on $S(f)$. Then we see the assertion for the cases $C = \{[2.1]\}$ and $C = \{[2.1], [2.2]\}$. We assume η satisfies [2.1]–[2.3]. By (2.1) and $c_{1n} = 1$ $(\neq 0)$ we see the assertion. □

It is clear that the conditions [2.2]–[2.4] do not depend on the choice of ξ, i.e., non-zero functional multiple and extension other than $S(f)$. Moreover, they do not depend on the non-zero functional multiple of η:

Lemma 2.4. *Let h be a non-zero function. If f and (ξ, η) satisfy the condition C, then f and $(\xi, \hat{\eta})$ satisfy C, where $\hat{\eta} = h\eta$ and C is the same as those in Lemma* 2.3.

Proof. Since $(h\eta)^i f$ is a linear combination of $\eta f, \ldots, \eta^i f$, and the coefficient of $\eta^i f$ is h^i, we see the assertion. □

A coordinate system (u, v) satisfying $S(f) = \{v = 0\}$, $\eta|_{S(f)} = \partial_v$ is said to be *adapted*.

Lemma 2.5. *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be a frontal-germ satisfying* $\operatorname{rank} df_0 = 1$. *We assume that the set of singular points $S(f)$ is a regular curve. For any null vector field η, there exists an adapted coordinate system (u, v) such that $\eta = \partial_v$ for any (u, v).*

In this lemma, we do not assume that f is an m-type edge.

Proof. Since $\operatorname{rank} df_0 = 1$, one can easily see that there exists a coordinate system (u, v) such that $\eta = \partial_v$ for any (u, v). Since $S(f)$ is a regular curve, and the tangent direction of it is not in $\ker df_0$, $S(f)$ can be parametrized as $(u, a(u))$. Define a new coordinate system $(\tilde{u}, \tilde{v})$ by $\tilde{u} = u$ and $\tilde{v} = v - a(u)$. Then $S(f) = \{\tilde{v} = 0\}$ and $\partial/\partial\tilde{v} = \partial/\partial v$ hold. This shows the assertion. □

Lemma 2.6. *If two null vector fields $\eta, \tilde{\eta}$ satisfy* [2.1], *and (ξ, η) satisfies C, then $(\xi, \tilde{\eta})$ also satisfies C. Here, C is the collection of the conditions $C = \{[2.2]\}$, $C = \{[2.2], [2.3]\}$, $C = \{[2.2]–[2.4]\}$.*

Proof. Let us assume that η and $\tilde{\eta}$ satisfy [2.1]. Since the assumption [2.1] and the assertion do not depend on the choice of the coordinate system on the source by Lemma 2.5, we take (u, v) an adapted coordinate system with $\eta = \partial_v$ for any (u, v) and $\xi = \partial_u$. Since $f_v = \cdots = f_{v^{m-1}} = 0$ on the u-axis, f_v has the form $f_v = v^{m-1}\psi(u, v)$. If the pair (ξ, η) satisfies [2.2], then $\operatorname{rank}(f_u, \psi) = 2$ on the u-axis. On the other hand, any null vector field is written as $a_1(u, v)\partial_u + a_2(u, v)\partial_v$, $(a_1(u, 0) = 0,\ a_2(u, v) \neq 0)$. By Lemma 2.4, dividing this by a_2, we may assume an extended null vector field $\tilde{\eta}$ is

$$\tilde{\eta} = va(u, v)\partial_u + \partial_v.$$

Since it holds that $\tilde{\eta}^2 f = 0$ on the u-axis (when $m > 2$) and $f_u(u, 0) \neq 0$, we have $a(u, 0) = 0$. Continuing this argument, we may assume

$$\tilde{\eta} = v^{m-1}a(u, v)\partial_u + \partial_v. \tag{2.2}$$

Thus, $\tilde{\eta}f = v^{m-1}(af_u + \psi)$ holds, and $\tilde{\eta}^m f = (m-1)!(af_u + \psi)$ holds on the u-axis. Therefore $(\xi, \tilde{\eta})$ satisfies [2.2]. We assume that the pair $(\xi, \eta = \partial_v)$ satisfies [2.1]-[2.3], and $(\xi, \tilde{\eta})$ satisfies [2.1], [2.2]. By this assumption, $\operatorname{rank}(f_u, \psi) = 2$, and $\operatorname{rank}(f_u, \psi, \psi_{v^i}) = 2$ $(0 < i < n-m)$. By the form of $\tilde{\eta}$, it holds that $\tilde{\eta}^{m+1} f = (m-1)!(a_v f_u + af_{uv} + \psi_v)$ on the u-axis. Since $f_v = 0$ on the u-axis, $f_{uv} = 0$ on the u-axis. Thus, $\operatorname{rank}(\xi f, \tilde{\eta}^m f, \tilde{\eta}^{m+1} f) = 2$ on the u-axis. Similarly, $f_{v^{m-1}} = 0$ on the u-axis, $f_{uv^2} = \cdots = f_{uv^{m-1}} = 0$ on the u-axis. Thus, if $i \le m-1$, then since $n < 2m$, we have $\tilde{\eta}^{m+i} f = (m-1)!(\psi_{v^i} + a_{v^i} f_u)$ on the u-axis. Thereby we have $\operatorname{rank}(\xi f, \tilde{\eta}^m f, \tilde{\eta}^{m+i} f) = 2$ $(i \le m-2)$ on the u-axis. The last assertion can be shown by the same calculation. $\square$

Proof of Proposition 2.2. We assume f satisfies the condition of the proposition, and (ξ, η) satisfies the conditions [2.1] and [2.2]. Then we take an adapted coordinate system (u, v) such that $\eta = \partial_v$. By the proof of Lemma 2.6, there exist $p(u)$ and $q(u, v)$ such that $f(u, v) = p(u) + v^m q(u, v)$, and $(p_1)_u(0, 0) \neq 0$, where $p = (p_1, p_2, p_3)$. We set $U = p_1(u)$, $V = v$. Then f has the form $(U, P_2(U), P_3(U)) + V^m Q(U, V)$. By a coordinate change on the target, f has the form $(U, 0, 0) + V^m Q(U, V)$, where $Q(U, V) = (0, Q_2(U, V), Q_3(U, V))$. Rewriting the notation, we may assume f is written as

$$f(u, v) = (u, v^m q_2(u, v), v^m q_3(u, v)).$$

On this coordinate system, ∂_v satisfies the condition [2.1], and it also satisfies [2.2] by Lemma 2.6. This implies $(q_2(0, 0), q_3(0, 0)) \neq (0, 0)$. So we assume $q_2(0, 0) \neq 0$. We set $U = u$, $V = vq_2(u, v)^{1/m}$. Rewriting the notation, we may assume f is written as $(u, v^m, v^m q_3(u, v))$. By a coordinate change on the target, we may assume f is written as $(u, v^m, v^{m+1} q_3(u, v))$. This proves the first assertion. We assume that η also satisfies [2.3] and [2.4]. We may assume f is written as $(u, v^m, v^{m+1} q_3(u, v))$. By Lemma 2.6, we may assume that ∂_v satisfies [2.3] and [2.4]. By [2.3], the function $q_3(u, v)$ satisfies $q_3 = (q_3)_v = \cdots (q_3)_{v^{n-m-1}} = 0$ on the u-axis. Thus, f is written as $(u, v^m, v^n q_4(u, v))$. By [2.4], it holds that $q_4 \neq 0$, and hence the assertion is proved. $\square$

By the proof of Lemma 2.6, we have the following property:

Corollary 2.7. *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be a frontal satisfying* $\operatorname{rank} df_0 = 1$*, and let the set of singular points $S(f)$ be a regular curve. Furthermore, assume η is a vector field satisfying* [2.1]*. Let (u, v) be an adapted coordinate system with ∂_v satisfying* [2.1]*. Then there exists ψ such that $\eta f(u, v) = v^{m-1}\psi(u, v)$.*

2.2. **Normal form of m- or (m, n)-type edges.** Given a curve-germ $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$, if there exists m such that $\gamma' = t^{m-1}\rho$ $(\rho(0) \neq 0)$, then γ at 0 is said to be of *finite multiplicity*, and such an m is called the *multiplicity* or the *order* of γ at 0. Moreover, if there exists n $(n > m$ and $n \neq km$, $k = 2, 3, \ldots)$ such that γ is $\mathcal{A}^n$-equivalent to (t^m, t^n), then γ is called of *(m, n)-type*. This (m, n) is well-defined since if γ is $\mathcal{A}^r$-equivalent to $(t^m, 0)$ then it is not $\mathcal{A}^r$-equivalent to (t^m, t^i) for $i \le r$, $i \neq km$ $(k = 1, 2, \ldots)$. We simplify a curve-germ of (m, n)-type and an (m, n)-type edge by coordinate changes on the source and by special orthonormal matrices on the target. Let (x, y) be the ordinary coordinate system of $(\mathbf{R}^2, 0)$. A coordinate system $(u, v) = (u(x, y), v(x, y))$ is *positive* if the determinant of the Jacobi matrix of $(u(x, y), v(x, y))$ is positive. We have the following results.

Lemma 2.8. *Let $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a curve germ satisfying $\gamma^{(i)}(0) = 0$ $(i = 1, \dots, m-1)$ and $\gamma^{(m)}(0) \neq 0$. Then there exist a parameter t and a special orthonormal matrix A on $\mathbf{R}^2$ such that*

$$A\gamma(t) = \left(t^m,\ t^{m+1}b(t)\right).$$

Let $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a curve germ of (m, n)-type. Then there exist a parameter t and a special orthonormal matrix A on $\mathbf{R}^2$ such that

$$A\gamma(t) = \left(t^m, \sum_{i=2}^{\lfloor n/m \rfloor} a_i t^{im} + t^n b(t)\right) \quad (b(0) \neq 0), \tag{2.3}$$

where $\lfloor k \rfloor$ is the greatest integer less than k (in our convention, n/m is not an integer).

Proof. One can easily see the first assertion. We assume that γ is a curve germ of (m, n)-type; then we may assume $\gamma(t) = (t^m, t^{m+1}b(t))$. If $t^{m+1}b(t)$ has a term t^i $(i < n, i \neq km)$, then $j^n\gamma(0)$ is not $\mathcal{A}^n$-equivalent to (t^m, t^n). This proves the assertion. □

Proposition 2.9. *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be an m-type edge. Then there exist a positive coordinate system (u, v) and a special orthonormal matrix A on $\mathbf{R}^3$ such that*

$$Af(u, v) = \left(u, \frac{u^2 a(u)}{2} + \frac{v^m}{m!}, \frac{u^2 b_0(u)}{2} + \frac{v^m}{m!} b_m(u, v)\right) \quad (b_m(0, 0) = 0). \tag{2.4}$$

Moreover, if f is an (m, n)-type edge, then there exist a positive coordinate system (u, v) and a special orthonormal matrix A on $\mathbf{R}^3$ such that

$$Af(u, v) = \left(u, \frac{u^2 a(u)}{2} + \frac{v^m}{m!}, \frac{u^2 b_0(u)}{2} + \sum_{i=2}^{\lfloor n/m \rfloor} \frac{v^{im}}{(im)!} b_{im}(u) + \frac{v^n b_n(u, v)}{n!}\right), \tag{2.5}$$

$b_n(0, 0) \neq 0$.

Proof. By the proof of Proposition 2.2, we may assume

$$f(u, v) = (u, u^2 a_2(u) + v^m a_{2m}(u, v), u^2 a_3(u) + v^m a_{3m}(u, v)).$$

By that proof again, $(a_{2m}(0, 0), a_{3m}(0, 0)) \neq (0, 0)$. By a rotation on $\mathbf{R}^3$, we may assume $a_{2m}(0, 0) > 0$ and $a_{3m}(0, 0) = 0$. By a coordinate change $v \mapsto v a_{2m}(u, v)^{1/m}$, we may assume $f(u, v) = (u, u^2 a_2(u) + v^m/m!, u^2 a_3(u) + v^m a_{3m}(u, v))$, $(a_{3m}(0, 0) = 0)$. This proves the first assertion. If f is an (m, n)-type edge, then the function $a_{3m}(u, v)$ can be expanded by

$$\sum_{i=0}^{n-1} v^i b_i(u) + v^n b_n(u, v).$$

Since f is an (m,n)-type edge, the curve $v \mapsto f(u,v)$ is of (m,n)-type for any u near 0. This implies that $b_i(u) = 0$ $(i \neq km,\ k \geq 1)$. By $a_{3m}(0,0) = 0$, $b_0(u) = 0$. This proves the assertion. □

Each form (2.4) and (2.5) is called the *normal form* of an m-type edge and an (m,n)-type edge, respectively. Looking at the first and the second components in (2.4) and (2.5), we remark that the m-jet of the coordinate system (u,v) which gives the normal form is uniquely determined up to $\pm$ when m is even. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be an m-type edge and η a null vector field which satisfies the condition [2.1]. Then the subspace $V_1 = df_0(T_0\mathbf{R}^2)$ and the subspace V_2 spanned by $df_0(T_0\mathbf{R}^2)$, $\eta^m f(0)$ do not depend on the choice of η. We assume that the representation $f = (f_1, f_2, f_3)$ of xyz-space $\mathbf{R}^3$ satisfies that V_1 is the x-axis and V_2 is the xy-plane. Then the coordinate system (u,v) gives the normal form (2.4) if and only if $f_1(u,v) = u$ and $(f_2)_{uv}$ is identically zero.

