ON THE ENESTRÖM–KAKEYA THEOREM AND ITS VARIOUS FORMS IN THE QUATERNIONIC SETTING

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ABSTRACT. We study the extensions of the classical Eneström–Kakeya theorem and its various generalizations regarding the distribution of zeros of polynomials from the complex to the quaternionic setting. We aim to build upon the previous work by various authors and derive zero-free regions of some special regular functions of a quaternionic variable with restricted coefficients, namely quaternionic coefficients whose real and imaginary components or moduli of the coefficients satisfy suitable inequalities. The obtained results for this subclass of polynomials and slice regular functions produce generalizations of a number of results known in the literature on this subject.

1. INTRODUCTION AND PRELIMINARIES

The study of the distribution of zeros of polynomials and related analytic functions in geometric function theory is a problem of interest both in mathematics and in application areas such as physical systems. In addition to having numerous applications, this study has been the inspiration for much further research from both theoretical and practical perspectives. Since the zeros of a polynomial are continuous functions of its coefficients, in general it is quite complicated to derive bounds on the norm of zeros of a general algebraic polynomial. Therefore, in order to attain better and sharp zero bounds, it is desirable to put some restrictions on the coefficients of the polynomial. In this connection, we state the following elegant result concerning the distribution of zeros of a polynomial when its coefficients are restricted, known in the literature as the Eneström–Kakeya theorem [12].

Theorem 1.1 (Eneström–Kakeya theorem). *If* $T(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n (where z is a complex variable) with real coefficients and satisfying*

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $T(z)$ lie in $|z| \leq 1$.

The above classical result is particularly important in the study of stability of numerical methods for differential equations and is the starting point of a rich literature concerning its extensions, generalizations and improvements in several directions; see, e.g. the papers [1], [3], [10], to mention only a few. For an exhaustive survey of its extensions and refinements, we refer the reader to the comprehensive books of Marden [12], Milovanović et al. [13] and Gardner and Taylor [5]. We get the following equivalent form of Theorem 1.1 by applying it to the polynomial $z^n T\left(\frac{1}{z}\right)$.

Theorem 1.2. *If* $T(z) = \sum\limits_{v=0}^{n} a_v z^v$ *is a polynomial of degree n (where z is a complex variable) with real coefficients and satisfying*

$$a_0 \geq a_1 \geq \ldots \geq a_{n-1} \geq a_n > 0,$$

then $T(z)$ *does not vanish in* $|z| < 1$.

From the above results on polynomials, analogues specifying a zero-free disk of a power series analytic in a given region can be deduced. The extension of Theorem 1.2 to a class of related analytic functions was established by Aziz and Mohammad [1] in the form of the following result.

Theorem 1.3. *Let* $f(z) = \sum\limits_{v=0}^{\infty} a_v z^v \not\equiv 0$ *be analytic in* $|z| \leq t$, $t > 0$. *If*

$$a_v > 0 \quad and \quad a_{v-1} - t a_v \geq 0, \ v = 1, 2, 3, \ldots,$$

then $f(z)$ *does not vanish in* $|z| < t$.

The goal of this paper is to extend the above results and their various generalizations in the quaternionic setting. We begin with some preliminaries on quaternions and regular functions of a quaternionic variable which will be useful in what follows. Quaternions are essentially a generalization of complex numbers to four dimensions (one real and three imaginary parts) and were first studied and developed by Sir William Rowan Hamilton in 1843. The number system of quaternions is denoted by $\mathbb{H}$ in his honor. This theory of quaternions is by now very well developed in many different directions, and we refer the reader to [17] for the basic features of quaternionic functions. The set of quaternions is a noncommutative division ring. It consists of elements of the form $q = \alpha + \beta i + \gamma j + \delta k$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, where the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Every element $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$ is composed of the real part $\mathrm{Re}(q) = \alpha$ and the imaginary part $\mathrm{Im}(q) = \beta i + \gamma j + \delta k$. The conjugate of q is denoted by $\overline{q}$ and is defined as $\overline{q} = \alpha - \beta i - \gamma j - \delta k$, and the norm of q is $|q| = \sqrt{q\overline{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The inverse of each nonzero element q of $\mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$. For $r > 0$, we define the ball $B(0, r) = \{q \in \mathbb{H} : |q| < r\}$. By $\mathbb{B}$ we denote the open unit ball in $\mathbb{H}$ centered at the origin, i.e.,

$$\mathbb{B} = \{q = \alpha + \beta i + \gamma j + \delta k : \alpha^2 + \beta^2 + \gamma^2 + \delta^2 < 1\},$$

and by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{q = \beta i + \gamma j + \delta k : \beta^2 + \gamma^2 + \delta^2 = 1\}.$$

The functions we consider in this paper are slice regular functions as polynomials of the form

$$T(q) = \sum_{v=0}^{n} q^v a_v$$

and power series of the form

$$f(q) = \sum_{v=0}^{\infty} q^v a_v$$

of the quaternionic variable q on the left and with quaternionic coefficients a_v on the right.

