

WAVE MAPS INTO THE SPHERE

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ABSTRACT. In this note we discuss some geometric analogs of the classical harmonic functions on \mathbb{R}^n and their associated evolutions.

Harmonic functions are ubiquitous in mathematics, with applications arising in complex analysis, potential theory, electrostatics, and heat conduction. A harmonic function u in \mathbb{R}^n solves the Laplace equation $\Delta u = 0$ in a domain Ω , where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$. The well-known Dirichlet principle says that harmonic functions are critical points of the Dirichlet energy $\frac{1}{2} \int_{\Omega} |\nabla u|^2$. The associated evolutions are the heat equation $\partial_t u - \Delta u = 0$ and the wave equation $\partial_t^2 u - \Delta u = 0$. The geometric analogs we will be discussing (which also have applications in physics) are harmonic maps: functions $u: \mathbb{R}^n \rightarrow M$, where M is a Riemannian manifold. Introduced in the early 1960s by Eells and Sampson [15], they are also critical points of the Dirichlet energy. The resulting Euler–Lagrange equation is a nonlinear PDE, because of the nonlinear constraint that $u(x) \in M$. When $M = S^2$ with the round metric, the equation becomes $\Delta u = -|\nabla u|^2 u$. The point of introducing harmonic maps was to use them as a tool to study the geometric and topological properties of the manifold M . For instance, Eells and Sampson showed, using the associated heat flow (the harmonic map heat flow), that smooth functions from \mathbb{R}^n to M can be deformed (under certain geometric conditions on M) into harmonic maps. This work inspired Hamilton [20] to introduce his Ricci flow, which eventually led to the proof of the Poincaré conjecture by Perelman [28]. The study of the singularities of these flows led to the notion of *bubbling* in singularity formation. It turns out that the bubbling phenomenon is universal, and is analogous to the soliton resolution which we will be considering later.

We now turn to wave maps, the wave flow associated with harmonic maps. The topic is vast, and I will concentrate only on aspects close to my interests. There are several ways to define wave maps $u: \mathbb{R} \times \mathbb{R}^n \rightarrow M$. A formal one is to consider the Lagrangian

$$\mathcal{L}(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^{1+n}} [-|\partial_t u|^2 + |\nabla u|^2] \, dt \, dx,$$

and its (formal) critical points. The corresponding Euler–Lagrange equation is the wave map equation. When $M = S^2$, $n = 2$, this becomes

$$(\partial_t^2 - \Delta)(u) = - [|\partial_t u|^2 - |\nabla u|^2] u,$$

the wave equation associated with harmonic maps. Harmonic maps are static solutions. The wave map equation is Hamiltonian and time reversible, with conserved energy

$$E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^2} [|\partial_t u(t)|^2 + |\nabla u(t)|^2] dx.$$

The equation is invariant under the scaling $u \mapsto u_\lambda(t, x) = u(\lambda t, \lambda x)$, and the conserved energy is also invariant under this scaling ($n = 2$). This is an energy-critical equation. It is the wave analog of the harmonic map heat flow. As mentioned earlier, this inspired the introduction of the Ricci flow. In the wave world, the analog is the system of Einstein vacuum field equations of general relativity. Wave maps are natural geometric wave equations. They also arise as physical models in particle physics in gauge theories (nonlinear sigma models). They are invariant under scaling, rotations, translations, and Lorentz transformations.

The theory of the Cauchy (initial value) problem for small data in the energy space was developed by Tataru [37] and by Tao [36], around 2000, while a theory for large data (short time) was developed by Sterbenz and Tataru [33] (2010). When M is a surface of revolution, $n = 2$, there are special classes of wave maps which respect symmetries. When $M = S^2$, we consider u such that $u \circ \rho = \rho^k \circ u$, the k -equivariant wave maps. Here, $k \in \mathbb{N}$ and ρ is a rotation of \mathbb{R}^2 , and of S^2 , about a fixed axis. Then, the wave map u is given, in polar coordinates (r, ω) for \mathbb{R}^2 and (η, θ) for S^2 (fixing the axis), as

$$(t, r, \omega) \xrightarrow{u} (t, \eta, \theta) = (t, \psi(r, t), k\omega).$$

The nonlinear wave equation for ψ is

$$\partial_t^2 \psi - \partial_r^2 \psi - \frac{\partial_r \psi}{\partial r} + \frac{k^2 \sin(2\psi)}{2r^2} = 0,$$

with conserved energy

$$E(\psi, \partial_t \psi) = \frac{1}{2} \int_0^\infty (\partial_t \psi)^2 r dr + \frac{1}{2} \int_0^\infty (\partial_r \psi)^2 r dr + \frac{k^2}{2} \int_0^\infty \frac{\sin^2(\psi)}{r^2} r dr.$$