2.3. **Geometric invariants.**

2.3.1. *Cuspidal curvatures.* Let f be an m-type edge. A pair of vector fields (ξ, η) is said to be *adapted* if ξ is tangent to $S(f)$, and η is a null vector field. We take an adapted pair of vector fields (ξ, η) such that η satisfies the condition [2.1], and (ξ, η) is positively oriented. One can show the existence of such a pair by the definition of m-type edge. We define

$$\omega_{m,m+1}(t) = \frac{|\xi f|^{(m+1)/m} \det(\xi f, \eta^m f, \eta^{m+1} f)}{|\xi f \times \eta^m f|^{(2m+1)/m}}(\mu(t)),$$

where μ is a parametrization of $S(f)$. We call $\omega_{m,m+1}$ the $(m, m+1)$-*cuspidal curvature*. We have the following proposition:

Proposition 2.10. *The function $\omega_{m,m+1}$ does not depend on the choice of (ξ, η) satisfying the condition* [2.1].

Proof. Since it does not appear in the formula, $\omega_{m,m+1}$ does not depend on the choice of the coordinate system. Let (ξ, η) be an adapted pair of vector fields satisfying the condition [2.1]. It is clear that the function $\omega_{m,m+1}$ does not depend on the choice of ξ. We take an adapted coordinate system (u,v) satisfying $\partial_v = \eta$. Then

$$\omega_{m,m+1}(u) = |f_u|^{(m+1)/m} \det(f_u, f_{v^m}, f_{v^{m+1}}) |f_u \times f_{v^m}|^{-(2m+1)/m}.$$

By Corollary 2.7, we have $f_v = v^{m-1}\psi$. Let $\tilde{\eta}$ be another null vector field satisfying the condition [2.1]. We see that $\omega_{m,m+1}$ does not depend on the non-zero functional multiples of η; we may assume $\tilde{\eta} = a(u,v)\partial_u + \partial_v$. By the proof of Lemma 2.6, we may assume that $\tilde{\eta}$ is

$$\tilde{\eta} = v^{m-1} a(u,v)\partial_u + \partial_v. \tag{2.6}$$

Then by $f_v = v^{m-1}\psi$,

$$\tilde{\eta} f = v^{m-1}(a f_u + \psi).$$

Thus,

$$\tilde{\eta}^m f = (m-1)!(a f_u + \psi) + (m-1)(m-1)! v\eta(a f_u + \psi) + v^2 g(u,v), \tag{2.7}$$

where g is a function, and

$$\tilde{\eta}^{m+1}f = (m-1)!\eta(af_u+\psi)+(m-1)(m-1)!\eta v\eta(af_u+\psi) = m!(\eta af_u+a\eta f_u+\eta\psi)$$

hold on the u-axis. Since $\psi = ((m-1)!)^{-1}f_{v^m}$ and $\psi_v = (m!)^{-1}f_{v^{m+1}}$, we have

$$\begin{aligned}\frac{|\xi f|^{(m+1)/m}\det(\xi f,\eta^m f,\eta^{m+1}f)}{|\xi f\times\eta^m f|^{(2m+1)/m}((m-1)!)^{1/m}}(u,0) &= \frac{|f_u|^{(m+1)/m}\det(f_u,\psi,\psi_v)}{|f_u\times\psi|^{(2m+1)/m}}(u,0)\\ &= \frac{|f_u|^{(m+1)/m}\det(f_u,f_{v^m},f_{v^{m+1}})}{|f_u\times f_{v^m}|^{(2m+1)/m}}(u,0).\end{aligned}$$

This shows the assertion. □

We have the following proposition.

Proposition 2.11. *Let $f\colon(\mathbf{R}^2,0)\to(\mathbf{R}^3,0)$ be an m-type edge. Then f at 0 is an $(m,m+1)$-type edge if and only if $\omega_{m,m+1}\neq 0$ at 0.*

Proof. Since f is an m-type edge, by Proposition 2.9, we may assume that f is given by the right-hand side of (2.4). Since $b_m(0,0)=0$, there exist $c_1(u)$ and $c_2(u,v)$ such that $b_m(u,v)=c_1(u)+vc_2(u,v)$. Since we can take $\eta=\partial_v$, the function $\omega_{m,m+1}$ is a non-zero functional multiple of $c_2(u,0)$. Then we see the assertion. □

It is easy to show that $(m,m+1)$-type edges are fronts and that an m-type edge is a front if and only if $\omega_{m,m+1}\neq 0$. In Appendix A, we define the (m,n)-cuspidal curvature for a curve germ of (m,n)-type, denoting it by $r_{m,n}$. An intersection curve of $(m,m+1)$-type edge f with T as in Lemma 2.1 is a curve-germ of $(m,m+1)$-type. The following holds.

Corollary 2.12. *Let $f\colon(\mathbf{R}^2,0)\to(\mathbf{R}^3,0)$ be a σ-edge, where σ is $\mathcal{A}$-equivalent to $v\mapsto(v^m,v^{m+1})$. Then the $(m,m+1)$-cuspidal curvature $\omega_{m,m+1}$ at 0 coincides with the $(m,m+1)$-cuspidal curvature $r_{m,m+1}$ of the intersection curve ρ of f with a plane P which is perpendicular to the tangent line to f at 0.*

Proof. By the assumption, we may assume that f is given by the normal form (2.4). Since $f_u(0,0)=(1,0,0)$ and $f_v(0,0)=(0,0,0)$, the plane P is given by $P=\{(0,y,z)\in\mathbf{R}^3 \mid y,z\in\mathbf{R}\}$. Thus, the intersection curve ρ can be parametrized by

$$\rho(v)=f(0,v)=\left(0,\frac{v^m}{m!},\frac{b_{m+1}(0,v)v^{m+1}}{(m+1)!}\right).$$

This can be considered as a normal form of a curve which is $\mathcal{A}^{m+1}$-equivalent to $v\to(v^m,v^{m+1})$. Hence we have the assertion by Example A.2. □

Let f be an m-type edge. We assume $\omega_{m,m+1}$ is identically zero on $S(f)$. Let $\mu(t)$ be a parametrization of $S(f)$. We define

$$\omega_{m,m+2}(t)=\frac{|\xi f|^{(m+2)/m}\det(\xi f,\eta^m f,\eta^{m+2}f)}{|\xi f\times\eta^m f|^{(2m+2)/m}}(\mu(t)).$$

We will see this does not depend on the choice of (ξ,η) which satisfies the conditions [2.1] and [2.2] in Proposition 2.2 and $\det(\xi f,\eta^m f,\eta^j f)=0$ $(j<m+2)$. Inductively, we define $\omega_{m,m+i}$ when $\omega_{m,m+j}=0$ $(j\le i-1)$ by

$$\omega_{m,m+i}=\frac{|\xi f|^{(m+i)/m}\det(\xi f,\eta^m f,\eta^{m+i}f)}{|\xi f\times\eta^m f|^{(2m+i)/m}}(\mu(t)).$$

We will also see this does not depend on the choice of (ξ,η) satisfying the conditions [2.1] and [2.2] in Proposition 2.2 and $\det(\xi f,\eta^m f,\eta^j f)=0$ $(j<m+i)$. If $i=m$, we set $\beta_{m,2m}=\omega_{m,2m}$.

Proposition 2.13. *Under the assumption $\omega_{m,m+1}=\cdots=\omega_{m,m+i-1}=0$, the function $\omega_{m,m+i}$ $(i=1,\dots,m-1)$ does not depend on the choice of the pair (ξ,η) which satisfies the conditions* [2.1] *and* [2.2] *in Proposition* 2.2 *and the condition* $\det(\xi f,\eta^m f,\eta^{m+j}f)=0$ $(1\le j<i)$.

Proof. We already showed the case $i=1$ in Proposition 2.10. Let (ξ,η) be a pair of vector fields satisfying the assumption of the lemma. We take an adapted coordinate system (u,v) such that $\partial_v=\eta$. By the proof of Lemma 2.6, we see that $f_v=v^{m-1}\psi$.

Moreover, we have:

Lemma 2.14. *There exist functions α,β, and a vector valued function θ such that*

$$\psi_v=\alpha f_u+\beta\psi+v^{i-1}\theta.$$

Proof. Since $f_{v^{m+1}}=(m-1)!\psi_v$ on the u-axis, $\omega_{m,m+1}=0$ implies that there exists $\alpha_1,\beta_1,\theta_1$ such that $\psi_v=\alpha_1 f_u+\beta_1\psi+v\theta_1$. We assume that there exist $\alpha_k,\beta_k,\theta_k$ such that $\psi_v=\alpha_k f_u+\beta_k\psi+v^k\theta_k$ $(k=1,\dots,i-2)$. Differentiating this equation, we have

$$\psi_{v^{k+1}}=\sum_{l=0}^{k}\binom{k}{l}\left((\alpha_k)_{v^l}f_{uv^{k-l}}+(\beta_k)_{v^l}\psi_{v^{k-l}}+(v^k)_{v^l}(\theta_k)_{v^{k-l}}\right).$$

Thus, since $f_{uv}=\cdots=f_{uv^{m-1}}=0$ and $\psi_{v^j}\in\langle f_u,\psi\rangle_{\mathbf{R}}$ $(j\le k)$ on the u-axis, we have $2=\operatorname{rank}(f_u,\psi,\psi_{v^{k+1}})=\operatorname{rank}(f_u,\psi,\theta_k)$ on the u-axis. Hence there exist functions α_{k+1},β_{k+1}, and a vector valued function θ_{k+1} such that $\theta_k=\alpha_{k+1}f_u+\beta_{k+1}\psi+v\theta_{k+1}$. This shows the assertion. □

We continue the proof of Proposition 2.13. Since the assertion holds by multiplying the null vector field by a non-zero function, we take a null vector field η as in the right-hand side of (2.6). By the same calculations in the proof of Proposition 2.10, we have $\eta f=v^{m-1}(af_u+\psi)$. Thus,

$$\eta^{m+i}f=\sum_{k=0}^{m+i-1}\binom{m+i-1}{k}\eta^k v^{m-1}\eta^{m+i-1-k}(af_u+\psi).$$

Since $\eta^k v^{m-1}=0$ if $k\ne m-1$ and $\eta^k v^{m-1}=(m-1)!$, we have

$$\eta^{m+i}f=\binom{m+i-1}{m-1}(m-1)!\eta^i(af_u+\psi).$$

Thus, $\eta^{m+i}f = vg(u,v) + (af_u + \psi)_{v^i}$, where g is a function. Since $f_{uv} = \cdots = f_{uv^{m-1}} = 0$, and $\psi_{v^j} \in \langle f_u, \psi\rangle_{\mathbf{R}}$ $(j \leq k)$ on the u-axis by Lemma 2.14, we have

$$\frac{|\xi f|^{(m+i)/m}\det(\xi f, \eta^m f, \eta^{m+i} f)}{|\xi f \times \eta^m f|^{(2m+i)/m}}(u,0) = \frac{|f_u|^{(m+i)/m}\det(f_u, f_{v^m}, f_{v^{m+i}})}{|f_u \times f_{v^{m+i}}|^{(2m+i)/m}}(u,0),$$

and this shows the assertion. □

We call $\omega_{m,m+i}$ the $(m, m+i)$-*cuspidal curvature* and $\beta_{m,2m}$ the $(m, 2m)$-*bias*. Note that $\beta_{m,2m}$ does not depend on the choice of (ξ, η) satisfying [2.1], [2.2] and $\langle \xi f, \eta^m f\rangle = 0$ at p by the same calculation. In this case, $a(0,0) = 0$ by the additional assumption. If f is an m-type edge, and written as (2.4), then $\omega_{m,m+1}(0) = (m+1)(b_m)_v(0,0)$. If f is an (m,n)-edge $(n < 2m)$, and written as (2.5), then $\omega_{m,n}(0) = b_n(0,0)$, and $\beta_{m,2m}(0,0) = b_{2m}(0)$. See Appendix A for geometric meanings of the terms b_{im} $(i = 2, \ldots, \lfloor n/m \rfloor)$.