Two quaternionic polynomials of this kind can be multiplied according to the convolution product (Cauchy multiplication rule): given $T_1(q) = \sum_{i=0}^{n} q^i a_i$ and $T_2(q) = \sum_{j=0}^{m} q^j b_j$, we define

$$(T_1 * T_2)(q) := \sum_{\substack{i=0,1,\ldots,n \\ j=0,1,\ldots,m}} q^{i+j} a_i b_j.$$

If T_1 has real coefficients, the so-called $*$ multiplication coincides with the usual pointwise multiplication. Notice that the $*$ product is associative and not, in general, commutative. Given two quaternionic power series $f(q) = \sum_{v=0}^{\infty} q^v a_v$ and $g(q) = \sum_{v=0}^{\infty} q^v b_v$ with radii of convergence greater than R, we define the regular product of f and g as the series

$$(f * g)(q) = \sum_{v=0}^{\infty} q^v c_v,$$

where $c_v = \sum_{k=0}^{v} a_k b_{v-k}$ for all v. Further, as observed in [4], [7], for each quaternionic power series $f(q) = \sum_{v=0}^{\infty} q^v a_v$ there exists a ball $B(0,R) = \{q \in \mathbb{H} : |q| < R\}$ such that f converges absolutely and uniformly on each compact subset of $B(0,R)$ and where the sum function of f is regular.

The regular functions of a quaternionic variable have been introduced and intensively studied in the past decade; they have proven to be a fertile topic in analysis, and their rapid development has been largely driven by the applications to operator theory. In the preliminary steps, the structure of the zero sets of a quaternionic regular function and the factorization property of zeros was described. In this regard, Gentili and Stoppato [7] (see also [9]) gave a necessary and sufficient condition for a regular quaternionic power series to have a zero at a point in the form of the following result.

Theorem 1.4. *Let* $f(q) = \sum\limits_{v=0}^{\infty} q^v a_v$ *be a quaternionic power series with radius of convergence* R, *and let* $p \in B(0, R)$. *Then* $f(p) = 0$ *if and only if there exists a quaternionic power series* $g(q)$ *with radius of convergence* R *such that*

$$f(q) = (q - p) * g(q).$$

This extends to quaternionic power series the theory presented in [11] for polynomials. The following result, which completely describes the zero sets of a regular product of two polynomials in terms of the zero sets of the two factors, is from [11] (see also [9] and [7]).

Theorem 1.5. *Let* f *and* g *be quaternionic polynomials. Then* $(f * g)(q_0) = 0$ *if and only if* $f(q_0) = 0$ *or* $f(q_0) \neq 0$ *implies* $g\left(f(q_0)^{-1} q_0 f(q_0)\right) = 0$.

Gentili and Struppa [8] introduced a maximum modulus theorem for regular functions, which includes convergent power series and polynomials, in the form of the following result.

Theorem 1.6 (Maximum modulus theorem)**.** *Let* $B = B(0, r)$ *be a ball in* $\mathbb{H}$ *with center* 0 *and radius* $r > 0$, *and let* $f : B \to \mathbb{H}$ *be a regular function. If* $|f|$ *has a relative maximum at a point* $a \in B$, *then* f *is a constant on* B.

In [9], [8], [7] the structure of the zeros of polynomials was used and a topological proof of the fundamental theorem of algebra was established. We point out that the fundamental theorem of algebra for regular polynomials with coefficients in $\mathbb{H}$ was already proved by Niven (for reference, see [14], [15]) by using different techniques. This lead to the complete identification of the zeros of polynomials in terms of their factorization; for reference, see [16]. Thus it became an interesting perspective to think about the regions containing some or all the zeros of a regular polynomial of a quaternionic variable. Very recently, Carney et al. [2] extended the Eneström–Kakeya theorem and its various generalizations from complex polynomials to quaternionic polynomials by making use of Theorems 1.5 and 1.6. Firstly, they established the following quaternionic analogue of Theorem 1.1.

