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ON FIBRATIONS AND MEASURES OF IRRATIONALITY OF HYPER-KÄHLER MANIFOLDS

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Abstract. We prove some results on the fibers and images of rational maps from a hyper-Kähler manifold. We study in particular the minimal genus of fibers of a fibration into curves. The last section of this paper is devoted to the study of the rational map defined by a linear system on a hyper-Kähler fourfold satisfying numerical conditions similar to those considered by O'Grady in his study of fourfolds numerically equivalent to $K3^{[2]}$. We extend his results to this more general context.

1. Introduction

Bastianelli et al. [4] introduced and discussed two numerical birational invariants of a projective variety X, the *covering gonality* $\operatorname{covgon}(X)$ and the *degree of irrationality* $\operatorname{irr}(X)$. The former is defined as the minimal gonality of a curve C, which is the general fiber of a family

$$\psi\colon \mathcal{C} \to B, \quad \phi\colon \mathcal{C} \to X$$

of curves covering X, that is, ϕ is dominant and nonconstant on the fibers of ψ. The second number is defined as the minimal degree of a dominant rational map $X \dashrightarrow \mathbb{P}^n$, $n = \dim X$. Obviously, one has $\operatorname{irr}(X) \geq \operatorname{covgon}(X)$ but the inequality is strict in many cases. For example, the covering gonality of a uniruled manifold is 1, while its irrationality is 1 only if it is rational. One can similarly introduce the *covering genus* $\operatorname{covgen}(X)$, namely the genus of a curve C, which is the general fiber of a family

$$\psi\colon \mathcal{C} \to B, \quad \phi\colon \mathcal{C} \to X$$

of curves covering X.

There are several similarly defined numbers that can be studied, namely the fibering gonality $\operatorname{fibgon}(X)$ and the fibering genus $\operatorname{fibgen}(X)$ defined as follows:

Definition 1.1. The *fibering gonality* of X is the minimal gonality of the general fiber of a fibration $X \dashrightarrow B$ into curves. The *fibering genus* of X is the minimal genus of the general fiber of a fibration $X \dashrightarrow B$ into curves.

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Here, the *general fiber* of a rational map $\phi\colon X \dashrightarrow B$ is defined as the general fiber of a resolution of singularities $\widetilde{\phi}\colon \widetilde{X} \to B$. Instead of studying coverings of X by varieties of a given type, we thus study fibrations, namely a dominant rational map $X \dashrightarrow B$ with connected fibers and $\dim B < \dim X$, with fibers of a given genus or gonality. There are obvious inequalities

$$\mathrm{covgon}(X) \leq \mathrm{fibgon}(X), \quad \mathrm{covgen}(X) \leq \mathrm{fibgen}(X). \tag{1.1}$$

Another simple comparison between the fibering genus and the fibering gonality of a projective variety X introduced in (1.1) is

$$\mathrm{fibgon}(X) \leq \left\lceil \frac{\mathrm{fibgen}(X)}{2} \right\rceil + 1,$$

which follows indeed from the Brill–Noether theory showing the existence of g^1_k on curves of genus $\leq 2k-2$. Note that, in the case of a surface, the fibering genus is called the Konno invariant [19]. Ein and Lazarsfeld [11] studied a different higher dimensional generalization of it, defined as the minimal geometric genus p_g of a fiber of a rational map to $\mathbb{P}^1$.

Ein and Lazarsfeld prove that the Konno invariant of a K3 surface with Picard group of rank 1 generated by a line bundle of self-intersection h grows like $\sqrt{h}$. This is in strong contrast with the covering genus which is always equal to 1. A beautiful construction by Kollár [18] shows that a rationally connected smooth projective manifold, hence of covering gonality 1 and covering genus 0, can be nonruled, hence can have fiber gonality at least 2 and fiber genus at least 1, so both inequalities in (1.1) are strict in general.

In the case of hyper-Kähler manifolds, the following question asked by Pacienza (oral communication) is still open.

Question 1.2. Let X be a hyper-Kähler manifold which is projective and very general in moduli. Is X swept-out by elliptic curves? Equivalently, is $\mathrm{covgen}(X) = 1$?

Here the assumptions on X mean that X is equiped with a given polarization (very ample line bundle) and, equipped with this polarization, is very general in the corresponding moduli space of polarized hyper-Kähler manifolds. In particular, we have $\rho(X) = 1$ by generalities on the period map. We expect that the answer to this question is no in some examples, but were not able to prove or disprove it even on some explicit examples like the Fano variety of lines on a cubic fourfold, although we described in [30] some consequences of the existence of a covering by elliptic curves. Note that, if $\rho(X) = 2$, the example of Hilbert schemes $S^{[n]}$ for any projective K3 surface S shows that we may have many such coverings. Indeed, it is well known that S itself has many coverings by 1-parameter families of elliptic curves E_t, and then $z \times E_t \subset S^{[n]}$ for any 0-dimensional subscheme $z \subset S$ of length $n-1$ not intersecting E is an elliptic curve in $S^{[n]}$ and these elliptic curves cover $S^{[n]}$.

In contrast, we will show in Section 2.1 that Question 1.2 has an easy negative answer if the covering genus is replaced by the fibering genus:

Proposition 1.3. *Let X be a hyper-Kähler manifold of dimension $2n$. Then if $n > 1$, one has*

$$\mathrm{fibgen}(X) \geq 3, \tag{1.2}$$

$$\mathrm{fibgon}(X) \geq 3. \tag{1.3}$$

The proofs are elementary. The inequality (1.2) is a consequence of the inequality $\mathrm{fibgen}(X) \geq 2$ and of (1.3). The inequality $\mathrm{fibgen}(X) \geq 2$ can be given several proofs. One of them generalizes to the case of fibrations by varieties birational to abelian varieties for which we prove the following result.

Theorem 1.4. *Let X be a hyper-Kähler manifold of dimension $2n$. Then if X admits a fibration $X \dashrightarrow B$ with general fiber birational to an abelian variety of dimension g, one has $g = n$, hence also $\dim B = n$, and the fibration is Lagrangian.*

Theorem 1.4 is wrong if we replace "fibrations" by "coverings". A counterexample is given by the variety $S^{[n]}$ above and its coverings by elliptic curves. In Section 2, we will give examples with $\rho(X) = 1$ of very general hyper-Kähler varieties of dimension 8 swept-out by varieties birational to abelian surfaces. Note that, if instead of studying rational maps, we consider actual morphisms from X to a smaller dimensional basis B, then we already know they are quite restricted when X is a hyper-Kähler manifold. Indeed, if B is not a point, Matsushita [24, 25] proves that they are given by Lagrangian fibrations and in particular the dimension of B is n.