2.3.2. *Singular, normal curvatures and cuspidal torsion.* Let f be an m-type edge, and $\mu(t)$ be a parametrization of the singular set. Let ν be a unit normal vector field of f, and set $\lambda = \det(f_u, f_v, \nu)$ for an oriented coordinate system (u, v) on $(\mathbf{R}^2, 0)$. We set $\hat{\mu} = f \circ \mu$. Then we define

$$\kappa_s(t) = \operatorname{sgn}\left(\delta\ \eta^{m-1}\lambda(\mu(t))\right)\frac{\det(\hat{\mu}', \hat{\mu}'', \nu(\mu))}{|\hat{\mu}'|^3}, \quad \kappa_\nu(t) = \frac{\langle \hat{\mu}'', \nu(\mu)\rangle}{|\hat{\mu}'|^2} \tag{2.8}$$

and

$$\kappa_t(t) = \frac{\det(\xi f, \eta^m f, \xi\eta^m f)}{|\xi f \times \eta^m f|^2}(\mu(t)) - \frac{\det(\xi f, \eta^m f, \xi^2 f)\,\langle \xi f, \eta^m f\rangle}{|\xi f|^2|\xi f \times \eta^m f|^2}(\mu(t)), \tag{2.9}$$

where $\delta = 1$ if (μ', η) agrees with the orientation of the coordinate system, and $\delta = -1$ if (μ', η) does not agree with the orientation. We call κ_s, κ_ν and κ_t *singular curvature*, *normal curvature* and *cuspidal torsion*, respectively. These definitions are direct analogies of [13, 10]. It is easy to see that the definitions (2.8) do not depend on the choice of parametrization of the singular curve. Moreover, κ_s does not depend on the choice of ν, nor the choice of η when m is even. To see the well-definedness of κ_t, we need the following proposition.

Proposition 2.15. *The definition* (2.9) *does not depend on the choice of the adapted vector fields* (ξ, η)*, where* η *satisfies* [2.1]*.*

Proof. One can easily check it does not depend on the choice of functional multiplications of η. Since the assertion does not depend on the choice of local coordinate system, one can choose an adapted coordinate system (u, v) with ∂_v satisfying [2.1]. Let η be a null vector field which satisfies [2.1]. Then by the proof of Lemma 2.6, we may assume η is given by (2.2). Then by (2.7), we see that $\eta^m f = (m-1)!(af_u + \psi)$ on the u-axis, where ψ is given in the proof of Lemma 2.6. Furthermore, by (2.7), we see that $\xi\eta^m f = (m-1)!(a_u f_u + af_{u^2} + \psi_u)$ on the u-axis. Substituting these formulas into the right-hand side of (2.9), we see it is

$$\frac{\det(f_u, \psi, \psi_u)}{|f_u \times \psi|^2}(u,0) - \frac{\det(f_u, \psi, f_{u^2})\,\langle f_u, \psi\rangle}{|f_u|^2|f_u \times \psi|^2}(u,0),$$

and since $f_{v^m} = (m-1)!\psi$, this shows the assertion. □

If an m-type edge f is given by the form (2.4), then $\kappa_s(0) = a(0)$, $\kappa_\nu(0) = b(0)$ and $\kappa_t(0) = (b_m)_u(0,0)$.

2.4. **Boundedness of Gaussian curvature and mean curvature near an m-type edge.** Here we study the behavior of the Gaussian and mean curvatures. Let $g : (\mathbf{R}^i, 0) \to \mathbf{R}$ be a function-germ ($i = 1, 2$). If there exists an integer n ($n \geq 1$) such that $g \in \mathcal{M}_i^n$ and $g \notin \mathcal{M}_i^{n+1}$, then g is said to be of *order* n, where $\mathcal{M}_i = \{g\colon (\mathbf{R}^i, 0) \to \mathbf{R} \mid g(0) = 0\}$ is the unique maximal ideal of the local ring of function-germs and $\mathcal{M}_i^n$ denotes the nth power of $\mathcal{M}_i$ (cf. [9, p. 46]). If $g \notin \mathcal{M}_i$, then the order of g is 0. The order of g is denoted by $\mathrm{ord}(g)$. If g is of order n ($n \geq 0$), then g is said to be of *finite order*. Let $g_1, g_2 : (\mathbf{R}^i, 0) \to \mathbf{R}$ be two function-germs such that g_i is of finite order. The *rational order* $\mathrm{ord}(f)$ of a function $f = g_1/g_2 : (\mathbf{R}^i \setminus Z, 0) \to \mathbf{R}$, where $Z = g_2^{-1}(0)$, is

$$\mathrm{ord}(f) = \mathrm{ord}(g_1) - \mathrm{ord}(g_2).$$

For a function $f = g_1/(|g_2| g_3) : (\mathbf{R}^i \setminus Z, 0) \to \mathbf{R}$, we define $\mathrm{ord}(f) = \mathrm{ord}(g_1) - \mathrm{ord}(g_2) - \mathrm{ord}(g_3)$, where $Z = g_2^{-1}(0) \cup g_3^{-1}(0)$. If $g_1 \in \mathcal{M}_i^\infty$, then we define $\mathrm{ord}(f) = \infty$. If $\mathrm{ord}(f) = 0$, then f is called *rationally bounded*, and if $\mathrm{ord}(f) = 1$, then f is called *rationally continuous* ([12, Definition 3.4]). If $i = 1$, this is the usual one.

Since the property $g \in \mathcal{M}_i^n$ does not depend on the choice of coordinate system, the order and the rational order does not depend on the choice of coordinate system.

Let $f\colon (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be an m-type edge, and let (u, v) be an adapted coordinate system with ∂_v satisfying [2.1]. We take $(m-1)!\psi$ in Corollary 2.7. Namely, here we set ψ by $f_v = v^{m-1}\psi/(m-1)!$. Since f is an m-type edge, f_u and ψ are linearly independent (Proposition 2.2 and the independence of the condition [2.2]). Thus, the unit normal vector ν of f can be taken as $\nu = \hat{\nu}/|\hat{\nu}|$ ($\hat{\nu} = f_u \times \psi$). Using f_u, ψ and ν, we define the following functions:

$$\widehat{E} = \langle f_u, f_u \rangle, \qquad \widehat{F} = \langle f_u, \psi \rangle, \qquad \widehat{G} = \langle \psi, \psi \rangle,$$
$$\widehat{L} = -\langle f_u, \widehat{\nu}_u \rangle, \qquad \widehat{M} = -\langle \psi, \widehat{\nu}_u \rangle, \qquad \widehat{N} = -\langle \psi, \widehat{\nu}_v \rangle.$$

We note that coefficients of the first and the second fundamental forms of σ-edges being of multiplicity m can be written as

$$E = \widehat{E}, \qquad F = \frac{v^{m-1}}{(m-1)!}\widehat{F}, \qquad G = \left(\frac{v^{m-1}}{(m-1)!}\right)^2 \widehat{G},$$
$$L = \frac{\widehat{L}}{|\widehat{\nu}|}, \qquad M = \frac{v^{m-1}}{|\widehat{\nu}|(m-1)!}\widehat{M}, \qquad N = \frac{v^{m-1}}{(m-1)!|\widehat{\nu}|}\widehat{N}.$$

Lemma 2.16. *The differentials ν_u and ν_v of ν are written as*

$$\nu_u = -\frac{\widehat{G}\widehat{L} - \widehat{F}\widehat{M}}{(\widehat{E}\widehat{G} - \widehat{F}^2)|\hat{\nu}|} f_u - \frac{\widehat{E}\widehat{M} - \widehat{F}\widehat{L}}{(\widehat{E}\widehat{G} - \widehat{F}^2)|\hat{\nu}|}\psi,$$

$$\nu_v = -\frac{\frac{v^{m-1}}{(m-1)!}\widehat{G}\widehat{M} - \widehat{F}\widehat{N}}{(\widehat{E}\widehat{G} - \widehat{F}^2)|\hat{\nu}|} f_u - \frac{\widehat{E}\widehat{N} - \frac{v^{m-1}}{(m-1)!}\widehat{F}\widehat{M}}{(\widehat{E}\widehat{G} - \widehat{F}^2)|\hat{\nu}|}\psi.$$

Proof. Since $\langle \nu_u, \nu\rangle = \langle \nu_v, \nu\rangle = 0$, there exist functions A, B, C, D on $(\mathbf{R}^2, 0)$ such that

$$\nu_u = Af_u + B\psi, \quad \nu_v = Cf_u + D\psi.$$

Considering $\langle \nu_u, f_u\rangle, \langle \nu_u, \psi\rangle, \langle \nu_v, f_u\rangle$ and $\langle \nu_v, \psi\rangle$, we have

$$-\frac{1}{|\hat{\nu}|}\begin{pmatrix}\widehat{L}\\ \widehat{M}\end{pmatrix} = \begin{pmatrix}\widehat{E} & \widehat{F}\\ \widehat{F} & \widehat{G}\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}, \quad -\frac{1}{|\hat{\nu}|}\begin{pmatrix}\frac{v^{m-1}}{(m-1)!}\widehat{M}\\ \widehat{N}\end{pmatrix} = \begin{pmatrix}\widehat{E} & \widehat{F}\\ \widehat{F} & \widehat{G}\end{pmatrix}\begin{pmatrix}C\\ D\end{pmatrix}.$$

Solving these equations, we have the assertion. □

By this lemma, ν_v can be written as

$$\nu_v = \frac{\widehat{N}}{(\widehat{E}\widehat{G} - \widehat{F}^2)|\hat{\nu}|}(\widehat{F}f_u - \widehat{E}\psi)$$

along the u-axis. Since f_u and ψ are linearly independent and $\widehat{E} \neq 0$, the condition $\nu_v(0) \neq 0$ is equivalent to $\widehat{N}(0) \neq 0$. To see this fact, we take the same setting in the proof of Proposition 2.10. Then we see that

$$\det(f_u, f_{v^m}, f_{v^{m+1}}) = m\det(f_u, \psi, \psi_v) = m\langle \hat{\nu}, \psi_v\rangle = m\widehat{N} \tag{2.10}$$

along the u-axis, where $\hat{\nu} = f_u \times \psi$ and $\widehat{N} = \langle \hat{\nu}, \psi_v\rangle = -\langle \hat{\nu}_v, \psi\rangle$. Since $\{f_u, \psi, \nu\}$ is a frame of $\mathbf{R}^3$ and $\langle f_u, \nu_v\rangle = \langle f_v, \nu_u\rangle = 0$, $\langle \nu, \nu_v\rangle = 0$, it holds that $\nu_v \neq 0$ if and only if $\langle \nu_v, \psi\rangle \neq 0$. Moreover, since $\langle \nu, \psi\rangle = 0$, it holds that $\langle \nu_v, \psi\rangle \neq 0$ is equivalent to $\langle \hat{\nu}_v, \psi\rangle \neq 0$. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be an (m, n)-type edge, and let us set

$$r = \min\left(\{n\} \cup \{im \,|\, b_{im}(0) \neq 0 \text{ in the form (2.5)}, i = 2, 3, \dots\}\right).$$

Theorem 2.17. *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be an (m, n)-type edge. Then the rational order of the mean curvature H is $r - 2m$. If the normal curvature does not vanish at 0, then the rational order of the Gaussian curvature K is $r - 2m$.*

Proof. We take an adapted coordinate system (u, v) such that ∂_v satisfies [2.1]. Since $\hat{L} = \langle f_{uu}, \nu\rangle$, the normal curvature does not vanish if and only if $\hat{L}(0) \neq 0$. The Gaussian curvature K and the mean curvature H of f are given by

$$K = \frac{(m-1)!}{v^{m-1}}\frac{\widehat{L}\widehat{N} - \frac{v^{m-1}}{(m-1)!}\widehat{M}^2}{|\hat{\nu}|^2(\widehat{E}\widehat{G} - \widehat{F}^2)}, \quad H = \frac{(m-1)!}{v^{m-1}}\frac{\widehat{E}\widehat{N} - 2\frac{v^{m-1}}{(m-1)!}\widehat{F}\widehat{M} + \frac{v^{m-1}}{(m-1)!}\widehat{G}\widehat{L}}{2|\hat{\nu}|(\widehat{E}\widehat{G} - \widehat{F}^2)}.$$

This and $\widehat{E} \neq 0$, $\widehat{E}\widehat{G} - \widehat{F}^2 \neq 0$ at 0, together with

$$\widehat{N} = \frac{v^{r-m-1}}{m(r-m-1)!}(b_r(0) + v\alpha(u,v)),$$

where α is a function, by using the form (2.5) and (2.10), give the assertion. □

By Theorem 2.17, the orders of K and H coincide. Moreover, since $n < 2m$, they are never bounded when the normal curvature does not vanish.