Theorem 1.7. *If* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a polynomial of degree* n *(where* q *is a quaternionic variable) with real coefficients and satisfying*

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $T(q)$ *lie in* $|q| \leq 1$.

In this form, the above theorem has been extensively studied and extended in various ways, even to quaternionic coefficients with restricted real and imaginary components. In the same paper, Carney et al. [2] established the following generalization of Theorem 1.7 to quaternionic coefficients.

Theorem 1.8. *If* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a polynomial of degree n (where q is a quaternionic variable) with quaternionic coefficients, where $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ for $v = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1 \geq \alpha_0,$$
$$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0,$$
$$\gamma_n \geq \gamma_{n-1} \geq \ldots \geq \gamma_1 \geq \gamma_0,$$
$$\delta_n \geq \delta_{n-1} \geq \ldots \geq \delta_1 \geq \delta_0,$$

then all the zeros of $T(q)$ lie in

$$|q| \leq \frac{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)}{|a_n|}.$$

In the meantime, Tripathi ([18, Theorem 3.1]) established the following generalization of Theorem 1.8.

Theorem 1.9. *Let* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *be a polynomial of degree n (where q is a quaternionic variable) with quaternionic coefficients, where $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ for $v = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_l,$$
$$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_l,$$
$$\gamma_n \geq \gamma_{n-1} \geq \ldots \geq \gamma_l,$$
$$\delta_n \geq \delta_{n-1} \geq \ldots \geq \delta_l$$

for $0 \leq l \leq n$. Then all the zeros of $T(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\Big[|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (\alpha_n - \alpha_l) + (\beta_n - \beta_l)$$
$$+ (\gamma_n - \gamma_l) + (\delta_n - \delta_l) + M_l\Big],$$

where

$$M_l = \sum_{v=1}^{l} \Big[|\alpha_v - \alpha_{v-1}| + |\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|\Big].$$

Very recently, Gardner and Taylor [6] established the following more general result giving a ring-shaped region containing all the zeros of a quaternionic polynomial with restricted real and imaginary components. As a consequence, it gives Theorem 1.8 and many related results as special cases.

Theorem 1.10. *Let $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ be a polynomial of degree n (where q is a quaternionic variable) with quaternionic coefficients, where $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ for $v = 0, 1, 2, \ldots, n$. If $a_n \neq 0$ and for some l, m, r, s and $t > 0$ the following relations hold:*

$$\alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^l\alpha_l \geq t^{l+1}\alpha_{l+1} \geq \cdots \geq t^n\alpha_n$$

$$\beta_0 \leq t\beta_1 \leq t^2\beta_2 \leq \cdots \leq t^m\beta_m \geq t^{m+1}\beta_{m+1} \geq \cdots \geq t^n\beta_n$$

$$\gamma_0 \leq t\gamma_1 \leq t^2\gamma_2 \leq \cdots \leq t^r\gamma_r \geq t^{r+1}\gamma_{r+1} \geq \cdots \geq t^n\gamma_n$$

$$\delta_0 \leq t\delta_1 \leq t^2\delta_2 \leq \cdots \leq t^s\delta_s \geq t^{s+1}\delta_{s+1} \geq \cdots \geq t^n\delta_n,$$

then $T(q)$ has all its zeros in $R_1 \leq |q| \leq R_2$, where

$$R_1 = \min\left\{ t\,|a_0|\,/\,\left(2\left(t^l\alpha_l + t^m\beta_m + t^r\gamma_r + t^s\delta_s\right) - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0)\right.\right.$$
$$\left.\left. -t^n\left(\alpha_n + \beta_n + \gamma_n + \delta_n - |a_n|\right)\right), t\right\}$$

and

$$R_2 = \max\left\{ \frac{1}{|a_n|}\left(|a_0|t^{n+1} - t^{n-1}(\alpha_0 + \beta_0 + \gamma_0 + \delta_0) - t(\alpha_n + \beta_n + \gamma_n + \delta_n)\right.\right.$$

$$+ (t^2 + 1)(t^{n-l-1}\alpha_l + t^{n-m-1}\beta_m + t^{n-r-1}\gamma_r + t^{n-s-1}\delta_s)$$

$$+ (t^2 - 1)\left(\sum_{j=1}^{l-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{m-1} t^{n-j-1}\beta_j + \sum_{j=1}^{r-1} t^{n-j-1}\gamma_j + \sum_{j=1}^{s-1} t^{n-j-1}\delta_j\right)$$

$$+ (1 - t^2)\left(\sum_{j=l+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=m+1}^{n-1} t^{n-j-1}\beta_j\right.$$

$$\left.\left.\left. + \sum_{j=r+1}^{n-1} t^{n-j-1}\gamma_j + \sum_{j=s+1}^{n-1} t^{n-j-1}\delta_j\right)\right), \frac{1}{t}\right\}.$$

We get the following generalization of Theorem 1.8 as a consequence of Theorem 1.10 for $t = 1$.

