ASYMPTOTIC MEAN VALUE FORMULAS FOR PARABOLIC NONLINEAR EQUATIONS

PABLO BLANC, FERNANDO CHARRO, JUAN J. MANFREDI, AND JULIO D. ROSSI

ABSTRACT. In this paper we characterize viscosity solutions to nonlinear parabolic equations (including parabolic Monge–Ampère equations) by asymptotic mean value formulas. Our asymptotic mean value formulas can be interpreted from a probabilistic point of view in terms of dynamic programming principles for certain two-player, zero-sum games.

1. INTRODUCTION

1.1. **Asymptotic mean value formulas for elliptic equations.** It is a well-known classical fact that a function u is harmonic (u is a solution to $\Delta u = \operatorname{trace}(D^2u) = 0$) in a domain $\Omega \subset \mathbb{R}^n$ if and only if u satisfies the mean value property

$$u(x) = \fint_{B_\varepsilon(x)} u(y)\, dy \tag{1.1}$$

for each $x \in \Omega$ and all $0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)$. In fact, a weaker statement, an asymptotic version of the mean value property, suffices to characterize harmonic functions. A continuous function u is harmonic in Ω if and only if

$$u(x) = \fint_{B_\varepsilon(x)} u(y)\, dy + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0; \tag{1.2}$$

see [7, 23, 33]. Moreover, the mean value property can be used to characterize sub- and superharmonic functions replacing the equality by the appropriate inequality in (1.1) and (1.2). A discrete version of the asymptotic mean value property also

holds: A continuous function u is harmonic if and only if

$$u(x) = \frac{1}{n} \sum_{j=1}^{n} \left\{ \frac{1}{2} u(x + \varepsilon e_j) + \frac{1}{2} u(x - \varepsilon e_j) \right\} + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0,$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. There are many other mean value formulas for linear elliptic operators other than the Laplacian (see [27]) and for degenerate elliptic equations (see [8]).

In recent years asymptotic mean value formulas were found for nonlinear operators such as the normalized (also called homogeneous) p-Laplacian,

$$\Delta_p^N u = |\nabla u|^{2-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = \Delta u + (p-2)\,\Delta_\infty^N u,$$

for $1 < p < \infty$. These mean value formulas come from the connection between probability (via the dynamic programming principle for tug-of-war games) and the normalized infinity Laplacian; see [24, 25, 31]. A nonlinear mean value property for p-harmonic functions first appeared in [28] motivated by the random tug-of-war games with noise in [32]. It was proved in [28] that p-harmonic functions are characterized by the fact that they satisfy the following asymptotic mean value formula:

$$u(x) = \left(\frac{p-2}{p+n}\right)\left(\frac{\max\limits_{B_\varepsilon(x)} u + \min\limits_{B_\varepsilon(x)} u}{2}\right) + \left(\frac{2+n}{p+n}\right) \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0,$$

$$(1.3)$$

in the viscosity sense. The asymptotic mean value formula (1.3) holds in the viscosity sense if whenever a smooth test function with non-vanishing gradient touches u from above (respectively below) at a point x, the mean value formula (1.3) is satisfied with $\leq$ (respectively $\geq$) for the test function at x. This is weaker than requiring the asymptotic formula to hold in the classical sense, yet enough to characterize p-harmonic functions. For mean value properties for the p-Laplacian in the Heisenberg group see [16], and for the standard variational p-Laplacian see [14]. See also [2] and the recently published book [5] for historical references and more general equations.

It is worth noting that the expression

$$\left(\frac{p-2}{p+n}\right)\left(\frac{\max\limits_{B_\varepsilon(x)} u + \min\limits_{B_\varepsilon(x)} u}{2}\right) + \left(\frac{2+n}{p+n}\right) \fint_{B_\varepsilon(x)} u(y)\,dy$$

in the asymptotic mean value property (1.3) has a game-theoretic interpretation for which it is essential that the coefficients are positive and add up to 1, so that they play the role of conditional probabilities.

For asymptotic mean value properties for the elliptic Monge–Ampère equation

$$\det D^2 u = f \quad \text{in } \Omega \tag{1.4}$$

we refer to our recent paper [4]. For the equation to fit into the framework of the theory of fully nonlinear elliptic equations, one must look for convex solutions u to ensure that $\det(D^2 u)$ is indeed a monotone function of $D^2 u$. Thus, one requires the

right-hand side $f(x)$ to be nonnegative; see [9, 10]. Moreover, the Monge–Ampère equation can be expressed as an infimum of a family of linear operators as follows:

$$\left(\det D^2 u(x) \right)^{1/n} = \frac{1}{n} \inf_{\det A=1} \operatorname{trace}(A^t D^2 u(x) A).$$

Let $\phi(\varepsilon)$ be a positive function such that

$$\lim_{\varepsilon \to 0} \phi(\varepsilon) = \infty \qquad \text{and} \qquad \lim_{\varepsilon \to 0} \varepsilon \, \phi(\varepsilon) = 0.$$

Then, a convex function $u \in C(\Omega)$ is a viscosity solution of the Monge–Ampère equation (1.4) if and only if for every $x \in \Omega$ we have

$$u(x) = \inf_{\substack{\det A=1 \\ A \le \phi(\varepsilon)I}} \left\{ \fint_{B_\varepsilon(0)} u(x + Ay) \, dy \right\} - \frac{n}{2(n+2)} \, (f(x))^{1/n} \, \varepsilon^2 + o(\varepsilon^2) \qquad (1.5)$$

as $\varepsilon \to 0$, in the viscosity sense; see [4]. We remark that classical C^2 solutions to the Monge–Ampère equation satisfy (1.5) in the standard pointwise sense.

1.2. **Asymptotic mean value formulas for parabolic equations.** In the linear case, u is a solution to the heat equation

$$\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t)$$

if and only if u satisfies the mean value formula

$$u(x, t) = \int_{E(x,t;r)} u(y, s) \frac{|x - y|^2}{(t - s)^2} \, dy ds,$$

where the integral is taken over the heat ball

$$E(x, t; r) = \left\{ (y, s) \in \mathbb{R}^{n+1} : s \le t, \, (4\pi(t - s))^{1/2} e^{\frac{(x-y)^2}{4n(t-s)}} \le r \right\}.$$

Concerning mean value formulas for the heat equation we refer also to [40] and [1]. For a version with variable coefficients see [15], and for a proof of the rigidity of the formula see [22, 34].

It turns out that there is a simpler asymptotic mean value formula where we only need to integrate over the parabolic cylinder $B_\varepsilon(x) \times (t - \frac{\varepsilon^2}{n+2}, t)$ and there is no need for a kernel; i.e., u is a solution to the heat equation if and only if u satisfies

$$u(x, t) = \fint_{t - \frac{\varepsilon^2}{n+2}}^{t} \fint_{B_\varepsilon(x)} u(y, s) \, dy \, ds + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

Equivalently, the asymptotic mean value formula

$$u(x, t) = \fint_{t - \varepsilon^2}^{t} \fint_{B_\varepsilon(x)} u(y, s) \, dy \, ds + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0,$$

holds in the viscosity sense if and only if u is a solution to

$$(n + 2) \, u_t(x, t) = \Delta u(x, t).$$

For the normalized parabolic p-Laplacian it is convenient to write the equation in the form

$$(n + p)\, u_t(x, t) = |\nabla u|^{2-p} \Delta_p u(x, t). \qquad (1.6)$$

This equation has been studied in [29], where a game approximation (that leads to an asymptotic mean value formula) was analyzed. Namely, a function u solves (1.6) if and only if the following expansion holds in the viscosity sense:

$$u(x, t) = \frac{1}{2} \left(\frac{p-2}{p+n} \right) \fint_{t-\varepsilon^2}^{t} \left(\max_{y \in B_\varepsilon(x)} u(y, s) + \min_{y \in B_\varepsilon(x)} u(y, s) \right) ds$$

$$+ \left(\frac{2+n}{p+n} \right) \fint_{t-\varepsilon^2}^{t} \fint_{B_\varepsilon(x)} u(y, s)\, dy\, ds + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

2. Main results

Our main goal is to obtain mean value formulas for parabolic versions of the Monge–Ampère equation and other nonlinear parabolic equations.

2.1. **Parabolic Monge–Ampère.** First we show that an asymptotic, nonlinear mean value formula holds for two different parabolic versions of the Monge–Ampère equation. The first one reads as follows:

$$\frac{\partial u}{\partial t}(x, t) = (\det(D^2 u(x, t)))^{1/n} + f(x, t), \qquad (2.1)$$

and has been studied in [11, 13, 17, 21] in relation to geometric evolution problems.

Our first result describes an asymptotic mean value formula for this equation.

Theorem 2.1. *Let $\phi(\varepsilon)$ be a positive function such that*

$$\lim_{\varepsilon \to 0} \phi(\varepsilon) = \infty \qquad and \qquad \lim_{\varepsilon \to 0} \varepsilon\, \phi(\varepsilon) = 0.$$

A function $u \in C(\Omega \times (t_1, t_2))$ that is convex in the spatial variables is a viscosity solution of the Monge–Ampère equation (2.1) *if and only if*

$$u(x, t) = \inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \left\{ \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x + Ay, s)\, dy\, ds \right\}$$

$$+ \frac{n}{2(n+2)}\, f(x, t)\, \varepsilon^2 + o(\varepsilon^2) \qquad (2.2)$$

as $\varepsilon \to 0$ for $x \in \Omega$, $t \in (t_1, t_2)$, in the viscosity sense.

For the precise definition of a viscosity solution and the statement of a mean value formula in the viscosity sense we refer to Section 3 (see also [12] and [28]). Informally, an equation or a mean value property holds in the viscosity sense when it holds with the appropriate inequality instead of an equality for smooth functions that touch u from above or from below at (x, t).

The mean value property (2.2) involves an average of u both in space and in time. Notice the parabolic character of the time average in formula (2.2), where we are integrating over a time interval of length comparable to ε^2.

Remark 2.2. We assume without loss of generality that the matrices A that appear throughout this paper are symmetric and positive definite. This assumption is not restrictive since we can use the (unique) left polar decomposition of A, namely $A = SQ$ (where Q is orthogonal and S is a positive definite symmetric matrix) and consider S instead of a general matrix A in our formulas.

Remark 2.3. The concept of a mean value formula in the viscosity sense is weaker than a mean value formula that holds in a pointwise sense. In fact, there are examples of asymptotic mean value formulas that hold in the viscosity sense but do not hold pointwise, like the ones that hold for the infinity Laplacian (see [28]) and for the elliptic Monge–Ampère equation (see [4]). When a solution to the involved equation is smooth, $u(x,t) \in C^{2,1}$, the mean value property holds pointwise. This is a consequence of the fact that for $C^{2,1}$ functions we have formula (2.2) in the classical pointwise sense.