Concerning the fibering genus, we will prove

Theorem 1.5. *Let X be a hyper-Kähler manifold of dimension $2n$ with $n \geq 3$ and $b_2(X)_{\mathrm{tr}} \geq 5$. Assume that the Mumford–Tate group of the Hodge structure on $H^2(X, \mathbb{Q})_{\mathrm{tr}}$ is maximal. Then if X admits a fibration $\phi\colon X \dashrightarrow B$, with $\dim B = 2n-1$, the general fiber of ϕ has genus $g \geq \mathrm{Inf}(n+2, 2^{\lfloor \frac{b_{2,\mathrm{tr}}-3}{2} \rfloor})$. In other words,*

$$\mathrm{fibgen}(X) \geq \mathrm{Inf}(n+2, 2^{\lfloor \frac{b_{2,\mathrm{tr}}-3}{2} \rfloor}).$$

Note that the bound in Theorem 1.5 is presumably not optimal. Looking at the proof, we see that a more natural bound would be $\mathrm{fibgen}(X) \geq \mathrm{Inf}(2n-1, 2^{\lfloor \frac{b_{2,\mathrm{tr}}-3}{2} \rfloor})$. (This is also the reason for the assumption $n \geq 3$ in Theorem 1.5.) For $n = 2$, we do not know what the correct bound is, but we can easily construct an example where the bound $g = n + 2$ is achieved. Indeed, let Y be a smooth cubic fourfold, and let $Y_H \subset Y$ be a hyperplane section. Let X be the variety of lines of Y. It admits a rational map

$$X \dashrightarrow Y_H$$

which to a general point $\delta \in X$ parameterizing a line $\Delta \subset Y$ associates the intersection point $y := \Delta \cap Y_H \in Y_H$. The fiber of this map over a general point $y \in Y_H$ is the curve of lines in Y passing through y, and this is well known (see [8]) to be a genus 4 curve, complete intersection of a quadric and a cubic in $\mathbb{P}^3$.

Proposition 1.3 and the example above leave open the following

Question 1.6. Are there hyper-Kähler fourfolds with fibgen $= 3$? Are there hyper-Kähler sixfolds with fibgen $= 5$?

We now turn to the measure of irrationality $\mathrm{irr}(X)$ mentioned at the beginning of this introduction. In the geometric context we are considering, namely hyper-Kähler manifolds, which in any case are not rational, there are two natural variants of this number, namely

$$\mathrm{RCirr}(X) := \mathrm{Inf} \deg \phi,$$

where ϕ runs through all the generically finite rational maps $X \dashrightarrow Y$, with Y smooth projective rationally connected, and

$$\mathrm{cohirr}(X) := \mathrm{Inf} \deg \phi, \tag{1.4}$$

where ϕ runs through all the generically finite rational maps $X \dashrightarrow Y$, with Y smooth projective with $H^0(Y, \Omega_Y^l) = 0$ for $l > 0$.

Remark 1.7. When X is a hyper-Kähler fourfold, it is equivalent in (1.4) to consider the smooth projective varieties Y with $H^0(Y, K_Y) = 0$, since the existence of a dominant generically finite rational map $\phi\colon X \dashrightarrow Y$ then implies that $H^0(Y, \Omega_Y^l) = 0$ for $l > 0$. Indeed, if Y has a holomorphic 2-form, it is generically nondegenerate since it pulls back to the holomorphic 2-form on X, hence $h^0(Y, K_Y) \neq 0$.

Obviously $\mathrm{cohirr}(X) \leq \mathrm{RCirr}(X) \leq \mathrm{irr}(X)$. The invariant $\mathrm{cohirr}(X)$ has been studied by Alzati and Pirola [2]. A particular case of their results is

Theorem 1.8. *If X is a hyper-Kähler manifold of dimension $2n$, then* $\mathrm{cohirr}(X) \geq n + 1$. *In particular, if* $\dim X \geq 4$, *one has* $\mathrm{cohirr}(X) \geq 3$*; if* $\dim X \geq 6$, *one has* $\mathrm{cohirr}(X) \geq 4$.

Combining Theorem 1.8 and Theorem 1.5, we will prove in Section 2.1

Corollary 1.9. *Let X be a hyper-Kähler manifold of dimension ≥ 6. Assume that $b_2(X)_{\mathrm{tr}} \geq 9$ and X is very general with given Picard number. Then* $\mathrm{fibgon}(X) \geq 4$.

It is likely that a better lower bound for Theorem 1.8 can be found, maybe depending on numerical data as in [23], which studies the case of abelian surfaces. In the case of dimension 4, we leave this as

Question 1.10. Let X be a hyper-Kähler fourfold which is very general with fixed Picard number. Is it true that $\mathrm{cohirr}(X) \geq 4$?

We prove one result in this direction in Section 3, namely Proposition 3.1 which is used in the last section of the paper. We establish there a generalization of a result of O'Grady (see [27] or Theorem 4.1). O'Grady studies the rational map $\phi_L \colon X \dashrightarrow \mathbb{P}^5$ induced by the complete linear system $|L|$, for a line bundle L of top self-intersection 12 on a compact Kähler fourfold X which is numerically equivalent to $\mathrm{K3}^{[2]}$. Assuming X is very general with Picard number 1, O'Grady proves that the image of ϕ_L is a hypersurface of degree ≥ 6. We prove a similar result (see Theorem 4.2) under different assumptions. First of all, X is only known to have the same Betti numbers, Chern numbers, and Fujiki constant as a hyper-Kähler fourfold of type $\mathrm{K3}^{[2]}$. Second, in our case, the line bundle is the sum $L+M$, where both L and M are numerically effective and satisfy the intersection conditions

$$L^4 = 0, \quad M^4 = 0, \quad L^2M^2 = 2, \tag{1.5}$$

which implies $(L+M)^4 = 12$. Our result is

Theorem 1.11. *Under the assumptions above, assuming X is very general with Picard number 2 and $h^0(X,L) = 0$, the image of $\phi_{L+M}\colon X \dashrightarrow \mathbb{P}^5$ is not rationally connected.*

Although this result may seem a bit specific, this statement is needed in order to conclude the proof of the main result in [9], namely that a hyper-Kähler fourfold X admitting two integral degree 2 cohomology classes l, m satisfying the condition (1.5) has to be of $\mathrm{K3}^{[2]}$ deformation type.