3. Curves passing through m-type edges

In this section, we consider geometric invariants of a curve γ passing through an m-type edge f. If $\hat{\gamma} = f \circ \gamma$ is non-singular, then the usual invariants can be defined in the same way as in the regular case. We consider the case when $\hat{\gamma}$ has a singular point, namely, γ passing through a singular point of f in the direction of a null vector.

3.1. **Normalized curvatures of singular curves.** Following [14, 4], we introduce normalized curvature on curves in $\mathbf{R}^2$. Let $\hat{\gamma} : (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)$ be a curve, and let 0 be a singular point. We assume that there exists k such that $\hat{\gamma}' = t^{k-1}\rho$ ($\rho(0) \neq 0$).

We set

$$s = \int |\hat{\gamma}'|\, dt \tag{3.1}$$

and

$$\tilde{s} = \operatorname{sgn}(s)|s|^{1/k}, \tag{3.2}$$

where we see $\tilde{s}$ is a C^∞ function and $d\tilde{s}/dt(0) > 0$. We call this parameter a *$1/k$-arc-length.*

Proposition 3.1. *The parameter t is a $1/k$-arc-length parameter of $\hat{\gamma}$ if and only if $|\hat{\gamma}'(t)| = k|t^{k-1}|$.*

Proof. If $|\hat{\gamma}'(t)| = k|t^{k-1}|$ and $s(t)$ as in (3.1), it holds that

$$s(t) = \int_0^t k|\xi^{k-1}|\, d\xi = \int_0^t \varepsilon\, k\xi^{k-1}\, d\xi = \varepsilon\, t^k \quad (\varepsilon = \operatorname{sgn}(t) \text{ if } k \text{ is even, } 1 \text{ if } k \text{ is odd}).$$

Since $\operatorname{sgn}(s) = \operatorname{sgn}(t)$, we have $|s| = |t^k|$, and therefore, $t = \operatorname{sgn}(s)|s|^{1/k}$.

Let us suppose now that t is the $1/k$-arc-length, i.e., $t = \operatorname{sgn}(s)|s|^{1/k}$, with $s(t)$ as in (3.1). Since $\operatorname{sgn}(s) = \operatorname{sgn}(t)$, we have $t^k = \operatorname{sgn}(s)^k|s| = \operatorname{sgn}(s)^{k+1}s$, and consequently, $s'(t) = \operatorname{sgn}(t)^{k+1}kt^{k-1} = k|t|^{k-1}$. Therefore, it holds that $|\hat{\gamma}'(t)| = k|t^{k-1}|$. □

Let us set $n = 2$. Then the curvature κ satisfies that

$$\tilde{\kappa} = |\tilde{s}^{k-1}|\kappa$$

is a C^∞ function. We call $\tilde{\kappa}$ the *normalized curvature.* This is originally introduced in [14] and generalized in [4]. Let $f(t)$ be a given C^∞-function, and $k \geq 2$ be an integer. Then similarly to [14, Theorem 1.1], one can show that there exists a unique plane curve up to isometries in $\mathbf{R}^2$ with normalized curvature given by $\tilde{\kappa}(t) = f(t)$, where t is the $1/k$-arc-length parameter.

Using the frame $\{\boldsymbol{e}, \boldsymbol{n}\}$ along $\hat{\gamma}$ defined by $\boldsymbol{e} = \rho/|\rho|$ and $\boldsymbol{n}$ the $\pi/2$-rotation of $\boldsymbol{e}$, the normalized curvature can be interpreted as follows: We define the function κ_1 by the equation

$$\begin{pmatrix} \boldsymbol{e}' \\ \boldsymbol{n}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 \\ -\kappa_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{n} \end{pmatrix}, \tag{3.3}$$

where the prime $'$ denotes differentiation with respect to the $1/k$-arc-length. Then we have:

Proposition 3.2. *Let $\{\boldsymbol{e}, \boldsymbol{n}\}$ be the above frame along $\hat{\gamma}(t)$ in the Euclidean plane $\mathbf{R}^2$ satisfying* (3.3)*, where t is the $1/k$-arc-length parameter. Then $\kappa_1 = k\tilde{\kappa}$ holds.*

Proof. Since $\hat{\gamma}'(t) = t^{k-1}\rho(t)$, where $\rho(0) \neq 0$ and the $1/k$-arc-length parameter t satisfies $|\hat{\gamma}'(t)| = k|t|^{k-1}$, we have $\hat{\gamma}''(t) = (k-1)t^{k-2}\rho(t) + t^{k-1}\rho'(t)$ and $|\rho(t)| = k$. Then

$$\kappa(t) = \frac{1}{k^3|t|^{k-1}} \det(\rho(t), \rho'(t)).$$

Consequently,

$$\tilde{\kappa}(t) = |t|^{k-1}\kappa(t) = \frac{1}{k^3} \det(\rho(t), \rho'(t)).$$

On the other hand, since $\kappa_1(t) = \boldsymbol{e}'(t) \cdot \boldsymbol{n}(t)$, where $\boldsymbol{e}(t) = \rho(t)/|\rho(t)| = \rho(t)/k$ and $\boldsymbol{n}(t)$ is the $\pi/2$-counterclockwise rotation of $\boldsymbol{e}(t)$, and the dot '$\cdot$' denotes the canonical inner product of $\mathbf{R}^2$, it holds that

$$\kappa_1(t) = \frac{1}{k}\rho'(t) \cdot \boldsymbol{n}(t) = \frac{1}{k^2} \det(\rho(t), \rho'(t)).$$

Thus, we have the assertion. □

3.2. **Normalized curvatures on frontals.** Following Section 3.1, we define the normalized curvatures for curves on a frontal. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be a frontal and ν a unit normal vector field of f. Let $\gamma\colon (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a curve. We set $\hat{\gamma} = f \circ \gamma$. We assume there exists k such that $\hat{\gamma}' = t^{k-1}\rho$ ($\rho(0) \neq 0$). The geodesic curvature κ_g, the normal curvature κ_n and the geodesic torsion τ_g are defined by

$$\kappa_g = \frac{\det(\hat{\gamma}', \hat{\gamma}'', \nu)}{|\hat{\gamma}'|^3}, \quad \kappa_n = \frac{\langle \hat{\gamma}'', \nu \rangle}{|\hat{\gamma}'|^2}, \quad \tau_g = \frac{\det(\hat{\gamma}', \nu, \nu')}{|\hat{\gamma}'|^2}$$

on regular points (see [1, p. 261]). These curvatures can be unbounded near singular points. Indeed, it holds that

$$\kappa_g = \frac{1}{|t|^{k-1}} \frac{\det(\rho, \rho', \nu)}{|\rho|^3}, \quad \kappa_n = \frac{1}{t^{k-1}} \frac{\langle \rho', \nu \rangle}{|\rho|^2}, \quad \tau_g = \frac{1}{t^{k-1}} \frac{\det(\rho, \nu, \nu')}{|\rho|^2}. \tag{3.4}$$

One can easily see that

$$\tilde{\kappa}_g = |\tilde{s}|^{k-1}\, \kappa_g, \quad \tilde{\kappa}_n = \tilde{s}^{k-1}\, \kappa_n, \quad \tilde{\tau}_g = \tilde{s}^{k-1}\, \tau_g \tag{3.5}$$

are C^∞ functions, where $\tilde{s}$ is the function given by (3.2) for $\hat{\gamma}$. We call $\tilde{\kappa}_g, \tilde{\kappa}_n, \tilde{\tau}_g$ *normalized* geodesic curvature, normal curvature, and geodesic torsion of $\tilde{\gamma}$, respectively. These satisfy:

Lemma 3.3. *It holds that*

$$\tilde{\kappa}_g = \frac{1}{k^2\, k!^{-1/k}} \frac{\det\left(\hat{\gamma}^{(k)}, \hat{\gamma}^{(k+1)}, \nu\right)}{|\hat{\gamma}^{(k)}|^{2+1/k}},$$

$$\tilde{\kappa}_n = \frac{1}{k^2\, k!^{-1/k}} \frac{\left\langle \hat{\gamma}^{(k+1)}, \nu \right\rangle}{|\hat{\gamma}^{(k)}|^{1+1/k}},$$

$$\tilde{\tau}_g = \frac{1}{k\, k!^{-1/k}} \frac{\det(\hat{\gamma}^{(k)}, \nu, \nu')}{|\hat{\gamma}^{(k)}|^{1+1/k}}$$

at $t = 0$*.*

Proof. Since $\hat{\gamma}'(t) = t^{k-1}\rho(t)$, we have $\rho(0) = \frac{\hat{\gamma}^{(k)}(0)}{(k-1)!}$, $\rho'(0) = \frac{\hat{\gamma}^{(k+1)}(0)}{k!}$ and $\rho_0 = |\rho(0)| = \frac{|\hat{\gamma}^{(k)}(0)|}{(k-1)!}$. Therefore, it holds that

$$\tilde{s}^{k-1} = t^{k-1}\left(\frac{\rho_0^{(k-1)/k}}{k^{(k-1)/k}} + tO(t)\right),$$

where $O(t)$ is a smooth function of t. Thus, by (3.4) and (3.5), we get at $t = 0$:

$$\begin{aligned}
\tilde{\kappa}_g &= \frac{\rho_0^{\frac{k-1}{k}} k^{1/k}}{k} \frac{\det(\rho, \rho', \nu)}{\rho_0^3} = \frac{k^{1/k}}{k} \frac{\det(\rho, \rho', \nu)}{\rho_0^{2+1/k}} \\
&= \frac{k^{1/k}(k-1)!^{2+1/k}}{k\,(k-1)!\,k!} \frac{\det(\hat{\gamma}^{(k)}, \hat{\gamma}^{(k+1)}, \nu)}{|\hat{\gamma}^{(k)}|^{2+1/k}} = \frac{k!^{1/k}}{k^2} \frac{\det(\hat{\gamma}^{(k)}, \hat{\gamma}^{(k+1)}, \nu)}{|\hat{\gamma}^{(k)}|^{2+1/k}}, \\
\tilde{\kappa}_n &= \frac{\rho_0^{(k-1)/k}}{k^{(k-1)/k}} \frac{\langle \rho', \nu \rangle}{\rho_0^2} = \frac{k^{1/k}}{k} \frac{\langle \rho', \nu \rangle}{\rho_0^{1+1/k}} \\
&= \frac{k^{1/k}(k-1)!^{1+1/k}}{k\,k!} \frac{\left\langle \hat{\gamma}^{(k+1)}, \nu \right\rangle}{|\hat{\gamma}^{(k)}|^{1+1/k}} = \frac{k!^{1/k}}{k^2} \frac{\left\langle \hat{\gamma}^{(k+1)}, \nu \right\rangle}{|\hat{\gamma}^{(k)}|^{1+1/k}}, \\
\tilde{\tau}_g &= \frac{\rho_0^{(k-1)/k}}{k^{(k-1)/k}} \frac{\det(\rho, \nu, \nu')}{\rho_0^2} = \frac{k^{1/k}}{k} \frac{\det(\rho, \nu, \nu')}{\rho_0^{1+1/k}} \\
&= \frac{k^{1/k}(k-1)!^{1+1/k}}{k(k-1)!} \frac{\det(\hat{\gamma}^{(k)}, \nu, \nu')}{|\hat{\gamma}^{(k)}|^{1+1/k}} = \frac{k!^{1/k}}{k} \frac{\det(\hat{\gamma}^{(k)}, \nu, \nu')}{|\hat{\gamma}^{(k)}|^{1+1/k}},
\end{aligned}$$

which show the assertion. □

Similar to the case of plane curves, these invariants can be interpreted as follows. Under the same assumption above, we set $e = \rho/|\rho|$, $\nu = \nu(\hat{\gamma})$ and $b = -e \times \nu$. Then $\{e, \nu, b\}$ is a frame along $\hat{\gamma}$. We define $\kappa_1, \kappa_2, \kappa_3$ by

$$\begin{pmatrix} e' \\ b' \\ \nu' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{pmatrix} \begin{pmatrix} e \\ b \\ \nu \end{pmatrix},$$

where $' = d/dt$ denotes differentiation with respect to the $1/k$-arc-length parameter. With the above notation, we get the following:

Proposition 3.4. *If t is the $1/k$-arc-length parameter, then*

$$\kappa_1 = k\tilde{\kappa}_g, \quad \kappa_2 = k\tilde{\kappa}_n \quad \text{and} \quad \kappa_3 = k\tilde{\tau}_g$$

hold for any t.