Remark 2.4. We can also obtain a mean value property evaluating at $t - \frac{n}{2(n+2)}\varepsilon^2$ instead of averaging in time; that is, it holds that u is a viscosity solution of the parabolic Monge–Ampère equation (2.1) if and only if

$$
u(x,t) = \inf_{\substack{\det(A)=1 \\ A \le \phi(\varepsilon)I}} \left\{ \fint_{B_\varepsilon(0)} u\left(x + Ay, t - \frac{n}{2(n+2)}\varepsilon^2\right) dy \right\}
$$
$$
+ \frac{n}{2(n+2)} f(x,t)\,\varepsilon^2 + o(\varepsilon^2),
$$

in the viscosity sense.

A different version of the parabolic Monge–Ampère equation reads

$$
-\frac{\partial u}{\partial t}(x,t) \cdot \det\left(D^2 u(x,t)\right) = f(x,t) \tag{2.3}
$$

and appears in connection with the movement of a hypersurface by Gauss–Kronecker curvature; see [36]. For the study of this equation we refer to [20, 41, 42] and concerning regularity of the solutions to [19, 35, 38, 39]. The asymptotic behavior at infinity of global solutions has been studied in [37].

In order for the general viscosity theory to work in this case, one has to restrict to solutions that are parabolically convex. A function $u : \Omega \times (t_1, t_2) \to \mathbb{R}$ is *parabolically convex* if it is continuous, convex in x, and non-increasing in t. Therefore, we assume that the right-hand side f is nonnegative.

Our next result shows that there is also a mean value formula in this case.

Theorem 2.5. *Let $\phi(\varepsilon)$ be a positive function such that*

$$
\lim_{\varepsilon \to 0} \phi(\varepsilon) = \infty \qquad and \qquad \lim_{\varepsilon \to 0} \varepsilon\,\phi(\varepsilon) = 0.
$$

A parabolically convex function $u \in C(\Omega \times (t_1, t_2))$ is a viscosity solution of the Monge–Ampère equation (2.3) if and only if

$$
u(x,t) = \inf_{\substack{\det(A) \times b = 1 \\ A \leq \phi(\varepsilon)I, \, b \leq \phi(\varepsilon)}} \left\{ \fint_{t - \frac{b^2}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x + Ay, s) \, dy \, ds \right\} \tag{2.4}
$$

$$
- \frac{n+1}{2(n+2)} \left(f(x,t) \right)^{\frac{1}{n+1}} \varepsilon^2 + o(\varepsilon^2)
$$

as $\varepsilon \to 0$ for $x \in \Omega$, $t \in (t_1, t_2)$, in the viscosity sense.

Notice that formula (2.4) is similar to (2.2), except for the extra parameter b that appears in the infimum, which is related to the time interval average.

Remark 2.6. We can again obtain a mean value property evaluating at $t - \frac{b^2}{2(n+2)}\varepsilon^2$ instead of averaging in time; that is, the parabolically convex function $u \in C(\Omega \times (t_1, t_2))$ is a viscosity solution of the Monge–Ampère equation (2.3) if and only if

$$
u(x,t) = \inf_{\substack{\det(A) \times b = 1 \\ A \leq \phi(\varepsilon)I, \, b \leq \phi(\varepsilon)}} \left\{ \fint_{B_\varepsilon(0)} u\left(x + Ay, t - \frac{b^2}{2(n+2)}\varepsilon^2 \right) dy \, ds \right\}
$$

$$
- \frac{n+1}{2(n+2)} \left(f(x,t) \right)^{\frac{1}{n+1}} \varepsilon^2 + o(\varepsilon^2)
$$

as $\varepsilon \to 0$ in the viscosity sense.

2.2. Bounded operators.

2.2.1. *Infimum operators.* Let us start with the differential operator $F : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^n(\mathbb{R}) \to \mathbb{R}$ given by

$$
F\left(x, t, \frac{\partial u}{\partial t}, D^2 u \right) = \inf_{(A,b) \in \mathcal{A}_{x,t}} \left\{ \operatorname{trace}(A^t D^2 u(x,t) A) - b \frac{\partial u}{\partial t}(x,t) \right\}. \tag{2.5}
$$

Here $\mathcal{A}_{x,t} \subset \mathbb{S}^n_+(\mathbb{R}) \times \mathbb{R}^+$ is a bounded subset for each point $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and $\mathbb{S}^n_+(\mathbb{R})$ denotes the set of symmetric positive semi-definite matrices. Examples of these operators include parabolic equations related to Pucci operators and evolution problems associated with the convex envelope. See Section 5 below for details.

Our next result gives an asymptotic mean value formula that characterizes viscosity solutions of the corresponding homogeneous equation.

Theorem 2.7. *A function $u \in C(\Omega \times (t_1, t_2))$ is a viscosity solution of the equation*

$$
F\left(x, t, \frac{\partial u}{\partial t}, D^2 u \right) = 0
$$

if and only if

$$
u(x,t) = \inf_{(A,b) \in \mathcal{A}_{x,t}} \fint_{t - \frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x + Ay, s) \, dy \, ds + o(\varepsilon^2)
$$

as $\varepsilon \to 0$ in the viscosity sense.

Observe that an analogous statement holds for suprema, and hence we can tackle operators of the form

$$F\Big(x,t,\frac{\partial u}{\partial t},D^2u\Big) = \sup_{(A,b)\in\mathcal{A}_{x,t}}\Big\{\mathrm{trace}(A^tD^2u(x,t)A) - b\frac{\partial u}{\partial t}(x,t)\Big\}.$$

Remark 2.8. Since the sets $\mathcal{A}_{x,t}$ are bounded, the formula is local. In fact, for every $x \in \Omega$, $t \in (t_1,t_2)$ there exists $C = C(x,t) > 0$ such that $A \leq CI$ and $b \leq C$ for every $(A,b) \in \mathcal{A}_{x,t}$ and we get

$$\mathrm{dist}(x+Ay,x) = |Ay| \leq C\varepsilon \to 0$$

and

$$0 \leq t - s \leq \frac{b}{n+2}\varepsilon^2 \leq \frac{C}{n+2}\varepsilon^2 \to 0$$

as $\varepsilon \to 0$ for every $y \in B_\varepsilon(0)$ and $s \in (t - \varepsilon^2\frac{bn}{n+2}, t)$. In particular observe that, for ε small enough, $(x+Ay,s) \in \Omega \times (t_1,t_2)$ for every $y \in B_\varepsilon(0)$ and $s \in (t - \varepsilon^2\frac{b}{n+2}, t)$.

2.2.2. *Supremum-infimum operators.* Our next step is to consider a special type of Isaacs operators where the supremum (or the infimum) is taken over a subset $\mathbb{A}_{x,t}$ of the parts of $\mathbb{S}_+^n(\mathbb{R}) \times \mathbb{R}^+$. To be more precise, let $\mathbb{A}_{x,t} \subset \mathcal{P}(\mathbb{S}_+^n(\mathbb{R}) \times \mathbb{R}^+)$ be a subset for each $x \in \Omega$, $t \in (t_1,t_2)$ such that

$$\bigcup \mathbb{A}_{x,t} = \Big\{(A,b) \in \mathbb{S}_+^n(\mathbb{R}) \times \mathbb{R}^+ : (A,b) \in \mathcal{A} \text{ for some } \mathcal{A} \in \mathbb{A}_{x,t}\Big\}$$

is bounded. We consider the differential operator $F : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^n(\mathbb{R}) \to \mathbb{R}$ given by

$$F\Big(x,t,\frac{\partial u}{\partial t},D^2u\Big) = \sup_{\mathcal{A}\in\mathbb{A}_{x,t}}\inf_{(A,b)\in\mathcal{A}}\Big\{\mathrm{trace}(A^tD^2u(x,t)A) - b\frac{\partial u}{\partial t}(x,t)\Big\}.$$

Examples of these operators include evolution problems for the eigenvalues of the Hessian. See Section 5 below for details.

Theorem 2.9. *A function $u \in C(\Omega \times (t_1,t_2))$ is a viscosity solution to*

$$F\Big(x,t,\frac{\partial u}{\partial t},D^2u\Big) = 0$$

if and only if

$$u(x,t) = \sup_{\mathcal{A}\in\mathbb{A}_{x,t}}\inf_{(A,t)\in\mathcal{A}} \int_{t-\frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x+Ay,s)\,dy\,ds + o(\varepsilon^2) \qquad (2.6)$$

as $\varepsilon \to 0$ in the viscosity sense.

Remark 2.10. In the previous two cases we can evaluate at $t - \frac{b}{2(n+2)}\varepsilon^2$ instead of taking averages in time and get a mean value formula of the form

$$u(x,t) = \sup_{\mathcal{A}\in\mathbb{A}_{x,t}}\inf_{(A,t)\in\mathcal{A}} \fint_{B_\varepsilon(0)} u\Big(x+Ay,t - \frac{b}{2(n+2)}\varepsilon^2\Big)\,dy\,ds + o(\varepsilon^2) \qquad (2.7)$$

as $\varepsilon \to 0$.

The paper is organized as follows: In Section 3 we gather some definitions and preliminary results; in Section 4 we prove the mean value formulas for the two versions of the parabolic Monge–Ampère equation; in Section 5 we deal with infimum and sup-inf operators; finally, Section 6 contains a brief discussion of the relation between these mean value formulas and dynamic programming principles from game theory.

3. Preliminaries

In this section we set the notation, recall basic results on the Monge–Ampère equation, and state some definitions. We begin by stating the definition of a viscosity solution to a fully nonlinear second order parabolic PDE. We refer to [12] for general results on viscosity solutions.

Given a continuous function

$$F : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^n(\mathbb{R}) \to \mathbb{R},$$

where $\mathbb{S}^n(\mathbb{R})$ denotes the set of symmetric $n \times n$ matrices, we consider the PDE

$$F\left(x, t, \frac{\partial u}{\partial t}(x,t), D^2 u(x,t)\right) = 0, \qquad x \in \Omega, \ t \in (t_1, t_2). \tag{3.1}$$

Viscosity solutions use the monotonicity of F in $D^2 u$ (ellipticity) and in $\frac{\partial u}{\partial t}$ in order to evaluate the equation for smooth test functions and obtain sub- and supersolution inequalities.

