REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 67, No. 1, 2024, Pages 317–325
Published online: June 3, 2024
https://doi.org/10.33044/revuma.3535

COFINITE MODULES AND COFINITENESS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let n be a non-negative integer, R a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, M a finitely generated R-module, and X an arbitrary R-module. In this paper, we first prove that if $\dim_R(M) \leq n+2$, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module and $\{\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) : \dim(R/\mathfrak{p}) \geq n\}$ is a finite set for all i. As a consequence, it follows that $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ is a finite set for all i when R is a semi-local ring and $\dim_R(M) \leq 3$. Then, we show that if $\dim(R/\mathfrak{a}) \leq n+1$, then $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i whenever $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq \dim_R(X) - n$. Finally, in the case that $\dim(R/\mathfrak{a}) \leq 2$, X is $\mathfrak{a}$-torsion, and $n > 0$ or $\mathrm{Supp}_R(X) \cap \mathrm{Var}(\mathfrak{a}) \cap \mathrm{Max}(R)$ is finite, we prove that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module when $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq 2-n$. We conclude with some ordinary $\mathfrak{a}$-cofiniteness results for local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(X)$.

1. Introduction

Throughout, let R denote a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, M a finite (i.e., finitely generated) R-module, X an arbitrary R-module which is not necessarily finite, and n a non-negative integer. We refer the reader to [9, 10, 26] for basic results, notations, and terminology not given in this paper.

The following questions are two important problems in local cohomology (see [17, First Question] and [20, Problem 4]). Recall that an $\mathfrak{a}$-torsion R-module X is said to be $\mathfrak{a}$-*cofinite* if $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i [17].

Question 1.1. *Is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $\mathfrak{a}$-cofinite R-module for all i?*

Question 1.2. *Is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ a finite set for all i?*

Hartshorne [17, Section 3] and Singh [27, Section 4] have given counterexamples to these questions. However, these questions have been studied by many authors and they were shown to have an affirmtive answer in some situations (see e.g., [17,

2020 *Mathematics Subject Classification.* 13D07, 13D45.

Key words and phrases. Associated prime ideals, cofinite modules, local cohomology modules.

The research of Alireza Vahidi was in part supported by a grant from Payame Noor University.

Corollary 7.7], [21, Theorem 4.1], [13, Theorem 3], [14, Theorem 1], [31, Theorem 1.1], [11, Theorem 1.4], [6, Theorem 2.3], [7, Theorem 2.6], [23, Theorem 8], and [2, Theorem 3.4]). In [24, Theorem 7.10] and [25, Theorem 2.10], Melkersson provided affirmative answers to these questions for the case that either $\dim(R) \leq 2$ or $\mathfrak{a}$ is an ideal of R with $\dim(R/\mathfrak{a}) \leq 1$. As a generalization of [24, Theorem 7.10], the answer to these questions is also yes if $\dim_R(M) \leq 2$ by [12, Corollary 5.2].

Recall that X is said to be an $\mathrm{FD}_{<n}$ (or *in dimension* $< n$) *R-module* if there exists a finite R-submodule X' of X such that $\dim_R(X/X') < n$ [2, 4]. The class of $\mathrm{FD}_{<n}$ R-modules is a Serre subcategory of the category of R-modules from [32, Theorem 2.3]. We say that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-module* if X is an $\mathfrak{a}$-torsion R-module and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i [3, Definition 4.1]. Note that X is a finite (resp. an $\mathfrak{a}$-cofinite) R-module if and only if X is an $\mathrm{FD}_{<0}$ (resp. $(\mathrm{FD}_{<0}, \mathfrak{a})$-cofinite) R-module. Therefore, as generalizations of Questions 1.1 and 1.2, we have the following questions (see [1, Question], [29, Questions 1.6 and 1.8], and [30, Questions 1.5 and 1.6]). In this paper, for a subset A of $\mathrm{Spec}(R)$, the set $\{\mathfrak{p} \in A : \dim(R/\mathfrak{p}) \geq n\}$ (resp. $\{\mathfrak{p} \in A : \dim(R/\mathfrak{p}) = n\}$) is denoted by $A_{\geq n}$ (resp. $A_{=n}$).