Theorem 1.11 is proved by a case-by-case analysis. As will be clear from the proof, a positive answer to Question 1.10 and a negative answer to Question 1.6 would greatly simplify the proof, since by Lemmas 4.5, 4.6, 4.7, 4.9 and 4.8 and Claim 4.11 the most difficult cases to exclude are those where X is fibered into curves of genus 3, or ϕ_{L+M} has degree 3 on its image.

2. Fibrations of hyper-Kähler manifolds by curves and abelian varieties

2.1. **Some general inequalities.** We start by establishing easy lower bounds for the fibering genus and gonality, and various irrationality invariants of hyper-Kähler manifolds.

Lemma 2.1 (See also [27])**.** *Let X be a hyper-Kähler manifold of dimension $2n$. If there exists a dominant rational map $\phi\colon X \dashrightarrow Y$ of degree 2, where Y is a smooth projective variety, then $h^0(Y,\Omega_Y^{4k}) = 1$ for $k \leq n$. In particular, the cohomological measure of irrationality* cohirr(X) *of a hyper-Kähler $2n$-fold is strictly greater than 2 for $n \geq 2$.*

Proof. We observe that, as ϕ has degree 2, there is a rational involution ι on X over Y. As $h^0(X,\Omega_X^2) = 1$, one has $\iota^*\sigma_X = \pm\sigma_X$. It follows that $\iota^*\sigma_X^{2k} = \sigma_X^{2k}$ for any integer k. Thus the $(4k,0)$-form σ_X^{2k} on X, which is nonzero for $2k \leq n$, descends to Y, proving the inequality $h^0(Y,\Omega_Y^{4k}) \geq 1$ for $k \leq n$. The inequality $h^0(Y,\Omega_Y^{4k}) \leq 1$ for $k \leq n$ follows from the fact that ϕ^* is injective on holomorphic forms. ☐

We now apply this result to the proof of Proposition 1.3.

Proof of Proposition 1.3. We first prove

Lemma 2.2. *Let X be a hyper-Kähler $2n$-fold with $n \geq 2$. Then X does not admit a fibration $\phi\colon X \dashrightarrow Y$ into elliptic curves, hence* fibgen$(X) \geq 2$.

Proof. Let $\tau : \widetilde{X} \to X$, $\tilde{\phi}\colon \widetilde{X} \to Y$ be a resolution of the indeterminacies of ϕ, with $\widetilde{X}$ smooth. Then, as the general fiber F of $\tilde{\phi}$ is elliptic, one has $K_{\widetilde{X}|F} = \mathcal{O}_F$.

But $K_{\widetilde{X}}$ has a section whose divisor has for support the exceptional divisor of τ. It follows that F does not intersect the exceptional divisor of τ. In other words, ϕ is quasiholomorphic. This contradicts a theorem of Matsushita [24] which says that a quasiholomorphic map from a hyper-Kähler $2n$-fold to a manifold of smaller dimension has image of dimension $\leq n$. ☐

Inequality (1.3) in Proposition 1.3 implies inequality (1.2) since curves of genus ≤ 2 have gonality ≤ 2. We now prove the inequality (1.3). Assume that X admits a fibration $\phi\colon X \dashrightarrow Y$ into hyperelliptic curves. By Lemma 2.2, the fibers have genus at least 2. The smooth projective variety Y obviously satisfies $h^0(Y, \Omega_Y^l) = 0$ for $l > 0$. Furthermore, there exists a relative hyperelliptic involution ι on X such that any smooth model Q of X/ι is a fibration into $\mathbb{P}^1$ over Y. Thus Q satisfies $h^0(Q, \Omega_Q^4) = 0$. This contradicts Lemma 2.1. ☐

Another easy result is the following

Lemma 2.3. *Let X be a hyper-Kähler manifold of dimension ≥ 4. Then*

$$\mathrm{RCirr}(X) \leq 2\,\mathrm{fibgen}(X) - 2.$$

Proof. Let $f\colon X \dashrightarrow B$ be a fibration realizing the fibering genus, so that the fibers have genus $g = \mathrm{fibgen}(X)$, and let $\tilde{f}\colon \widetilde{X} \to B$ be a resolution of the indeterminacies of f. By Lemma 2.2, we know that $g \geq 2$. By [22], the base B is rationally connected. We now choose a rank 2 subsheaf $\mathcal{F}$ of the sheaf $R^0\tilde{f}_*K_{\widetilde{X}/B}$. The variety $\mathbb{P}(\mathcal{F})$ is generically a $\mathbb{P}^1$-bundle over B, hence is rationally connected, and there is a natural rational map

$$\psi : \widetilde{X} \dashrightarrow \mathbb{P}(\mathcal{F})$$

over B, which is of degree $\leq 2g - 2$. ☐

Remark 2.4. By Theorem 1.8, Lemma 2.3 implies that, for any hyper-Kähler fourfold X, one has $\mathrm{fibgen}(X) \geq \frac{n+3}{2}$, a statement to be compared with Theorem 1.5.

We finally combine the results above to prove

Proposition 2.5. *Let X be a projective hyper-Kähler manifold of dimension ≥ 4. Assume that the fibering gonality of X is 3. Then one of the following possibilities holds:*

(i) $\mathrm{fibgen}(X) = 3$ *and* $\mathrm{RCirr}(X) \leq 4$.
(ii) $\mathrm{fibgen}(X) = 4$ *and* $\mathrm{RCirr}(X) \leq 6$.
(iii) $\mathrm{fibgen}(X) > 4$ *and* $\mathrm{RCirr}(X) = 3$.