Definition 3.1. A lower semi-continuous function u is a *viscosity supersolution* of (3.1) if for every $\phi \in C^{2,1}$ such that ϕ touches u at $(x,t) \in \Omega \times (t_1, t_2)$ strictly from below (that is, $u - \phi$ has a strict minimum at (x,t) with $u(x,t) = \phi(x,t)$), we have

$$F\left(x, t, \frac{\partial \phi}{\partial t}(x,t), D^2 \phi(x,t)\right) \geq 0.$$

An upper semi-continuous function u is a *subsolution* of (3.1) if for every $\phi \in C^{2,1}$ such that ϕ touches u at $(x,t) \in \Omega \times (t_1, t_2)$ strictly from above (that is, $u - \phi$ has a strict maximum at (x,t) with $u(x,t) = \phi(x,t)$), we have

$$F\left(x, t, \frac{\partial \phi}{\partial t}(x,t), D^2 \phi(x,t)\right) \leq 0.$$

Finally, u is a *viscosity solution* of (3.1) if it is both a super- and a subsolution.

To apply this definition to the parabolic Monge–Ampère equation (2.1) we have to consider the operator

$$F\left(x, t, \frac{\partial u}{\partial t}(x,t), D^2 u(x,t)\right)$$
$$= \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - (\det(D^2 u(x,t)))^{1/n} - f(x,t) & \text{if } D^2 u(x,t) \geq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

This is equivalent to requiring the function to be convex in the space variable and restricting the test functions to paraboloids convex in space. Similarly, for solutions

to equation (2.3) we require the function to be parabolically convex (convex in x and non-increasing in t) and restrict the test functions to parabolically convex paraboloids, defined next.

Definition 3.2. A $C^{2,1}$ function $P(x,t)$ is a *parabolic paraboloid* if and only if it coincides with its second order Taylor expansion in x and first order in t, i.e., we have

$$P(x,t) = P(x_0,t_0) + \frac{\partial P}{\partial t}(x_0,t_0)(t-t_0)$$
$$+ \langle \nabla P(x_0,t_0), (x-x_0)\rangle + \frac{1}{2}\langle D^2 P(x_0,t_0)(x-x_0),(x-x_0)\rangle$$

for any given (x_0,t_0). Furthermore, $P(x,t)$ is a *parabolically convex paraboloid* if and only if $\frac{\partial P}{\partial t} \geq 0$ and $D^2 P \geq 0$.

We will also need the definition of an asymptotic mean value formula in the viscosity sense. First, recall that given a constant c and a real function g we write

$$c \leq g(\varepsilon) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0$$

whenever we have

$$\lim_{\varepsilon \to 0} \frac{[c - g(\varepsilon)]^+}{\varepsilon^2} = 0,$$

and

$$c \geq g(\varepsilon) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0$$

whenever we have

$$\lim_{\varepsilon \to 0} \frac{[c - g(\varepsilon)]^-}{\varepsilon^2} = 0.$$

In the next definition, $M(u,\varepsilon)(x)$ stands for a mean value operator (that depends on the parameter ε) applied to u at the point x. As an example, consider

$$M(u,\varepsilon)(x,t) = \inf_{(A,b)\in\mathcal{A}_{x,t}} \fint_{t-b\,\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x + Ay, s)\, dy\, ds.$$

Definition 3.3. A continuous function u satisfies

$$u(x,t) = M(u,\varepsilon)(x,t) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0,$$

in the viscosity sense if

(1) for every $\phi \in C^{2,1}$ such that $u - \phi$ has a strict minimum at the point $(x,t) \in \Omega \times (t_1,t_2)$ with $u(x,t) = \phi(x,t)$, we have

$$0 \geq -\phi(x,t) + M(\phi,\varepsilon)(x,t) + o(\varepsilon^2);$$

(2) for every $\phi \in C^{2,1}$ such that $u - \phi$ has a strict maximum at the point $(x,t) \in \Omega \times (t_1,t_2)$ with $u(x,t) = \phi(x,t)$, we have

$$0 \leq -\phi(x,t) + M(\phi,\varepsilon)(x,t) + o(\varepsilon^2).$$

The following elementary fact will be used several times in what follows.

Lemma 3.4. *Let M be a square matrix of dimension n. Then,*

$$\text{trace}(M) = \frac{n}{\varepsilon^2} \fint_{\partial B_\varepsilon(0)} \langle My, y \rangle \, d\mathcal{H}^{n-1}(y), \tag{3.2}$$

$$= \frac{n+2}{\varepsilon^2} \fint_{B_\varepsilon(0)} \langle My, y \rangle \, dy. \tag{3.3}$$

For symmetric square matrices, $A > 0$ means positive definite and $A \geq 0$ means positive semidefinite. We will denote by $\lambda_i(A)$ the eigenvalues of A; in particular $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the smallest and largest eigenvalues, respectively. We have the following linear algebra facts.

Lemma 3.5. *Let M be symmetric and $M \geq 0$. Then,*

$$\inf_{\det A = 1} \text{trace}(A^t M A) = n(\det(M))^{1/n}.$$

On the other hand, if M has negative eigenvalues, then the infimum is $-\infty$.

For a proof of Lemma 3.5 we refer to [4].

Lemma 3.6. *Let $M > 0$. Then, for every*

$$\theta > \theta_0 := \left(\frac{(\det M)^{1/n}}{\lambda_{\min}(M)} \right)^{1/2},$$

we have

$$\inf_{\det A = 1} \text{trace}\left(A^t M A\right) = \inf_{\substack{\det A = 1 \\ A \leq \theta I}} \text{trace}(A^t M A).$$

Proof. In general, the right-hand side is larger than the left-hand side since we are taking infimum over a smaller set. The infimum is realized for

$$A^* = (\det M)^{\frac{1}{2n}} M^{-1/2}.$$

Then the result follows since $A^* \leq \theta_0 I$. $\qquad\qquad\square$

Lemma 3.7. *For $M > 0$ we have*

$$(\det(M \pm \eta I))^{1/n} = (\det M)^{1/n} + O(\eta) \qquad \text{as } \eta \to 0. \tag{3.4}$$

For $M \geq 0$ we have

$$(\det(M + \eta I))^{1/n} = (\det M)^{1/n} + O(\eta) \qquad \text{as } \eta \to 0. \tag{3.5}$$

Proof. To see this, first notice that

$$\det(M \pm \eta I) = \det M + \sum_{k=1}^{n} (\pm \eta)^k \sigma_{n-k}(M),$$

where the coefficients $\sigma_{n-k}(M)$ in the expansion are given by the elementary symmetric polynomials on the eigenvalues of M, which are positive. Therefore,

$$(\det M - C\eta)^{1/n} \leq (\det(M \pm \eta I))^{1/n} \leq (\det M + C\eta)^{1/n}$$

for some $C > 0$. Then, the Mean Value Theorem applied to $g(t) = t^{1/n}$, with $a = \det M$ and $b = \det M + C\eta$, gives that there exists $\xi \in (a, b)$ such that

$$(\det M + C\eta)^{1/n} = g(b) = g(a) + g'(\xi)(b - a) \leq (\det M)^{1/n} + C\eta.$$

Similarly, we obtain that

$$(\det M - C\eta)^{1/n} \geq (\det M)^{1/n} - C\eta$$

and (3.4) follows.

It remains to prove (3.5) in the case $\det M = 0$. In this case the upper bound obtained is still at our disposal. Therefore we conclude the proof by observing that $(\det (M + \eta I))^{1/n} \geq 0$. $\qquad\square$

Lemma 3.8. *For $M > 0$ we have*

$$\inf_{\substack{\det A = 1 \\ A \leq \phi(\varepsilon)I}} \operatorname{trace}(A^t(M \pm \eta I)A) \to n (\det M)^{1/n} \qquad \text{as } \varepsilon, \eta \to 0.$$

For $M \geq 0$ we have

$$\inf_{\substack{\det A = 1 \\ A \leq \phi(\varepsilon)I}} \operatorname{trace}(A^t(M + \eta I)A) \to n (\det M)^{1/n} \qquad \text{as } \varepsilon, \eta \to 0.$$

Proof. For $M > 0$, we consider $\eta < \min\{1, \lambda_{\min}(M)/2\}$, and we have

$$\left(\frac{(\det M \pm \eta I)^{1/n}}{\lambda_{\min}(M \pm \eta I)} \right)^{1/2} \leq \left(\frac{(\det M + I)^{1/n}}{\lambda_{\min}(M)/2} \right)^{1/2} = \theta_0.$$

Then, for ε_0 such that for every $\varepsilon < \varepsilon_0$ we have $\phi(\varepsilon) > \theta_0$, combining Lemma 3.6 and Lemma 3.5 we get

$$\inf_{\substack{\det A = 1 \\ A \leq \phi(\varepsilon)I}} \operatorname{trace}(A^t(M \pm \eta I)A) = n (\det M \pm \eta I)^{1/n}.$$

The result follows from equation (3.4).

It remains to prove the case $\det(M) = 0$. Given $\delta > 0$, by Lemma 3.5 there exists A_0 with $\det(A_0) = 1$ such that

$$\operatorname{trace}(A_0^t M A_0) < \delta.$$

Then, for ε such that $A_0 \leq \phi(\varepsilon)I$ we have

$$\inf_{\substack{\det A = 1 \\ A \leq \phi(\varepsilon)I}} \operatorname{trace}(A^t(M \pm \eta I)A) \leq \operatorname{trace}(A_0^t(M + \eta I)A_0)$$

and by (3.5) for η small enough we get

$$\operatorname{trace}(A_0^t(M + \eta I)A_0) < \delta$$

and the result follows. $\qquad\square$

4. Parabolic Monge–Ampère

4.1. **First version.** Let us start with the equation

$$\frac{\partial u}{\partial t}(x,t) = (\det(D^2 u(x,t)))^{1/n} + f(x,t). \tag{4.1}$$

Our goal is to show that u is a viscosity solution of the Monge–Ampère equation (4.1) if and only if

$$u(x,t) = \inf_{\substack{\det(A)=1 \\ A \le \phi(\varepsilon)I}} \left\{ \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x+Ay,s)\,dy\,ds \right\}$$

$$+ \frac{n}{2(n+2)}\,f(x,t)\,\varepsilon^2 + o(\varepsilon^2)$$

as $\varepsilon \to 0$ for $x \in \Omega$ in the viscosity sense.

We first prove the asymptotic mean value property under the restriction that solutions of the Monge–Ampère equation are classical ($C^{2,1}$ solutions) and then we deal with the general case (viscosity solutions). We start by proving the result for smooth strictly convex functions (with strictly positive definite Hessian). We proceed with the proof of Theorem 2.1.