Question 1.3. *Is* $\mathrm{H}^i_{\mathfrak{a}}(M)$ *an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-module for all i?*

Question 1.4. *Is* $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))_{\geq n}$ *a finite set for all i?*

The first author and Morsali, in [29, Corollary 4.5], provided affirmative answers to Questions 1.3 and 1.4 for the case that $\dim(R/\mathfrak{a}) \leq n+1$, which is a generalization of Melkersson's result [25, Theorem 2.10] (see also [1, Theorems 2.5 and 2.10]). Also, the first author and Papari-Zarei, in [30, Corollary 3.2], proved that the answer to Questions 1.3 and 1.4 is yes if $\dim(R) \leq n+2$, which is a generalization of Melkersson's result [24, Theorem 7.10]. In the first main result of this paper, as generalizations of [12, Corollary 5.2] and [30, Corollary 3.2], we show that the answer to Questions 1.3 and 1.4 is yes if $\dim_R(M) \leq n+2$. As a consequence, we provide an affirmative answer to Question 1.2 for the case that R is a semi-local ring and $\dim_R(M) \leq 3$.

By [8, Corollary 2.6], $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i when $\dim(R/\mathfrak{a}) = 1$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq \dim_R(X)$. In the second main result, we generalize and improve [8, Corollary 2.6] by showing that if $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<n+2}$ R-module for all $i < \dim_R(X) - n$ (e.g., $\dim(R/\mathfrak{a}) \leq n+1$) and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq \dim_R(X) - n$, then $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i. As a consequence, it follows that if $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq \dim_R(X) - n$, then the i-th Bass number and the i-th Betti number of X with respect to $\mathfrak{p}$ are finite for every integer i and every prime ideal $\mathfrak{p}$ of $\mathrm{Var}(\mathfrak{a})_{\geq n}$. Here, we denote $\mathrm{Var}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$.

From [8, Theorem 3.5], X is an $\mathfrak{a}$-cofinite R-module whenever R is a local ring with $\dim(R/\mathfrak{a}) \leq 2$ and X is an $\mathfrak{a}$-torsion R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq 2$ (see also [22, Theorem 2.6] and [19, Theorem 3.3]). Assume that $\dim(R/\mathfrak{a}) \leq 2$, t is a non-negative integer, and $n > 0$ or $\mathrm{Supp}_R(X) \cap \mathrm{Var}(\mathfrak{a}) \cap \mathrm{Max}(R)$ is a finite set. In the third main result, as a generalization and improvement of [8, Theorem 3.5], we prove that if X is an $\mathfrak{a}$-torsion R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \le 2-n$, then X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. This result shows that when $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \le t+1$, then $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(X))$ is a finite R-module for all $i \le t$ if and only if $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite R-module for all $i < t$, which improves [8, Theorem 3.7]. It also shows that if $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module and $\mathrm{H}^{2i}_{\mathfrak{a}}(X)$ (or $\mathrm{H}^{2i+1}_{\mathfrak{a}}(X)$) is an $\mathfrak{a}$-cofinite R-module for all i, then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite R-module for all i, which improves [8, Theorem 3.8].

2. Main results

In [24, Theorem 7.10], Melkersson proved that Questions 1.1 and 1.2 have affirmative answers in the case that $\dim(R) \le 2$. As a generalization of this result, in [12, Corollary 5.2] it is shown that the answer to these questions is yes if $\dim_R(M) \le 2$. The first author and Papari-Zarei, in [30, Corollary 3.2], proved that Questions 1.3 and 1.4 have affirmative answers if $\dim(R) \le n+2$, which is a generalization of Melkersson's result [24, Theorem 7.10]. In the first main result of this paper, we generalize [12, Corollary 5.2] and improve [30, Corollary 3.2] by showing that the answer to Questions 1.3 and 1.4 is yes if $\dim_R(M) \le n+2$. Note that, for an ideal $\mathfrak{b}$ of R with $\mathfrak{b}X = 0$, X is an $\mathrm{FD}_{<n}$ R-module if and only if X is an $\mathrm{FD}_{<n}$ $R/\mathfrak{b}$-module.

