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CLIQUE COLORING EPT GRAPHS ON BOUNDED DEGREE TREES

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ABSTRACT. The edge-intersection graph of a family of paths on a host tree is called an EPT graph. When the host tree has maximum degree h, we say that the graph is $[h, 2, 2]$. If the host tree also satisfies being a star, we have the corresponding classes of EPT-star and $[h, 2, 2]$-star graphs. In this paper, we prove that $[4, 2, 2]$-star graphs are 2-clique colorable, we find other classes of EPT-star graphs that are also 2-clique colorable, and we study the values of h such that the class $[h, 2, 2]$-star is 3-clique colorable. If a graph belongs to $[4, 2, 2]$ or $[5, 2, 2]$, we prove that it is 3-clique colorable, even when the host tree is not a star. Moreover, we study some restrictions on the host trees to obtain subclasses that are 2-clique colorable.

1. INTRODUCTION

An *EPT representation* of a graph G is a pair $\langle \mathscr{P}, T \rangle$, where T is a tree and $\mathscr{P}$ is a family $(P_v)_{v \in V(G)}$ of paths of T satisfying that two vertices v and v' are adjacent in G if and only if their corresponding paths P_v and $P_{v'}$ have edge intersection. The tree T is called a *host tree*. We say that G is an *EPT graph* if it has an EPT representation. Moreover, when the host tree is a star, we say that the graph is *EPT-star*. The EPT class was first investigated by Golumbic and Jamison [16, 17].

When the maximum degree of the host tree is at most h, the EPT representation of G is called an $(h, 2, 2)$*-representation*. The class of graphs that admit an $(h, 2, 2)$-representation is denoted by $[h, 2, 2]$. This definition clearly implies that $[h', 2, 2]$ is a subclass of $[h, 2, 2]$ for $h' < h$. We say that a graph is $[h, 2, 2]$*-star* if it has an $(h, 2, 2)$-representation where the host tree is a star. It is known that $[3, 2, 2]$ is the class of EPT Chordal graphs [19], while $[4, 2, 2]$ is the class of EPT Weakly Chordal graphs [18]. A complete hierarchy of related graph classes emerging by imposing restrictions on the tree representation is published in [16]. The class $[\infty, 2, 2]$ is the class of EPT graphs, where ∞ means that there is no restriction in the maximum degree of the host tree. Recognizing EPT graphs is an NP-complete problem [16].

EPT graphs are used in network applications, where the problem of scheduling undirected calls in a tree network is equivalent to the problem of coloring an EPT

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graph (see [14, 16]). The communication network is represented as an undirected interconnection graph, where each edge is associated with a physical link between two nodes. An undirected call is a path in the network. When the network is a tree, this model is clearly an EPT representation. Coloring the EPT graph such that two adjacent vertices have different colors implies that paths sharing at least one common edge in the EPT representation have different colors, meaning that undirected calls that share a physical link are scheduled in different times. Another application is assigning wavelengths to connections in an optical network, where virtual connections share physical links by wavelength-division multiplexing. This problem is equivalent to the problem of coloring an EPT graph as follows: in the optical interconnection graph, every edge is associated with an optical link between two vertices. Virtual connections that share the same optical link must have different wavelengths. The coloring may be associated with assignment of wavelengths such that connections of the same color can be assigned the same wavelength. For a survey, see [5, 15].

Let $\langle \mathscr{P}, T\rangle$ be an EPT representation of G. For an edge e of T, we define $\mathscr{P}[e] = \{P \in \mathscr{P} \mid e \in P\}$. For any induced subgraph H of T isomorphic to $K_{1,3}$, let $\mathscr{P}[H] = \{P \in \mathscr{P} \mid P$ contains two edges of $H\}$. Clearly, each $\mathscr{P}[e]$ and each $\mathscr{P}[H]$ corresponds to a complete set of G. A clique of the form $\mathscr{P}[e]$ is called an *edge clique*, and one of the form $\mathscr{P}[H]$, a *claw clique*. Thus we have:

Theorem 1.1 ([16]). *Let G be an EPT graph, and let $\langle \mathscr{P}, T\rangle$ be an EPT representation of G. Every clique C of G corresponds to a subcollection of the form $\mathscr{P}[e]$ or $\mathscr{P}[H]$ for some edge e in T or some induced claw H in T.*

Note that the condition of being edge clique or claw clique depends on the given EPT representation.

A family $\mathscr{F}$ satisfies the *Helly property* if every pairwise intersecting subfamily of $\mathscr{F}$ has an element in common, that is, for any subfamily $\mathscr{F}'$ of $\mathscr{F}$ we have $F_i \cap F_j \neq \emptyset$ for all $F_i, F_j \in \mathscr{P}'$ implies that $\bigcap_{F_i \in \mathscr{F}'} F_i \neq \emptyset$.

A *Helly-EPT representation* is an EPT representation $\langle \mathscr{P}, T\rangle$ such that the family $(E(P))_{P\in\mathscr{P}}$ satisfies the Helly property. We say that a graph is *EPT-Helly* if it has an EPT-Helly representation. As a consequence of these definitions, in an EPT-Helly representation all cliques are edge cliques.

For an integer $k > 2$, a *pie* of size k in an EPT representation $\langle \mathscr{P}, T\rangle$ is a star subgraph of T with center q and neighbours $q_1, \dots, q_k$, and a subfamily of paths $P_1, \dots, P_k$ such that $\{q_i, q, q_{i+1}\} \subseteq V(P_i)$ for $i \in [1, k-1]$ (the natural interval $\{1, 2, \dots, k-1\}$), and $\{q_k, q, q_1\} \subseteq V(P_k)$. In that case, we say that the paths $P_1, \dots, P_k$ of $\mathscr{P}$ form a pie. Golumbic and Jamison introduced the notion of pie to describe the EPT representations of induced cycles.

Theorem 1.2 ([17]). *Let $\langle \mathscr{P}, T\rangle$ be an EPT representation of a graph G. If G contains an induced cycle C_k, with $k > 3$, then $\langle \mathscr{P}, T\rangle$ contains a pie of size k whose paths are in one to one correspondence with the vertices of C_k.*

A *proper k-coloring* of a graph G is a function $f : V(G) \to \{1, 2, \dots, k\}$ such that if $v, w \in V(G)$ are adjacent in G, then $f(v) \neq f(w)$. A graph G is *k-colorable* if it has a proper k-coloring. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G is k-colorable.

A *k-clique coloring* of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that no clique of G with size at least two has all its vertices with the same color (we usually say that a set is monochromatic when all its elements have the same color). A graph G is *k-clique colorable* if it has a k-clique coloring. The *clique chromatic number* of G, denoted by $\chi_c(G)$, is the smallest integer k such that G is k-clique colorable. The clique coloring can also be seen as coloring the clique hypergraph of a graph. The question of coloring clique hypergraphs was proposed in [13].

Clique coloring has some similarities with usual coloring. For example, every proper k-coloring is also a k-clique coloring, and $\chi(G)$ and $\chi_c(G)$ coincide if G is triangle-free. However, there are also major differences. For example, a clique coloring of a graph need not be a clique coloring for its subgraphs. Indeed, subgraphs may have a greater clique chromatic number than the original graph. Another difference is that even a 2-clique colorable graph can contain an arbitrarily large clique. It is known that the 2-clique coloring problem is NP-complete, even under different constraints [3, 20]. Many families of graphs are 3-clique colorable, for example, comparability graphs, co-comparability graphs, circular arc graphs and the k-powers of cycles [7, 8, 9, 13, 12]. In [3], Bacsó et al. proved that almost all perfect graphs are 3-clique colorable and conjectured that all perfect graphs are 3-clique colorable. This conjecture was recently disproved by Charbit et al. [10], who showed that there exist perfect graphs with arbitrarily large clique chromatic number. Previously known families of graphs having unbounded clique chromatic number are, for example, EPT graphs, triangle-free graphs, and line graphs [3, 9, 22]. It has been proved that chordal graphs, and in particular interval graphs, are 2-clique colorable [23].