Proof of Theorem 2.1. As we have mentioned, first we prove the result for classical solutions. Assume that u is $C^{2,1}$, convex in space and a solution to

$$\frac{\partial u}{\partial t}(x,t) = (\det(D^2 u(x,t)))^{1/n} + f(x,t).$$

We use the Taylor expansion of $u(y,s)$, given by

$$u(y,s) = u(x,t) + \frac{\partial u}{\partial t}(x,t)(s-t) + \langle \nabla u(x,t), (y-x) \rangle$$

$$+ \frac{1}{2}\langle D^2 u(x,t)(y-x), (y-x) \rangle + o(|t-s| + |x-y|^2),$$

to define the parabolic paraboloid

$$P(y,s) = u(x,t) + \frac{\partial u}{\partial t}(x,t)(s-t) + \langle \nabla u(x,t), (y-x) \rangle + \frac{1}{2}\langle D^2 u(x,t)(y-x), (y-x) \rangle.$$

Since $u \in C^{2,1}$, we have

$$u(y,s) - P(y,s) = o(|y-x|^2 + |t-s|) \qquad \text{as } y \to x,\ s \to t,$$

which means that for every $\eta > 0$, there exists $\delta > 0$ such that

$$P(y,s) - \eta\left(\frac{|y-x|^2}{2} + |t-s|\right) \le u(y,s) \le P(y,s) + \eta\left(\frac{|y-x|^2}{2} + |t-s|\right)$$

for every $(y,s) \in B_\delta(x) \times (t-\delta, t+\delta)$. For convenience, we denote

$$P_\eta^\pm(y,s) = P(y,s) \pm \eta\left(\frac{|y-x|^2}{2} + |t-s|\right).$$

Let us assume first that $D^2u(x,t) > 0$. We have

$$\fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} P_\eta^\pm(x+Ay, s)\, dy\, ds - u(x,t)$$

$$= \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \left(\frac{\partial u}{\partial t}(x,t)(s-t) \pm \eta|t-s| \right) dy\, ds$$

$$+ \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \left(\langle \nabla u(x,t), Ay \rangle + \frac{1}{2}\langle D^2u(x,t)Ay, Ay \rangle \pm \frac{\eta}{2}|Ay|^2 \right) dy\, ds$$

$$= \left(-\frac{\partial u}{\partial t}(x,t) \pm \eta \right) \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} (s-t)\, ds + \frac{1}{2}\fint_{B_\varepsilon(0)} \langle A^t(D^2u(x,t) \pm \eta I)Ay, y \rangle\, dy$$

$$= \frac{n}{2(n+2)}\varepsilon^2 \left(-\frac{\partial u}{\partial t}(x,t) \pm \eta \right) + \frac{1}{2}\fint_{B_\varepsilon(0)} \langle A^t(D^2u(x,t) \pm \eta I)Ay, y \rangle\, dy.$$

By Lemma 3.4 we get

$$\fint_{B_\varepsilon(0)} \frac{1}{2}\langle A^t \left(D^2u(x,t) \pm \eta I \right) Ay, y \rangle\, dy = \frac{\varepsilon^2}{2(n+2)} \operatorname{trace}\left(A^t \left(D^2u(x,t) \pm \eta I \right) A \right),$$

and hence we obtain

$$\fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} P_\eta^\pm(x+Ay, s)\, dy\, ds - u(x,t)$$

$$= \frac{n}{2(n+2)}\varepsilon^2 \left(-\frac{\partial u}{\partial t}(x,t) \pm \eta \right) + \frac{\varepsilon^2}{2(n+2)} \operatorname{trace}(A^t \left(D^2u(x,t) \pm \eta I \right) A).$$

Therefore,

$$\inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} P_\eta^\pm(x+Ay, s)\, dy\, ds - u(x,t)$$

$$= \frac{n}{2(n+2)}\varepsilon^2 \left(-\frac{\partial u}{\partial t}(x,t) \pm \eta \right)$$

$$+ \frac{\varepsilon^2}{2(n+2)} \inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \operatorname{trace}\left(A^t \left(D^2u(x,t) \pm \eta I \right) A \right).$$

Observe that $A \leq \phi(\varepsilon)I$ and $|y| \leq \varepsilon$ imply $x + Ay \in B_\delta(x)$ for $\varepsilon < \varepsilon_0$ (since $\varepsilon\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$). Therefore, given $\eta > 0$ if $\varepsilon < \varepsilon_0$, we have

$$P_\eta^-(x+Ay, s) \leq u(x+Ay, s) \leq P_\eta^+(x+Ay, s) \qquad \text{for every } y \in B_\varepsilon(0).$$

Then, we get

$$
\frac{n\,\varepsilon^2}{2(n+2)} \left\{ \left(-\frac{\partial u}{\partial t}(x,t) - \eta \right) + \frac{1}{n} \inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \text{trace}\left(A^t \left(D^2 u(x,t) - \eta I \right) A \right) \right\}
$$

$$
\leq \inf_{\substack{\det A=1 \\ A \leq \phi(\varepsilon)I}} \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \left(u(x+Ay,s) - u(x,t) \right) dy\, ds
$$

$$
\leq \frac{n\,\varepsilon^2}{2(n+2)} \left\{ \left(-\frac{\partial u}{\partial t}(x,t) + \eta \right) + \frac{1}{n} \inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \text{trace}\left(A^t \left(D^2 u(x,t) + \eta I \right) A \right) \right\}.
$$

From here, the result follows from Lemma 3.8 since both the upper and lower bound involve an expression that converges to

$$
-\frac{\partial u}{\partial t}(x,t) + (\det\left(D^2 u(x,t) \right))^{1/n} = -f(x,t)
$$

as $\eta \to 0$.

Now, if $\det(D^2 u(x,t)) = 0$, then we use a minor modification of the above argument. The upper bound is still at our disposal but we have to obtain a new lower bound. Here we use the convexity in space of u to get

$$
\inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \left\{ \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x+Ay,s)\, dy\, ds \right\} - u(x,t)
$$

$$
\geq \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} u(x,s)\, ds - u(x,t) \geq -\frac{n}{2(n+2)}\varepsilon^2 \left(\frac{\partial u}{\partial t}(x,t) \right) + o(\varepsilon^2),
$$

and the result follows.

For the viscosity case, it is enough to use convex (in space) smooth functions as test functions in the definition of a viscosity solution. Let $u(x,t)$ be a viscosity solution to

$$
\frac{\partial u}{\partial t}(x,t) = (\det\left(D^2 u(x,t) \right))^{1/n} + f(x,t).
$$

Take $\phi(x,t)$ a $C^{2,1}$ function, convex in space, that touches u from above at (x,t), that is, we have

$$
u(y,s) - \phi(y,s) \leq u(x,t) - \phi(x,t), \qquad \text{for every } |y-x| \leq a,\ t-a \leq s \leq t.
$$

The fact that u is a viscosity solution implies that

$$
\frac{\partial \phi}{\partial t}(x,t) - (\det(D^2 \phi(x,t)))^{1/n} \leq f(x,t).
$$

On the other hand, from our previous computations, using that $\phi \in C^{2,1}$, we obtain

$$
\inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \phi(x + Ay, s)\, dy\, ds - \phi(x,t)
$$

$$
= -\frac{n}{2(n+2)}\, \varepsilon^2 \left(\frac{\partial \phi}{\partial t}(x,t) - (\det(D^2\phi(x,t)))^{1/n} \right) + o(\varepsilon^2)
$$

$$
\geq -\frac{n}{2(n+2)}\, \varepsilon^2 f(x,t) + o(\varepsilon^2).
$$

An analogous computation shows that a $C^{2,1}$ convex function $\phi(x,t)$ that touches u from below at (x,t) satisfies

$$
\inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \phi(x + Ay, s)\, dy\, ds - \phi(x,t) \leq -\frac{n}{2(n+2)}\, \varepsilon^2 f(x,t) + o(\varepsilon^2).
$$

This proves that a viscosity solution to the PDE satisfies the asymptotic mean value formula in the viscosity sense.

For the converse, let u be a convex function that satisfies the asymptotic mean value property in the viscosity sense. Take $\phi(x,t)$ a convex smooth function that touches u from above at (x,t); then we have

$$
u(y,s) - u(x,t) \leq \phi(y,s) - \phi(x,t), \qquad \text{for every } |y - x| \leq a,\ t - a \leq s \leq t.
$$

Since u satisfies the asymptotic mean value property in the viscosity sense, we have

$$
\fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \phi(x + Ay, s)\, dy\, ds - \phi(x,t) \geq -\frac{n}{2(n+2)}\, \varepsilon^2 f(x,t) + o(\varepsilon^2).
$$

Using again our previous computations for a $C^{2,1}$ function, we obtain

$$
-\frac{n}{2(n+2)}\, \varepsilon^2 \left(\frac{\partial \phi}{\partial t}(x,t) - (\det(D^2\phi(x,t)))^{1/n} \right) \geq -\frac{n}{2(n+2)}\, \varepsilon^2 f(x,t) + o(\varepsilon^2).
$$

Taking the limit as $\varepsilon \to 0$ we obtain

$$
\left(\frac{\partial \phi}{\partial t}(x,t) - (\det(D^2\phi(x,t)))^{1/n} + f(x,t) \right) \geq 0.
$$

This proves that u is a viscosity subsolution to the parabolic Monge–Ampère equation.

To prove that u is a viscosity supersolution we proceed in a similar way, touching u from below with a convex smooth function ϕ and using the asymptotic mean value property with the reverse inequality. $\qquad \square$

Remark 4.1. Using (3.2),

$$
\operatorname{trace}(M) = \frac{n}{\varepsilon^2} \fint_{\partial B_\varepsilon(0)} \langle My, y \rangle\, d\mathcal{H}^{n-1}(y),
$$

instead of (3.3), we can obtain an asymptotic mean value formula involving means over surfaces of ellipsoids: u is a viscosity solution of the parabolic Monge–Ampère equation (4.1) if and only if

$$u(x,t) = \inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \left\{ \fint_{t-\varepsilon^2}^t \fint_{\partial B_\varepsilon(0)} u(x + Ay, s)\, dy\, ds \right\} + \frac{1}{2}\, f(x,t)\, \varepsilon^2 + o(\varepsilon^2)$$

as $\varepsilon \to 0$ for $x \in \Omega$ and $t \in (t_1, t_2)$, in the viscosity sense.