For general wave maps from \mathbb{R}^2 into S^2 , the static solutions are the harmonic maps from \mathbb{R}^2 into S^2 , which were completely described in 1976 in a work of Eells and Wood [16]. The traveling wave solutions are the Lorentz transformations of the harmonic maps. They are the *bubbles* for wave maps. The k -equivariant ones correspond to the static solutions of the k -equivariant equation. These are

$$Q_k(r) = 2 \arctan(r^k) + \ell\pi, \quad \ell \in \mathbb{Z}.$$

The one of least energy (which actually corresponds to the harmonic map of least

energy among all non-constant harmonic maps) is $Q_1(r) = Q(r) = 2 \arctan(r) + \ell\pi$. This is called the *ground state*. Since it is co-rotational, this lays particular emphasis on the co-rotational case.

The well-posedness theory in the energy space for the k -equivariant case is much simpler than in the general case, and goes back to work of Shatah, Tahvildar and Zadeh [32] (1994). The issue of singularity formation was open for many years. In the equivariant case, blow-up can only occur at the origin and in an energy concentration scenario (bubbling). A pioneering work of Struwe [35] (2003) showed that this can only happen by the bubbling-off of a non-constant harmonic map. Blow-up solutions were constructed by Krieger, Schlag and Tataru [26] (2008) ($k = 1$), Rodnianski and Sterbenz [30] (2010) ($k \geq 4$), and Raphaël and Rodnianski [29] (2012) ($k \geq 1$). The threshold conjecture for general wave maps was proved by Sterbenz and Tataru [34] (2010): If the energy of the wave map is smaller than that of the ground state Q , the wave map exists for all times and scatters (has linear behavior as time goes to infinity). What can one say for energy above Q ? For general wave maps? For k -equivariant wave maps? For co-rotational wave maps?

An approach to these questions is suggested by the soliton resolution conjecture. This predicts that a global-in-time solution of a nonlinear dispersive equation evolves asymptotically, as time goes to infinity, as a sum of decoupled solitons (a coherent structure) and a radiative term (typically a solution to a linear equation). Solitons are traveling wave solutions, which are well localized and travel at constant (possibly zero) speed. For finite-time blow-up solutions, a similar decomposition should hold, depending on the nature of the blow-up.

This conjecture arose in the 1970s from numerical simulations and the theory of integrable equations. In a numerical simulation carried out at Los Alamos in the early 1950s (the birth of scientific computation), Fermi, Pasta and Ulam [18] encountered a puzzling paradox. In the mid 1960s, Kruskal [27] found an explanation for this paradox, from the existence of solitons for the Korteweg–de Vries (KdV) equation, $\partial_t u + \partial_x^3 u + u \partial_x u = 0$. After this discovery, Zabusky and Kruskal [39] (1965) conducted another influential numerical simulation, which indicated the emergence of solitons and multisolitons (a superposition of solitons) for the KdV equation. This simulation led to the soliton resolution conjecture, and to the theory of integrable nonlinear equations, to explain the observed elastic collision of solitons. Integrable nonlinear equations can be solved by a reduction to a collection of linear problems. This is an important class, but non-generic. Soliton resolution has been proved in a few integrable cases like KdV. The proofs are challenging, with issues still unresolved. There are also results for non-integrable cases, in perturbative regimes near solitons. The phenomenon seems to be universal. It has been observed numerically and experimentally, for instance, in dynamics of gas bubbles in a compressible fluid and in the formation of black holes in gravitational collapse. The mechanism for relaxation to a coherent structure, observed numerically and experimentally, is the radiation of excess energy to spatial infinity. Proving this in non-integrable settings is a major goal in the area of nonlinear dispersive and wave equations.