Theorem 1.11. *If $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ is a quaternionic polynomial of degree n, where $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ for $v = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_l \geq \alpha_{l-1} \geq \ldots \geq \alpha_1 \geq \alpha_0, \quad 0 \leq l \leq n$$

$$\beta_n \leq \beta_{n-1} \leq \ldots \leq \beta_m \geq \beta_{m-1} \geq \ldots \geq \beta_1 \geq \beta_0, \quad 0 \leq m \leq n$$

$$\gamma_n \leq \gamma_{n-1} \leq \ldots \leq \gamma_r \geq \gamma_{r-1} \geq \ldots \geq \gamma_1 \geq \gamma_0, \quad 0 \leq r \leq n$$

$$\delta_n \leq \delta_{n-1} \leq \ldots \leq \delta_s \geq \delta_{s-1} \geq \ldots \geq \delta_1 \geq \delta_0, \quad 0 \leq s \leq n,$$

then all the zeros of $T(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\left[(2\alpha_l - \alpha_n + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_n + |\beta_0| - \beta_0)\right.$$
$$\left. + (2\gamma_r - \gamma_n + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_n + |\delta_0| - \delta_0)\right].$$

From Theorem 1.11, we obtain the following generalization of Theorem 1.7 for polynomials with quaternionic coefficients.

Theorem 1.12. *If* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree n, where* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ *for* $v = 0, 1, 2, \ldots, n$, *satisfying*

$$0 < \alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_l \geq \alpha_{l-1} \geq \ldots \geq \alpha_1 \geq \alpha_0 \geq 0$$
$$0 \leq \beta_n \leq \beta_{n-1} \leq \ldots \leq \beta_m \geq \beta_{m-1} \geq \ldots \geq \beta_1 \geq \beta_0 \geq 0$$
$$0 \leq \gamma_n \leq \gamma_{n-1} \leq \ldots \leq \gamma_r \geq \gamma_{r-1} \geq \ldots \geq \gamma_1 \geq \gamma_0 \geq 0$$
$$0 \leq \delta_n \leq \delta_{n-1} \leq \ldots \leq \delta_s \geq \delta_{s-1} \geq \ldots \geq \delta_1 \geq \delta_0 \geq 0$$

for some nonnegative integers l, m, r and s, then all the zeros of $T(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} \Big[2(\alpha_l + \beta_m + \gamma_r + \delta_s) - (\alpha_n + \beta_n + \gamma_n + \delta_n) \Big].$$

It is natural to study the geometric properties of quaternionic polynomials and slice regular functions in general, as well as the regional location of their zeros. Slice regular functions are now a widely studied topic, important especially in replicating many properties of holomorphic functions of a complex variable. The main purpose of this paper is to extend various results of Eneström–Kakeya type from the complex to the quaternionic setting and to obtain zero-free regions of some special slice regular functions of a quaternionic variable with restricted coefficients. We shall make use of the recently established maximum modulus theorem (Theorem 1.6), the structure of the zero sets of the regular product of two polynomials (Theorem 1.5) and the factorization theorem (Theorem 1.4) to get the desired results. The obtained results also produce generalizations and refinements of Theorems 1.3, 1.11, 1.12 and many other related results.

2. Main results

In this section, we state our main results. Their proofs are given in the next section. We start with the following refinement of Theorem 1.11.

Theorem 2.1. *If* $T(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree n, where* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ *for* $v = 0, 1, 2, \ldots, n$, *satisfying*

$$\alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_l \geq \alpha_{l-1} \geq \ldots \geq \alpha_1 \geq \alpha_0, \quad 0 \leq l \leq n - 1$$
$$\beta_n \leq \beta_{n-1} \leq \ldots \leq \beta_m \geq \beta_{m-1} \geq \ldots \geq \beta_1 \geq \beta_0, \quad 0 \leq m \leq n - 1$$
$$\gamma_n \leq \gamma_{n-1} \leq \ldots \leq \gamma_r \geq \gamma_{r-1} \geq \ldots \geq \gamma_1 \geq \gamma_0, \quad 0 \leq r \leq n - 1$$
$$\delta_n \leq \delta_{n-1} \leq \ldots \leq \delta_s \geq \delta_{s-1} \geq \ldots \geq \delta_1 \geq \delta_0, \quad 0 \leq s \leq n - 1,$$

then all the zeros of $T(q)$ lie in

$$\left| q + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \Big[(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0)$$
$$+ (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0) \Big].$$

To show that Theorem 2.1 is an improvement of Theorem 1.11, we show that the region defined by Theorem 2.1 is contained in the region defined by Theorem 1.1.