Proof. Let $\phi\colon X \dashrightarrow B$ be a fibration realizing the fibering gonality, so that the fibers of f are trigonal curves. We know by Proposition 1.3 that the genus of the fibers is at least 3. If the genus of the fibers is 3 or 4, then we apply Lemma 2.3 and get the inequalities in (i) and (ii). If the genus of the fibers is ≥ 5, we recall that a curve of genus ≥ 5 which is trigonal admits a unique g_3^1, unless it is hyperelliptic, which is excluded by Proposition 1.3. It follows that there exists a fibration $P \dashrightarrow B$ into $\mathbb{P}^1$'s and a rational map of degree 3

$$\psi\colon X \dashrightarrow P$$

over B, which induces the trigonal map on the fibers of f. As B is rationally connected, P is rationally connected and thus $\mathrm{RCirr}(X) = 3$. $\square$

2.2. **Proof of Theorem 1.4.** Let X be a hyper-Kähler manifold of dimension $2n$ admitting a fibration $f\colon X \dashrightarrow B$ with general fiber birational to an abelian variety of dimension g. Let L be an ample line bundle on X. The restriction to the general fiber $\widetilde{X}_t$ of a resolution $\tilde{f}\colon \widetilde{X} \to B$ of the indeterminacies of f has top self-intersection $D := \deg L^g_{|\widetilde{X}_t}$. We will denote by Z_t the 0-cycle $L^g_{|\widetilde{X}_t} \in \mathrm{CH}_0(\widetilde{X}_t)$.

As $\widetilde{X}_t$ is birational to its Albanese variety, there is a rational action by translation

$$\widetilde{X}_t \times \mathrm{Alb}\,\widetilde{X}_t \dashrightarrow \widetilde{X}_t$$
$$(x, u) \mapsto x + u$$

of $\mathrm{Alb}\,\widetilde{X}_t$ on $\widetilde{X}_t$.

For any integer k, we can construct a rational self-map

$$\Psi_k\colon X \dashrightarrow X$$

preserving f, that is, acting fiberwise, and defined by

$$\Psi_k(x) = x + k\,\mathrm{alb}_{\widetilde{X}_t}(Dx - Z_t), \quad x \in \widetilde{X}_t. \tag{2.1}$$

Lemma 2.6. *The degree of Ψ_k is $(kD+1)^{2g}$.*

Proof. As Ψ_k acts in a fiberwise way with respect to f, its degree is equal to the degree of its restriction to the fibers $\widetilde{X}_t$. By (2.1), this restriction is birationally conjugate to the multiplication by $kD+1$ on a g-dimensional abelian variety, which proves the result. $\square$

We next have

Lemma 2.7. *Let $\sigma_X \in H^0(X, \Omega^2_X)$ be a generator. We have either $\Psi_k^*\sigma_X = (kD+1)\sigma_X$ or $\Psi_k^*\sigma_X = (kD+1)^2\sigma_X$. In the first case, the fibers $\widetilde{X}_t$ are isotropic for σ_X.*

Proof. As $\Psi_k^*\sigma_X$ is a nonzero holomorphic 2-form on X, it must be a nonzero multiple of σ_X, so $\Psi_k^*\sigma_X = \mu\sigma_X$. As Ψ_k acts in a fiberwise way, we have

$$(\Psi_k^*\sigma_X)_{|\widetilde{X}_t} = \Psi^*_{k|\widetilde{X}_t}(\sigma_{X|\widetilde{X}_t}). \tag{2.2}$$

As $\Psi^*_{k|\widetilde{X}_t}$ acts as multiplication by $(kD+1)^2$ on the transcendental degree 2 cohomology of $\widetilde{X}_t$, (2.2) implies that $\mu = (kD+1)^2$ if the fibers $\widetilde{X}_t$ are not isotropic for σ_X. If the fibers $\widetilde{X}_t$ are isotropic for σ_X, then σ_X (or rather its pull-back $\tau^*\sigma_X$ on a model $\widetilde{X}$ where f is well defined) maps to an element σ_t of $H^0(\widetilde{X}_t, \Omega_{\widetilde{X}_t}) \otimes \Omega_{B,t}$ which is nonzero for generic t, as otherwise $\tau^*\sigma_X$ would be everywhere degenerate. As $\Psi^*_{k|\widetilde{X}_t}$ acts as multiplication by $kD+1$ on 1-forms on $\widetilde{X}_t$, we get in this case

$\mu = kD+1$, using the fact that the action of Ψ_k^* on σ_t is induced by the action of $\Psi^*_{k|\widetilde{X}_t}$ on the space $H^0(\widetilde{X}_t, \Omega_{\widetilde{X}_t})$. □

Proof of Theorem 1.4. We have $\Psi_k^*\sigma_X = \mu\sigma_X$, which we write in the form

$$\widetilde{\Psi}_k^*(\sigma_X) = \mu\tau^*\sigma_X, \tag{2.3}$$

where

$$\tau : \widetilde{X} \to X, \quad \widetilde{\Psi}_k \colon \widetilde{X} \to X$$

is a desingularization of $\Psi_k \colon X \dashrightarrow X$. As we are now working with morphisms in (2.3) and μ is a real number by Lemma 2.7, it follows that

$$\widetilde{\Psi}_k^*(\sigma_X^n \wedge \overline{\sigma_X}^n) = \mu^{2n}\tau^*(\sigma_X^n \wedge \overline{\sigma_X}^n).$$

Integrating both sides over $\widetilde{X}$, we get $\deg \widetilde{\Psi}_k = \deg \Psi_k = \mu^{2n}$. By Lemma 2.6, we deduce that

$$\mu^{2n} = (kD+1)^{2g}, \tag{2.4}$$

while by Lemma 2.7, we have $\mu = kD+1$ or $\mu = (kD+1)^2$. If $\mu = (kD+1)^2$, we get by (2.4) that $4n = 2g$, which contradicts the fact that $g < 2n$. Hence $\mu = kD+1$, which implies by (2.4) that $n = g$. Furthermore, the fibers are isotropic in this case by Lemma 2.7. □

If instead of a fibration we consider a covering by varieties birational to abelian varieties of dimension g, we can conclude that they are isotropic, assuming that the Mumford–Tate group of the Hodge structure on $H^2(X,\mathbb{Q})_{\mathrm{tr}}$ is maximal and

$$g < 2^{\lfloor \frac{b_{2,\mathrm{tr}}-3}{2} \rfloor}, \tag{2.5}$$

by applying the result in [6] (or [29] if $b_2(X)_{\mathrm{tr}} \geq 5$). Indeed, these results say that the Hodge structure on $H^2(X,\mathbb{Q})_{\mathrm{tr}}$, which is simple, cannot be realized as a Hodge substructure of $H^2(A)$ for any abelian variety of dimension g if g satisfies (2.5). Note that, without the inequality (2.5), one can construct coverings by abelian subvarieties which are not isotropic, as shows the example of the generalized Kummer $K_n(A)$ which is swept out by copies of surfaces birational to A.