In [9], EPT-Helly graphs are called UEH graphs and it is proved that, even though the clique chromatic number is unbounded for EPT graphs, there is a bound for the clique chromatic number of UEH graphs.

Theorem 1.3 ([9])**.** *If G is an UEH graph, then $\chi_C(G) \leq 3$.*

Since triangle-free EPT graphs are EPT-Helly graphs (the existence of a claw clique in a representation would imply the existence of a triangle), we have the following corollary.

Corollary 1.4. *Let G be a triangle-free EPT graph. Then $\chi_C(G) \leq 3$.*

Motivated by the applications of coloring EPT graphs, and the difficulty (and unboundedness) of this problem in general, we begin to study the topic by considering restrictions to the EPT representations, mostly based on the maximum degree, in search for bounds in these restricted cases. The paper is organized as follows.

Section 2 contains the necessary definitions and preliminary results. In Section 3, we show that $[4, 2, 2]$-star graphs, $[5, 2, 2]$-star graphs different from C_5, and diamond-free EPT-star graphs different from an odd cycle are all 2-clique colorable. Moreover, we deduce that the graphs in $[16, 2, 2]$-star are 3-clique colorable.

In Section 4, we prove that C_5 and the prism graph are the only minimal graphs in $[5, 2, 2] - [4, 2, 2]$, and we prove that $[5, 2, 2]$ graphs are 3-clique colorable. In Section 5, we show that 3 is still a tight bound for the clique chromatic number of $[4, 2, 2]$ graphs, and we introduce some restrictions on the host tree to obtain 2-clique colorable subclasses. Finally, in Section 6 we present conclusions and some open questions.

2. Preliminaries

The graphs considered in this work are simple and finite. Many of the definitions and notations that we use are considered standard in graph theory (see [6, 21]), and they will be used without a previous definition.

For a *graph* $G = (V, E)$, V is the set of *vertices* and E is the set of *edges*. If $|V(G)| = 1$, G is called a *trivial graph*. The *open neighborhood* of a vertex v, denoted by $N_G(v)$, is the set of vertices which are adjacent to v. Its *closed neighborhood*, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. A vertex v is *isolated* when $N_G(v) = \emptyset$, and it is *universal* when $N_G[v] = V(G)$. The *degree* of v, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. To simplify, when there is no confusion we omit the subscript G and simply write $N(v)$, $N[v]$ or $d(v)$. Two vertices u, $v \in V(G)$ are called *true twins* (or simply *twins*) if $N[u] = N[v]$, and they are called *false twins* if $N(u) = N(v)$ and u is not adjacent to v in G.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If G is a graph and X is a nonempty subset of vertices of G, the subgraph of G *induced* by X is the subgraph H such that $V(H) = X$ and $E(H)$ is the set of edges of G that have both endpoints in X, and it is denoted by $G[X]$. We say that H is an induced subgraph of G when $H = G[X]$ for some subset X of vertices of G.

A graph is *complete* if all its vertices are pairwise adjacent. We denote by K_n the complete graph with n vertices. In an abuse of notation, we say that a set of vertices of G is complete when they induce a complete subgraph. A *clique* is a maximal complete set, that is, it is a complete set contained in no other complete set of the graph. We call $\mathscr{C}(G)$ the family of cliques of G. In the literature, it is more usual to define a clique without the maximality condition. The reader must be aware that, for the purposes of this paper, whenever we refer to a clique we assume that it is maximal.

The *diamond* is the graph obtained by removing one edge from K_4.

A *path* P is a nonempty sequence $v_0, v_1, \ldots, v_k$ of different vertices of G such that, for all i, $1 \leq i \leq k$, v_{i-1} and v_i are adjacent in G. We say that P is a path between v_0 and v_k. These two vertices are called the endpoints of P. The edges of P are the ones between its consecutive vertices. We denote the set of edges of P by $E(P)$. The number k of edges in the path is called the *length* of P. When it is more convenient, we will express a path P as the sequence $e_1, e_2, \ldots, e_k$ of its edges, where, for $1 \leq i \leq k$, $e_i = v_{i-1}v_i$. Two paths P_1 and P_2 *share an edge* e if both P_1 and P_2 have e as an edge. In this case, we say that P_1 and P_2 have *edge intersection*, or that P_1 *edge-intersects* P_2.

A *cycle* C is a nonempty sequence $v_0, v_1, \ldots, v_k$, where v_0 and v_k are the only equal vertices and such that, for all i, $1 \leq i \leq k$, v_{i-1} and v_i are adjacent in G. We also refer to k as the *length* of C. A cycle is *even* or *odd* when its length is even or odd, respectively. A *chord* of a cycle is an edge connecting nonconsecutive vertices of the cycle. An *induced cycle* of a graph is a cycle that has no chords. We denote the induced cycle of n vertices by C_n.

A graph G is *connected* if, for every pair of different vertices of G, there exists a path between them. A *connected component* of G is a maximal connected subgraph of G. A graph G is *disconnected* if it is not connected. The graphs in this paper are assumed to be connected, unless stated otherwise. Thus, when the graph is nontrivial, its cliques have size at least 2.

A *tree* is a connected graph that has no cycles. A connected subgraph of a tree is called a *subtree*. A vertex of degree one in a tree is called a *leaf*. We say that a *tree has degree h* when the maximum degree of its vertices is h. An edge of the tree T is a *pending edge* if it is incident on a leaf of T. A vertex v of a tree that is not a leaf is an *external vertex* if all the edges incident on it, with the possible exception of one, are pending edges.

The *star* of size n, denoted by S_n, is the complete bipartite graph $K_{1,n}$, that is, it is a tree with $n+1$ vertices such that one of them is universal and the other n are leaves. The star of size 3, $K_{1,3}$, is known as a *claw graph*.

A graph is *chordal* if it has no induced cycles C_n, with $n \geq 4$.

A vertex v of G is *simplicial* if $N_G[v]$ is a clique. Then we have:

Theorem 2.1 ([11])**.** *Let G be a chordal graph. Then G has a simplicial vertex. If G is not complete, then it has two nonadjacent simplicial vertices.*

As a consequence of this theorem, we can infer that a graph is chordal if and only if every induced subgraph of it has a simplicial vertex.

A graph G is *weakly chordal* if, for every $n \geq 5$, G has neither C_n nor its complement as an induced subgraph.

3. Clique coloring EPT-star graphs

In this section, we give some results related to the clique coloring of EPT-star graphs. To start, we show that it is easy to 2-clique-color $[4, 2, 2]$-star graphs.

Proposition 3.1. *Let G be a $[4, 2, 2]$-star graph. Then $\chi_c(G) \leq 2$.*

Proof. If G has no induced C_4, then G is chordal and it is 2-clique colorable [23]. If G has an induced C_4, we color its vertices alternately with colors 0 and 1. We have that every extension of this coloring using the same colors 0 and 1 is a clique coloring, because every clique of G, being an edge clique or claw clique, contains an edge of the induced C_4, so it contains a vertex with color 0 and a vertex with color 1. Hence, G is 2-clique colorable. □

We do not have the same bound for $[5, 2, 2]$-star graphs because C_5 is in the class and its clique-chromatic number is 3.

To find the values of h for which an upper bound of 3 stays true, we can connect this question to line graphs and Ramsey theory.

Recall that, given a graph G, the *line graph* $L(G)$ of G has $E(G)$ as vertex set such that two different edges of G are adjacent in $L(G)$ if and only if they share at least one endpoint.

Golumbic and Jamison proved in [17] that EPT-star graphs are just the line graphs of multigraphs (graphs that admit loops and multiple edges).

To establish the equality, given a multigraph G, one can take a star whose leaves are just the vertices of G. To build an EPT-star representation of $L(G)$, an edge of G with endpoints u and v is represented by the path in the star connecting them, while a loop of G on the vertex v is represented by the path of the star that consists solely of the edge incident on v.

Conversely, given an EPT-star representation for a graph G, we can apply the inverse procedure to obtain a multigraph H such that $L(H) = G$.