Proof. The $1/k$-arc-length parameter t satisfies $|\hat{\gamma}'(t)| = k|t^{k-1}|$. Then $|\rho(t)| = k$, and $e = \rho/k$. So, putting $\tilde{s} = t$ at (3.5) and using (3.4), it holds that

$$\kappa_1 = \langle e', b\rangle = \frac{1}{k^2}\det(\rho, \rho', \nu) = k\tilde{\kappa}_g,$$

$$\kappa_2 = \langle e', \nu\rangle = \frac{1}{k}\langle \rho', \nu\rangle = k\tilde{\kappa}_n\,,$$

$$\kappa_3 = -\langle \nu', b\rangle = \frac{1}{k}\det(\rho, \nu, \nu') = k\tilde{\tau}_g.$$

Thus, the assertion holds. □

3.3. **Behaviors of κ_g, κ_n and τ_g passing through an m-type edge.** In this section we shall study the orders of the geodesic and normal curvatures and the geodesic torsion of a curve passing through an m-type edge, concluding on boundedness. Describing the condition, we use the curvature of such curve. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be an m-type edge, $m \geq 2$, and $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a regular curve such that $\gamma'(0)$ is a null vector of f at 0. Let (u, v) be a coordinate system which gives the form (2.4), and $\tilde{\gamma}(t) = (u(t), v(t))$ be a parametrization of γ, where the coordinate system on the target space is (u, v), and the orientation of $\tilde{\gamma}$ agrees the direction of v at 0. Since such coordinate system is unique (unique up to $(u, v) \mapsto (u, -v)$ if m is even), the order of contact of $\tilde{\gamma}$ with the v-axis at 0 and the curvature $\tilde{\kappa}$ of $\tilde{\gamma}$ is well-defined as a curve on f. We call such order of contact the *order of contact with the normalized null direction*, and we call $\tilde{\kappa}$ the *curvature written in the normal form*. If $\tilde{\gamma}(t) = (t^l c(t), t)$ $(c(0) \neq 0)$, then the order of contact with the normalized null direction is l, and $\tilde{\kappa}^{(l-2)}(0) = -l!c(0)$ and $\tilde{\kappa}^{(l-1)}(0) = -(l+1)!c'(0)$ hold.

Theorem 3.5. *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be an m-type edge, $m \geq 2$, and $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a regular curve with order of contact $l \geq 2$ with the null direction of f at 0 and $\tilde{\kappa}$ the curvature of γ written in the normal form of f. Then it holds that:*

(1) *The case $l \geq m$. For κ_g,*
- *if $m < l \leq 2m$, then $\operatorname{ord}\kappa_g = l - 2m$;*
- *if $l > 2m$, then $\operatorname{ord}\kappa_g \geq 1$, and $\operatorname{ord}\kappa_g = 1$ is equivalent to*

$$\begin{cases} (l-1)!\kappa_t(0)\omega_{m,m+1}(0) - m!(m+1)!\tilde{\kappa}^{(l-2)}(0) \neq 0 & \text{if } l = 2m+1, \\ \kappa_t(0)\omega_{m,m+1}(0) \neq 0 & \text{if } l > 2m+1; \end{cases}$$

- *if $l = m$, then $\operatorname{ord}\kappa_g \geq 1 - m$, and $\operatorname{ord}\kappa_g = 1 - m$ if and only if $\tilde{\kappa}^{(l-1)}(0) \neq 0$.*

For κ_n, it holds that $\operatorname{ord}\kappa_n \geq 1 - m$, and $\operatorname{ord}\kappa_n = 1 - m$ if and only if $\omega_{m,m+1}(0) \neq 0$. For τ_g,

- *if* $l < 2m$, *then* $\operatorname{ord}\tau_g \geq l - 2m + 1$, *and* $\operatorname{ord}\tau_g = l - 2m + 1$ *is equivalent to*

$$\begin{cases} \omega_{m,m+1}(0) \neq 0 & \text{if } l < 2m-1, \\ m(l-1)!\kappa_t(0) + (m-1)!^2\, \tilde{\kappa}^{(l-2)}(0)\, \omega_{m,m+1}(0) \neq 0 & \text{if } l = 2m-1; \end{cases}$$

- *if* $l \geq 2m$, *then* $\operatorname{ord}\tau_g \geq 0$, *and* $\operatorname{ord}\tau_g = 0$ *if and only if* $\kappa_t(0) \neq 0$.

(2) *The case* $m/2 < l < m$. *For this case, it holds that* $\operatorname{ord}\kappa_g = m - 2l$, $\operatorname{ord}\kappa_n \geq m - 2l + 1$, *and* $\operatorname{ord}\kappa_n = m - 2l + 1$ *is equivalent to*

$$\begin{cases} \omega_{m,m+1}(0) \neq 0 & \text{if } l > (m+1)/2, \\ (m+1)!(m-l+1)\kappa_\nu(0)(\tilde{\kappa}^{(l-2)})^2(0) + 2l!^2\omega_{m,m+1}(0) \neq 0 & \text{if } l = (m+1)/2. \end{cases}$$

For τ_g, *it holds that* $\operatorname{ord}\tau_g \geq 1 - l$, *and* $\operatorname{ord}\tau_g = 1 - l$ *is equivalent to* $\omega_{m,m+1}(0) \neq 0$.

(3) *The case* $l \leq m/2$. *In this case, it holds that* $\operatorname{ord}\kappa_g \geq 0$, *and* $\operatorname{ord}\kappa_g = 0$ *is equivalent to*

$$\begin{cases} \kappa_s(0) \neq 0 & \text{if } l < m/2, \\ m!\kappa_s(0)(\tilde{\kappa}^{(l-2)})^2(0) + 2l!^2 \neq 0 & \text{if } l = m/2. \end{cases}$$

For κ_n *it holds that* $\operatorname{ord}\kappa_n \geq 0$, *and* $\operatorname{ord}\kappa_n = 0$ *if and only if* $\kappa_n(0) \neq 0$. *For* τ_g, *it holds that* $\operatorname{ord}\tau_g \geq 1 - l$, *and* $\operatorname{ord}\tau_g = 1 - l$ *if and only if* $\omega_{m,m+1}(0) \neq 0$.

If m is even and (u, v) is a coordinate system that gives the form (2.4), then $(u, -v)$ also gives (2.4). In this case, changing (u, v) to $(u, -v)$, the signs of $\tilde{\kappa}$ and $\omega_{m,m+1}$ reverse, and those of κ_t and κ_s do not change. So, when m is even, none of the conditions

$$\begin{aligned} (l-1)!\kappa_t(0)\omega_{m,m+1}(0) - m!(m+1)!\tilde{\kappa}^{(l-2)}(0) \neq 0, \\ m(l-1)!\kappa_t(0) + (m+1)!^2\, \tilde{\kappa}^{(l-2)}(0)\, \omega_{m,m+1}(0) \neq 0, \\ m!\kappa_s(0)(\tilde{\kappa}^{(l-2)})^2(0) + 2l!^2 \neq 0 \end{aligned}$$

change under the coordinate change (u, v) to $(u, -v)$.

Proof. Let $\hat{\gamma} = f \circ \gamma$. One can assume that f is given by the form (2.4) and, since ∂v is a null vector of f, one can take $\gamma(t) = (x(t), t)$, with $x(0) = x'(0) = 0$. Then $x(t)$ is of order l and we set $\gamma(t) = (t^l c(t), t)$ $(c(0) \neq 0)$.

In the normal form (2.4), since $b_m(0, 0) = 0$, we may further assume f is given by $f(u, v) = (u, u^2 a(u)/2 + v^m/m!, u^2 b_0(u)/2 + (v^m/m!)(u b_{m1}(u) + v b_{m2}(u, v)))$. We recall that $\kappa_s(0) = a(0), \kappa_\nu(0) = b(0)$ and $\kappa_t(0) = (b_m)_u(0, 0)$. Furthermore, it holds that $\kappa_t(0) = b_{m1}(0)$, $\omega_{m,m+1} = (m+1)b_{m2}(0, 0)$, $\tilde{\kappa}^{(l-2)}(0) = -l!c(0)$ and $\tilde{\kappa}^{(l-1)}(0) = -(l+1)!c'(0)$. We set φ by $f_v = v^{m-1}\varphi/(m-1)!$. Then $\tilde{\nu}_2 = f_u \times \varphi$ gives a non-zero normal vector field to f.

(1). Assume $l \geq m$. By (2.4) we get $\hat{\gamma} = t^m \tilde{\rho}$, where

$$\begin{aligned}
\tilde{\rho}(t) &= (t^{l-m} c(t), g_2(t), t g_3(t)),\\
g_2(t) &= \frac{m! t^{2l-m} a(t) c(t)^2 + 2}{2m!},\\
g_3(t) &= \frac{m! t^{2l-m-1} b_0(t) c(t)^2 + 2 b_{m2}(t) + 2 t^{l-1} b_{m1}(t) c(t)}{2m!}.
\end{aligned}$$

Then $\hat{\gamma} = t^{m-1}\rho$, where $\rho = m\tilde{\rho} + t\tilde{\rho}'$. Note that $\rho(0) \neq 0$. Setting $\nu_2(t) = \tilde{\nu}_2(\gamma(t))$, we can show that $\nu_2(t) = (t^m d(t), t e(t), 1)$, where

$$\begin{aligned}
d(t) = \frac{1}{2mm!}\Big(&- 2m b_{m1} + 2m t^{2l-m} a b_{m1} c^2 m! - 2m t^{l-m} b_0 c m!\\
&+ 2 t^{1+l-m} a b_{m2} c m! + 2m t^{1+l-m} a b_{m2} c m! + m t^{3l-m} b_{m1} c^3 m! a'\\
&+ t^{1+2l-m} b_{m2} c^2 m! a' + m t^{1+2l-m} b_{m2} c^2 m! a' - m t^{2l-m} c^2 m! b_0'\\
&+ 2 t^{2+l-m} a c m! b_{m2,v} + t^{2+2l-m} c^2 m! a' b_{m2,v} - 2m t^l c b_{m1}' - 2m t b_{m2,u}\Big),
\end{aligned}$$

$$e(t) = \frac{-1}{m}\Big(m t^{l-1} b_{m1} c + (1+m) b_{m2} + t b_{m2,v}\Big). \tag{3.6}$$

We abbreviate the variable, namely $a = a(t)$, $b_{m2} = b_{m2}(\gamma(t))$, for instance, and $(b_{m2})_v = b_{m2,v}$. Here, we see that

$$\begin{aligned}
&g_2(0) = \frac{1}{m!},\ g_2'(0) = 0, \quad g_3(0) = \frac{b_{m2}(0)}{m!}, \quad d(0) = \frac{-b_{m1}(0)}{m!} \text{ (if } m < l),\\
&d(0) = \frac{-m! b_0(0) c(0) - b_{m1}(0)}{m!} \text{ (if } m = l), \quad e(0) = -\frac{(m+1) b_{m2}(0)}{m}.
\end{aligned}$$

To see the rational order of the invariants κ_g, κ_n, τ_g at 0, we may use $\nu_2(t)$ instead of $\nu \circ \gamma(t)$ in (3.4). Since $g_2'(0) = 0$, we can write $g_2' = t\tilde{g}$. We see that

$$\rho = (l t^{l-m} c + t^{l-m+1} c', m g_2 + t^2 \tilde{g}_2, (m+1) t g_3 + t^2 g_3'), \tag{3.7}$$

$$\rho' = \begin{cases} \big(l(l-m) t^{l-m-1} c + t^{l-m} O(1), t O(1), (m+1) g_3 + t O(1)\big) & (l > m)\\ \big((m+1) c' + t O(1), t O(1), (m+1) g_3 + t O(1)\big) & (m = l), \end{cases} \tag{3.8}$$

where $O(1)$ means a smooth function depending on t. Then we see that $|\rho, \rho', \nu_2|$, where $|\cdot| = \det(\cdot)$, is, for $l > m$,

$$\begin{vmatrix} l t^{l-m} c + t^{l-m+1} O(1) & l(l-m) t^{l-m-1} c + t^{l-m} O(1) & t^m d\\ m g_2 + t O(1) & t O(1) & t e\\ (m+1) t g_3 + t^2 O(1) & (m+1) g_3 + t O(1) & 1 \end{vmatrix}. \tag{3.9}$$

If $2m - l + 1 > 0$, then (3.9) is $t^{l-m-1} A_1(t)$, where

$$A_1(0) = \begin{vmatrix} 0 & l(l-m) c(0) & 0\\ m g_2(0) & 0 & 0\\ 0 & (m+1) g_3(0) & 1 \end{vmatrix} = -\frac{l(l-m)}{(m-1)!} c(0).$$

If $2m-l+1=0$, then (3.9) is $t^m A_2(t)$, where

$$A_2(0)=\begin{vmatrix} 0 & l(m+1)c(0) & d(0)\\ mg_2(0) & 0 & 0\\ 0 & (m+1)g_3(0) & 1\end{vmatrix} = -m(m+1)g_2(0)(lc-dg_3)(0)$$
$$= -\frac{m+1}{(m-1)!}\left(\frac{b_{m1}b_{m2}}{(m!)^2}+lc\right)(0).$$

If $2m-l+1<0$, then $l>m$ and (3.9) is $t^m A_3(t)$, where

$$A_3(0)=\begin{vmatrix} 0 & 0 & d(0)\\ mg_2(0) & 0 & 0\\ 0 & (m+1)g_3(0) & 1\end{vmatrix} = m(m+1)d(0)g_2(0)g_3(0)$$
$$= -\frac{m(1+m)}{(m!)^3}b_{m1}(0)b_{m2}(0).$$

If $m=l$, by (3.7) and (3.8), we see the assertion, once $\operatorname{ord}|t|^{m-1}=m-1$ and $\operatorname{ord}|\rho|^3=0$. This shows the assertion for κ_g.