Let q be any point belonging to the region defined by Theorem 2.1; then

$$\left| q + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \Big[(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0)$$
$$+ (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0) \Big].$$

Now

$$|q| \leq \left| q + \frac{a_{n-1}}{a_n} - 1 \right| + \left| \frac{a_n - a_{n-1}}{a_n} \right|$$

$$\leq \frac{1}{|a_n|} \Big[(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0)$$
$$+ (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0) \Big]$$
$$+ \frac{1}{|a_n|} \Big[(\alpha_{n-1} - \alpha_n) + (\beta_{n-1} - \beta_n) + (\gamma_{n-1} - \gamma_n) + (\delta_{n-1} - \delta_n) \Big]$$

$$= \frac{1}{|a_n|} \Big[(2\alpha_l - \alpha_n + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_n + |\beta_0| - \beta_0)$$
$$+ (2\gamma_r - \gamma_n + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_n + |\delta_0| - \delta_0) \Big],$$

which shows that the point q belongs to the region defined by Theorem 1.1.

We now turn to study the zero-free regions of some special slice regular functions of the form $\sum_{v=0}^{\infty} q^v a_v$ with restricted coefficients, regular in the ball $B(0, R)$, $R > 0$. In this direction, we first prove the following quaternionic analogue of Theorem 1.3.

Theorem 2.2. *Let $f : B(0, R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{v=0}^{\infty} q^v a_v$ for all $q \in B(0, R)$. If a_v, $v = 0, 1, 2, \ldots$, are real and positive satisfying*

$$a_0 \geq t a_1 \geq t^2 a_2 \geq \ldots,$$

where $0 < t < R$, then $f(q)$ does not vanish in $|q| < t$.

Instead of proving Theorem 2.2, we prove the following more general result which includes the above one as a consequence.

Theorem 2.3. *Let $f : B(0, R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{v=0}^{\infty} q^v a_v$ for all $q \in B(0, R)$. If a_v, $v = 0, 1, 2, \ldots$, are real and positive satisfying*

$$\lambda a_0 \geq t a_1 \geq t^2 a_2 \geq \ldots$$

for some $\lambda \geq 1$ and $0 < t < R$, then $f(q)$ does not vanish in

$$\left| q - \frac{(\lambda - 1)t}{2\lambda - 1} \right| < \frac{\lambda t}{2\lambda - 1}.$$

Remark 2.4. Taking $\lambda = 1$ in Theorem 2.3, we recover Theorem 2.2.

Finally, we shall prove the following result for slice regular power series with quaternionic coefficients when we have information about the modulli of the coefficients.

Theorem 2.5. *Let* $f : B(0, R) \to \mathbb{H}$ *be a regular power series in the quaternionic variable* q, *i.e.,* $f(q) = \sum\limits_{v=0}^{\infty} q^v a_v$ *for all* $q \in B(0, R)$. *Let* a_v, $v = 0, 1, 2, \ldots$, *be such that for some* $\lambda \geq 1$ *we have*

$$\lambda|a_0| \geq t|a_1| \geq t^2|a_2| \geq \ldots,$$

where $0 < t < R$. *Let* b *be a nonzero quaternion and suppose that* $\angle(a_v, b) \leq \theta \leq \frac{\pi}{2}$ *for some* θ *and for* $v = 0, 1, 2, \ldots$. *Then* $f(q)$ *does not vanish in*

$$\left| q - \frac{(\lambda - 1)t}{E^2 - (\lambda - 1)^2} \right| < \frac{Et}{E^2 - (\lambda - 1)^2},$$

where $E = \lambda(\cos\theta + \sin\theta) + \frac{2\sin\theta}{|a_0|} \sum\limits_{v=1}^{\infty} |a_v| t^v$.

Remark 2.6. Taking $\theta = 0$ and assuming b to be a positive real number in Theorem 2.5, we recover Theorem 2.3.