Theorem 2.1. *Suppose that M is a finite R-module such that $\dim_R(M) \le n+2$. Then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i.*

Proof. Set $\overline{R} = R/\operatorname{Ann}_R(M)$ and $\overline{\mathfrak{a}} = (\mathfrak{a} + \operatorname{Ann}_R(M))/\operatorname{Ann}_R(M)$. Since $\dim_R(M) \le n+2$, we have $\dim(\overline{R}) \le n+2$ and so $\mathrm{H}^i_{\overline{\mathfrak{a}}}(M)$ is an $(\mathrm{FD}_{<n}, \overline{\mathfrak{a}})$-cofinite $\overline{R}$-module for all i from [30, Corollary 3.2]. That is, $\operatorname{Ext}^j_{\overline{R}}(R/(\mathfrak{a}+\operatorname{Ann}_R(M)), \mathrm{H}^i_{\overline{\mathfrak{a}}}(M))$ is an $\mathrm{FD}_{<n}$ $\overline{R}$-module for all j and all i. Assume that i and j are two integers. There exists a spectral sequence

$$\mathrm{E}_2^{p,q} := \operatorname{Ext}^p_{\overline{R}}\big(\operatorname{Tor}^R_q(\overline{R}, R/\mathfrak{a}), \mathrm{H}^i_{\overline{\mathfrak{a}}}(M)\big) \underset{p}{\Longrightarrow} \operatorname{Ext}^{p+q}_R(R/\mathfrak{a}, \mathrm{H}^i_{\overline{\mathfrak{a}}}(M))$$

by [26, Theorem 10.74]. For all k, $0 \le k \le j$, $\mathrm{E}_2^{j-k,k}$ is an $\mathrm{FD}_{<n}$ $\overline{R}$-module from [18, Proposition 3.4 (i)] and so $\mathrm{E}_\infty^{j-k,k}$ is an $\mathrm{FD}_{<n}$ $\overline{R}$-module because $\mathrm{E}_\infty^{j-k,k} = \mathrm{E}_{j+2}^{j-k,k}$ and $\mathrm{E}_{j+2}^{j-k,k}$ is a subquotient of $\mathrm{E}_2^{j-k,k}$. There is a finite filtration

$$0 = \phi^{j+1}H^j \subseteq \phi^j H^j \subseteq \cdots \subseteq \phi^1 H^j \subseteq \phi^0 H^j = \operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\overline{\mathfrak{a}}}(M))$$

such that for all k, $0 \le k \le j$, $\mathrm{E}_\infty^{j-k,k} \cong \phi^{j-k}H^j/\phi^{j-k+1}H^j$. For all k, $0 \le k \le j$, by the short exact sequence

$$0 \longrightarrow \phi^{j-k+1}H^j \longrightarrow \phi^{j-k}H^j \longrightarrow \mathrm{E}_\infty^{j-k,k} \longrightarrow 0,$$

$\phi^{j-k}H^j$ is an $\mathrm{FD}_{<n}$ $\overline{R}$-module whenever $\phi^{j-k+1}H^j$ is an $\mathrm{FD}_{<n}$ $\overline{R}$-module. Therefore $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\overline{\mathfrak{a}}}(M))$ is an $\mathrm{FD}_{<n}$ $\overline{R}$-module and so an $\mathrm{FD}_{<n}$ R-module. Thus $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ is an $\mathrm{FD}_{<n}$ R-module. Hence $\mathrm{H}^i_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i, as we desired. □

Corollary 2.2. *Suppose that M is a finite R-module such that $\dim_R(M) \leq n+2$. Then $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))_{\geq n}$ is a finite set for all i.*

Proof. For all i, by Theorem 2.1, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ is an $\mathrm{FD}_{<n}$ R-module and so the set $\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M)))_{\geq n}$ is finite. Thus $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))_{\geq n}$ is a finite set for all i form [10, Exercise 1.2.28]. □

Corollary 2.3 (see [12, Corollary 5.2])**.** *Suppose that M is a finite R-module such that $\dim_R(M) \leq 2$. Then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-cofinite R-module and $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ is a finite set for all i.*

Proof. Put $n = 0$ in Theorem 2.1 and Corollary 2.2. □

The next result shows that Question 1.2 has an affirmative answer in the case that R is a semi-local ring and $\dim_R(M) \leq 3$. Recall that X is said to be a *weakly Laskerian R-module* if the set of associated prime ideals of any quotient module of X is finite [15, Definition 2.1]. Also, we say that X is an $\mathfrak{a}$-*weakly cofinite R-module* if X is an $\mathfrak{a}$-torsion R-module and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a weakly Laskerian R-module for all i [16, Definition 2.4].