A far-reaching conjecture in general relativity, the final state conjecture, states that the evolution of general sets of initial values for the Einstein vacuum field equations leads to the superposition of a finite number of Kerr black holes (the solitons) plus a decaying radiative term. Currently, this is out of reach, but it motivates the corresponding problem for wave maps.

In the last twelve years, in collaboration with T. Duyckaerts and F. Merle, and also with H. Jia, Y. Martel, R. Côte, A. Lawrie, and W. Schlag, we have been studying these issues for wave maps, and for the closely connected energy-critical nonlinear wave equation

$$\begin{cases} \partial_t u - \Delta u = |u|^{4/(N-2)}u, & x \in \mathbb{R}^N, t \in \mathbb{R}; \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N); \\ \partial_t u|_{t=0} = u_1 \in L^2(\mathbb{R}^N). \end{cases} \tag{NLW}$$

The conserved energy does not have a sign! For this, Duyckaerts, Kenig and Merle [9] introduced the *energy channel method* to give a mathematical way to measure the radiation of excess energy to spatial infinity. For the connection between radial (NLW) and co-rotational wave maps $\partial_t^2 \psi - \partial_r^2 \psi + \frac{\sin(2\psi)}{2r^2} = 0$, choose $N = 4$, so that $|u|^{4/(N-2)}u = u^3$, and let

$$u(r, t) = \sqrt{\frac{2}{3}} \frac{\psi(r, t)}{r}.$$

Then u satisfies

$$\partial_t^2 u - \partial_r^2 u - \frac{3}{r} \partial_r u = \frac{1}{r^3} \Lambda(ru),$$

where $\Lambda(U) = U - \frac{\sin(\sqrt{6}U)}{\sqrt{6}}$. Λ is odd, smooth, and $\Lambda(U) \sim U^3$ for U small. The radial stationary solutions for (NLW) in $\mathbb{R} \times \mathbb{R}^N$ are $\pm \lambda^{-(N-2)/2} W(\frac{x}{\lambda})$, where $W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-(N-2)/2}$. The first breakthrough result for (NLW) was in the radial case, $N = 3$. Duyckaerts, Kenig and Merle [11] (2013) obtained the decomposition with $\pm W_\lambda = \lambda^{-(N-2)/2} W(\frac{x}{\lambda})$, $N = 3$, as the solitons, for all radial solutions when $T_+ = \infty$, and when $T_+ < \infty$, all radial solutions u with

$$\sup_{0 < t < T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty. \tag{\dagger}$$

The key point of the proof was: If u is a radial solution of (NLW), $N = 3$, $u \not\equiv 0, \pm W_\lambda$, and $t \in (-\infty, \infty)$, then there exist $R > 0$ and $\eta > 0$ such that, for all $t > 0$ or for all $t \leq 0$,

$$\int_{|x| \geq |t|+R} |\nabla_{x,t} u(x, t)|^2 dx \geq \eta \tag{Energy Channel}. \tag{1}$$

The relation (1) captures the observed radiation of energy to spatial infinity. The proof relies on the outer energy lower bound for radial solutions v of the linear

wave equation on $\mathbb{R} \times \mathbb{R}^3$: for all $t \geq 0$ or for all $t \leq 0$, we have

$$\int_{|x| > |t| + r_0} |\nabla_{x,t} v|^2 \, dx \geq \frac{1}{2} \|\Pi_{r_0}^\perp(v_0, v_1)\|_{\dot{H}^1 \times L^2(r, r_0)}^2 \tag{2}$$

([9], 2009), where $\Pi_{r_0}^\perp$ is the projection onto the orthogonal complement of $P_{r_0} = \{(ar^{-1}, 0) : a \in \mathbb{R}, r > r_0\} \subset \dot{H}^1\{r > r_0\}$. To use (2) in the nonlinear case (1), a key point is that $W(r) = \left(\frac{1}{1+r^2/6}\right)^{1/2} \sim c/r$ for r large. The relation (2) has a non-radial variant, when $r_0 = 0$, valid for all odd dimensions N ([10], 2012): for all $t \geq 0$ or for all $t \leq 0$,

$$\int_{|x| \geq |t|} |\nabla_{x,t} v|^2 \, dx \geq \frac{1}{2} \int |\nabla v_0|^2 + v_1^2. \tag{3}$$