For an example of the construction, refer to Figure 1.

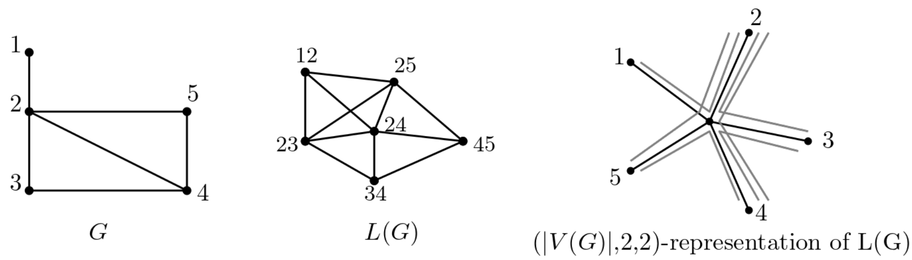

FIGURE 1. Example of the construction of an EPT-star representation of $L(G)$.

We denote by $R_k(3)$ the minimum number n such that every k-edge coloring of K_n has a monochromatic triangle. The following result concerning the clique chromatic number of line graphs is derived from the work in [2, 4].

Proposition 3.2 ([4]). *Let G be a graph, $G \neq C_5$, such that $\chi(G) = M$. Then $M < R_k(3)$ implies that $\chi_C(L(G)) \leq k$. Conversely, for the case of complete graphs, we have that $\chi_c(L(K_M))$ is the minimum number k such that $M < R_k(3)$.*

By the previous discussion, we know that every $[5, 2, 2]$-star graph is the line graph of a multigraph with at most five vertices. Additionally, we have $R_2(3) = 6$, which leads to the following conclusion.

Proposition 3.3. *Let G be a $[5, 2, 2]$-star graph, $G \neq C_5$. Then $\chi_c(G) \leq 2$.*

Furthermore, it is known that the Ramsey number $R_3(3)$ is equal to 17, so Proposition 3.2 now yields the following conclusion.

Proposition 3.4. *Every graph in $[16, 2, 2]$-star is 3-clique colorable. Additionally, $L(K_{17})$ is a $[17, 2, 2]$-star graph whose clique chromatic number is 4.*

Now we focus on finding conditions that make an EPT-star graph 2-clique colorable and that do not depend on the maximum degree of the host tree.

We include the following simple result as a proposition since it will be used frequently in proofs.

Proposition 3.5. *Let G be a graph, $v, w \in V(G)$, $v \neq w$, such that $N[v] \subseteq N[w]$ and $k \geq 2$. If $G - v$ is k-clique colorable, then G is also k-clique colorable.*

Proof. Consider a clique coloring of $G - v$ using colors $1, 2, \ldots, k$. Extend this coloring by giving v a color different from that of w.

Consider now a clique C of G. If $v \notin C$, then C is a clique of $G - v$ and hence is not monochromatic. If $v \in C$, then the condition $N[v] \subseteq N[w]$ implies that w is also in C. Since v and w have different colors, C is not monochromatic. Therefore, the coloring of G is a k-clique coloring. □

Particular cases of this proposition are when v is a simplicial vertex or v has a twin vertex. When one looks at the representation rather than the graph, particular cases are when the representation has two identical paths, or when it has a path of length one. The reason for this is that vertices whose paths are of length 1 are simplicial vertices and vertices with equal paths are twins.

Lemma 3.6. *If G is a graph that contains as a subgraph a tree T such that, for all $C \in \mathscr{C}(G)$ (with at least two vertices), T has an edge contained in C, then $\chi_c(G) \leq 2$.*

Proof. We know that $\chi(T) \leq 2$. Any extension of a proper 2-coloring of T (using those two colors only) provides a clique coloring of G. In fact, every clique C of G with at least two vertices contains an edge e of T whose endpoints have different color. □

The following result can be derived from [9, Theorem 2.2]. As no proof of it is given there, we include a proof that is adjusted to our needs.

Lemma 3.7. *Let G be an EPT-Helly-star graph. If $G \neq C_{2n+1}$, $n > 1$, then $\chi_c(G) \leq 2$.*

Proof. The result is trivial if G has just one vertex, so we assume that G has at least two vertices. If G is an even cycle, then the usual proper 2-coloring of it is also a clique coloring. We will prove this result for the case where G is not a cycle of length at least four using Lemma 3.6.

Let $\mathcal{G}$ be the family that contains every connected spanning subgraph of G such that, for all $C \in \mathscr{C}(G)$, we have that the subgraph has an edge contained in C. Among the subgraphs in $\mathcal{G}$ with the minimum amount of edges, we take H without cycles or, if that is not possible, with the minimum girth. Note that every vertex v of G is contained in at most two cliques because, for some EPT-Helly-star representation of G, every clique is an edge clique, and the path P_v has at most two edges. Additionally, if u is adjacent to v and they are not twins, then $P_u \neq P_v$, and the intersection of these paths consists of a single edge, and hence there is a unique clique containing the edge uv.

Let us see that H is a tree. Suppose, on the contrary, that H has some cycle, and let C be one of minimum length in H. Let e_1 and e_2 be any two consecutive edges of the cycle with vertices x_1, v and x_2, v, respectively.

We prove that x_1 and x_2 are not adjacent in G. Suppose, on the contrary, that they are adjacent. We cannot have both a clique containing e_1 but not containing x_2 and a clique containing e_2 and not containing x_1, because together with a clique containing $\{x_1, x_2, v\}$, we would get the contradiction that v is contained in at least three cliques. Therefore, every clique containing e_1 also contains e_2, or every clique containing e_2 also contains e_1. In the first case, we conclude that $H - e_1$ is in $\mathcal{G}$. In the second case, we conclude that $H - e_2$ is in $\mathcal{G}$. Both cases contradict the minimality of H. Therefore, x_1 and x_2 are not adjacent. Particularly, C cannot be a triangle.

As G is different from C_n, $n \geq 4$, it has an edge e_3 that is not in C. We now check that e_3 can be chosen so that it is a chord of C or it is an edge of H with one of its endpoints in C. If C has no chord in G, then every edge of G not in C has at least one endpoint not in C. Let w be a vertex of G not in C. Since H is connected, it contains one path from w to a vertex of C. If among these paths we consider one of minimum length, then the final edge of the path has one endpoint in C and the other endpoint not in C.

Call v one of the endpoints of e_3 in C, and define e_1, e_2, x_1 and x_2 as before. Let C_1 and C_2 be the cliques containing e_1 and e_2, respectively. Since v_1 and v_2 are not adjacent, these cliques are different and are the only cliques containing v. Furthermore, C_1 is the only clique containing e_1, and C_2 is the only clique containing e_2.

Let C_3 be a clique containing e_3. By the previous paragraph, it follows that $C_3 = C_1$ or $C_3 = C_2$. Suppose without loss of generality that $C_3 = C_1$.

If e_3 is a chord of the cycle, then it cannot be an edge of H. Since $C_3 = C_1$, $H + e_3 - e_1$ is in $\mathcal{G}$, and its girth is less than that of H, which is a contradiction.

If e_3 is not a chord and is in H, then $H - e_1$ is in $\mathcal{G}$, which contradicts the minimality of H.

Therefore, H cannot have a cycle. Since it is connected, it is a tree. The final conclusion now comes from Lemma 3.6. □

Thus, we have the following corollary about diamond-free graphs.

Corollary 3.8. *Let G be an EPT-star graph. If G is not an odd cycle and it is diamond-free, then $\chi_c(G) \leq 2$.*

Proof. It is enough to show that G is EPT-Helly. This is trivial when G has at most 3 vertices.

Assume now that G has more vertices, and consider an EPT-star representation of it.

If this representation has a claw clique C and no vertex outside C, then G is complete, and hence EPT-Helly.

If there exists a vertex v not in C, we can choose v so that it is adjacent to at least one vertex of C. As a consequence, the path P_v contains only one edge of the claw associated to C. Let u_1, u_2 and u_3 be three vertices of C with different paths in the representation. Thus, v, u_1, u_2 and u_3 induce a diamond, which is a contradiction.