Remark 4.2. We can also deal with equations with coefficients such as

$$a(x,t)\frac{\partial u}{\partial t}(x,t) = b(x,t)(\det(D^2u(x,t)))^{1/n} + c(x,t)f(x,t)$$

(see [13]). In this case we obtain an asymptotic mean value property of the form

$$u(x,t) = \inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \left\{ \fint_{t-k_1(x,t)\varepsilon^2}^t \fint_{B_\varepsilon(0)} u(x + Ay, s)\, dy\, ds \right\}$$

$$+ \frac{n}{2(n+2)}\, k_2(x,t) f(x,t)\, \varepsilon^2 + o(\varepsilon^2)$$

as $\varepsilon \to 0$ for $x \in \Omega$ and $t \in (t_1, t_2)$ with an appropriate choice of $k_1(x,t)$ and $k_2(x,t)$.

4.2. **Second version.** Now we deal with

$$-\frac{\partial u}{\partial t}(x,t) \cdot \det\left(D^2u(x,t)\right) = f(x,t) \tag{4.2}$$

and our goal is to prove Theorem 2.5, that is, to show that viscosity solutions to (4.2) are characterized by the asymptotic mean value formula

$$u(x,t) = \inf_{\substack{\det(A)\times b=1 \\ A \leq \phi(\varepsilon)I,\, b \leq \phi(\varepsilon)}} \left\{ \fint_{t-\frac{b^2}{n+2}\varepsilon^2}^t \fint_{B_\varepsilon(0)} u(x + Ay, s)\, dy\, ds \right\}$$

$$- \frac{n+1}{2(n+2)}\, f(x,t)^{\frac{1}{n+1}}\, \varepsilon^2 + o(\varepsilon^2)$$

as $\varepsilon \to 0$ for $x \in \Omega$ in the viscosity sense.

Proof of Theorem 2.5. Let us first prove the result for classical solutions. Assume that u is $C^{2,1}$, convex in space, non-increasing in t, and a solution to

$$-\frac{\partial u}{\partial t}(x,t) \cdot \det\left(D^2u(x,t)\right) = f(x,t).$$

As in the proof of Theorem 2.1, we define the parabolic paraboloid

$$P(y,s) = u(x,t) + \frac{\partial u}{\partial t}(x,t)(s-t) + \langle \nabla u(x,t), (y-x)\rangle + \frac{1}{2}\langle D^2u(x,t)(y-x), (y-x)\rangle.$$

Since $u \in C^{2,1}$, the Taylor expansion of $u(y,s)$ yields

$$u(y,s) - P(y,s) = o(|y-x|^2 + |t-s|) \qquad \text{as } y \to x,\ s \to t,$$

which means that for every $\eta > 0$, there exists $\delta > 0$ such that

$$P(y,s) - \eta \left(\frac{|y-x|^2}{2} + |t-s| \right) \le u(y,s) \le P(y,s) + \eta \left(\frac{|y-x|^2}{2} + |t-s| \right) \quad (4.3)$$

for every $(y,s) \in B_\delta(x) \times (t-\delta, t+\delta)$. For convenience, we denote

$$P_\eta^\pm(y,s) = P(y,s) \pm \eta \left(\frac{|y-x|^2}{2} + |t-s| \right).$$

We assume first that $D^2 u(x,t) > 0$ and that $-\frac{\partial u}{\partial t}(x,t) > 0$. In this case we have

$$\fint_{t-\frac{b^2}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} P_\eta^\pm(x + Ay, s) \, dy \, ds - u(x,t)$$

$$= \fint_{t-\frac{b^2}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \left(\frac{\partial u}{\partial t}(x,t)(s-t) \pm \eta|t-s| \right) dy \, ds$$

$$+ \fint_{t-\frac{b^2}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \left(\langle \nabla u(x,t), Ay \rangle + \frac{1}{2} \langle D^2 u(x,t) Ay, Ay \rangle \pm \frac{\eta}{2}|Ay|^2 \right) dy \, ds$$

$$= \left(-\frac{\partial u}{\partial t}(x,t) \pm \eta \right) \fint_{t-\frac{b^2}{n+2}\varepsilon^2}^{t} (t-s) \, ds + \frac{1}{2} \fint_{B_\varepsilon(0)} \langle A^t(D^2 u(x,t) \pm \eta I) Ay, y \rangle \, dy$$

$$= \frac{b^2}{2(n+2)} \varepsilon^2 \left(-\frac{\partial u}{\partial t}(x,t) \pm \eta \right) + \frac{1}{2} \fint_{B_\varepsilon(0)} \langle A^t(D^2 u(x,t) \pm \eta I) Ay, y \rangle \, dy.$$

By Lemma 3.4 we have that

$$\fint_{B_\varepsilon(0)} \langle A^t(D^2 u(x,t) \pm \eta I) Ay, y \rangle \, dy = \frac{\varepsilon^2}{n+2} \, \text{trace} \left(A^t \left(D^2 u(x,t) \pm \eta I \right) A \right)$$

and we obtain

$$\inf_{\substack{\det(A) \times b = 1 \\ A \le \phi(\varepsilon)I, \, b \le \phi(\varepsilon)}} \left\{ \fint_{t-\frac{b^2}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} P_\eta^\pm(x + Ay, s) \, dy \, ds \right\} - u(x,t)$$

$$= \frac{\varepsilon^2}{2(n+2)}$$

$$\times \inf_{\substack{\det(A) \times b = 1 \\ A \le \phi(\varepsilon)I, \, b \le \phi(\varepsilon)}} \left\{ b^2 \left(-\frac{\partial u}{\partial t}(x,t) \pm \eta \right) + \text{trace} \left(A^t \left(D^2 u(x,t) \pm \eta I \right) A \right) \right\}.$$

Observe that $A \le \phi(\varepsilon)I$ and $|y| \le \varepsilon$ imply $x + Ay \in B_\delta(x)$ for $\varepsilon < \varepsilon_0$ (since $\varepsilon\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$). Similarly, $b \le \phi(\varepsilon)$ implies $\left(t - \frac{b^2}{n+2}\varepsilon^2, t \right) \subset (t-\delta, t+\delta)$ for $\varepsilon < \varepsilon_0$. Therefore, (4.3) implies that given $\eta > 0$ there is ε_0 such that if $\varepsilon < \varepsilon_0$, then

$$P_\eta^-(x + Ay, s) \le u(x + Ay, s) \le P_\eta^+(x + Ay, s)$$

for every $(y,s) \in B_\varepsilon(0) \times \left(t - \frac{b^2}{n+2}\varepsilon^2, t\right)$. Consequently, we obtain

$$
\frac{\varepsilon^2}{2(n+2)} \inf_{\substack{\det(A)\times b=1 \\ A\leq\phi(\varepsilon)I,\, b\leq\phi(\varepsilon)}} \left\{ b^2\left(-\frac{\partial u}{\partial t}(x,t) - \eta\right) + \operatorname{trace}\left(A^t\left(D^2 u(x,t) - \eta I\right)A\right)\right\}
$$

$$
\leq \inf_{\substack{\det(A)\times b=1 \\ A\leq\phi(\varepsilon)I,\, b\leq\phi(\varepsilon)}} \fint_{t-\frac{b^2}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} (u(x+Ay,s) - u(x,t))\, dy\, ds
$$

$$
\leq \frac{\varepsilon^2}{2(n+2)}
$$

$$
\times \inf_{\substack{\det(A)\times b=1 \\ A\leq\phi(\varepsilon)I,\, b\leq\phi(\varepsilon)}} \left\{ b^2\left(-\frac{\partial u}{\partial t}(x,t) + \eta\right) + \operatorname{trace}\left(A^t\left(D^2 u(x,t) + \eta I\right)A\right)\right\}.
$$

Now, we notice that

$$
b^2\left(-\frac{\partial u}{\partial t}(x,t) \pm \eta\right) + \operatorname{trace}\left(A^t\left(D^2 u(x,t) \pm \eta I\right)A\right)
$$

$$
= \operatorname{trace}\left(C_{b,A}^t \begin{bmatrix} -\frac{\partial u}{\partial t}(x,t) \pm \eta & 0 \\ 0 & D^2 u(x,t) \pm \eta I \end{bmatrix} C_{b,A}\right),
$$

where the matrix $C_{b,A} \in \mathbb{S}^{n+1}$ is given by

$$
C_{b,A} = \begin{bmatrix} b & 0 \\ 0 & A \end{bmatrix}.
$$

Then, from Lemma 3.8 (notice that we assumed that $D^2 u(x,t)$ is strictly positive and $-\frac{\partial u}{\partial t}(x,t)$ is also strictly positive; in this case we have $f(x,t) \neq 0$) we get that

$$
\lim_{\eta\to 0} \left\{ \inf_{\substack{\det(A)\times b=1 \\ A\leq\phi(\varepsilon)I,\, b\leq\phi(\varepsilon)}} \operatorname{trace}\left(C_{b,A}^t \begin{bmatrix} -\frac{\partial u}{\partial t}(x,t) \pm \eta & 0 \\ 0 & D^2 u(x,t) \pm \eta I \end{bmatrix} C_{b,A}\right)\right\}
$$

$$
= \lim_{\eta\to 0} \left\{ \inf_{\det(A)\times b=1} \operatorname{trace}\left(C_{b,A}^t \begin{bmatrix} -\frac{\partial u}{\partial t}(x,t) \pm \eta & 0 \\ 0 & D^2 u(x,t) \pm \eta I \end{bmatrix} C_{b,A}\right)\right\}
$$

$$
= (n+1)\left(-\frac{\partial u}{\partial t}(x,t) \cdot \det\left(D^2 u(x,t)\right)\right)^{\frac{1}{n+1}}
$$

for ε small enough, and the result follows.

Now, if

$$
-\frac{\partial u}{\partial t}(x,t) \cdot \det(D^2 u(x,t)) = 0
$$

(i.e., $f(x,t) = 0$), the parabolic convexity of u and a minor modification of the previous arguments yield

$$\inf_{\substack{\det(A) \times b = 1 \\ A \leq \phi(\varepsilon)I,\, b \leq \phi(\varepsilon)}} \left\{ \fint_{t - \frac{b^2}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x + Ay, bs)\, dy\, ds \right\} - u(x,t) \geq o(\varepsilon^2).$$

Since the upper bound is still at our disposal, the result follows.