3. Proofs of the main results

Proof of Theorem 2.1. Consider the polynomial

$$T(q) * (1 - q) = a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \ldots + q^n(a_n - a_{n-1}) - q^{n+1}a_n$$
$$= \psi(q) + q^n(a_n - a_{n-1}) - q^{n+1}a_n,$$

where $\psi(q) = a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \ldots + q^{n-1}(a_{n-1} - a_{n-2})$.

By Theorem 1.5, $T(q) * (1 - q) = 0$ if and only if either $T(q) = 0$ or $T(q) \neq 0$ implies $T(q)^{-1}qT(q) - 1 = 0$, that is $T(q)^{-1}qT(q) = 1$. Thus, if $T(q) \neq 0$, this implies $q = 1$, so the only zero of $T(q) * (1 - q)$ are $q = 1$ and the zeros of $T(q)$.

We first note that

$$|a_v - a_{v-1}| = |(\alpha_v - \alpha_{v-1}) + (\beta_v - \beta_{v-1})i + (\gamma_v - \gamma_{v-1})j + (\delta_v - \delta_{v-1})k|$$
$$\leq |\alpha_v - \alpha_{v-1}| + |\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|.$$

For $|q| = 1$, we have

$$
\begin{aligned}
|\psi(q)| &\le |a_0| + \sum_{v=1}^{n-1} |a_v - a_{v-1}| \\
&\le |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| \\
&\quad + \sum_{v=1}^{n-1} \left(|\alpha_v - \alpha_{v-1}| + |\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}| \right) \\
&= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{v=1}^{l} (\alpha_v - \alpha_{v-1}) + \sum_{v=l+1}^{n-1} (\alpha_{v-1} - \alpha_v) \\
&\quad + \sum_{v=1}^{m} (\beta_v - \beta_{v-1}) + \sum_{v=m+1}^{n-1} (\beta_{v-1} - \beta_v) + \sum_{v=1}^{r} (\gamma_v - \gamma_{v-1}) \\
&\quad + \sum_{v=r+1}^{n-1} (\gamma_{v-1} - \gamma_v) + \sum_{v=1}^{s} (\delta_v - \delta_{v-1}) + \sum_{v=s+1}^{n-1} (\delta_{v-1} - \delta_v) \\
&= 2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0 + 2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0 \\
&\quad + 2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0 + 2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0.
\end{aligned}
$$

Notice that we have

$$
\max_{|q|=1} \left| q^n * \psi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} \left| q^n \psi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} \left| \psi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} |\psi(q)|.
$$

It is clear that $q^n * \psi\left(\frac{1}{q}\right)$ has the same bound on $|q| = 1$ as ψ, that is,

$$
\begin{aligned}
\left| q^n * \psi\left(\frac{1}{q}\right) \right| &\le (2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0) \\
&\quad + (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0) \quad \text{for } |q| = 1.
\end{aligned}
$$

Since $q^n * \psi\left(\frac{1}{q}\right)$ is a polynomial and hence is regular in $|q| \le 1$, it follows by the maximum modulus theorem (Theorem 1.6), that

$$
\begin{aligned}
\left| q^n * \psi\left(\frac{1}{q}\right) \right| = \left| q^n \psi\left(\frac{1}{q}\right) \right| &\le (2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0) \\
&\quad + (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0) \quad \text{for } |q| \le 1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left| \psi\left(\frac{1}{q}\right) \right| &\le \frac{1}{|q^n|} \Big[(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0) \\
&\quad + (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0) \Big] \quad \text{for } |q| \le 1.
\end{aligned}
$$

Replacing q by $\frac{1}{q}$, we see that

$$|\psi(q)| \leq \Big[(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0)$$

$$+ (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0)\Big]|q|^n \quad \text{for } |q| \geq 1.$$

$$(3.1)$$

For $|q| \geq 1$, we have

$$|T(q) * (1-q)| = \big|\psi(q) + q^n(a_n - a_{n-1}) - q^{n+1}a_n\big|$$

$$\geq |q|^n |a_n| \left|q + \frac{a_{n-1}}{a_n} - 1\right| - |\psi(q)|$$

$$\geq |q|^n |a_n| \left[\left|q + \frac{a_{n-1}}{a_n} - 1\right| - \frac{1}{|a_n|}\big\{(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0)\right.$$

$$+ (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0) + (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0)$$

$$\left. + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0)\big\}\right] \quad \text{(by (3.1))}.$$