Corollary 2.4. *Suppose that R is a semi-local ring and that M is a finite R-module such that $\dim_R(M) \leq 3$. Then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-weakly cofinite R-module and $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ is a finite set for all i.*

Proof. Consider [5, Theorem 3.3] and take $n = 1$ in Theorem 2.1 and Corollary 2.2. □

If $\dim(R/\mathfrak{a}) = 1$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq \dim_R(X)$, then $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i from [8, Corollary 2.6]. We generalize and improve this result in the second main result of this paper.

Theorem 2.5. *Suppose that X is an arbitrary R-module such that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<n+2}$ R-module for all $i < \dim_R(X) - n$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq \dim_R(X) - n$. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i.*

Proof. We first prove that for a non-negative integer t, $\mathrm{H}^t_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<\dim_R(X)-t+1}$ R-module. Suppose, on the contrary, that $\mathrm{H}^t_{\mathfrak{a}}(X)$ is not an $\mathrm{FD}_{<\dim_R(X)-t+1}$ R-module. Then $\dim_R(\mathrm{H}^t_{\mathfrak{a}}(X)) > \dim_R(X) - t$ and so there is a prime ideal $\mathfrak{p}$ of $\mathrm{Supp}_R(\mathrm{H}^t_{\mathfrak{a}}(X))$ such that $\dim(R/\mathfrak{p}) > \dim_R(X) - t$. Thus $\mathrm{H}^t_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \neq 0$ and $t > \dim_{R_{\mathfrak{p}}}(X_{\mathfrak{p}})$, which contradicts [9, Theorem 6.1.2]. To prove that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i, by [3, Theorem 4.2], it is enough to show that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i. From the first part of the proof, $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<n}$ R-module and so is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i > \dim_R(X) - n$. Also, by [29, Theorem 4.2 (i)], $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i < \dim_R(X) - n$. Thus $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{\dim_R(X)-n}_{\mathfrak{a}}(X))$ is an $\mathrm{FD}_{<n}$ R-module from [3, Theorem 2.3]. On the other hand, again by the first part of the proof, $\mathrm{H}^{\dim_R(X)-n}_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<n+1}$ R-module. Hence $\mathrm{H}^{\dim_R(X)-n}_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from [29, Lemma 2.1]. □

As immediate applications of the above theorem, we have the following corollaries.

Corollary 2.6. *Suppose that X is an arbitrary R-module such that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<2}$ R-module for all $i < \dim_R(X)$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq \dim_R(X)$. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i.*

Proof. Put $n = 0$ in Theorem 2.5. □

Corollary 2.7. *Suppose that $\dim(R/\mathfrak{a}) \leq n + 1$ and X is an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq \dim_R(X) - n$. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i.*

Proof. This follows from Theorem 2.5. □

Corollary 2.8 (see [8, Corollary 2.6]). *Suppose that $\dim(R/\mathfrak{a}) \leq 1$ and X is an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq \dim_R(X)$. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i.*

Proof. Take $n = 0$ in Corollary 2.7. □

Corollary 2.9. *Suppose that X is an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq \dim_R(X) - n$. Then, for every integer i and every prime ideal $\mathfrak{p}$ of $\mathrm{Var}(\mathfrak{a})_{\geq n}$, the i-th Bass number and the i-th Betti number of X with respect to $\mathfrak{p}$ are finite.*

Proof. Let $\mathfrak{p} \in \mathrm{Var}(\mathfrak{a})_{\geq n}$. Then $\mathrm{Ext}^i_R(R/\mathfrak{p}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq \dim_R(X) - n$ by [18, Proposition 3.4 (i)] and so $\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, X_\mathfrak{p})$ is a finite $R_\mathfrak{p}$-module for all $i \leq \dim_{R_\mathfrak{p}}(X_\mathfrak{p})$. Thus $\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, X_\mathfrak{p})$ is a finite $R_\mathfrak{p}$-module for all i from Corollary 2.8. Hence $\mathrm{Tor}^{R_\mathfrak{p}}_i(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, X_\mathfrak{p})$ is a finite $R_\mathfrak{p}$-module for all i from [24, Theorem 2.1]. □