Concerning the statement about the dimension, the following is an example of a covering of a hyper-Kähler manifold of dimension 8 with $\rho = 1$ by varieties birational to abelian surfaces.

Example 2.8. Let Y be a cubic fourfold, and let X be the LLSvS 8-fold of Y (see [21]). This is an 8-fold which is deformation equivalent to $\mathrm{K3}^{[4]}$ (see [1]). Furthermore, if Y is very general, one has $\rho(X) = 1$. Let $F_1(Y)$ be the variety of lines in Y. There exists a dominant rational map (see [31])

$$\psi \colon F_1(Y) \times F_1(Y) \dashrightarrow X.$$

Next, the hyper-Kähler manifold $F_1(Y)$ is itself covered by surfaces birational to abelian surfaces. Indeed, consider the surfaces of lines Σ_{Y_H} contained in a hyperplane section Y_H of Y. It is a classical fact that, when Y_H has one singular point y, Σ_H is birational to the symmetric product $C^{(2)}_{y,H}$, where $C_{y,H}$ is the curve of lines contained in Y_H and passing through y. This curve is of genus 4 when Y_H has one ordinary quadratic singularity at y and is smooth otherwise. When Y_H has two more singular points y' and y'', the curve $C_{y,H}$ becomes singular at these points, and its geometric genus decreases to 2. It is clear that $F_1(Y)$ is covered by these surfaces $\Sigma_{y,H}$ birational to the symmetric product $C_{y,H}^{(2)}$ of a curve of genus 2, hence to abelian surfaces, and using the morphism ψ, it follows that X is covered by the surfaces $\psi(x \times \Sigma_{y,H})$, which are birational to abelian surfaces.

2.3. **Proof of Theorem 1.5.** Let X be a hyper-Kähler $2n$-fold and let

$$f : \widetilde{X} \to B, \quad \tau : \widetilde{X} \to X, \tag{2.6}$$

where τ is birational and $\widetilde{X}$ is smooth projective, be a fibration into curves of genus g over a base B of dimension $2n-1$. We have $h^0(B, \Omega_B^l) = 0$ for any $l > 0$ and in fact B is rationally connected (see [22]). Let $b \in B$ be a general point so that the fiber $\widetilde{X}_b$ is smooth. Consider the natural morphism

$$\sigma_b \colon T_{B,b} \to H^0(\widetilde{X}_b, \Omega_{\widetilde{X}_b})$$

induced by the vector bundle morphism

$$T_{\widetilde{X}_b} \to f^*\Omega_{B,b}$$

defined by contraction with the holomorphic 2-form $\tau^*\sigma_X$ along $\widetilde{X}_b$.

Lemma 2.9. *The morphism σ_b has rank $\geq n$.*

Proof. Over the open set B^0 of B of regular values of f, we have the relative Albanese (or Jacobian) fibration $J_f \to B^0$. The $(2,0)$-form σ_X on $\widetilde{X}^0$ induces a $(2,0)$-form

$$\sigma_J := \mathcal{P}^*\sigma_X$$

on J_f, where $\mathcal{P} \subset J_f \times_{B^0} \widetilde{X}^0$ is a universal divisor, satisfying the assumption that, for some nonzero integer d,

$$\mathrm{alb}_{\widetilde{X}_b}(\mathcal{P}_y) = dy$$

for any $y \in J_{f,b} = \mathrm{Pic}^0(\widetilde{X}_b)$.

We also have the Albanese embedding (up to isogeny)

$$\mathrm{alb}_f \colon \widetilde{X}^0 \to J_f,$$

which maps $x \in \widetilde{X}_t$ to $\mathrm{alb}_{\widetilde{X}_t}((2g-2)x - K_{\widetilde{X}_t})$. We have

$$\mathrm{alb}_f^*\,\sigma_J = d(2g-2)\sigma_{\widetilde{X}} \tag{2.7}$$

since by definition of σ_J, $\mathrm{alb}_f^*\,\sigma_J = \Gamma^*\sigma_X$, where Γ is the self-correspondence

$$x \mapsto d((2g-2)x - K_{\widetilde{X}_t}), \quad t = f(x)$$

of X over B, which induces multiplication by $d(2g-2)$ on $\mathrm{CH}_0(X)_{\mathrm{hom}}$ because B is rationally connected. It follows from (2.7) that we have the inequality of generic ranks

$$\mathrm{rank}\,\sigma_J \geq \mathrm{rank}\,\sigma_{\widetilde{X}},$$

that is,

$$\operatorname{rank}\sigma_J \geq 2n. \tag{2.8}$$

By construction, the $(2,0)$-form σ_J vanishes identically on the fibers $J_b = J(\widetilde{X}_b)$ of $\pi\colon J \to B^0$, hence induces a contraction map $\sigma_{J,b}\colon T_{B^0,b} \to H^0(J_b, \Omega_{J_b})$, and, by (2.7), we clearly have a commutative diagram

$$\begin{array}{ccc} T_{B^0,b} & \stackrel{\sigma_{J,b}}{\to} & \mathrm{alb}^*_{\widetilde{X}_b}\,\Omega_{J_b} \\ \| & & a\downarrow \\ T_{B^0,b} & \stackrel{\sigma_b}{\to} & \Omega_{\widetilde{X}_b} \end{array}$$

of morphisms of vector bundles on $\widetilde{X}_b$, where $a := \frac{1}{d(2g-2)}\,\mathrm{alb}^*_{\widetilde{X}_b}$. Taking global sections, we get

$$\begin{array}{ccc} T_{B^0,b} & \stackrel{\sigma_{J,b}}{\to} & H^0(J_b, \Omega_{J_b}) \\ \| & & a\downarrow \\ T_{B^0,b} & \stackrel{\sigma_b}{\to} & H^0(\widetilde{X}_b, \Omega_{\widetilde{X}_b}), \end{array} \tag{2.9}$$