Hence, if

$$\left|q + \frac{a_{n-1}}{a_n} - 1\right| > \frac{1}{|a_n|}\big[(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0)$$

$$+ (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0)\big],$$

then $|T(q) * (1-q)| > 0$, that is $T(q) * (1-q) \neq 0$. Therefore, it follows that all the zeros of $T(q) * (1-q)$ whose norm is greater than or equal to one lie in

$$\left|q + \frac{a_{n-1}}{a_n} - 1\right| \leq \frac{1}{|a_n|}\big[(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0)$$

$$+ (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0)\big].$$

$$(3.2)$$

We now show that all the zeros of $T(q) * (1-q)$ whose norm is less than or equal to one also satisfy (3.2). Let $q \in \mathbb{H}$ be such that $|q| \leq 1$; then

$$\left|q + \frac{a_{n-1}}{a_n} - 1\right| \leq 1 + \left|\frac{a_{n-1}}{a_n} - 1\right|$$

$$= \frac{|a_n| + |a_{n-1} - a_n|}{|a_n|}$$

$$\leq \frac{1}{|a_n|}\big[\alpha_n + \beta_n + \gamma_n + \delta_n + \alpha_{n-1} - \alpha_n + \beta_{n-1} - \beta_n + \gamma_{n-1} - \gamma_n$$

$$+ \delta_{n-1} - \delta_n + |\alpha_0| - \alpha_0 + |\beta_0| - \beta_0 + |\gamma_0| - \gamma_0 + |\delta_0| - \delta_0\big]$$

$$= \frac{1}{|a_n|}\big[\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} + \delta_{n-1}$$

$$+ |\alpha_0| - \alpha_0 + |\beta_0| - \beta_0 + |\gamma_0| - \gamma_0 + |\delta_0| - \delta_0\big]. \quad (3.3)$$

Now, by hypothesis,

$$\alpha_{n-1} \leq \alpha_l, \qquad 0 \leq l \leq n-1,$$
$$\beta_{n-1} \leq \beta_m, \qquad 0 \leq m \leq n-1,$$
$$\gamma_{n-1} \leq \gamma_r, \qquad 0 \leq r \leq n-1,$$
$$\delta_{n-1} \leq \delta_s, \qquad 0 \leq s \leq n-1;$$

therefore, we have

$$2(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} + \delta_{n-1}) \leq 2(\alpha_l + \beta_m + \gamma_r + \delta_s).$$

Equivalently,

$$\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} + \delta_{n-1}$$
$$\leq 2(\alpha_l + \beta_m + \gamma_r + \delta_s) - (\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} + \delta_{n-1}).$$

Using this in (3.3), we get

$$\left| q + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \Big[2(\alpha_l + \beta_m + \gamma_r + \delta_s) - (\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} + \delta_{n-1})$$
$$+ (|\alpha_0| - \alpha_0) + (|\beta_0| - \beta_0) + (|\gamma_0| - \gamma_0) + (|\delta_0| - \delta_0) \Big],$$

which is exactly the region defined by (3.2).

Since the only zeros of $T(q) * (1-q)$ are $q = 1$ and the zeros of $T(q)$, we have that $T(q) \neq 0$ for

$$\left| q + \frac{a_{n-1}}{a_n} - 1 \right| > \frac{1}{|a_n|} \Big[(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0)$$
$$+ (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0) \Big].$$

In other words, all the zeros of $T(q)$ lie in

$$\left| q + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \Big[(2\alpha_l - \alpha_{n-1} + |\alpha_0| - \alpha_0) + (2\beta_m - \beta_{n-1} + |\beta_0| - \beta_0)$$
$$+ (2\gamma_r - \gamma_{n-1} + |\gamma_0| - \gamma_0) + (2\delta_s - \delta_{n-1} + |\delta_0| - \delta_0) \Big].$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.3. Consider the power series

$$F(q) = (t - q) * f(q)$$
$$= (t - q) * (a_0 + qa_1 + q^2 a_2 + \dots)$$
$$= ta_0 - \{ q(a_0 - ta_1) + q^2(a_1 - ta_2) + \dots \}$$
$$= ta_0 - qa_0 + q\lambda a_0 - q\psi(q),$$

where $\psi(q) = (\lambda a_0 - ta_1) + \sum_{v=2}^{\infty} q^{v-1}(a_{v-1} - ta_v)$.