Therefore, G is EPT-Helly. By Lemma 3.7, G is 2-clique colorable. □

It is possible to extend Lemma 3.7 to EPT-star graphs that have claw cliques in their representations under special conditions.

Theorem 3.9. *Let G be an EPT-star graph and $G \neq C_{2n+1}$, $n > 1$, with a representation such that its claw cliques (if any) are pairwise disjoint. Then $\chi_c(G) \leq 2$.*

Proof. By Proposition 3.5, it is enough to prove it for graphs with star representations that do not have identical paths or paths of length 1. Thus, every claw clique consists of exactly three paths.

If G is EPT-Helly, then G is 2-clique colorable. Otherwise, let $C_1, \dots, C_n$ be the claw cliques of the EPT representation of G. We construct a subgraph G' as follows: as G is connected, there is a vertex u that is adjacent to all but one vertex v_1 of C_1. We first remove v_1 from G. For $2 \leq i \leq n$, remove one vertex in C_i different from u. As a result of this construction, G' is an EPT-Helly-star graph which contains a triangle (consisting of u and the remaining vertices of C_1), and hence $G' \neq C_n$, $n \geq 4$ (see Figure 2).

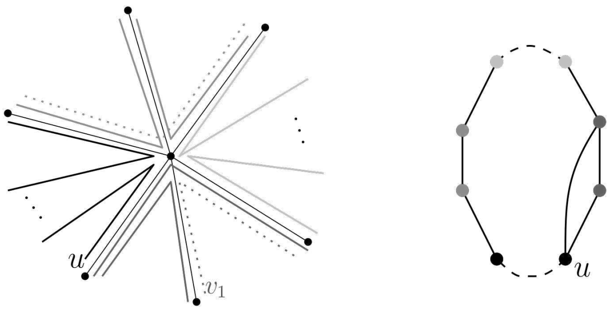

FIGURE 2. Construction of G'. The dotted paths correspond to the vertices that are removed.

Let e be an edge of the host tree of G such that the paths of the representation that contain e form an edge clique of G. We prove that the paths corresponding to vertices of G' that contain e form an edge clique of G'.

Suppose first that the representation has exactly two paths that contain e. If there is a claw clique that contains one of those paths, then it should contain both, contradicting that we have an edge clique. Thus, such a claw clique cannot exist and the edge clique of G is also an edge clique of G'.

If there are at least three paths containing e, then the representation of G' keeps the paths that are not in any claw clique and at least one path per claw clique such that the claw contains e. This ensures that the representation of G' has at least two paths containing e, which determine an edge clique of G'.

By the proof of Lemma 3.7, there is a spanning tree T of G' such that, for every clique C of G', there exists an edge of T contained in C. For each claw clique C_i of G, we add v_i to T and we make it adjacent to another vertex in C_i. The additions ensure that we get a tree T' such that for every claw clique C there is an edge of T' contained in C. The same was true for edge cliques and T, so it is true for T' as well. By Lemma 3.6, G is 2-clique colorable. □

4. Clique coloring $[5, 2, 2]$ graphs

In this section, we prove that the graphs belonging to the class $[5, 2, 2]$ are 3-clique colorable. To do it, it is helpful to find what type of graphs are $[5, 2, 2]$ and are not in $[4, 2, 2]$.

Definition 4.1 ([1])**.** Let n_1, n_2 and n_3 be positive integers. A *general prism* F_{n_1,n_2,n_3} consists of two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, and three disjoint chordless paths Q_1, Q_2 and Q_3 such that, for $1 \leq i \leq 3$, Q_i is an a_ib_i-path of length n_i.

The *prism graph* is the general prism graph $F_{1,1,1}$ (see Figure 3). General prisms are all EPT graphs, and we have the following results concerning the degrees of the trees that can be used to represent them.

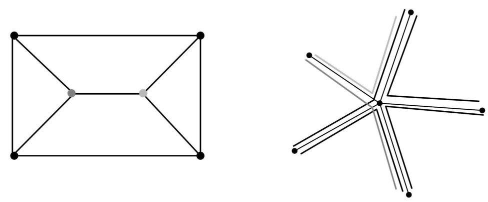

Figure 3. Prism graph and its EPT representation.

Lemma 4.2 ([1])**.** *The general prism F_{n_1,n_2,n_3} is an $[h, 2, 2]$ graph for $h = n_1 + n_2 + n_3 + 2$.*

Theorem 4.3 ([1])**.** *Let $h = n_1 + n_2 + n_3 + 1$. The general prism F_{n_1,n_2,n_3} is $\{C_n : n > h\}$-free and it is not an $[h, 2, 2]$ graph.*

Remark 4.4. By Lemma 4.2 and Theorem 4.3, we know that the prism graph is in $[5, 2, 2] - [4, 2, 2]$. That is, it belongs to the class $[5, 2, 2]$ but not to the class $[4, 2, 2]$.

We now find all the minimal graphs that are in $[5, 2, 2] - [4, 2, 2]$.

Theorem 4.5. *The graph C_5 and the prism graph are the only minimal graphs that are in $[5, 2, 2] - [4, 2, 2]$.*

Proof. The graph C_5 is in $[5, 2, 2] - [4, 2, 2]$. It is minimal because if we remove any vertex of C_5, the resulting graph is a path, and paths are $[4, 2, 2]$ graphs. By the previous results, we also know that the prism is in $[5, 2, 2] - [4, 2, 2]$. It is also a minimal graph not in $[4,2,2]$ because if a single vertex is removed, the resulting graph consists of an induced cycle of length four plus a fifth vertex that is adjacent to just two adjacent vertices of the cycle. This graph can be represented by adding to the representation of C_4 on a star of size 4 a path that consists of a single edge of the star.

Now suppose that G is a $[5,2,2]$ graph with no induced C_5 and no induced prism. We will prove that G is also $[4,2,2]$. Consider a $(5,2,2)$-representation of G with a host tree T with the minimum amount of vertices of degree 5. We will prove that T has actually no vertex of degree five.

Suppose, on the contrary, that T has a vertex x of degree 5. Call T_x to the subtree of T which is the star of size 5 around x. Call $\mathscr{P}'$ to the subfamily of paths in $\mathscr{P}$ that contain two edges of T_x. Given two edges e_1 and e_2 of T_x, we say that e_1 is $\mathscr{P}'$-dominated by e_2 if every path in $\mathscr{P}'$ that contains e_1 also contains e_2. We show by contradiction that such domination relationship cannot take place. Let the endpoints of e_1 and e_2 different from x be y_1 and y_2, respectively. If e_1 is $\mathscr{P}'$-dominated by e_2, we create a new $(5,2,2)$-representation for G in the following way:

We remove the edge e_1 from T, subdivide the edge e_2 (thus creating a new vertex z) and we make y_1 adjacent to z to obtain a new tree T'. For every path P in $\mathscr{P}$, P changes according to the following rules:

If it contains e_1, we replace x with z in the sequence of vertices of the path.

If it contains e_2 and not e_1, then we replace e_2 with its subdivision.

In any other case, P does not change.

This is also a representation for G, because two paths of $\mathscr{P}$ share the edge e_1 if and only if their corresponding modified paths share the edge y_1z; and two paths of $\mathscr{P}$ share the edge e_2 if and only if the modified paths share the edge zy_2. In this representation, the degree of x in the host tree is 4, the degrees of y_1 and y_2 are the same as in T and the degree of z is 3. Thus, we have one less vertex of degree 5 than in T, which contradicts the way that T was chosen. As a consequence, no edge of T_x can be $\mathscr{P}'$-dominated by another.

Let G' be the subgraph of G induced by the paths of $\mathscr{P}'$. We consider two cases.

Case 1: G' has an induced C_4.

Consider an induced C_4 in G'. Let e be the only edge of T_x that is not used by the paths in $\mathscr{P}'$ that induce the C_4, and let y be the endpoint of e different from x. As e is not $\mathscr{P}'$-dominated by another edge, there are two other edges e' and e'' in T_x such that $\mathscr{P}'$ has a path P_1 containing e and e' and a path P_2 containing e and e''.