Finally, for the viscosity version of our asymptotic mean value formulas, we proceed as in the proof of Theorem 2.1, by touching u from above or below with a parabolically convex, smooth test function ϕ and using our previous computations that showed that a smooth parabolically convex function satisfies an inequality like

$$-\frac{\partial \phi}{\partial t}(x,t) \cdot \det(D^2 \phi(x,t)) \geq f(x,t)$$

if and only if

$$\phi(x,t) \leq \inf_{\substack{\det(A) \times b = 1 \\ A \leq \phi(\varepsilon)I,\, b \leq \phi(\varepsilon)}} \left\{ \fint_{t - \frac{b^2}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \phi(x + Ay, s)\, dy\, ds \right\}$$
$$- \frac{n+1}{2(n+2)} \left(f(x,t) \right)^{\frac{1}{n+1}} \varepsilon^2 + o(\varepsilon^2),$$

proving our characterization of viscosity solutions to (4.2).   □

Remark 4.3. With the same ideas, from the formula

$$\mathrm{trace}(M) = \frac{n}{\varepsilon^2} \fint_{\partial B_\varepsilon(0)} \langle My, y \rangle \, d\mathcal{H}^{n-1}(y),$$

we can obtain an asymptotic mean value formula involving means over surfaces of ellipsoids: u is a viscosity solution to the parabolic Monge–Ampère equation

$$-\frac{\partial u}{\partial t}(x,t) \cdot \det\left(D^2 u(x,t) \right) = f(x,t)$$

if and only if

$$u(x,t) = \inf_{\substack{\det(A) \times b = 1 \\ A \leq \phi(\varepsilon)I,\, b \leq \phi(\varepsilon)}} \left\{ \fint_{t - \frac{b^2}{n}\varepsilon^2}^{t} \fint_{\partial B_\varepsilon(0)} u(x + Ay, s)\, dy\, ds \right\}$$
$$+ \frac{n+1}{2n} \left(f(x,t) \right)^{\frac{1}{n+1}} \varepsilon^2 + o(\varepsilon^2)$$

as $\varepsilon \to 0$ in the viscosity sense.

5. Infimum and inf-sup operators

5.1. **Infimum operators.** In this section we deal with equations of the form

$$F\left(x, t, \frac{\partial u}{\partial t}, D^2 u\right) = \inf_{(A,b) \in \mathcal{A}_{x,t}} \left\{ \mathrm{trace}(A^t D^2 u(x,t) A) - b \frac{\partial u}{\partial t}(x,t) \right\} = 0. \tag{5.1}$$

Here $\mathcal{A}_{x,t} \subset \mathbb{S}_+^n(\mathbb{R}) \times \mathbb{R}^+$ are bounded subsets for each $x \in \mathbb{R}$ and $t \in [t_1, t_2]$.

Observe that we are assuming that the sets $\mathcal{A}_{x,t}$ are bounded. This has to be contrasted to the previous case, the Monge–Ampère case, where the set of relevant matrices $\mathcal{A}_x = \{A \in \mathbb{S}^n_+(\mathbb{R}) : \det(A) = 1\}$ is unbounded.

The fact that we are taking a bounded set of coefficients in (5.1) is equivalent to assuming that F is well defined and finite.

Lemma 5.1. *The operator*

$$F(x, t, z, M) = \inf_{(A,b)\in\mathcal{A}_{x,t}} \left\{ \operatorname{trace}(A^t M A) - bz \right\}$$

is finite for every $M \in \mathbb{S}^n(\mathbb{R})$ and every $z \in \mathbb{R}$ if and only if $\mathcal{A}_{x,t}$ is bounded.

Proof. Suppose that $\mathcal{A}_{x,t}$ is not bounded. First, assume that there exists a sequence of matrices $A_k \in \mathcal{A}_x$ such that their largest eigenvalue λ_k diverges. Let v_k be the corresponding unitary eigenvectors. Since v_k are unitary vectors, we can extract a subsequence (still denoted by v_k) that has a limit, that is, $v_k \to v$. Let M be the symmetric matrix with eigenvector v of eigenvalue -1 and eigenvalue 0 with multiplicity $n - 1$. Then

$$\operatorname{trace}(A_k^t M A_k) \leq -\lambda_k^2 \| \operatorname{proj}_v(v_k) \| + \lambda_k^2 \| \operatorname{proj}_{v^\perp}(v_k) \|,$$

which is a contradiction since the right-hand side goes to $-\infty$.

Next, consider the case where the set $\mathcal{A}_{x,t}$ contains an unbounded set of b. If $b_k \to +\infty$, we just take $z = -1$ and if $b_k \to -\infty$ then take $z = 1$, to obtain the contradiction. $\qquad\square$

Now we are ready to prove Theorem 2.7.

Proof of Theorem 2.7. As in the previous proofs, let us use the Taylor expansion of $u(y, s)$,

$$u(y, s) = u(x, t) + \frac{\partial u}{\partial t}(x, t)(s - t) + \langle \nabla u(x, t), (y - x) \rangle$$
$$+ \frac{1}{2} \langle D^2 u(x, t)(y - x), (y - x) \rangle + o(|t - s| + |x - y|^2).$$

Notice that now, since the sets of involved coefficients (matrices) are bounded, the error terms in the Taylor expansion are uniform; therefore, we can avoid the use of paraboloids here.

As before, we start with the proof of the case in which the solution u is a classical $C^{2,1}$ solution. We have

$$\fint_{t-\frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x + Ay, s) \, dy \, ds - u(x, t)$$

$$= \fint_{t-\frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \frac{\partial u}{\partial t}(x, t)(s - t) \, dy \, ds$$

$$+ \fint_{t-\frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} \langle \nabla u(x, t), Ay \rangle + \frac{1}{2} \langle D^2 u(x, t) Ay, Ay \rangle \, dy \, ds + o(\varepsilon^2)$$

$$= \frac{\partial u}{\partial t}(x,t) \int_{t-\frac{b}{n+2}\varepsilon^2}^{t} (s-t)\,ds + \fint_{B_\varepsilon(0)} \frac{1}{2}\langle A^t(D^2u(x,t))Ay, y\rangle\,dy + o(\varepsilon^2)$$

$$= -\frac{b}{2(n+2)}\varepsilon^2 \frac{\partial u}{\partial t}(x,t) + \fint_{B_\varepsilon(0)} \frac{1}{2}\langle A^t(D^2u(x,t))Ay, y\rangle\,dy + o(\varepsilon^2).$$

Now, we use one more time Lemma 3.4 to obtain

$$\fint_{B_\varepsilon(0)} \frac{1}{2}\langle A^t(D^2u(x,t))Ay, y\rangle\,dy = \frac{\varepsilon^2}{n+2}\,\mathrm{trace}(A^t(D^2u(x,t))A).$$

Therefore, we have

$$\int_{t-\frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x+Ay, s)\,dy\,ds - u(x,t)$$

$$= \frac{\varepsilon^2}{2(n+2)}\left(-b\frac{\partial u}{\partial t}(x,t) + \mathrm{trace}(A^t(D^2u(x,t))A)\right) + o(\varepsilon^2).$$

Hence, we conclude that

$$\inf_{(A,b)\in\mathcal{A}_{x,t}} \int_{t-\frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x+Ay, s)\,dy\,ds - u(x,t)$$

$$= \frac{\varepsilon^2}{2(n+2)} \inf_{(A,b)\in\mathcal{A}_{x,t}} \left(-b\frac{\partial u}{\partial t}(x,t) + \mathrm{trace}(A^t(D^2u(x,t))A)\right) + o(\varepsilon^2)$$

$$= \frac{\varepsilon^2}{2(n+2)} F\left(x,t,\frac{\partial u}{\partial t}, D^2u\right) + o(\varepsilon^2)$$

$$= o(\varepsilon^2),$$

proving the asymptotic mean value formula in a pointwise sense for smooth solutions.

As a consequence of the mean value property for smooth functions we can obtain the characterization for viscosity solutions in Theorem 2.7. The details are left to the reader. $\square$

Remark 5.2. The previous proof shows that in fact viscosity supersolutions (subsolutions) to

$$\inf_{(A,b)\in\mathcal{A}_{x,t}} \left\{ \mathrm{trace}(A^t D^2u(x,t)A) - b\frac{\partial u}{\partial t}(x,t) \right\} = 0$$

are characterized by

$$\inf_{(A,b)\in\mathcal{A}_{x,t}} \int_{t-\frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x+Ay, s)\,dy\,ds - u(x,t) \le o(\varepsilon^2) \quad (\ge o(\varepsilon^2))$$

as $\varepsilon \to 0$ in the viscosity sense.

5.1.1. *Examples.* Let us mention some examples of operators and the corresponding set of matrices $\mathcal{A}$ such that the previous results apply. To this end, we denote by $\lambda_1(M) \le \lambda_2(M) \le \cdots \le \lambda_n(M)$ the eigenvalues of the matrix M, or simply by λ_i when the matrix involved is clear from the context.

Parabolic equations related to Pucci operators. An important example under these assumptions are the parabolic Pucci operators. For given $0 < \theta < \Theta$, consider the second order, uniformly parabolic operators,

$$\mathcal{M}^-_{\theta,\Theta}\Big(\frac{\partial u}{\partial t}, D^2 u\Big) = \frac{\partial u}{\partial t} - \Big(\theta \sum_{\lambda_i(D^2 u) > 0} \lambda_i(D^2 u) + \Theta \sum_{\lambda_i(D^2 u) < 0} \lambda_i(D^2 u)\Big)$$

and

$$\mathcal{M}^+_{\theta,\Theta}\Big(\frac{\partial u}{\partial t}, D^2 u\Big) = \frac{\partial u}{\partial t} - \Big(\Theta \sum_{\lambda_i(D^2 u) > 0} \lambda_i(D^2 u) + \theta \sum_{\lambda_i(D^2 u) < 0} \lambda_i(D^2 u)\Big).$$

In this case, we have as the involved set of matrices

$$\mathcal{A} = \Big\{A \in \mathbb{S}^n_+(\mathbb{R}) : \sqrt{\theta} \le \lambda_i(A) \le \sqrt{\Theta}\Big\},$$

which is bounded uniformly in x. In fact, we can write

$$\Big(\theta \sum_{\lambda_i(M) > 0} \lambda_i(M) + \Theta \sum_{\lambda_i(M) < 0} \lambda_i(M)\Big) = \inf_{A \in \mathcal{A}} \operatorname{trace}(A^t M A)$$

and

$$\Big(\theta \sum_{\lambda_i(M) > 0} \lambda_i(M) + \Theta \sum_{\lambda_i(M) < 0} \lambda_i(M)\Big) = \inf_{A \in \mathcal{A}} \operatorname{trace}(A^t M A) = \sup_{A \in \mathcal{A}} \operatorname{trace}(A^t M A).$$

For these operators we have: u is a viscosity solution to

$$\frac{\partial u}{\partial t}(x,t) - \Big(\theta \sum_{\lambda_i(D^2 u(x,t)) > 0} \lambda_i(D^2 u(x,t)) + \Theta \sum_{\lambda_i(D^2 u(x,t)) < 0} \lambda_i(D^2 u(x,t))\Big) = 0$$

if and only if

$$u(x,t) = \inf_{\sqrt{\theta} \le \lambda_i(A) \le \sqrt{\Theta}} \fint_{t - \frac{\varepsilon^2}{n+2}}^{t} \fint_{B_\varepsilon(0)} u(x + Ay, s)\, dy\, ds + o(\varepsilon^2)$$

as $\varepsilon \to 0$ in the viscosity sense.