For $|q| = t$, we have

$$|\psi(q)| \leq |\lambda a_0 - ta_1| + \sum_{v=2}^{\infty} |q|^{v-1} |a_{v-1} - ta_v|$$

$$= (\lambda a_0 - ta_1) + \sum_{v=2}^{\infty} t^{v-1}(a_{v-1} - ta_v)$$

$$= \lambda a_0.$$

Since $\psi(q)$ is regular in $|q| \leq t$, it follows by the maximum modulus theorem (Theorem 1.6), that

$$|\psi(q)| \leq \lambda a_0 \quad \text{for } |q| \leq t. \tag{3.4}$$

For $|q| \leq t$, we have

$$|F(q)| = |ta_0 - qa_0 + q\lambda a_0 - q\psi(q)|$$

$$\geq |ta_0 - qa_0 + q\lambda a_0| - |q||\psi(q)|$$

$$\geq |a_0|[|q(\lambda - 1) + t| - |q|\lambda] \quad \text{(by (3.4))}$$

$$> 0$$

if

$$|q|\lambda < |q(\lambda - 1) + t|,$$

i.e., if

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - \frac{2t(\lambda - 1)\alpha}{2\lambda - 1} < \frac{t^2}{2\lambda - 1},$$

or

$$\left[\alpha - \left(\frac{\lambda - 1}{2\lambda - 1}\right)t\right]^2 + \beta^2 + \gamma^2 + \delta^2 < \left(\frac{\lambda t}{2\lambda - 1}\right)^2,$$

which is precisely the disk

$$\left\{ q : \left| q - \left(\frac{\lambda - 1}{2\lambda - 1}\right)t \right| < \frac{\lambda t}{2\lambda - 1} \right\}. \tag{3.5}$$

Since by Theorem 1.4, the only zeros of $(t - q) * f(q)$ are $q = t$ and the zeros of $f(q)$, it follows that $f(q)$ does not vanish in the disk defined by (3.5). This proves Theorem 2.3.

We need the following auxiliary result due to Carney et al. [2] for the proof of Theorem 2.5.

Lemma 3.1. *Let $q_1, q_2 \in \mathbb{H}$ with $q_1 = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k$ and $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$, $\measuredangle(q_1, q_2) = 2\theta' \leq 2\theta$ and $|q_1| \leq |q_2|$. Then*

$$|q_2 - q_1| \leq (|q_2| - |q_1|)\cos\theta + (|q_2| + |q_1|)\sin\theta.$$

Proof of Theorem 2.5. Again consider the power series

$$(t - q) * f(q) = (t - q) * (a_0 + qa_1 + q^2 a_2 + \dots)$$
$$= ta_0 - qa_0 + q\lambda a_0 - q\psi(q),$$

where $\psi(q) = (\lambda a_0 - ta_1) + \sum_{v=2}^{\infty} q^{v-1}(a_{v-1} - ta_v)$. For $|q| = t$, we have

$$|\psi(q)| \leq |\lambda a_0 - ta_1| + \sum_{v=2}^{\infty} |q|^{v-1}|a_{v-1} - ta_v|$$

$$\leq (\lambda|a_0| - t|a_1|)\cos\theta + (\lambda|a_0| + t|a_1|)\sin\theta$$

$$+ \sum_{v=2}^{\infty} t^{v-1}\{(|a_{v-1}| - t|a_v|)\cos\theta + (|a_{v-1}| + t|a_v|)\sin\theta\} \quad \text{(by Lemma 3.1)}$$

$$= \lambda|a_0|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{v=1}^{\infty} |a_v|t^v$$

$$= |a_0|E,$$

where $E = \lambda(\cos\theta + \sin\theta) + \frac{2\sin\theta}{|a_0|} \sum_{v=1}^{\infty} |a_v|t^v$.

Now, proceeding similarly as in the proof of Theorem 2.3, it follows that

$$|F(q)| = |(t - q) * f(q)| > 0$$

if

$$|q|E < |q(\lambda - 1) + t|,$$

i.e., if

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - \frac{2t(\lambda - 1)\alpha}{E^2 - (\lambda - 1)^2} < \frac{t^2}{E^2 - (\lambda - 1)^2},$$

or

$$\left[\alpha - \frac{(\lambda - 1)t}{E^2 - (\lambda - 1)^2}\right]^2 + \beta^2 + \gamma^2 + \delta^2 < \left(\frac{Et}{E^2 - (\lambda - 1)^2}\right)^2,$$

which is precisely the disk

$$\left\{q : \left|q - \frac{(\lambda - 1)t}{E^2 - (\lambda - 1)^2}\right| < \frac{Et}{E^2 - (\lambda - 1)^2}\right\}. \tag{3.6}$$

Since by Theorem 1.4, the only zeros of $(t-q)*f(q)$ are $q = t$ and the zeros of $f(q)$, it follows that $f(q)$ does not vanish in the disk defined by (3.6). This completes the proof of Theorem 2.5.

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