If e' and e'' are both contained in one of the paths that induce the C_4, then G' has an induced C_5, which is a contradiction. Otherwise, the paths in the pie corresponding to the C_4, P_1, and P_2 induce a prism, another contradiction.

Case 2: G' has no induced C_4.

G' is a chordal graph. Then, it has a simplicial vertex. Let P be the path in $\mathscr{P}'$ corresponding to that vertex. If the paths of $\mathscr{P}'$ that edge-intersect P form an edge clique, let e be an edge of T_x contained in all these paths, and let e' be the other edge of P in T_x. It follows that e' is $\mathscr{P}'$-dominated by e, which is impossible.

As a consequence of the previous paragraph, the paths of $\mathscr{P}'$ that edge-intersect P form a claw clique. Let e_1, e_2, and e_3 be the edges of T_x in the claw, in such a way that P contains e_1 and e_2. Let e_4 and e_5 be the other edges of T_x. By the simpliciality, every path in $\mathscr{P}'$ that contains e_4 or e_5 cannot contain e_1 or e_2. We take advantage of this fact to build a new representation for G as follows:

Using a similar notation as before, we call y_i the endpoint of e_i that is different from x for $1 \leq i \leq 5$.

In T, we subdivide e_3 (creating a new vertex z), we remove the edges e_4 and e_5, and we add the edges y_4z and y_5z to obtain a new tree.

Every path P in $\mathscr{P}$ changes according to the following rules:

If it contains e_4 or e_5, then x is replaced with z.

If it contains e_3 but not e_4 or e_5, then e_3 is replaced with its subdivision.

Every other path stays the same.

Arguing like before, this procedure results in a new representation for G with a host tree that has one less vertex of degree 5 than T, which results in a contradiction.

As every case led to a contradiction, we conclude that T is necessarily a tree with maximum degree four, and hence G is $[4, 2, 2]$. ☐

A close examination of Theorem 4.5 reveals that it can be derived from the following result, which is proved using the same arguments:

Proposition 4.6. *Let G be a $[5, 2, 2]$ graph with $(5, 2, 2)$-representation $\langle \mathscr{P}, T \rangle$ and x be a vertex of T. If x has degree 5 and the subgraph of G induced by the paths in $\mathscr{P}$ that contain x has no induced C_5 and no induced prism, then there exists another $(5, 2, 2)$-representation for G whose host tree has less vertices of degree 5 than T.*

Now, we prove that $[5, 2, 2]$ graphs are 3-colorable. To do it, we indicate how to deal with induced C_5's and induced prisms.

Theorem 4.7. *Let G be a $[5, 2, 2]$ graph. Then $\chi_C(G) \leq 3$.*

Proof. We do it by induction on the number of vertices. The property is trivial for graphs with at most 3 vertices. Suppose that the property holds for every graph with at most k vertices and let G have $k + 1$ vertices.

Suppose, without loss of generality, that G is connected. Let $\langle \mathscr{P}, T \rangle$ be a $(5, 2, 2)$-representation of G such that the number of vertices of degree 5 in T is minimum. Additionally, among the possible host trees with minimum amount of vertices of degree 5, choose T to have the minimum number of edges.

If there are vertices x and y such that $N[x] \subseteq N[y]$, then we can apply the induction hypothesis on $G - x$ and apply Proposition 3.5 to obtain the desired conclusion. Suppose from now on that there are no vertices like this. This implies that there are neither simplicial nor twin vertices.

If T is a star, then, by Proposition 3.3 and the fact that C_5 is 3-clique colorable, we infer that G is 3-clique colorable.

Otherwise, consider an external vertex w in T. Let v be the neighbour of w that is not a leaf, and let $e = vw$. Consider T' and T'' the subtrees of T such that T' is induced by v, w and all the vertices of T that are not neighbours of w, and T'' be the star induced by w and its neighbours in T. Let $\mathscr{P}'$ and $\mathscr{P}''$ be the subsets of $\mathscr{P}$ such that P is in $\mathscr{P}'$ if and only if P has an edge in T', and P is in $\mathscr{P}''$ if and only if P has an edge in T''. Let G' and G'' be the edge intersection graphs of $\mathscr{P}'$ and $\mathscr{P}''$, respectively.

By the minimality of T, $\mathscr{P}''$ must have a path P that does not contain the edge e. All paths of the representation have length at least two by the absence of simplicial vertices. For that reason, T'' cannot be a star of size 2. Neither can the size of the star be 3, because, otherwise, P would correspond to a simplicial vertex of G. Thus, T'' is a star of size at least 4.

Suppose initially that T'' is a star of size exactly 4. Let u_1, u_2, and u_3 be the leaves of T adjacent to w. Consider the path P in the previous paragraph, and suppose without loss of generality that P is the path u_1wu_2. Similarly, let P' be a path in $\mathscr{P}$ that contains the edge wu_3 and does not contain the edge vw, and suppose without loss of generality that P' is the path u_1wu_3. Let x and y be the vertices of G corresponding to P and P', respectively.

Since $N[x]$ is not contained in $N[y]$, there exists a vertex x' that is adjacent to x but not to y. Similarly, there exists a vertex y' that is adjacent to y but not to x. This is only possible if the paths $P_{x'}$ and $P_{y'}$ contain the edge e, so x' and y' are adjacent vertices of G', and $xyy'x'x$ is an induced C_4 in G''.

By the induction hypothesis, G' is 3-clique colorable. Consider a clique coloring of G' using colors 0, 1 and 2. To extend it to G, we consider two cases.

If x' and y' have different colors, suppose 0 and 1, we assign color 0 to y, color 1 to x and any color to every other vertex not yet colored, if any. As in the proof of Proposition 3.1, it follows that no clique of G'' is monochromatic.

If x' and y' have the same color, say 0, we assign color 1 to x, color 2 to y and any color in $\{1, 2\}$ to every other vertex not yet colored, if any. This coloring works because every clique of G'' contains an edge of the induced C_4, $\{x', y'\}$ is contained in a larger clique of G (otherwise, they would not have received the same color in the clique coloring of G'), and hence every clique of G'' containing $\{x', y'\}$ that is also a clique of G contains a vertex of color 1 or 2.

Suppose from now on that $d_T(w) = 5$. We now consider three cases:

Case 1: G'' has no induced C_5 and no induced prism.

This is impossible. By Proposition 4.6, it would be possible to build a new representation where the host tree has fewer vertices of degree 5, contradicting the minimality condition imposed on T.

Case 2: G'' has an induced C_5.

Consider an induced C_5 consisting of the vertices a_1, a_2, a_3, a_4 and a_5 (in that order), and suppose without loss of generality that P_{a_1} and P_{a_2} are the paths in $\mathscr{P}[e]$. Take a color different from that of a_1 and a_2 and assign it to both a_3 and a_5. If there exists a vertex a_6 in $\mathscr{P}[e]$ and adjacent to a_4, color a_4 with a color different from those of a_3, a_5 and a_6. If such a vertex does not exist, then just color a_4 with a color different from that of a_3 and a_5. As a result of this, every pair of adjacent vertices in the cycle (with the possible exception of a_1 and a_2) receive different colors. Thus, every edge clique of G'' different from $\mathscr{P}[e]$ is not monochromatic. If $\mathscr{P}[e]$ is a clique, then it is a clique of G' and hence is not monochromatic.

For any vertex x of G'' not colored yet, we give it a color according to two cases.

If the vertices of the cycle adjacent to x were colored using one or two colors, give x a color different from them.

Otherwise, consider its corresponding path P_x contained in T''. By the absence of twin vertices, P_x must be different from every path representing a vertex of the cycle. There exist two claws in T'' such that the corresponding claw clique contains P_x and exactly one path from the cycle. Let u_1 and u_2 be the vertices of the cycle in these two claw cliques. Color x with a color different from that of u_1 and u_2.