The evolution problem associated with the convex envelope, $\lambda_1(D^2 u)$. Notice, however, that the operators that satisfy (2.5) do not need to be uniformly elliptic. For example, the operator

$$\frac{\partial u}{\partial t} - \lambda_1(D^2 u),$$

studied in [3] in connection with the convex envelope of a function, corresponds to the set of matrices

$$\mathcal{A} = \big\{A \in \mathbb{S}^n_+(\mathbb{R}) : \lambda_1 = \cdots = \lambda_{n-1} = 0 \text{ and } \lambda_n = 1\big\}.$$

In this case the asymptotic mean value formula reads

$$u(x,t) = \inf_{|v|=1} \fint_{t - \frac{\varepsilon^2}{n+2}}^{t} \fint_{B_\varepsilon(0)} u\big(x + \langle y, v\rangle v, s\big)\, dy\, ds + o(\varepsilon^2).$$

5.2. **Sup-inf operators.** We also consider a special type of Isaacs operators where for every x the supremum (or the infimum) is taken over a subset $\mathbb{A}_{x,t}$ of the parts of $S_+^n(\mathbb{R}) \times \mathbb{R}^+$. More precisely, let $\mathbb{A}_{x,t} \subset \mathcal{P}(S_+^n(\mathbb{R}) \times \mathbb{R}^+)$ be a subset for each $x \in \mathbb{R}^n$ and $t > 0$ such that

$$\bigcup \mathbb{A}_{x,t} = \left\{ (A, b) \in \mathbb{S}_+^n(\mathbb{R}) \times \mathbb{R}^+ \; : \; (A, b) \in \mathcal{A} \text{ for some } \mathcal{A} \in \mathbb{A}_{x,t} \right\}$$

is bounded.

We consider the differential operator $F : \Omega \times \mathbb{R} \times \mathbb{R} \times S^n(\mathbb{R}) \to \mathbb{R}$ given by

$$F\left(x, t, \frac{\partial u}{\partial t}, D^2 u\right) = \sup_{\mathcal{A} \in \mathbb{A}_{x,t}} \inf_{(A,b) \in \mathcal{A}} \left\{ \operatorname{trace}(A^t D^2 u(x,t) A) - b \frac{\partial u}{\partial t}(x,t) \right\}.$$

Since $\cup \mathbb{A}_{x,t}$ is bounded, Theorem 2.9 follows as before, and we have the asymptotic mean value characterization of viscosity solutions of

$$F\left(x, t, \frac{\partial u}{\partial t}, D^2 u\right) = 0$$

as those continuous functions that satisfy

$$\sup_{\mathcal{A} \in \mathbb{A}_{x,t}} \inf_{(A,b) \in \mathcal{A}} \fint_{t-\frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u(x + Ay, s) \, dy \, ds - u(x,t) = o(\varepsilon^2)$$

as $\varepsilon \to 0$ in the sense of viscosity. We leave the details to the reader.

5.2.1. *Examples.* As examples of operators such that the previous results apply we mention the following.

Evolution problems associated with eigenvalues of the Hessian. This allows us to prove asymptotic mean value formulas for degenerate parabolic operators such as

$$\frac{\partial u}{\partial t} = \lambda_k(D^2 u)$$

that where studied in [3]. Here $\lambda_k(D^2 u)$ stands for the k-th smallest eigenvalue of the Hessian, given by the Courant–Fischer min-max principle

$$\lambda_k\big(D^2 u(x)\big) = \min_{\dim(V)=n-k+1} \; \min_{\substack{v \in V \\ |v|=1}} \left\{ \langle D^2 u(x) v, v \rangle \right\}.$$

For example, for the operator $\lambda_2(D^2 u)$ we have the set of sets of matrices

$$\mathcal{A} = \big\{ \{ A \in \mathbb{S}_+^n(\mathbb{R}) : \lambda_1 = \cdots = \lambda_{n-1} = 0, \; \lambda_n = 1 \text{ and } v_n \in V \}$$
$$: V \text{ a subspace of dimension } n - 1 \big\}.$$

In this case the asymptotic mean value formula is given by

$$u(x,t) = \sup_{\dim(V)=n-k+1} \; \inf_{\substack{v \in V \\ |v|=1}} \fint_{t-\frac{\varepsilon^2}{n+2}}^{t} \fint_{B_\varepsilon(0)} u\big(x + \langle y, v \rangle v, s\big) \, dy \, ds + o(\varepsilon^2).$$

For example, for the equation

$$\frac{\partial u}{\partial t}(x,t) - \lambda_2(D^2 u(x,t)) = 0,$$

we can write the following asymptotic mean value formula:

$$u(x,t) = \sup_{\substack{|w|=1}} \inf_{\substack{\langle v,w\rangle=0 \\ |v|=1}} \fint_{t-\frac{\varepsilon^2}{n+2}}^{t} \fint_{B_\varepsilon(0)} u\big(x + \langle y,v\rangle v, s\big)\, dy\, ds + o(\varepsilon^2).$$

6. A PROBABILISTIC INTERPRETATION

The asymptotic mean value formulas that we obtained can be interpreted in terms of dynamic programming principles for two-player zero-sum games.

First, we describe a closely related random walk. For a small $\varepsilon > 0$ and a fixed matrix A consider the following random walk in a bounded domain $\Omega_T = \Omega \times (0, T)$. From (x,t) the next spacial position is given by $x + Ay$ with $y \in B_\varepsilon$ chosen with uniform distribution in the ball and the new time is $t - \frac{b}{2(n+2)}\varepsilon^2$. The process continues until the spacial position leaves Ω or when the time becomes nonpositive. We call τ the stopping time given by the number of plays until the position of the game leaves Ω_T and call (x_τ, t_τ) the last position of the process. We have $(x_\tau, t_\tau) \notin \Omega_T$; furthermore, $(x_\tau, t_\tau) \in \Omega_T^c := \mathbb{R}^n \times (-\infty, T) \setminus \Omega_T$.

We fix a final payoff function $g : \Omega_T^c \to \mathbb{R}$ and we define

$$v^\varepsilon(x,t) = \mathbb{E}^{x,t}[g(x_\tau, t_\tau)].$$

Then, it follows that v^ε satisfies

$$v^\varepsilon(x,t) = \fint_{B_\varepsilon(0)} v^\varepsilon\Big(x + Ay, t - \frac{b}{2(n+2)}\varepsilon^2\Big)\, dy$$

for $x \in \Omega$, $t > 0$. That is, the expected final payoff starting at (x,t) is equal to the average of the expectation over all the possible next positions.

Next we describe the two-player zero-sum game. We are given $\mathbb{A}_{x,t} \subset \mathcal{P}(S_+^n(\mathbb{R}) \times (0, +\infty))$ for each $(x,t) \in \Omega_T$. The game starts at $(x_0, t_0) \in \Omega_T$; the first player, Player I, chooses a set of matrices and scalars $\mathcal{A} \in \mathbb{A}_{x,t}$ (she chooses a casino, in probabilistic terms) and next the second player, Player II, chooses a matrix and a scalar $(A, b) \in \mathcal{A}$ (she chooses a game to play in the casino chosen by Player I). The next position of the game is given by the previously described random walk, that is,

$$x_1 = x_0 + Ay$$

with $y \in B_\varepsilon$ being chosen according to the uniform distribution in B_ε and

$$t_1 = t_0 - b\frac{1}{2(n+2)}\varepsilon^2.$$

The game continues from x_1 accordingly to the same rules. The game finishes at the first time at which the position (x_τ, t_τ) leaves Ω_T. At this point, Player I gets $g(x_\tau, t_\tau)$ and Player II gets $-g(x_\tau, t_\tau)$ (one can think that Player II pays to Player I the amount given by the final payoff).

When the two players decide what to play at each turn (they choose their strategies), we can compute the expected payoff (the expected earnings for Player I) playing with strategies S_{I} and S_{II} starting at $(x,t) \in \Omega_T$ as

$$\mathbb{E}^{x,t}_{S_{\mathrm{I}}, S_{\mathrm{II}}}[g(x_\tau, t_\tau)].$$

Then, the extreme values for this game are given by

$$u^{\mathrm{I}}(x,t) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{x,t}_{S_{\mathrm{I}},S_{\mathrm{II}}}[g(x_\tau, t_\tau)]$$

and

$$u^{\mathrm{II}}(x,t) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x,t}_{S_{\mathrm{I}},S_{\mathrm{II}}}[g(x_\tau, t_\tau)].$$

We are taking sup and inf over the strategies for Player I and Player II respectively, since Player I wants to maximize the value of $g(x_\tau, t_\tau)$ while Player II is trying to minimize it. When these two extreme values coincide, we say that the game has a value given by

$$u^\varepsilon(x,t) := u^{\mathrm{II}}(x,t) = u^{\mathrm{I}}(x,t).$$

It turns out that $u^\varepsilon(x,t)$ is a solution to the dynamic programming principle associated to this game that reads

$$u^\varepsilon(x,t) = \sup_{\mathcal{A} \in \mathbb{A}_{x,t}} \inf_{A \in \mathcal{A}} \fint_{B_\varepsilon(0)} u^\varepsilon \left(x + Ay, t - b\frac{1}{2(n+2)}\varepsilon^2 \right) dy.$$

Notice that this formula is just our asymptotic mean value formula (2.7) without the error term.

When the next time position is also chosen with uniform probability in the interval $\left(t - b\frac{1}{n+1}\varepsilon^2, t \right)$ we obtain

$$u^\varepsilon(x,t) = \sup_{\mathcal{A} \in \mathbb{A}_x} \inf_{A \in \mathcal{A}} \fint_{t-\frac{b}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u^\varepsilon(x + Ay, s) \, dy \, ds,$$

where the reader can recognize the left-hand side of (2.6) without the error term.