Corollary 2.10 (see [8, Corollary 2.7]). *Suppose that X is an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq \dim_R(X)$. Then, for every integer i and every prime ideal $\mathfrak{p}$ of $\mathrm{Var}(\mathfrak{a})$, the i-th Bass number and the i-th Betti number of X with respect to $\mathfrak{p}$ are finite.*

Proof. Put $n = 0$ in Corollary 2.9. □

If R is a local ring with $\dim(R/\mathfrak{a}) \leq 2$ and X is an $\mathfrak{a}$-torsion R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq 2$, then X is an $\mathfrak{a}$-cofinite R-module by [8, Theorem 3.5] (see also [22, Theorem 2.6] and [19, Theorem 3.3]). In the third main result of this paper, we improve and generalize this result.

Theorem 2.11. *Suppose that* $\mathfrak{a}$ *is an ideal of* R *with* $\dim(R/\mathfrak{a}) \leq 2$ *and* X *is an* $\mathfrak{a}$*-torsion* R*-module such that* $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ *is an* $\mathrm{FD}_{<n}$ R*-module for all* $i \leq 2-n$*. Suppose also that one of the following conditions holds:*

(a) $n = 0$ *and* $\operatorname{Min}_R(\operatorname{Ext}^i_R(R/\mathfrak{a}, X)/Y_i)_{=0}$ *is a finite set for all* $i > 2$ *and for all finite* R*-submodules* Y_i *of* $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ *(e.g.,* $\operatorname{Supp}_R(X) \cap \operatorname{Var}(\mathfrak{a}) \cap \operatorname{Max}(R)$ *is a finite set);*
(b) $n > 0$.

Then X *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-module.*

Proof. We first assume that $n = 0$. Let t be a non-negative integer such that $t \geq 3$ and set $Y := \operatorname{Ext}^t_R(R/\mathfrak{a}, X)$. We prove that Y is a finite R-module. From [28, Lemma 2.1], it is enough to show that $\operatorname{Min}_R(Y/Y_t)$ is a finite set for all finite R-submodules Y_t of Y and $Y_\mathfrak{p}$ is a finite $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Since $\dim(R/\mathfrak{a}) \leq 2$ and X is an $\mathfrak{a}$-torsion R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is (a finite and so) an $\mathrm{FD}_{<1}$ R-module for all $i \leq 1$, Y is an $\mathrm{FD}_{<1}$ R-module by putting $n = 1$ in Corollary 2.7. Thus Y/Y_t is an $\mathrm{FD}_{<1}$ R-module and so $\operatorname{Min}_R(Y/Y_t)_{\geq 1}$ is a finite set. Hence the set $\operatorname{Min}_R(Y/Y_t)$ is finite by assumption. Let $\mathfrak{p}$ be a prime ideal of R. We have $\dim(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}) \leq 2$ and $\operatorname{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}, X_\mathfrak{p})$ is a finite $R_\mathfrak{p}$-module for all $i \leq 2$. Thus, from [8, Theorem 3.5], $Y_\mathfrak{p} \cong \operatorname{Ext}^t_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}, X_\mathfrak{p})$ is a finite $R_\mathfrak{p}$-module.

Now, assume that $n = 1$ (resp. $n = 2$). Since $\dim(R/\mathfrak{a}) \leq 2$ (resp. $\dim(R/\mathfrak{a}) \leq 3$) and X is an $\mathfrak{a}$-torsion R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ (resp. $\mathrm{FD}_{<2}$) R-module for all $i \leq 1$ (resp. $i \leq 0$), $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ (resp. $\mathrm{FD}_{<2}$) R-module for all i from taking $n = 1$ (resp. $n = 2$) in Corollary 2.7. For $n > 2$, since $\dim(R/\mathfrak{a}) \leq 2$, $\dim_R(\operatorname{Ext}^i_R(R/\mathfrak{a}, X)) \leq 2$ and so $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i. □

With respect to Question 1.1, we have the following two results which improve [8, Theorems 3.7 and 3.8] to the rings which are not necessarily local and to the modules which are not necessarily finite.