Now we prove that this way to color also ensures that every clique of G'' that is a claw clique is not monochromatic.

Suppose that the claw clique contains the edge a_1a_2. If a_1 and a_2 have different colors, then it is clearly not monochromatic. If a_1 and a_2 have the same color, consider the vertex x in the clique such that P_x does not contain e. The vertices of the cycle adjacent to x are a_1, a_2, a_3 and a_5, which in this case are colored with two colors. It follows from the coloring rules that x has a color different from that of all these vertices and hence the clique is not monochromatic.

If the clique contains any other edge of the cycle, then it is not monochromatic, because the edges of the cycle different from a_1a_2 are not monochromatic.

If the claw clique does not contain an edge of the cycle, then it contains exactly one vertex of the cycle. If the edge e is a part of the claw of T'' and a_4 is in the clique, then it is not monochromatic by the way a_4 was colored. For every other claw clique containing just one vertex of the cycle, take a vertex x of it that is neither in the cycle nor in $\mathscr{P}[e]$. By the two cases that define how x was colored, the clique is not monochromatic.

Therefore, in this extended coloring the cliques of G' and the cliques of G'' are not monochromatic, so we have a 3-clique coloring of G.

Case 3: G'' has an induced prism.

The representation of the induced prism in T'' is like the one of Figure 3. In this representation of the prism, there are two edges of the host tree that are contained in three paths (let us call them edges of type 1), and three edges of the host tree that are contained in two paths (let us call them edges of type 2). Every other path of length 2 in $\mathscr{P}''$ and not in $\mathscr{P}'$ (if any) should contain the two edges of type 1, because any other possibility leads to the creation of an induced C_5 in G'' or the presence of identical paths. A path like that would edge-intersect every other path in $\mathscr{P}''$. As a consequence, G'' is a prism, the prism plus a universal vertex, or a graph obtained from any of these two graphs by adding twin vertices (twin vertices should have paths that contain the edge e), which implies that G'' has as many cliques as its induced prism (every clique of the prism is contained in exactly one clique of G''). We can infer from this that every 3-coloring of G'' extending that of G' where the cliques of the prism, with the possible exception of the one that is contained in $\mathscr{P}[e]$ (like in Case 2, if this is a clique it is not monochromatic after coloring G'), are not monochromatic leads to a 3-clique coloring of G. It is always possible to extend the coloring of G' in that way as seen in Figures 4 and 5. In case the edge e is of type 2, the vertices that were already colored (in black) are adjacent and not in the same triangle. By symmetry, it is enough to consider just one of these edges. It is also enough to consider a case where those two vertices have the same color and a case where they have different colors. Otherwise, when the edge e is of type 1, the vertices that were already colored (in black) are in one triangle of the prism. By symmetry, it is enough to consider just one of these triangles. It is also enough to consider one case where the vertices of that triangle all have the same color, one where just two have the same color and one where they all have different colors. Therefore, G is 3-clique colorable. □

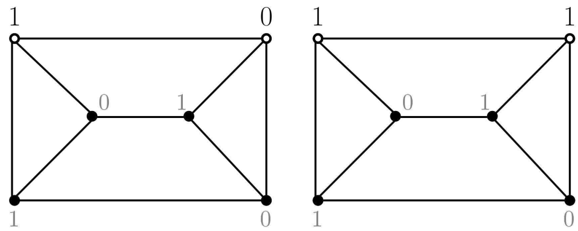

FIGURE 4. Extension of the coloring of G' when the edge e is of type 2.

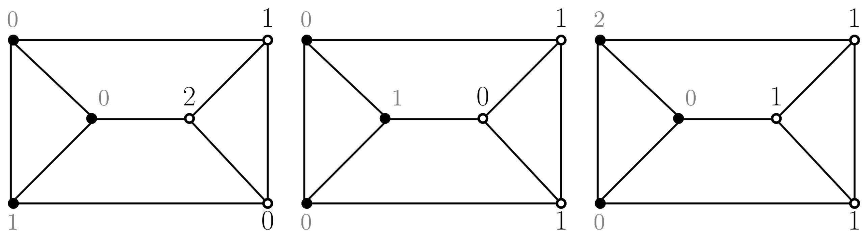

FIGURE 5. Extension of the coloring of G' when the edge e is of type 1.

5. Clique coloring subclasses of $[4, 2, 2]$ graphs

Given that $[5, 2, 2]$ graphs are 3-clique colorable, we first show that this bound on the clique chromatic number cannot be reduced for $[4, 2, 2]$ graphs. In Figure 6, we give some examples of graphs which are in this class, are not 2-clique colorable, and are minimal with this property (that is, if we delete any of their vertices they become 2-clique colorable).

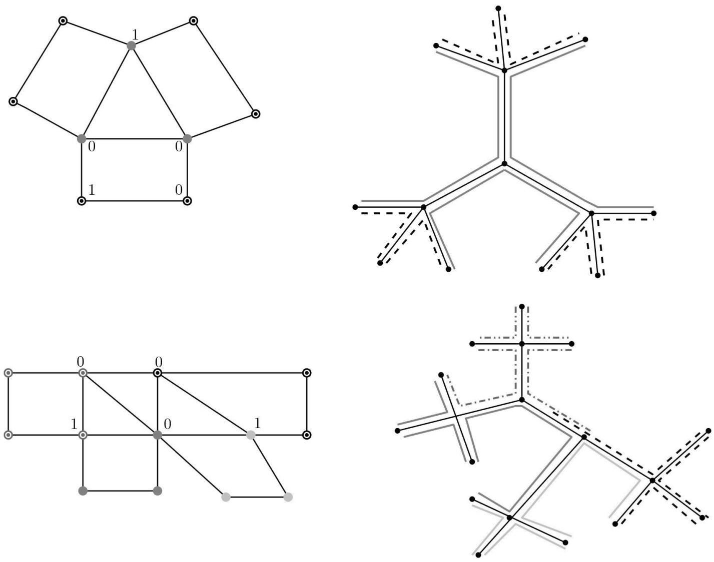

FIGURE 6. Examples of [4, 2, 2] graphs which are not 2-clique colorable.

If in the first graph of Figure 6 we 2-clique color the central claw clique, with colors 0 and 1, and we consider the C_4 which contains the vertices that received the same color (suppose without loss of generality that it is 0), we see that one of the cliques contained in it will be monochromatic. Therefore, this graph is not 2-clique colorable, but it is 3-clique colorable.

For the second graph of Figure 6, the existence of a 2-clique coloring of it would force adjacent vertices contained in a C_4 not forming a full clique to have different color. However, that would make the triangle in the middle be monochromatic. As a result of this contradiction, we conclude that the graph is not 2-clique colorable. It is easy to find a 3-clique coloring for it.

The purpose of this section is to find a restriction on the host tree that makes [4, 2, 2] graphs 2-clique colorable.

Call T_n ($n \geq 1$) the tree consisting of a path with $n + 2$ vertices such that every inner vertex of the path is adjacent to two vertices outside the path that are leaves.

Proposition 5.1. *Let G be an EPT graph that has a representation with T_n as a host tree for some n. Then $\chi_C(G) \leq 2$.*

Proof. Assume that n is minimum such that G can be represented with T_n as a host tree. The case $n = 1$ corresponds to a [4, 2, 2]-star graph, which is proven, so we assume in this proof that $n \geq 2$. We prove the property by induction on the amount of vertices of the graph. The case where G has at most two vertices is trivial. Suppose now that the proposition is true for every graph with at most k vertices and let G have $k + 1$ vertices.

If G has two vertices v and w such that $N[v] \subseteq N[w]$, then a 2-clique coloring of $G - v$ can be extended to G by applying Proposition 3.5.

From now on, we assume that G does not have a vertex that contains the neighborhood of another. This condition implies that no path in the representation consists of a single edge and that there are not identical paths.

If G has a clique C of size two whose removal disconnects the graph into components $G_1, \ldots, G_k$, we apply the induction hypothesis to get a 2-clique coloring of $G[V(G_i) \cup C]$ for each $1 \leq i \leq k$. As C has size two, the vertices of C have a different color in every coloring. We can suppose without loss of generality that the vertices of C receive the same color every time (if that is not the case, then switching colors fixes it). Thus, we can combine the colorings to entirely 2-clique color G.