Since one can easily see that $\langle \rho', \nu_2\rangle = (m+1)b_{m2}/m!$ at 0, the assertion for κ_n is proved. Next we see that $|\rho, \nu_2, \nu_2'|$ is

$$\begin{vmatrix} lt^{l-m}c+t^{l-m+1}O(1) & t^m d & mt^{m-1}d+t^mO(1)\\ mg_2+tO(1) & te & e+tO(1)\\ (m+1)tg_3+t^2O(1) & 1 & 0\end{vmatrix}. \tag{3.10}$$

If $2m-l-1>0$, then (3.10) is $t^{l-m}B_1(t)$, where

$$B_1(0)=\begin{vmatrix} lc(0) & 0 & 0\\ mg_2(0) & 0 & e(0)\\ 0 & 1 & 0\end{vmatrix} = -lc(0)e(0) = \frac{l(m+1)}{m}b_{m2}(0)c(0).$$

If $2m-l-1=0$, then $m<l$ and (3.10) is $t^{m-1}B_2(t)$, where

$$B_2(0)=\begin{vmatrix} lc(0) & 0 & md(0)\\ mg_2(0) & 0 & e(0)\\ 0 & 1 & 0\end{vmatrix} = (-lce+m^2dg_2)(0)$$
$$= -\frac{b_{m1}(0)}{(m-1)!^2}+\frac{l(m+1)b_{m2}(0)c(0)}{m}.$$

If $2m-l-1<0$, then $m<l$, and (3.10) is $t^{m-1}B_3(t)$, where

$$B_3(0)=\begin{vmatrix} 0 & 0 & md(0)\\ mg_2(0) & 0 & e(0)\\ 0 & 1 & 0\end{vmatrix} = m^2d(0)g_2(0) = -\frac{b_{m1}(0)}{(m-1)!^2}.$$

This shows the assertion for τ_g.

(2) and (3). We assume $l < m$ and we shall use the same notation of case (1). Setting $\nu_2(t) = \tilde{\nu}_2(\gamma(t))$, we can show that $\nu_2(t) = (t^l d(t), te(t), 1)$, where

$$\begin{aligned} d(t) = \frac{1}{2mm!}\big(&- 2mt^{m-l}b_{m1} + 2mm!t^l ab_{m1}c^2 - 2mm!b_0 c + 2m!tab_{m2}c + 2mm!tab_{m2}c \\ &+ mm!t^{2l}b_{m1}c^3 a' + m!t^{l+1}b_{m2}c^2 a' + mm!t^{l+1}b_{m2}c^2 a' - mt^l c^2 m! b_0' \\ &+ 2m!t^2 acb_{m2,v} + m!t^{l+2}c^2 a' b_{m2,v} - 2mt^m cb_{m1}' - 2mt^{m-l+1}b_{m2,u}\big) \end{aligned}$$

and e is the same as in (3.6). We assume $l \leq m/2$. Then $\hat{\gamma} = t^l \tilde{\rho}$, where $\tilde{\rho}(t) = (c(t), t^l g_2(t), t^l g_3(t))$ and

$$\begin{aligned} g_2(t) &= \frac{2t^{m-2l} + m!a(t)c(t)^2}{2m!}, \\ g_3(t) &= \frac{2t^{m-l}b_{m1}(t)c(t) + 2t^{m-2l+1}b_{m2}(\gamma(t)) + m!b_0(t)c(t)^2}{2m!}. \end{aligned}$$

Then $\hat{\gamma}' = t^l \rho$, where $\rho = l\tilde{\rho} + t\tilde{\rho}'$ with $\rho(0) \neq 0$. Since $\hat{\gamma}$ has multiplicity l, we need to replace $m-1$ in equations (3.4) by $l-1$. Here, we see that

$$\begin{aligned} &g_2(0) = a(0)c(0)^2/2 \text{ (if } l < m/2), \ g_2(0) = a(0)c(0)^2/2 + 1/m! \text{ (if } m = 2l), \\ &g_3(0) = b_0(0)c(0)^2/2, \ e(0) = -(1+m)b_{m2}(0)/m. \end{aligned}$$

To see the order, we may use $\nu_2(t)$ instead of $\nu \circ \gamma(t)$ in (3.4). We see that

$$\begin{aligned} \rho &= \big(lc + tc', t^l(2lg_2 + tg_2'), t^l(2lg_3 + tg_3')\big), \\ \rho' &= \big((l+1)c' + tO(1), t^{l-1}(2l^2 g_2 + tO(1)), t^{l-1}(2l^2 g_3 + tO(1))\big). \end{aligned} \tag{3.11}$$

By applying the formula

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ kx_{21} & x_{22} & x_{23} \\ kx_{31} & x_{32} & x_{33} \end{vmatrix} = \begin{vmatrix} x_{11} & kx_{12} & kx_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

for $k = t^{l-1}$, we see that $|\rho, \rho', \nu_2|$ is

$$\begin{aligned} &\begin{vmatrix} lc + tO(1) & (l+1)c' + tO(1) & t^l d \\ t^l(2lg_2 + tO(1)) & t^{l-1}(2l^2 g_2 + tO(1)) & te \\ t^l(2lg_3 + tO(1)) & t^{l-1}(2l^2 g_3 + tO(1)) & 1 \end{vmatrix} \\ &\qquad = \begin{vmatrix} lc + tO(1) & t^{l-1}(l+1)c' + tO(1) & t^{2l-1}d \\ t(2lg_2 + tO(1)) & t^{l-1}(2l^2 g_2 + tO(1)) & te \\ t(2lg_3 + tO(1)) & t^{l-1}(2l^2 g_3 + tO(1)) & 1 \end{vmatrix} \\ &\qquad = t^{l-1}C_1(t) \end{aligned} \tag{3.12}$$

with

$$C_1(t) = \begin{vmatrix} lc + tO(1) & (l+1)c' + tO(1) & t^{2l-1}d \\ t(2lg_2 + tO(1)) & 2l^2 g_2 + tO(1) & te \\ t(2lg_3 + tO(1)) & 2l^2 g_3 + tO(1) & 1 \end{vmatrix}.$$

Then $C_1(0) = 2l^3 c(0) g_2(0)$. This shows the assertion for κ_g. By (3.11), we see that $\langle \rho', \nu_2 \rangle = t^{l-1}(2l^2 g_3 + tO(1))$ and $|\rho, \nu_2, \nu_2'|(0) = l(m+1)c(0)b_{m2}/m$. This shows the assertions for κ_n and τ_g.

Next we assume $l > m/2$. In this case, $\hat{\gamma} = t^l(c(t), t^{m-l}g_2(t), t^{m-l+1}g_3(t))$. We set $\tilde{\rho}(t) = (c(t), t^{m-l}g_2(t), t^{m-l+1}g_3(t))$ and

$$g_2(t) = \frac{2 + m!\, t^{2l-m} a(t) c(t)^2}{2m!},$$
$$g_3(t) = \frac{2t^{l-1} b_{m1}(t) c(t) + 2b_{m2}(\gamma(t)) + m!\, t^{2l-m-1} b_0(t) c(t)^2}{2m!}.$$

Here, it holds that

$$g_2(0) = 1/m!,\ e(0) = -(1+m)b_{m2}(0)/m,$$
$$g_3(0) = b_{m2}(0)/m! \text{ (if } 2l - m - 1 > 0),$$
$$g_3(0) = b_0(0)c(0)^2/2 + b_{m2}(0)/m! \text{ (if } m = 2l - 1).$$

It holds that $\hat{\gamma}' = t^{l-1}\rho$ with $\rho = l\tilde{\rho} + t\tilde{\rho}'$ and $\rho(0) \neq 0$. We see that

$$\rho(t) = \Big(lc + tc', t^{m-l}(mg_2 + tg_2'), t^{m-l+1}((m+1)g_3 + tg_3')\Big),$$
$$\rho'(t) = \Big((l+1)c' + tO(1), t^{m-l-1}(m(m-l)g_2 + tO(1)),$$
$$t^{m-l}((m+1)(m-l+1)g_3 + tO(1))\Big).$$

Using a similar method to (3.12), we see that $|\rho, \rho', \nu_2|$ is

$$\begin{vmatrix} lc + tO(1) & (1+l)c' + tO(1) & t^l d \\ t^{m-l}(mg_2 + tO(1)) & t^{m-l-1}(m(m-l)g_2 + tO(1)) & te \\ t^{m-l+1}((m+1)g_3 + tO(1)) & t^{m-l}((m+1)(m-l+1)g_3 + tO(1)) & 1 \end{vmatrix}$$
$$= \begin{vmatrix} lc + tO(1) & t^{m-l-1}((1+l)c' + tO(1)) & t^{m-1}d \\ t(mg_2 + tO(1)) & t^{m-l-1}(m(m-l)g_2 + tO(1)) & te \\ t^2((m+1)g_3 + tO(1)) & t^{m-l}((m+1)(m-l+1)g_3 + tO(1)) & 1 \end{vmatrix}$$
$$= t^{m-l-1} C_2(t),$$

with

$$C_2(t) = \begin{vmatrix} lc + tO(1) & (1+l)c' + tO(1) & t^{m-1}d \\ t(mg_2 + tO(1)) & m(m-l)g_2 + tO(1) & te \\ t^2((m+1)g_3 + tO(1)) & t((m+1)(m-l+1)g_3 + tO(1)) & 1 \end{vmatrix}.$$

Then $C_2(0) = l(m-l)mc(0)g_2(0) = l(m-l)c(0)/(m-1)!$ and, replacing $m-1$ by $l-1$ in equations (3.4), this shows the assertion for κ_g. By (3.11), we see that $\langle \rho', \nu_2 \rangle = t^{m-l}C_3(t)$, where $C_3(t) = m(m-l)g_2(t)e_2(t) + (m+1)(m-l+1)g_3(t) + tO(1)$. It holds that

$$C_3(0) = \begin{cases} \dfrac{(m+1)b_{m2}(0)}{m!} & (m < 2l-1), \\ (m+1)\left(\dfrac{lb_0(0)c(0)^2}{2(l!)^2} + \dfrac{b_{m2}(0)}{m!}\right) & (m = 2l-1), \end{cases}$$

and $|\rho, \nu_2, \nu_2'|(0) = -lc(0)e(0) = l(m+1)c(0)b_{m2}(0)/m$. This shows the assertions for κ_n and τ_g. □

In particular, we have the following corollary on boundedness directly obtained from Theorem 3.5.