where the second vertical map is an isomorphism. We now have

Claim 2.10. *We have the equality of rank along J_b*

$$\operatorname{rank}\sigma_J = 2\operatorname{rank}\sigma_{J,b}. \tag{2.10}$$

Proof. The torsion points of J_b are dense in J_b for the Zariski or Euclidean topology, so it suffices to prove the equality at a torsion point $y \in J_b$. Through such a point, there is a torsion multisection $Z_y \subset J$, which is transverse to the fiber J_b. The $(2,0)$-form σ_J vanishes on Z_y, because torsion points are rationally equivalent (up to torsion) in the fibers to the origin $0_b \in J_b$ and all points in the 0-section are rationally equivalent in $\widetilde{X}$ since the base B is rationally connected. It follows that the matrix of σ_J at y in a basis adapted to the decomposition $T_{J,y} = T_{Z_y,y} \oplus T_{J_b,y}$, where $T_{Z_y,y} \cong T_{B,b}$, takes the block form

$$\begin{pmatrix} 0 & M_{\sigma_{J,b}} \\ -{}^tM_{\sigma_{J,b}} & 0 \end{pmatrix},$$

where $M_{\sigma_{J,b}}$ is the matrix of $\sigma_{J,b}$. $\square$

Using the identifications (2.9), Lemma 2.9 follows from (2.8) and (2.10). $\square$

Corollary 2.11. *One has $g \geq n$.*

Proof. Indeed, this follows from Lemma 2.9 since the rank of σ_b is not greater than $g = \dim H^0(\widetilde{X}_b, \Omega_{\widetilde{X}_b})$. $\square$

Remark 2.12. For $n = 2$, this gives a third proof of Lemma 2.2.

Let $\overline{\nabla}_b\colon T_{B,b} \to \mathrm{Hom}(H^0(\widetilde{X}_b, \Omega_{\widetilde{X}_b}), H^1(\widetilde{X}_b, \mathcal{O}_{\widetilde{X}_b}))$ be the infinitesimal variation of Hodge structure of the family of curves (2.6) at b. We will use the following classical symmetry result due to Donagi and Markman [10] (see also [3]).

Lemma 2.13. *The bilinear map* $T_{B,b} \otimes T_{B,b} \to H^1(\widetilde{X}_b, \mathcal{O}_{\widetilde{X}_b})$,

$$(u, v) \mapsto \overline{\nabla}_u(\sigma_b(v))$$

is symmetric in u *and* v.

Proof of Theorem 1.5. In the situation above, assume that $n \geq 3$ and $g = n$ or $g = n + 1$. Then $2n - 1 > n + 1 \geq g$. It follows that, at a general point $b \in B$, the morphism σ_b has a nontrivial kernel $K_b \subset T_{B,b}$. Moreover, by Lemma 2.9, the morphism σ_b is either surjective or has for image a hyperplane in $H^0(\widetilde{X}_b, K_{\widetilde{X}_b})$.

Case $g = n$ *or* $n + 1$ *and* σ_b *is surjective.* We first prove

Lemma 2.14. *The kernel* K_b *of* σ_b *is contained in the kernel of the Kodaira–Spencer map* $\rho_b \colon T_{B,b} \to H^1(\widetilde{X}_b, T_{\widetilde{X}_b})$.

Proof. We apply Lemma 2.13. It thus follows that for $u \in K_b$, and any $v \in T_{B,b}$, we have

$$\overline{\nabla}_u(\sigma_b(v)) = \overline{\nabla}_v(\sigma_b(u)) = 0.$$

As σ_b is surjective, this implies that $\overline{\nabla}_u \colon H^0(\widetilde{X}_b, K_{\widetilde{X}_b}) \to H^1(\widetilde{X}_b, \mathcal{O}_{\widetilde{X}_b})$ is identically 0. However, we know by Proposition 1.3 that the fibers $\widetilde{X}_b$ are not hyperelliptic, hence the map

$$H^1(\widetilde{X}_b, T_{\widetilde{X}_b}) \to \mathrm{Hom}(H^0(\widetilde{X}_b, K_{\widetilde{X}_b}), H^1(\widetilde{X}_b, \mathcal{O}_{\widetilde{X}_b}))$$

is injective. Therefore $\rho_b(u) = 0$. $\square$

Let $m \colon B \dashrightarrow \mathcal{M}_g$ be the moduli map, which to a general point $b \in B$ associates the isomorphism class of the curve $\widetilde{X}_b$. By Lemma 2.14, the vector space K_b is tangent to the fiber of m, hence it follows that the map m has positive dimensional fibers. We thus have, after Stein factorization, a fibration $m' \colon B \dashrightarrow B'$ with connected positive dimensional fibers, having the property that, restricted to a general fiber of m', the fibration f becomes isotrivial. Denoting by $f' : \widetilde{X} \dashrightarrow B'$ the composition $m' \circ f$, we can assume by modifying $\widetilde{X}$ that f' is a morphism, and prove

Lemma 2.15. *Assume that* X *is very general with fixed Picard number, that* $b_2(X)_{\mathrm{tr}} \geq 5$ *and that* $g < 2^{\lfloor \frac{b_{2,\mathrm{tr}}-3}{2} \rfloor}$. *Then the general fiber of* f' *is isotropic for* $\tau^*\sigma_X$.

Remark 2.16. Lemma 2.15 says in particular that $\operatorname{Ker} \rho_b \subset K_b$, hence $\operatorname{Ker} \rho_b = K_b$ by Lemma 2.14. In particular, $\dim B' = g$.

Proof of Lemma 2.15. As the fibration f is isotrivial after restriction to the general fiber $B_{b'} \subset B$ of m', the fiber $\widetilde{X}_{b'} := f'^{-1}(b')$ is rationally dominated by a product $C_{b'} \times \widetilde{B_{b'}}$, where $\widetilde{B_{b'}}$ is a generically finite cover of $B_{b'}$, and $C_{b'}$ is isomorphic to the fibers of the restricted family, so in particular has genus g. The fact that X is very general with fixed Picard group implies that the Mumford–Tate group of the Hodge structure on $H^2(X,\mathbb{Q})_{\mathrm{tr}}$ is the orthogonal group of the Beauville–Bogomolov form, and as proved in [29], this implies that, if the composite map

$$H^2(X,\mathbb{Q})_{\mathrm{tr}} \to H^2(\widetilde{X}_{b'},\mathbb{Q}) \to H^1(C_{b'},\mathbb{Q}) \otimes H^1(\widetilde{B_{b'}},\mathbb{Q})$$