Suppose that we have the host tree drawn such that the path of $n + 2$ vertices is horizontal and $x_1, x_2, \ldots, x_n$ are its inner vertices from left to right. We call this path H. Let $e_1, e_2, \ldots, e_{n+1}$ be the edges of H also from left to right. For $1 \leq i \leq n$, let u_i and l_i be the edges incident on x_i that are not in the path (we assume that they are drawn vertically), S_i be the star of T_n centered at x_i, and $\mathscr{C}_i$ be the family of cliques of G that are either edge cliques or claw cliques for some edge or claw of S_i, respectively. Additionally, the horizontal path H_i is the one consisting of the edges e_i and e_{i+1}, while the vertical path V_i is the one consisting of the edges u_i and l_i. Given an inner vertex x_i, we say that a path of the representation is to the left of x_i (or to the left of V_i) if it is not contained in V_i, and it does not contain any vertex to the right of x_i in the drawing of T_n. Similarly, we say that the path is to the right of x_i (or to the right of V_i) if it is not contained in V_i and it does not contain any vertex to the left of x_i in the drawing of T_n.

We will not work with the vertices of G, but directly with the paths that represent them. We will usually refer to them as paths of G.

Consider now the case where, for some i between 1 and n, G has neither a path containing e_i and u_i nor a path containing e_{i+1} and l_i, but does contain V_i. As a consequence of this assumption, there is no claw clique containing V_i. Then both edge cliques $\mathscr{P}[l_i]$ and $\mathscr{P}[u_i]$ exist; otherwise, V_i would correspond to a simplicial vertex of G. As as consequence of this, G must have a path P_1 containing e_i and l_i and one path P_2 containing e_{i+1} and u_i. Consider all the paths of G, except for P_1, P_2 and V_i, plus a path P obtained from the merger of $P_1 - l_i$ and $P_2 - u_i$. Apply the induction hypothesis to 2-clique color these paths. To obtain a 2-clique coloring of all the paths of G, keep the color of the paths of G that already have one, give P_1 and P_2 the same color as P, and give V_i the other color.

The case where, for some i between 1 and n, G has neither a path containing e_i and l_i nor a path containing e_{i+1} and u_i, but does contain V_i, is analogous.

Suppose now (assuming that $n > 2$) that, for some i between 2 and $n - 1$, we have that G does not have a path that contains H_i.

If the vertical path V_i is in the representation, we apply the induction hypothesis to 2-clique color V_i and the paths to the left of it, and to separately 2-clique color V_i and the paths to the right of it. If V_i received the same color in both, we combine them to get a coloring of G. Otherwise, we switch one of the colorings before we do the combination.

The only way for this coloring not to be a 2-clique coloring is that one of $\mathscr{P}[u_i]$ or $\mathscr{P}[l_i]$ is a clique, it has paths to both sides of V_i, and it is monochromatic. Suppose without loss of generality that $\mathscr{P}[u_i]$ is a monochromatic clique, with all its paths (including V_i) receiving color 0. We obtain a new coloring by switching the colors of the paths to the right of V_i, keeping the color that V_i has. Now $\mathscr{P}[u_i]$ is not monochromatic and neither are the cliques entirely to the left or entirely to the right of V_i. Another clique that is not monochromatic (if it exists) is the claw clique associated to the claw with edges e_i, l_i and u_i, since its paths did not change the color with respect to the previous coloring. Furthermore, if the claw clique with associated claw having edges e_{i+1}, l_i and u_i exists, then it is not monochromatic because V_i has color 0 and there is now a path containing u_i and e_{i+1} with color 1.

Finally, suppose that $\mathscr{P}[l_i]$ is a clique. Then it contains paths to both sides of V_i (otherwise, it would be contained in a claw clique) and hence there exists the claw clique associated to the claw with edges e_i, l_i and u_i. This claw clique is not monochromatic and, by our assumption, there exists a path containing e_i and l_i with color 1. Since V_i has color 0, $\mathscr{P}[l_i]$ is not monochromatic.

Now suppose that there is not a path in the representation containing H_i and that V_i is not in the representation, either. If the paths of G that intersect S_i induce a chordal graph, then only one of $\mathscr{P}[u_i]$, $\mathscr{P}[l_i]$ is an edge clique. Suppose without loss of generality that $\mathscr{P}[u_i]$ is an edge clique. Apply the induction hypothesis twice to 2-clique color the paths of the representation to the left of x_i and to 2-clique color the paths of the representation to the right of x_i. Combine these two colorings (possibly switching one of them to ensure that $\mathscr{P}[u_i]$ is not monochromatic) to obtain a 2-clique coloring of G.

Now suppose that S_i has a pie of size 4 that does not include V_i. By the minimality of n, there exists at least one path of G that does not contain x_i and is to the left of it (the same with right instead of left). Apply the induction hypothesis to 2-clique color the paths of G to the left of x_i and the paths $e_{i+1}u_i$ and $e_{i+1}l_i$ (these two do not necessarily have to be present in the original representation). Consider the restriction of this coloring to the paths to the left of x_i. By the construction, and forced by the paths $e_{i+1}u_i$ and $e_{i+1}l_i$, there are one path containing u_i, e_i and one path containing l_i, e_i that receive different colors.

Now apply the induction hypothesis to 2-clique color the paths of G to the right of x_i and the paths e_iu_i and e_il_i. Reasoning like before, the restriction of this coloring to the paths to the right of x_i has one path containing u_i, e_{i+1} and one path containing l_i, e_{i+1} that receive different colors. Combining the two restricted colorings (possibly switching one of them to ensure that $\mathscr{P}[u_i]$ and $\mathscr{P}[l_i]$ are not monochromatic), we obtain a 2-clique coloring of G.

Now consider the case $i = 1$. By the minimality of n, G has a path that contains e_1, l_1 or u_1. Among those paths, at least one must contain e_2 as well if G is connected. If necessary, we can rearrange the edges to have a path that contains e_1 and e_2. Similarly, we can have a path of G that contains e_n and e_{n+1}.

Assume from now on that, for every $1 \leq i \leq n$, G has a path that contains H_i. To complete the proof, we will construct a set Q such that (1) Q is a clique cover of G and (2) no clique of G is fully contained in Q, unless we fall on some case that we have already considered. If Q satisfies both conditions, a 2-clique coloring of G is obtained by giving color 0 to the elements of Q and color 1 to the elements not in Q.

The construction of Q is as follows:

Step 1: Let P_1 be a path of G that contains e_1 and e_2 with the additional requirement that it reaches as far to the right as possible (this is determined by looking at the vertices of H that the path contains).

Step 2 (iteration step): While P_j does not contain e_{n+1}, we do the following to choose the path P_{j+1} of G.

When P_j ends to the right at one horizontal edge e_i: Let P_{j+1} be a path of G containing H_i and extending as far to the right as possible.

When P_j ends to the right at one vertical edge u_i: If P_j edge-intersects every path edge-intersecting S_i and $i = n$, go to step 3. If P_j edge-intersects every path intersecting S_i and $i < n$, let P_{j+1} be a path of G containing H_{i+1} and extending as far to the right as possible. If P_j does not edge-intersect every path edge-intersecting S_i, choose a path of G containing l_i and e_{i+1} and extending as far to the right as possible. If such a path does not exist, choose a path containing e_{i+1} and not containing e_i, u_i and extending as far to the right as possible (from now on we will refer to this as the rightmost condition).

When P_j ends to the right at one vertical edge l_i: It is symmetric with the previous case. We can exchange u_i and l_i in the argument.

Step 3: Let Q consist of the paths chosen in the previous two steps. For every $1 \leq i \leq n$, if Q does not cover $\mathscr{C}_i$, then add V_i to Q.

Let us prove that Q is a clique cover of G.