Corollary 3.6. *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be an m-type edge with $m \geq 2$, and $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a regular curve with order of contact $l \geq 2$ with the null direction of f at 0.*

(1) *The case $l \geq m$. For κ_g,*
   - *if $l \geq 2m$, then κ_g is bounded at 0;*
   - *if $m < l < 2m$, then κ_g is unbounded at 0;*
   - *if $m = l$ and $\tilde{\kappa}^{(l-1)}(0) \neq 0$, then κ_g is unbounded at 0.*

   For κ_n, if $\omega_{m,m+1}(0) \neq 0$, then κ_n is unbounded at 0. For τ_g,
   - *if $m \leq l < 2m-1$ and $\omega_{m,m+1}(0) \neq 0$, then τ_g is unbounded at 0;*
   - *if $l = 2m-1$ and $m(l-1)!\kappa_t(0) + (m-1)!^2\,\tilde{\kappa}^{(l-2)}(0)\,\omega_{m,m+1}(0) \neq 0$, then τ_g is bounded at 0;*
   - *if $l > 2m-1$, then τ_g is bounded at 0.*

(2) *The case $m/2 < l < m$. In this case, κ_g is unbounded at 0. If $l = (m+1)/2$, then κ_n is bounded at 0. If $m > l > (m+1)/2$ and $\omega_{m,m+1}(0) \neq 0$, then κ_n is unbounded at 0. If $\omega_{m,m+1}(0) \neq 0$, then τ_g is unbounded at 0.*

(3) *The case $l \leq m/2$. In this case, κ_g and κ_n are bounded at 0. If $\omega_{m,m+1}(0) \neq 0$, then τ_g is unbounded at 0.*

We consider the case where $f\colon (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ is a cuspidal edge. By definition, it is a $(2,3)$-edge, in particular, a 2-type edge. Then by Theorem 3.5, the following assertion holds.

Corollary 3.7. *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be a cuspidal edge, and let $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a regular curve with order of contact $l \geq 2$ with the null direction of f at 0 and $\tilde{\kappa}$ the curvature of γ written in the normal form of f. Then, it holds that: For κ_g,*

- *if $l = 2$, then $\operatorname{ord}\kappa_g \geq -1$, and $\operatorname{ord}\kappa_g = -1$ if and only if $\tilde{\kappa}^{(l-1)}(0) \neq 0$;*
- *if $l = 3$ or 4, then $\operatorname{ord}\kappa_g = l-4$;*
- *if $l \geq 5$, then $\operatorname{ord}\kappa_g \geq 1$, and $\operatorname{ord}\kappa_g = 1$ is equivalent to*

$$\begin{cases} (l-1)!\kappa_t(0)\omega_{2,3}(0) - 12\tilde{\kappa}^{(l-2)}(0) \neq 0 & \text{if } l = 5, \\ \kappa_t(0)\omega_{2,3}(0) \neq 0 & \text{if } l > 5. \end{cases}$$

For κ_n*, it holds that* $\operatorname{ord}\kappa_n = -1$.
For τ_g*,*

- *if* $l = 2$ *or* 3*, then* $\operatorname{ord}\tau_g \geq l-3$*, and* $\operatorname{ord}\tau_g = l-3$ *is equivalent to*

$$\begin{cases} \omega_{2,3}(0) \neq 0 & \text{if } l < 3, \\ 2(l-1)!\kappa_t(0) + \tilde{\kappa}^{(l-2)}(0)\,\omega_{2,3}(0) \neq 0 & \text{if } l = 3; \end{cases}$$

- *if* $l \geq 4$*, then* $\operatorname{ord}\tau_g \geq 0$*, and* $\operatorname{ord}\tau_g = 0$ *if and only if* $\kappa_t(0) \neq 0$.

Proof. Since $\omega_{2,3}$ corresponds to the cuspidal curvature κ_c and it does not vanish at 0 ([12, Proposition 3.11]), we have the assertion by Theorem 3.5. □

About the boundedness, we have the following immediate corollary from Theorem 3.7.

Corollary 3.8. *Under the same assumption of Corollary* 3.7*, we have the following:*

(1) *For the geodesic curvature* κ_g*,*
  - *if* $l \geq 4$*, then* κ_g *is bounded at* 0*;*
  - *if* $l = 3$*, then* κ_g *is unbounded at* 0*;*
  - *if* $l = 2$ *and* $\tilde{\kappa}'(0) \neq 0$*, then* κ_g *is unbounded at* 0.

(2) *The normal curvature* κ_n *is unbounded at* 0.

(3) *For the geodesic torsion* τ_g*,*
  - *if* $l = 2$*, then* τ_g *is unbounded at* 0*;*
  - *if* $l = 3$ *and* $4\kappa_t(0) + \tilde{\kappa}'(0)\,\kappa_c(0) \neq 0$*, then* τ_g *is bounded at* 0*;*
  - *if* $l \geq 4$*, then* τ_g *is bounded at* 0*,*

  where κ_c *is the cuspidal curvature (cf.* [12]*) corresponding to* $\omega_{2,3}$.

Note that $\operatorname{ord}\kappa_g \geq -1$ for $l \geq 2$ is pointed out in [2, Proposition 2.19].

We observe that although in the above results we could not guarantee that the three invariants are bounded at the same time near a singular point, it is easy to find an example where it happens: taking $f = (u, \frac{v^2}{2}, v^5)$ and $\gamma(t) = (t^4, t)$, it holds that $m = 2$, $l = 4$, $\operatorname{ord}\kappa_g = 0$, $\operatorname{ord}\kappa_n = 1$, $\operatorname{ord}\tau_g = 3$ (see Figure 1). Thus, these three invariants are bounded at 0 (cf. Corollary 3.6). For the cuspidal edge $f(u,v) = (u, v^2, v^3)$ and the same γ, we see that κ_g and τ_g are bounded, but κ_n is unbounded at 0 (cf. Corollary 3.8). Figure 2 shows the graphs of these invariants near 0.

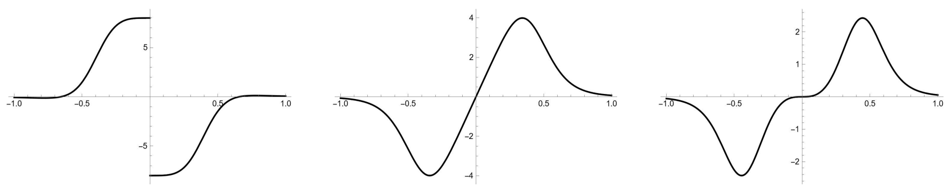

FIGURE 1. The graphs of κ_g (left), κ_n (middle) and τ_g (right) of the curve $\hat{\gamma}(t) = f(\gamma(t))$, where $f = (u, \frac{v^2}{2}, v^5)$ and $\gamma(t) = (t^4, t)$.

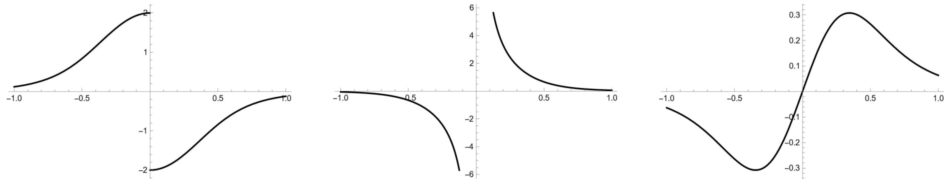

FIGURE 2. The graphs of κ_g (left), κ_n (middle) and τ_g (right) of the curve $\hat{\gamma}(t) = f(\gamma(t))$, where $f(u,v) = (u, v^2, v^3)$ and $\gamma(t) = (t^4, t)$.

APPENDIX A. GENERALIZED BIASES FOR A PLANE CURVE

Let $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a curve-germ of (m,n)-type which is given by the form (2.3) in the xy-plane $(\mathbf{R}^2, 0)$. The terms a_i $(i = 2, \dots, \lfloor n/m \rfloor)$ measures the bias of γ near a singular point. We call a_{i+1} the (m, im)-*bias* $(i = 2, \dots, \lfloor n/m \rfloor)$ of γ at 0, and it is denoted by $\beta_{m,im}$. We call $b(0)$ the (m,n)-*cuspidal curvature* as in [8], and it is denoted by $r_{m,n}$.

If m and n are even, then it is a half part of a curve of $(m/2, n/2)$-type, and we consider the following cases: (1) both m, n are odd, (2) m is odd and n is even, and (3) m is even and n is odd. Moreover, let a_k denote the first non-zero term of a_i $(i = 2, \dots, \lfloor n/m \rfloor)$. We consider the cases (1) and (2). Then γ passes through the origin tangent to the x-axis. In the case (1), if k is odd, it also passes across the x-axis. If k is even, it approaches the origin from one side of the x-axis and goes away into the same side of the x-axis, and if there does not exist such k (namely, the bias is zero), it passes through the x-axis. In the case (2), if the bias is zero, it approaches the origin from one side of the x-axis and goes away into the same side of the x-axis. Figure 3 shows the images of the curves $\gamma_1 : t \mapsto (t^3, a_1 t^6 + a_2 t^9 + t^{11})$ with $(a_1, a_2) = (1,0), (0,1), (0,0)$ from left to right. Figure 4 shows the images of the curves $\gamma_2 : t \mapsto (t^3, a_1 t^6 + a_2 t^9 + t^{14})$ with $(a_1, a_2) = (1,0), (0,1), (0,0)$ from left to right.

FIGURE 3. The images of the curves γ_1.

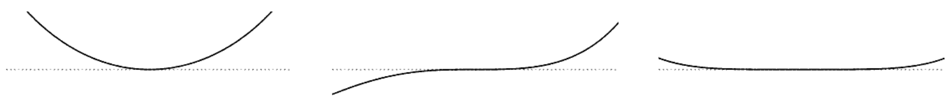

FIGURE 4. The images of the curves γ_2.

We consider the case (3). Then γ approaches the origin from a direction of the x-axis, makes a cusp, and goes back in the same direction. If k is both odd and even, it approaches the origin from one side of the x-axis and goes away into the same side of the x-axis. If the bias is zero, it passes through the x-axis. Figure 5 shows the images of the curves $\gamma_3 : t \mapsto (t^4, a_1t^8 + a_2t^{12} + t^{13})$ with $(a_1, a_2) = (1, 0), (0, 1), (0, 0)$ from left to right.

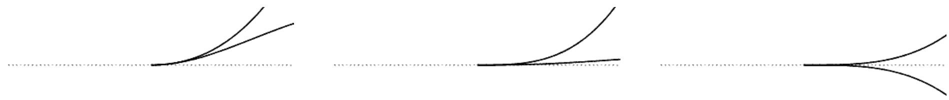

FIGURE 5. The images of the curves γ_3.

Example A.1. Let γ be a curve-germ $\mathcal{A}^3$-equivalent to $(t^3, 0)$. We set

$$\tilde{a}_i = \frac{\gamma^{(3)}(0) \cdot \gamma^{(i)}(0)}{i!|\gamma^{(3)}(0)|}, \quad \tilde{b}_i = \frac{\det(\gamma^{(3)}(0), \gamma^{(i)}(0))}{i!|\gamma^{(3)}(0)|}.$$

One can calculate the invariants up to 10 degrees as follows. The $(3, 4)$-cuspidal curvature $r_{3,4}$ is

$$r_{3,4} = \frac{\tilde{b}_4}{\tilde{a}_3^{4/3}}. \tag{A.1}$$

If $r_{3,4} \neq 0$, i.e., $\tilde{b}_4 \neq 0$, then γ is $\mathcal{A}$-equivalent to (t^3, t^4). We assume $\tilde{b}_4 = 0$. Then the $(3, 5)$-cuspidal curvature $r_{3,5}$ is

$$r_{3,5} = \frac{\tilde{b}_5}{\tilde{a}_3^{5/3}}. \tag{A.2}$$

If $r_{3,5} \neq 0$, i.e., $\tilde{b}_5 \neq 0$, then γ is $\mathcal{A}$-equivalent to (t^3, t^5). We assume $\tilde{b}_5 = 0$. Then the $(3, 6)$-bias $\beta_{3,6}$ and the $(3, 7)$-cuspidal curvature $r_{3,7}$ are

$$\beta_{3,6} = \frac{\tilde{b}_6}{\tilde{a}_3^2}, \tag{A.3}$$

$$r_{3,7} = \frac{-7\tilde{a}_4\tilde{b}_6 + 2\tilde{a}_3\tilde{b}_7}{2\tilde{a}_3^{10/3}}. \tag{A.4}$$

If $r_{3,7} \neq 0$, i.e., $-7\tilde{a}_4\tilde{b}_6 + 2\tilde{a}_3\tilde{b}_7 \neq 0$, then γ is $\mathcal{A}^7$-equivalent to (t^3, t^7). We assume $r_{3,7} = 0$, i.e., $\tilde{b}_7 = 7\tilde{a}_4\tilde{b}_6/(2\tilde{a}_3)$. Then the $(3, 8)$-cuspidal curvature $r_{3,8}$ is

$$r_{3,8} = \frac{-35\tilde{a}_4^2\tilde{b}_6 + 2\tilde{a}_3(-28\tilde{a}_5\tilde{b}_6 + 5\tilde{a}_3\tilde{b}_8)}{10\tilde{a}_3^{14/3}}. \tag{A.5}$$