is nontrivial, then the Hodge structure on $H^1(C_{b'},\mathbb{Q})$ contains a simple factor of the Kuga–Satake weight 1 Hodge structure of $H^2(X,\mathbb{Q})_{\mathrm{tr}}$, hence in particular $g \geq 2^{\lfloor \frac{b_{2,\mathrm{tr}}-3}{2} \rfloor}$. This is excluded by assumption and it follows that the form $\tau^*\sigma_{X|\widetilde{X}_{b'}}$ is either 0 or the pull-back of a holomorphic 2-form $\tau_{b'}$ on the fiber $B_{b'}$. In the first case, the lemma is proved. In the second case, there is a nonzero locally constant holomorphic 2-form $\eta_{b'} \in H^{2,0}(B_{b'})$ whose pull-back to $\widetilde{X}_{b'}$ is $\tau^*\sigma_{X|\widetilde{X}_{b'}}$ and Deligne's global invariant cycle theorem then implies that there is a holomorphic 2-form η on B whose restriction to $B_{b'}$ is $\eta_{b'}$. This is impossible, since otherwise $f^*\eta$ would provide a nonzero holomorphic 2-form on X of rank $< 2n$. $\square$

Let B'_0 be the Zariski open set of B' over which the morphism $f' : \widetilde{X} \to B'$ is smooth and let $A \to B'_0$ be the Albanese fibration of f'. There is a rational map

$$\psi\colon \widetilde{X} \dashrightarrow A,$$

which is constructed as follows: we define ψ as the composition of the relative Abel or Albanese map up to isogeny

$$\mathrm{alb}\colon \widetilde{X} \dashrightarrow J(\widetilde{X}/B), \tag{2.11}$$

which we used previously and which to $c \in \widetilde{X}_b$ associates $\mathrm{alb}_{\widetilde{X}_b}((2g-2)c - K_{\widetilde{X}_b})$, and the natural rational map

$$\psi_{\mathrm{ab}}\colon J(\widetilde{X}/B) \dashrightarrow A,$$

inducing over a general $b \in B$ the morphism

$$\psi_{\mathrm{ab},b}\colon J(\widetilde{X}_b) = \mathrm{Alb}(\widetilde{X}_b) \to \mathrm{Alb}(\widetilde{X}_{b'}), \quad b' = m'(b) \tag{2.12}$$

of abelian varieties.

Remark 2.17. The rational map ψ might be different from any relative Albanese map for f' constructed using a multisection of f'. More precisely, it may differ from it by translation by a rational section of A over B, that is, a rational map $B \dashrightarrow A$ over B'.

We have

Lemma 2.18. *The assumptions being as in Lemma 2.15, the image* $Y := \mathrm{Im}\,\psi \subset A$ *has dimension* $\dim B' + 1$ *and there is a holomorphic* 2*-form* η *on any smooth projective birational model of* Y*, whose pull-back to* X *under* $\psi_Y\colon X \dashrightarrow Y$ *is a nonzero multiple of* σ_X*.*

Proof. We first claim that for general $b \in B$, with $m'(b) = b'$, the morphism of abelian varieties (2.12) is an isogeny on its image. By Lemma 2.15, the general fibers of $f' : \widetilde{X} \to B'$ are isotropic for $\tau^*\sigma_X$, hence there is a morphism

$$\sigma'_{b'} : T_{B',b'} \to H^0(\widetilde{X}_{b'}, \Omega_{\widetilde{X}_{b'}})$$

of contraction with $\tau^*\sigma_X$. By Remark 2.16, the morphism

$$\sigma_b\colon T_{B,b} \to H^0(\widetilde{X}_b, \Omega_{\widetilde{X}_b})$$

(which is surjective by assumption in the case we are considering) factors through an isomorphism

$$\overline{\sigma}_b\colon T_{B',b'} \to H^0(\widetilde{X}_b, \Omega_{\widetilde{X}_b}).$$

It is immediate to check that the following diagram is commutative:

$$\begin{array}{ccc} T_{B',b'} & \stackrel{\sigma'_{b'}}{\to} & H^0(\widetilde{X}_{b'}, \Omega_{\widetilde{X}_{b'}}) \\ \| & & \psi^*_{\mathrm{ab},b}\downarrow \\ T_{B',b'} & \stackrel{\overline{\sigma}_b}{\to} & H^0(\widetilde{X}_b, \Omega_{\widetilde{X}_b}). \end{array}$$

This implies that $\psi^*_{\mathrm{ab},b}$ is surjective, thus proving the claim. It follows from the claim that the image $\operatorname{Im}\psi_{ab}$ is a family of abelian varieties $J' \to B'$ over B' which descends up to isogeny the family $J(\widetilde{X}/B) \to B$. The image of the curve $\widetilde{X}_b$ in $J'_{b'}$ via ψ obviously does not depend on the point b in the fiber $m'^{-1}(b') \subset B$, since by construction of ψ this is, up to isogeny, the curve $\widetilde{X}_b$ canonically embedded via the Abel map (2.11). This proves that $\dim Y = \dim B' + 1$. It remains to construct a nonzero holomorphic 2-form η on Y_{reg} satisfying the desired property. Recall that $Y \subset A := \operatorname{Alb}(\widetilde{X}/B')$. Using a multisection of $\widetilde{X} \dashrightarrow B'$, we get a relative Albanese map (or rather a multiple, depending on the degree of the multisection) $a_{\widetilde{X}/B'}\colon \widetilde{X} \dashrightarrow A$. Furthermore, by Lemma 2.15, using [3] or [30], the relative Albanese variety A admits a holomorphic 2-form σ_A (which extends to a smooth projective compactification of A) with the property that

$$a^*_{\widetilde{X}/B'}\sigma_A = \sigma_{\widetilde{X}}. \tag{2.13}$$

Let $\eta := \sigma_{A|Y_{\mathrm{reg}}}$. This $(2,0)$-form clearly extends to a smooth projective compactification of Y_{reg}. It remains to prove that the pull-back $\psi_Y^*\eta$, which by definition of η equals $\psi^*\sigma_A$, is nonzero on $\widetilde{X}$. This follows in fact directly from (2.13), by observing that the $(2,0)$-forms

$$\psi^*\sigma_A, \quad a^*_{\widetilde{X}/B'}\sigma_A$$

on $\widetilde{X}$ differ by a holomorphic $(2,0)$-form on B (see Remark 2.17) and B has no nonzero holomorphic 2-form. ☐

Lemma 2.18 provides us with a contradiction, since $\dim Y = \dim B' + 1 = g + 1 \leq n + 2 < 2n$ because $n \geq 3$ and thus the pull-back of η to $\widetilde{X}$ provides a holomorphic 2-form on $\widetilde{X}$ which is everywhere degenerate. This case is thus excluded.