Relations (2) and (3) fail for all even dimensions (Côte, Kenig and Schlag [5], 2014). The relation (3) holds (radial case) for $(v_0, 0)$, $N = 4, 8, 12, \dots$, $(0, v_1)$, $N = 6, 10, 14, \dots$. The relation (2) fails in the non-radial case $N \geq 3$; an analog of (2) holds in the radial case, N odd, with P_{r_0} replaced by a higher dimensional exceptional subspace, with $\dim \rightarrow \infty$ as $N \rightarrow \infty$ (see [24, 25]). Thus the validity of (1) and the decomposition were in question in all cases other than $N = 3$, radial, for a long time. Nevertheless, weaker results were obtained. This was done by using monotonicity after time averaging, inspired by the harmonic map heat flow. This gave decompositions, but only for well-chosen sequences of time: in the radial case of (NLW), for all $N \geq 3$, assuming (\dagger) when $T_+ < \infty$, and assuming

$$\sup_{0 < t < \infty} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty \tag{\ddagger}$$

when $T_+ = \infty$ (for N odd, C. Rodriguez [31] 2014; for $N = 4$, Côte, Kenig, Lawrie and Schlag [4] 2014; for even N , Jia and Kenig [23] 2017). In the non-radial case of (NLW), $3 \leq N \leq 6$, decompositions were obtained by Duyckaerts, Jia, Kenig and Merle [6] (2016) assuming conditions (\dagger) and (\ddagger) .

Remark 1. The method of proof using monotonicity after time averaging cannot give the decomposition for all times. In the case of the harmonic map heat flow into a manifold M , Topping [38] (1997) showed that the decomposition for all times does not hold. It is conjectured that the resolution does hold for $M = S^2$. Topping’s example shows that the resolution can fail for certain target manifolds.

In a sequence of three papers [13, 14, 12] (2019), Duyckaerts, Kenig and Merle proved the full decomposition for radial (NLW) for all odd N , assuming (\dagger) and (\ddagger) , by showing the inelastic collision of solitons. Returning to wave maps into the sphere, note that here conditions (\dagger) and (\ddagger) automatically hold since the conserved energy is positive. The first works in the direction of soliton resolution were due to Côte, Kenig, Lawrie, and Schlag [2, 3] in 2015, who established the decomposition for co-rotational (1-equivariant) wave maps into S^2 , under an energy constraint that implied the decomposition had only one bubble. Côte [1] (2015) extended this to general data, but only for well-chosen sequences of times. Jia and Kenig [23]

(2017) extended this to all k -equivariant cases, still for well-chosen sequences of times. These works used monotonicity after time averages. Engelstein and Mendelson [17] (2020) showed that, when M is not the sphere, an example analogous to Topping’s [38] for the harmonic map heat flow also exists for wave maps. Recently, in the k -equivariant case, Jendrej and Lawrie [21] (2020) established the decomposition for all times when at most two bubbles appear. For general wave maps into the sphere, very little is known so far. Grinis [19] (2017) obtained a decomposition for well-chosen sequences of times, but the error term is only small in much weaker norms than the energy norm. Duyckaerts, Jia, Kenig and Merle [7] (2018) studied the case of data an epsilon bigger in energy than the ground state Q (the energy in the threshold conjecture), and established the decomposition in this case, when $T_+ < \infty$.

We next turn to our very recent result, establishing the full decomposition for co-rotational wave maps into S^2 . This is the first unconditional soliton resolution for a wave-type equation, in an energy-critical setting. Let

$$E = \left\{ (\psi_0, \psi_1) : \frac{1}{2} \int_0^\infty \psi_1^2(r) r \, dr + \frac{1}{2} \int_0^\infty (\partial_r \psi_0)^2 r \, dr + \frac{1}{2} \int_0^\infty \frac{\sin^2(\psi_0(r))}{r^2} r \, dr < \infty \right\}.$$

Recall that $\partial_t^2 \psi - \partial_r^2 \psi - (\partial_r \psi)/r + \sin(2\psi)/2r^2 = 0$, and that $E(\psi_0, \psi_1) = E(\psi, \partial_t \psi)$. It is easy to see that if $(\psi_0, \psi_1) \in E$, then there exists $(\ell, m) \in \mathbb{Z}^2$ such that

$$\lim_{r \rightarrow 0} \psi_0(r) = \ell\pi \quad \text{and} \quad \lim_{r \rightarrow \infty} \psi_0(r) = m\pi. \tag{*}$$