If $r_{3,8} \neq 0$, then γ is $\mathcal{A}^8$-equivalent to (t^3, t^8). We assume $r_{3,8} = 0$, i.e., $\tilde{b}_8 = 7(5\tilde{a}_4^2 + 8\tilde{a}_3\tilde{a}_5)\tilde{b}_6/(10\tilde{a}_3^2)$. Then the $(3, 9)$-bias $\beta_{3,9}$ and the $(3, 10)$-cuspidal curvature

$r_{3,10}$ are

$$\beta_{3,9} = -\frac{63\tilde{a}_4\tilde{a}_5\tilde{b}_6 + 42\tilde{a}_3\tilde{a}_6\tilde{b}_6 - 5\tilde{a}_3^2\tilde{b}_9}{5\tilde{a}_3^5}, \tag{A.6}$$

$$r_{3,10} = \frac{(10\tilde{a}_3^3\tilde{b}_{10} + 945\tilde{a}_4^2\tilde{a}_5\tilde{b}_6 - 42\tilde{a}_3(3\tilde{a}_5^2 - 10\tilde{a}_4\tilde{a}_6)\tilde{b}_6 - 15\tilde{a}_3^2(8\tilde{a}_7\tilde{b}_6 + 5\tilde{a}_4\tilde{b}_9))}{10\tilde{a}_3^{19/3}}. \tag{A.7}$$

Proof of Example A.1. By rotating γ in $\mathbf{R}^3$, we can write

$$\gamma(t) = \left(\sum_{i=3}^{10}\frac{\tilde{a}_i}{i!}t^i,\ \sum_{i=4}^{10}\frac{\tilde{b}_i}{i!}t^i\right) + O(10).$$

We set

$$\varphi(t) = t\left(6\sum_{i=3}^{10}\frac{\tilde{a}_i}{i!}t^{i-3}\right)^{1/3},$$

and the inverse function of $s = \varphi(t)$ as $t = \psi(s)$. We set $\psi(s) = \sum_{i=1}^{10}\psi_i s^i/i! + O(10)$. Then we have:

$$\begin{aligned}
\psi_1 &= 1/\tilde{a}_3^{1/3},\\
\psi_2 &= -\tilde{a}_4/(6\tilde{a}_3^{5/3}),\\
\psi_3 &= (5\tilde{a}_4^2 - 4\tilde{a}_3\tilde{a}_5)/(40\tilde{a}_3^3),\\
\psi_4 &= (-175\tilde{a}_4^3 + 252\tilde{a}_3\tilde{a}_4\tilde{a}_5 - 72\tilde{a}_3^2\tilde{a}_6)/(1080\tilde{a}_3^{13/3}),\\
\psi_5 &= (13475\tilde{a}_4^4 - 27720\tilde{a}_3\tilde{a}_4^2\tilde{a}_5 + 10080\tilde{a}_3^2\tilde{a}_4\tilde{a}_6 + 432\tilde{a}_3^2(14\tilde{a}_5^2 - 5\tilde{a}_3\tilde{a}_7))/(45360\tilde{a}_3^{17/3}),\\
\psi_6 &= (-1575\tilde{a}_4^5 + 4200\tilde{a}_3\tilde{a}_4^3\tilde{a}_5 - 1680\tilde{a}_3^2\tilde{a}_4^2\tilde{a}_6 + 96\tilde{a}_3^2\tilde{a}_4(-21\tilde{a}_5^2 + 5\tilde{a}_3\tilde{a}_7)\\
&\quad + 16\tilde{a}_3^3(42\tilde{a}_5\tilde{a}_6 - 5\tilde{a}_3\tilde{a}_8))/(2240\tilde{a}_3^7),\\
\psi_7 &= (475475\tilde{a}_4^6 - 1556100\tilde{a}_3\tilde{a}_4^4\tilde{a}_5 + 655200\tilde{a}_3^2\tilde{a}_4^3\tilde{a}_6 - 42120\tilde{a}_3^2\tilde{a}_4^2(-28\tilde{a}_5^2 + 5\tilde{a}_3\tilde{a}_7)\\
&\quad + 3240\tilde{a}_3^3\tilde{a}_4(-182\tilde{a}_5\tilde{a}_6 + 15\tilde{a}_3\tilde{a}_8)\\
&\quad - 1296\tilde{a}_3^3(91\tilde{a}_5^3 - 60\tilde{a}_3\tilde{a}_5\tilde{a}_7 + 5\tilde{a}_3(-7\tilde{a}_6^2 + \tilde{a}_3\tilde{a}_9)))/(233280\tilde{a}_3^{25/3}),\\
\psi_8 &= (-155520\tilde{a}_{10}\tilde{a}_3^6 + 11(-4447625\tilde{a}_4^7 + 17243100\tilde{a}_3\tilde{a}_4^5\tilde{a}_5 - 7497000\tilde{a}_3^2\tilde{a}_4^4\tilde{a}_6\\
&\quad + 2570400\tilde{a}_3^2\tilde{a}_4^3(-7\tilde{a}_5^2 + \tilde{a}_3\tilde{a}_7) - 45360\tilde{a}_3^3\tilde{a}_4^2(-238\tilde{a}_5\tilde{a}_6 + 15\tilde{a}_3\tilde{a}_8)\\
&\quad + 15552\tilde{a}_3^4(-98\tilde{a}_5^2\tilde{a}_6 + 20\tilde{a}_3\tilde{a}_6\tilde{a}_7 + 15\tilde{a}_3\tilde{a}_5\tilde{a}_8)\\
&\quad + 5184\tilde{a}_3^3\tilde{a}_4(833\tilde{a}_5^3 - 420\tilde{a}_3\tilde{a}_5\tilde{a}_7 + 5\tilde{a}_3(-49\tilde{a}_6^2 + 5\tilde{a}_3\tilde{a}_9)))))/(6998400\tilde{a}_3^{29/3}),\\
\psi_9 &= (17920\tilde{a}_{10}\tilde{a}_4\tilde{a}_3^6 + 2480625\tilde{a}_4^8 - 11113200\tilde{a}_3\tilde{a}_4^6\tilde{a}_5 + 4939200\tilde{a}_3^2\tilde{a}_4^4(3\tilde{a}_5^2 + \tilde{a}_4\tilde{a}_6)\\
&\quad - 70560\tilde{a}_3^3\tilde{a}_4^2(84\tilde{a}_5^3 + 140\tilde{a}_4\tilde{a}_5\tilde{a}_6 + 25\tilde{a}_4^2\tilde{a}_7) + 4032\tilde{a}_3^4(84\tilde{a}_5^4 + 840\tilde{a}_4\tilde{a}_5^2\tilde{a}_6\\
&\quad + 600\tilde{a}_4^2\tilde{a}_5\tilde{a}_7 + 25\tilde{a}_4^2(14\tilde{a}_6^2 + 5\tilde{a}_4\tilde{a}_8)) + 2560\tilde{a}_3^6(12\tilde{a}_7^2 + 21\tilde{a}_6\tilde{a}_8 + 14\tilde{a}_5\tilde{a}_9)\\
&\quad - 4480\tilde{a}_3^5(72\tilde{a}_5^2\tilde{a}_7 + \tilde{a}_5(84\tilde{a}_6^2 + 90\tilde{a}_4\tilde{a}_8) + 5\tilde{a}_4(24\tilde{a}_6\tilde{a}_7 + 5\tilde{a}_4\tilde{a}_9)))/(89600\tilde{a}_3^{11}),\\
\psi_{10} &= 13(-16865646875\tilde{a}_4^9 + 85717170000\tilde{a}_3\tilde{a}_4^7\tilde{a}_5 + 19595520\tilde{a}_{10}\tilde{a}_3^6(-10\tilde{a}_4^2 + 3\tilde{a}_3\tilde{a}_5)\\
&\quad - 38710980000\tilde{a}_3^2\tilde{a}_4^6\tilde{a}_6 + 2844072000\tilde{a}_3^2\tilde{a}_4^5(-49\tilde{a}_5^2 + 5\tilde{a}_3\tilde{a}_7)
\end{aligned}$$

$$
\begin{aligned}
&- 1422036000\tilde{a}_3^3\tilde{a}_4^4(-70\tilde{a}_5\tilde{a}_6 + 3\tilde{a}_3\tilde{a}_8)\\
&+ 372314880\tilde{a}_3^4\tilde{a}_4^2(-154\tilde{a}_5^2\tilde{a}_6 + 20\tilde{a}_3\tilde{a}_6\tilde{a}_7 + 15\tilde{a}_3\tilde{a}_5\tilde{a}_8)\\
&+ 206841600\tilde{a}_3^3\tilde{a}_4^3(385\tilde{a}_5^3 - 132\tilde{a}_3\tilde{a}_5\tilde{a}_7 + \tilde{a}_3(-77\tilde{a}_6^2 + 5\tilde{a}_3\tilde{a}_9))\\
&- 1119744\tilde{a}_3^4\tilde{a}_4(10241\tilde{a}_5^4 - 7980\tilde{a}_3\tilde{a}_5^2\tilde{a}_7 + 150\tilde{a}_3^2(4\tilde{a}_7^2 + 7\tilde{a}_6\tilde{a}_8)\\
&+ 70\tilde{a}_3\tilde{a}_5(-133\tilde{a}_6^2 + 10\tilde{a}_3\tilde{a}_9)) + 186624\tilde{a}_3^5(22344\tilde{a}_5^3\tilde{a}_6 - 10080\tilde{a}_3\tilde{a}_5\tilde{a}_6\tilde{a}_7\\
&- 3780\tilde{a}_3\tilde{a}_5^2\tilde{a}_8 + 5\tilde{a}_3(-392\tilde{a}_6^3 + 135\tilde{a}_3\tilde{a}_7\tilde{a}_8 + 105\tilde{a}_3\tilde{a}_6\tilde{a}_9)))/(1763596800\tilde{a}_3^{37/3}).
\end{aligned}
$$

Substituting $t = \psi(s)$ into $\gamma(t)$, and by a straightforward calculation, we see that $\gamma(\psi(s)) = (s^3/6, r_{3,4}s^4/4!) + O(4)$, and we have (A.1). Under the condition $r_{3,4} = 0$, we have $\gamma(\psi(s)) = (s^3/6, r_{3,5}s^5/5!) + O(5)$, and we have (A.2). We assume $r_{3,4} = r_{3,5} = 0$; then we see that $\gamma(\psi(s)) = (s^3/6, \beta_{3,6}s^6/6! + \beta_{3,7}s^7/7!) + O(7)$, and we have (A.3) and (A.4). We assume $r_{3,7} = 0$; then we see that $\gamma(\psi(s)) = (s^3/6, \beta_{3,6}s^6/6! + \beta_{3,8}s^8/8!) + O(8)$, and we have (A.5). We assume $r_{3,8} = 0$; then we see that $\gamma(\psi(s)) = (s^3/6, \beta_{3,6}s^6/6! + \beta_{3,9}s^9/9! + r_{3,10}s^{10}/10!) + O(10)$, and we have (A.6) and (A.7). □

Example A.2. Let γ be a curve-germ $\mathcal{A}^{m+1}$-equivalent to (t^m, t^{m+1}). We set

$$\gamma(t) = \left(\sum_{i=m}^{m+1} \frac{a_i}{i!}t^i, \sum_{i=m}^{m+1} \frac{b_i}{i!}t^i\right) + O(m+1) \quad ((a_m, b_m) \neq (0,0)).$$

Then by a standard rotation A in $\mathbf{R}^2$ and a parameter change

$$t \mapsto \bar{a}^{-1/m}\left(t - \frac{\bar{a}_{m+1}}{m(m+1)a^{(m+1)/m}}t^2\right),$$

we see that

$$A\gamma(t) = \left(\frac{t^m}{m!}, \frac{r_{m,m+1}}{(m+1)!}t^{m+1}\right) \quad \left(r_{m,m+1} = \frac{\bar{b}_{m+1}}{\bar{a}_m^{(m+1)/m}}\right).$$

Thus, the $(m, m+1)$-cuspidal curvature is $r_{m,m+1}$. Here, $\bar{a}_i$ and $\bar{b}_i$ are

$$\bar{a}_i = \frac{\gamma^{(m)}(0) \cdot \gamma^{(i)}(0)}{i!|\gamma^{(m)}(0)|}, \quad \bar{b}_i = \frac{\det(\gamma^{(m)}(0), \gamma^{(i)}(0))}{i!|\gamma^{(m)}(0)|}.$$

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