Corollary 2.12. *Suppose that* $\mathfrak{a}$ *is an ideal of* R *with* $\dim(R/\mathfrak{a}) \leq 2$*,* X *is an arbitrary* R*-module, and* t *is a non-negative integer such that* $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ *is a finite* R*-module for all* $i \leq t+1$*. Suppose also that* $\operatorname{Min}_R(\operatorname{Ext}^i_R(R/\mathfrak{a}, \mathrm{H}^j_\mathfrak{a}(X))/Y_{ij})_{=0}$ *is a finite set for all* $i > 2$*, for all* $j < t$*, and for all finite* R*-submodules* Y_{ij} *of* $\operatorname{Ext}^i_R(R/\mathfrak{a}, \mathrm{H}^j_\mathfrak{a}(X))$ *(e.g.,* $\operatorname{Supp}_R(X) \cap \operatorname{Var}(\mathfrak{a}) \cap \operatorname{Max}(R)$ *is a finite set). Then the following statements are equivalent:*

(i) $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^j_\mathfrak{a}(X))$ *is a finite* R*-module for all* $j \leq t$*;*
(ii) $\mathrm{H}^j_\mathfrak{a}(X)$ *is an* $\mathfrak{a}$*-cofinite* R*-module for all* $j < t$*.*

Proof. (i) ⇒ (ii). We use induction on t. There is nothing to prove in the case that $t = 0$. Assume that $t = 1$. From [18, Corollary 2.4 (i) (c)] and the short exact sequence

$$0 \longrightarrow \Gamma_\mathfrak{a}(X) \longrightarrow X \longrightarrow X/\Gamma_\mathfrak{a}(X) \longrightarrow 0,$$

we get the exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(X)) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, X) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^1_{\mathfrak{a}}(X))$$
$$\longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(X)) \longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{a}, X) \longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{a}, X/\Gamma_{\mathfrak{a}}(X)),$$

which shows that $\operatorname{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(X))$ and $\operatorname{Ext}^2_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(X))$ are finite R-modules. Thus $\Gamma_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite R-module by Theorem 2.11. Now, suppose that $t > 1$ and that $t-1$ is settled. It is enough to show that $\mathrm{H}^{t-1}_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite R-module. Since $\mathrm{H}^j_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite R-module for all $j < t-1$, we have that $\operatorname{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(X))$ and $\operatorname{Ext}^2_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(X))$ are finite R-modules from [3, Theorem 2.3]. Thus $\mathrm{H}^{t-1}_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite R-module by Theorem 2.11.

(ii) ⇒ (i). Follows from [3, Theorem 2.3]. □

Corollary 2.13. *Suppose that* $\mathfrak{a}$ *is an ideal of* R *with* $\dim(R/\mathfrak{a}) \leq 2$ *and* X *is an arbitrary* R*-module such that* $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ *is a finite* R*-module for all* i*. Suppose also that* $\operatorname{Min}_R(\operatorname{Ext}^i_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(X))/Y_{ij})_{=0}$ *is a finite set for all* $i > 2$*, for all* j*, and for all finite* R*-submodules* Y_{ij} *of* $\operatorname{Ext}^i_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(X))$ *(e.g.,* $\operatorname{Supp}_R(X) \cap \operatorname{Var}(\mathfrak{a}) \cap \operatorname{Max}(R)$ *is a finite set). Then the following statements hold true:*

(i) $\mathrm{H}^j_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* R*-module for all* j *when* $\mathrm{H}^{2j}_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* R*-module for all* j*;*

(ii) $\mathrm{H}^j_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* R*-module for all* j *whenever* $\mathrm{H}^{2j+1}_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* R*-module for all* j*.*

Proof. This follows by [3, Theorem 2.3], Theorem 2.11, and using an induction argument on j. □

Acknowledgments

The authors would like to thank the referee for the invaluable comments on the manuscript.

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Received: August 14, 2022
Accepted: November 29, 2022