We let $H_{\ell,m} = \{(\psi_0, \psi_1) \in E : (*) \text{ holds}\}$, which is an affine space in $H = H_{0,0} = \{(\psi_0, \psi_1) \in E : \|(\psi_0, \psi_1)\|_H < \infty\}$, where

$$\|(\psi_0, \psi_1)\|_H^2 = \int_0^\infty \left[(\partial_r \psi_0)^2 + \frac{(\psi_0(r))^2}{r^2} \right] r \, dr + \int_0^\infty \psi_1^2 r \, dr < \infty.$$

Note that $Q(r) = 2 \arctan(r) \in H_{0,1}$. All other stationary solutions are $\ell\pi \pm Q(\lambda r)$.

Theorem 1 (Duyckaerts, Kenig, Martel, Merle [8], 2021). *Let ψ be a co-rotational wave map in the energy space E , with maximal time of existence T_+ . Then one of the following holds (with ‘ \ll ’ meaning that the ratio goes to 0):*

- (i) **Blow-up:** $T_+ < \infty$, and there exist
 - $(\phi_0, \phi_1) \in E$ and
 - $J \in \mathbb{N} \setminus \{0\}$ such that for all $j \in \{1, \dots, J\}$ there exists a sign $i_j \in \{\pm 1\}$ and $0 < \lambda_j(t)$ defined near T_+ such that $\lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_J(t) \ll (T_+ - t)$ as $t \rightarrow T_+$

such that

$$\lim_{t \rightarrow T_+} \left\| (\psi(t), \partial_t \psi(t)) - \left(\phi_0 + \sum_{j=1}^J i_j Q \left(\frac{\cdot}{\lambda_j(t)} \right), \phi_1 \right) \right\|_H = 0.$$

(ii) **Global solution:** $T_+ = \infty$, and there exist

- $\ell \in \mathbb{Z}$,
- a solution ψ_L of $\partial_t^2 \psi_L - \partial_r^2 \psi_L - \frac{\partial_r \psi_L}{r} - \frac{\psi_L}{r^2} = 0$ with initial data in H , and
- $J \in \mathbb{N}$ such that, for all $j \in \{1, \dots, J\}$, there exist $i_j \in \{\pm 1\}$, $0 < \lambda_j(t)$ such that $\lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_J(t) \ll t$ as $t \rightarrow \infty$

such that

$$\lim_{t \rightarrow \infty} \left\| \left(\psi, \partial_t \psi \right) - \left(\psi_L + \ell \pi + \sum_{j=1}^J i_j Q \left(\frac{\cdot}{\lambda_j} \right), \partial_t \psi_L \right) \right\|_H = 0.$$

Remark 2. If $T_+ < \infty$, $(\psi_0, \psi_1) \in H_{\ell, m}$, and $(\phi_0, \phi_1) \in H_{\ell', m'}$, then $\ell = \ell'$ and $m = m' + \sum_{j=1}^J i_j$. If $T_+ = \infty$, then $(\psi_0, \psi_1) \in H_{\ell, m}$, where ℓ is as in (ii) and $m = \ell + \sum_{j=1}^J i_j$.

Theorem 1 follows from the next result, which is a rigidity theorem analog of (1).

Theorem 2. Let $(\psi_0, \psi_1) \in E$ (so that $(\psi_0, \psi_1) \in H_{\ell, m}$). Assume that (ψ_0, ψ_1) is not a stationary solution, i.e., if $\ell = m$ then $(\psi_0, \psi_1) \neq (m\pi, 0)$, and otherwise for all $\lambda > 0$, $(\psi_0 - m\pi, \psi_1) \neq (\pm(\pi - Q(\lambda)), 0)$. Then there exists $\eta > 0$ such that, for all $t > 0$ or for all $t \leq 0$,

$$\int_{|t|}^{\infty} \left([\partial_t \psi(t, r)]^2 + [\partial_r \psi(t, r)]^2 + \frac{\sin^2(\psi(t, r))}{r^2} \right) r \, dr > \eta.$$

Corresponding results for (NLW), $N = 4$, were also proved in the same paper [8]. Even more recently, Jendrej and Lawrie [22] extended Theorem 1 (but not Theorem 2) to the k -equivariant case with $k > 1$.

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