For the mean value formulas associated with Monge–Ampère operators we just consider a game with only one player (a controller) that chooses at each point the parameters involved in the random walk (the controller chooses the matrix A involved in the random walk). We add a running payoff (at each point the player pays $\frac{n}{2(n+2)} f(x,t)\varepsilon^2$). In this case the value of the game is given by

$$u^\varepsilon(x,t) = \inf_{S_{\mathrm{I}}} \mathbb{E}^{x,t}_{S_{\mathrm{I}}} \left[g(x_\tau, t_\tau) + \sum_{j=0}^{\tau-1} \frac{n}{2(n+2)} f(x_j, t_j)\varepsilon^2 \right],$$

and the associated dynamic programming principle reads

$$u^\varepsilon(x,t) = \inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon)I}} \left\{ \fint_{t-\frac{n}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u^\varepsilon(x + Ay, s) \, dy \, ds \right\} + \frac{n}{2(n+2)} f(x,t) \, \varepsilon^2.$$

When the controller also chooses the coefficient b involved in the time step and the running payoff is given by $-\frac{n+1}{2(n+2)} (f(x,t))^{\frac{1}{n+1}} \varepsilon^2$, we get

$$u^\varepsilon(x,t) = \inf_{\substack{\det(A) \times b=1 \\ A \leq \phi(\varepsilon)I, \, b \leq \phi(\varepsilon)}} \left\{ \fint_{t-\frac{b^2}{n+2}\varepsilon^2}^{t} \fint_{B_\varepsilon(0)} u^\varepsilon(x + Ay, s) \, dy \, ds \right\}$$

$$- \frac{n+1}{2(n+2)} (f(x,t))^{\frac{1}{n+1}} \varepsilon^2.$$

Further details on this program involving games and PDEs can be found in [6, 30, 31], the book [5], and references therein.

References

[1] H. Aimar, I. Gómez and B. Iaffei, Parabolic mean values and maximal estimates for gradients of temperatures, *J. Funct. Anal.* **255** (2008), no. 8, 1939–1956. MR 2462582.

[2] A. R. Arroyo García, *Nonlinear Mean Value Properties Related to the p-Laplacian*, Ph.D. thesis, Universitat Autònoma de Barcelona, 2017. https://www.tdx.cat/handle/10803/405316.

[3] P. Blanc, C. Esteve and J. D. Rossi, The evolution problem associated with eigenvalues of the Hessian, *J. Lond. Math. Soc. (2)* **102** (2020), no. 3, 1293–1317. MR 4186128.

[4] P. Blanc, F. Charro, J. D. Rossi and J. J. Manfredi, A nonlinear mean value property for the Monge-Ampère operator, *J. Convex Anal.* **28** (2021), no. 2, 353–386. MR 4245872.

[5] P. Blanc and J. D. Rossi, *Game Theory and Partial Differential Equations*, Berlin, Boston: De Gruyter, 2019.

[6] P. Blanc and J. D. Rossi, Games for eigenvalues of the Hessian and concave/convex envelopes, *J. Math. Pures Appl. (9)* **127** (2019), 192–215. MR 3960142.

[7] W. Blaschke, Ein Mittelwertsatz und eine kennzeichnende Eigenschaft des logarithmischen Potentials, *Ber. Verh. Sächs. Akad. Wiss. Leipzig* **68** (1916), 3–7. JFM 46.0742.01.

[8] A. Bonfiglioli and E. Lanconelli, Subharmonic functions in sub-Riemannian settings, *J. Eur. Math. Soc. (JEMS)* **15** (2013), no. 2, 387–441. MR 3017042.

[9] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge–Ampère equation, *Comm. Pure Appl. Math.* **37** (1984), no. 3, 369–402. MR 0739925.

[10] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian, *Acta Math.* **155** (1985), no. 3-4, 261–301. MR 0806416.

[11] D. Chopp, L. C. Evans and H. Ishii, Waiting time effects for Gauss curvature flows, *Indiana Univ. Math. J.* **48** (1999), no. 1, 311–334. MR 1722203.

[12] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 1–67. MR 1118699.

[13] P. Daskalopoulos and O. Savin, $C^{1,\alpha}$ regularity of solutions to parabolic Monge-Ampère equations, *Amer. J. Math.* **134** (2012), no. 4, 1051–1087. MR 2956257.

[14] F. del Teso and E. Lindgren, A mean value formula for the variational p-Laplacian, *NoDEA Nonlinear Differential Equations Appl.* **28** (2021), no. 3, Paper No. 27, 33 pp. MR 4241463.

[15] E. B. Fabes and N. Garofalo, Mean value properties of solutions to parabolic equations with variable coefficients, *J. Math. Anal. Appl.* **121** (1987), no. 2, 305–316. MR 0872228.

[16] F. Ferrari, Q. Liu and J. Manfredi, On the characterization of p-harmonic functions on the Heisenberg group by mean value properties, *Discrete Contin. Dyn. Syst.* **34** (2014), no. 7, 2779–2793. MR 3177660.

[17] W. J. Firey, Shapes of worn stones, *Mathematika* **21** (1974), 1–11. MR 0362045.

[18] W. Fulks, A mean value theorem for the heat equation, *Proc. Amer. Math. Soc.* **17** (1966), 6–11. MR 0192200.

[19] C. E. Gutiérrez and Q. Huang, $W^{2,p}$ estimates for the parabolic Monge-Ampère equation, *Arch. Ration. Mech. Anal.* **159** (2001), no. 2, 137–177. MR 1857377.

[20] C. E. Gutiérrez and Q. Huang, A generalization of a theorem by Calabi to the parabolic Monge-Ampère equation, *Indiana Univ. Math. J.* **47** (1998), no. 4, 1459–1480. MR 1687122.

[21] R. S. Hamilton, Worn stones with flat sides, in *A Tribute to Ilya Bakelman (College Station, TX, 1993)*, 69–78, Discourses Math. Appl., 3, Texas A & M Univ., College Station, TX, 1994. MR 1423370.

[22] A. E. Kogoj, E. Lanconelli and G. Tralli, An inverse mean value property for evolution equations, *Adv. Differential Equations* **19** (2014), no. 7-8, 783–804. MR 3252902.

[23] Ü. Kuran, On the mean-value property of harmonic functions, *Bull. London Math. Soc.* **4** (1972), 311–312. MR 0320348.

[24] E. Le Gruyer and J. C. Archer, Harmonious extensions, *SIAM J. Math. Anal.* **29** (1998), no. 1, 279–292. MR 1617186.

[25] E. Le Gruyer, On absolutely minimizing Lipschitz extensions and PDE $\Delta_\infty(u) = 0$, *NoDEA Nonlinear Differential Equations Appl.* **14** (2007), no. 1-2, 29–55. MR 2346452.

[26] P. Lindqvist and J. Manfredi, On the mean value property for the *p*-Laplace equation in the plane, *Proc. Amer. Math. Soc.* **144** (2016), no. 1, 143–149. MR 3415584.

[27] W. Littman, G. Stampacchia and H. F. Weinberger, Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **17** (1963), 43–77. MR 0161019.

[28] J. J. Manfredi, M. Parviainen and J. D. Rossi, An asymptotic mean value characterization for *p*-harmonic functions, *Proc. Amer. Math. Soc.* **138** (2010), no. 3, 881–889. MR 2566554.

[29] J. J. Manfredi, M. Parviainen and J. D. Rossi, An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games, *SIAM J. Math. Anal.* **42** (2010), no. 5, 2058–2081. MR 2684311.

[30] J. J. Manfredi, M. Parviainen and J. D. Rossi, On the definition and properties of *p*-harmonious functions, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **11** (2012), no. 2, 215–241. MR 3011990.

[31] Y. Peres, O. Schramm, S. Sheffield and D. B. Wilson, Tug-of-war and the infinity Laplacian, *J. Amer. Math. Soc.* **22** (2009), no. 1, 167–210. MR 2449057.

[32] Y. Peres and S. Sheffield, Tug-of-war with noise: a game-theoretic view of the *p*-Laplacian, *Duke Math. J.* **145** (2008), no. 1, 91–120. MR 2451291.

[33] I. Privaloff, Sur les fonctions harmoniques, *Mat. Sb.* **32** (1925), no. 3, 464–471. JFM 51.0363.02.

[34] N. Suzuki and N. A. Watson, A characterization of heat balls by a mean value property for temperatures, *Proc. Amer. Math. Soc.* **129** (2001), no. 9, 2709–2713. MR 1838795.

[35] L. Tang, Regularity results on the parabolic Monge–Ampère equation with *VMO* type data, *J. Differential Equations* **255** (2013), no. 7, 1646–1656. MR 3072666.

[36] K. Tso, Deforming a hypersurface by its Gauss–Kronecker curvature, *Comm. Pure Appl. Math.* **38** (1985), no. 6, 867–882. MR 0812353.

[37] B. Wang and J. Bao, Asymptotic behavior on a kind of parabolic Monge–Ampère equation, *J. Differential Equations* **259** (2015), no. 1, 344–370. MR 3335929.

[38] R. H. Wang and G. L. Wang, On existence, uniqueness and regularity of viscosity solutions for the first initial-boundary value problems to parabolic Monge-Ampère equation, *Northeast. Math. J.* **8** (1992), no. 4, 417–446. MR 1210195.

[39] R. H. Wang and G. L. Wang, The geometric measure theoretical characterization of viscosity solutions to parabolic Monge-Ampère type equation, *J. Partial Differential Equations* **6** (1993), no. 3, 237–254. MR 1234574.

[40] N. A. Watson, A theory of subtemperatures in several variables, *Proc. London Math. Soc. (3)* **26** (1973), 385–417. MR 0315289.

[41] J. Xiong and J. Bao, On Jörgens, Calabi, and Pogorelov type theorem and isolated singularities of parabolic Monge–Ampère equations, *J. Differential Equations* **250** (2011), no. 1, 367–385. MR 2737847.

[42] W. Zhang, J. Bao and B. Wang, An extension of Jörgens–Calabi–Pogorelov theorem to parabolic Monge–Ampère equation, *Calc. Var. Partial Differential Equations* **57** (2018), no. 3, Paper No. 90, 36 pp. MR 3800271.

Pablo Blanc
Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35, FI-40014 Jyväskylä, Finland
pblanc@dm.uba.ar

Fernando Charro
Department of Mathematics, Wayne State University, 656 W. Kirby, Detroit, MI 48202, USA
fcharro@wayne.edu

Juan J. Manfredi
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
manfredi@pitt.edu

Julio D. Rossi✉
Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Pabellón I, Ciudad Universitaria, 1428 Buenos Aires, Argentina
jrossi@dm.uba.ar