P_1 contains H_1. If P_1 is not enough to cover $\mathscr{C}_1$, it means that there is a path of G in S_1 that does not edge-intersect P_1, which can only be V_1. If necessary, V_1 is added to Q in step 3, and P_1 and V_1 do cover $\mathscr{C}_1$.

Consider now $i > 1$. If there is a path in Q that contains H_i, then we can conclude that Q covers $\mathscr{C}_i$ reasoning like in the previous paragraph. Otherwise, the construction implies that there is a path P_j in Q that contains the edges e_i and u_i or that contains e_i and l_i. We only consider the first possibility, as the other one is similar.

If P_j is not enough to cover $\mathscr{C}_i$, consider P_{j+1}. If P_{j+1} contains e_{i+1} and l_i, then P_j and P_{j+1} clearly cover $\mathscr{C}_i$. Otherwise, P_{j+1} is a path containing e_{i+1} and containing none of e_i, u_i, l_i. If there is a clique C in $\mathscr{C}_i$ that is not covered by P_j and P_{j+1}, then C is $\mathscr{P}[l_i]$ or C is a claw clique where the claw has the edges e_i, e_{i+1}, l_i or l_i, u_i, e_{i+1}. If C is $\mathscr{P}[l_i]$, then it should have a path that does not edge-intersect P_j. As G does not have paths of a single edge, that path must have the edge e_{i+1}. Thus C has a path that contains l_i and e_{i+1}. This is also true if C is one of the claw cliques, which contradicts the choice of P_{j+1} according to the details of step 2. Therefore, P_j and P_{j+1} cover the cliques in $\mathscr{C}_i$.

Now we prove that Q does not fully contain any clique of G (unless we have a case that was already considered).

Suppose, to the contrary, that there exists a clique C of G contained in Q. We consider the different types of cliques that C can be:

• C is a claw clique where the claw has edges e_i, l_i, u_i for some i: This is impossible. By construction, Q cannot have both a path containing e_i, l_i and a path containing e_i, u_i.

• C is a claw clique where the claw has edges l_i, u_i, e_{i+1} for some i: Let P_j be a path of Q containing l_i, and e_{i+1} and $P_{j'}$ be a path of Q containing u_i and e_{i+1}. The presence of V_i in Q implies that Q also has a path P_k containing H_i, which precedes P_j and $P_{j'}$ in step 2 (if P_k is not the first one, then the second and the third of the three would contradict the rightmost condition). Furthermore, P_j and $P_{j'}$ arise subsequently in step 2 when choosing paths that contain two consecutive horizontal edges. If P_j precedes $P_{j'}$, then the rightmost condition is contradicted at the moment of choosing P_j (the existence of $P_{j'}$ does not make P_j an eligible path). We get the same type of contradiction if $P_{j'}$ precedes P_j.

• C is a claw clique where the claw has edges e_i, e_{i+1}, u_i for some i: By the construction, i is different from 1 and n. Additionally, the path containing e_i, u_i precedes the others in C in the construction of Q. Let P_j be the path of C containing e_i, e_{i+1}, and $P_{j'}$ be one path in C containing e_{i+1}, u_i. Considering which path precedes the other, we get the same contradiction as in the previous case.

• C is a claw clique where the claw has edges e_i, e_{i+1}, l_i for some i: Analogous to the previous case.

• $C = \mathscr{P}[u_i]$ for some i: If $\mathscr{P}[u_i]$ does not contain V_i, then, by the construction of Q, it consists of one path containing e_i and one one path containing e_{i+1}. It follows that C is contained in the claw clique whose claw has the edges e_i, e_{i+1}, u_i, which is a contradiction.

If Q has a path that contains e_i and u_i, then, by the construction of Q, V_i is not in it. It follows from this and the previous paragraph that C consists of V_i and a path containing the edges e_{i+1} and u_i, and we do not have a path containing e_i and u_i in G. Furthermore, we do not have a path containing e_{i+1} and l_i, because, otherwise, C would be contained in the claw clique whose claw contains e_{i+1}, l_i, and u_i. This is one of the cases where we have already proved that G is 2-clique colorable.

The case where $C = \mathscr{P}[l_i]$ is analogous.

• $C = \mathscr{P}[e_i]$ for some i between 1 and $n+1$: By the rightmost condition, Q has only one path that contains e_1. Looking at how step 2 ends, we conclude that Q has one or no path that contains e_{i+1}. Thus, n must be larger than 1, and i must be between 2 and n. By the minimality of n, $G - C$ is not connected. In this context, we know that G is 2-clique colorable if C has size 2. Thus, we assume that C has larger size.

Let P_j, $P_{j'}$, and $P_{j''}$ be the first three elements of C according to the order of Q. Then, by the construction, $P_{j'}$ and $P_{j''}$ are chosen to contain two consecutive edges of H, such that these edges are to the right of e_i or e_i is the leftmost of the two.

By the rightmost condition, the presence of $P_{j''}$ contradicts the moment when $P_{j'}$ is chosen.

All the cases have been considered. Hence, G is 2-clique colorable. □

T_n is a graph that has all its vertices of degree four contained in a path. We now prove that every $[4, 2, 2]$ graph with a representation that has a host tree satisfying this condition is 2-clique colorable.

Theorem 5.2. *Let G be a graph with a $(4, 2, 2)$-representation such that the vertices of degree four in the host tree are contained in a path. Then $\chi_C(G) \leq 2$.*

Proof. The proof will be by induction on the number of vertices. The case where G has at most two vertices is trivial. Suppose that the theorem is true for graphs with at most k vertices, and let G have $k + 1$ vertices. Let $\langle \mathscr{P}, T\rangle$ be a $(4, 2, 2)$-representation of G, where T has the vertices of degree four contained in a path. Choose the representation so that T has the minimum number of edges under these conditions.

If every leaf of T is adjacent to a vertex of degree four, then T is a subgraph of T_n for some n, and by the previous proposition, G is 2-clique colorable. Otherwise, let v be a leaf of T at maximum distance from the set of vertices of degree four. By the definition, the vertex w adjacent to v has degree less than four, is adjacent to a vertex w' that is not a leaf, and every other vertex adjacent to it is a leaf. By the minimality of T, $\mathscr{P}$ must have a path P that contains vw and does not contain ww' (otherwise, G could be represented using $T - vw$ as a host tree). If $\mathscr{P}$ has a path of length one, then G has a simplicial vertex. Otherwise, P itself corresponds to a simplicial vertex of G. In either case, G has a simplicial vertex x. As we did in previous proofs, we can 2-clique color $G - x$ and and apply Proposition 3.5 to conclude that G is 2-clique colorable. □

A conclusion that one can immediately derive from Theorem 5.2 is that every $[4, 2, 2]$ graph with a $(4, 2, 2)$-representation where the host tree has at most two vertices of degree 4 is 2-clique colorable.

6. Conclusions and open questions

Lately, the classes of intersection graphs have been heavily studied and numerous results are known. We chose to study the classes of edge intersection graphs of a family of paths in a tree.

We proved that if G is an $[h, 2, 2]$-star graph with $h \leq 16$, then G is 3-clique colorable, and we found subclasses of EPT-star graphs that are 2-clique colorable and do not have degree restrictions. If the host tree is not a star and the graph G belongs to $[4, 2, 2]$ or $[5, 2, 2]$, we proved that it is 3-clique colorable, also showing that this bound is tight. Finally, we presented some subclasses of $[4, 2, 2]$ which are 2-clique colorable.

For future work, we are interested in studying the graphs in the class $[6, 2, 2]$ to determine whether they are 3-clique colorable. If the answer is positive, we are also interested in finding the minimum h such that the entire class $[h, 2, 2]$ is not

3-clique colorable. As we consider larger values of h, another goal will be to define new subclasses of $[h, 2, 2]$ which have smaller upper bounds for the clique-chromatic number.

A final question to consider is to determine whether it is true that, for every h, the least upper bound for the clique chromatic number of $[h, 2, 2]$ graphs and the least upper bound for the clique chromatic number of $[h, 2, 2]$-star graphs differ by at most 1.

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