Case $g = n + 1$ and σ_b has for image a hyperplane in $H^0(\widetilde{X}_b, \Omega_{\widetilde{X}_b})$. We use the same notation as before, that is, $K_b \subset T_{B,b}$ is the kernel of the contraction map $\sigma_b\colon T_{B,b} \to H^0(\widetilde{X}_b, \Omega_{\widetilde{X}_b})$. In this case, we first have the following variant of Lemma 2.14:

Lemma 2.19. *At a general point $b \in B$, the rank of the map*

$$\rho_b \colon K_b \to H^1(\widetilde{X}_b, T_{\widetilde{X}_b})$$

is at most 1.

Proof. By the same arguments as in the proof of Lemma 2.14, we find that $\rho_b(K_b)$ is orthogonal with respect to Serre duality to $H^0(\widetilde{X}_b, K_{\widetilde{X}_b}) \cdot \operatorname{Im} \sigma_b \subset H^0(\widetilde{X}_b, 2K_{\widetilde{X}_b})$. As we know by Proposition 1.3 that the general fiber $\widetilde{X}_b$ is not hyperelliptic, and by assumption $\operatorname{Im} \sigma_b \subset H^0(\widetilde{X}_b, K_{\widetilde{X}_b})$ is a hyperplane, $H^0(\widetilde{X}_b, K_{\widetilde{X}_b}) \cdot \operatorname{Im} \sigma_b$ has codimension at most 1 in $H^0(\widetilde{X}_b, 2K_{\widetilde{X}_b})$, which proves the lemma. ☐

As $\operatorname{rank} \sigma_b = n$, we have $\dim K_b = n - 1 \geq 2$, and it follows from Lemma 2.19 that $\operatorname{Ker} \rho_b \neq 0$, that is, the moduli map has positive dimensional general fiber. The rest of the proof works as in the previous case, except that the morphism of abelian varieties $\psi_{\mathrm{ab},b}$ of (2.12) can now have a 1-dimensional kernel, so that only a $(g-1)$-dimensional quotient of the Jacobian fibration descends to B'. ☐

We conclude this section with the proof of Corollary 1.9.

Proof of Corollary 1.9. Let X be a very general hyper-Kähler $2n$-fold with $n \geq 3$ and $b_2(X)_{\mathrm{tr}} \geq 9$. By Theorem 1.3, one has $\operatorname{fibgen}(X) \geq 5$ and by Theorem 1.8, one has $\operatorname{cohirr}(X) \geq 4$, hence a fortiori $\operatorname{RCirr}(X) \geq 4$. It thus follows from Proposition 2.5 that $\operatorname{fibgon}(X) \geq 4$. ☐

3. Measure of irrationality

We do not know if (maybe under some assumptions on $b_{2,\mathrm{tr}}(X)$) the cohomological irrationality of a hyper-Kähler fourfold X is at least 4, which would greatly simplify the proof of Theorem 4.2, but we can prove a weaker statement that will be used in the next section.

Proposition 3.1. *Let X be a hyper-Kähler fourfold such that any big divisor on X is ample. Then there exists no quasifinite morphism $f \colon X \to Y$ of degree* 3*, where Y is projective, normal and $-K_{Y_{\mathrm{reg}}}$ is the restriction to the regular locus Y_{reg} of a big line bundle on Y.*

Here, by a "big line bundle on Y" we mean the sum of an ample line bundle and an effective divisor. Our assumptions on $K_{Y_{\mathrm{reg}}}$ thus mean that there exists an ample line bundle $\mathcal{L}$ on Y, and an effective divisor E in Y, such that

$$K_{Y_{\mathrm{reg}}} = \mathcal{L}^{-1}(-E)_{|Y_{\mathrm{reg}}}. \tag{3.1}$$

Proof. We argue by contradiction. We first observe that, under our assumptions, $h^0(\widetilde{Y}, K_{\widetilde{Y}}) = 0$ for any desingularization $\widetilde{Y}$ of Y. Indeed, using (3.1), we get that

$$H^0(Y_{\mathrm{reg}}, K_{Y_{\mathrm{reg}}}) \subset H^0(Y_{\mathrm{reg}}, \mathcal{L}^{-1}_{|Y_{\mathrm{reg}}})$$

and the right hand side is zero, since Y is normal and $\mathcal{L}$ is ample. This fact also implies that $h^{2,0}(\widetilde{Y}) = 0$, since otherwise the 2-form on X would be pulled back from Y, hence also its $(4,0)$-form, while we know that $h^{4,0}(\widetilde{Y}) = 0$.

The ramification divisor R of f, which is well defined on $f^{-1}(Y_{\rm reg})$, belongs to the linear system $|f^*(-K_Y)|$, hence is big on $f^{-1}(Y_{\rm reg})$. There is a second effective divisor R' in $f^{-1}(Y_{\rm reg}) \subset X$, namely $f^{-1}(f(R)) - 2R$. The divisor R' is not empty, since its image in $Y_{\rm reg}$ is equal to $f(R)$. We now prove

Lemma 3.2. *The locus defined as the intersection*

$$S := R \cap R' \tag{3.2}$$

in $X^0 := X \setminus f^{-1}(Y_{\rm sing})$ is isotropic for the 2-form σ_X.

Proof. We observe that, due to the fact that the map f is quasifinite (hence finite over the smooth locus of Y), the locus (3.2) consists of points $x \in X$ such that the length of the fiber $f^{-1}(f(x))$ at x is at least 3, hence equal to 3 since the degree of f is 3. For all these points $x \in X^0$, the class $3x \in {\rm CH}_0(X)$ is thus the inverse image of a 0-cycle of Y. It follows from Mumford's theorem [26] that the restriction of σ_X to any desingularization of S is the pull-back of a 2-form defined on $Y_{\rm reg}$, and in fact on a desingularization $\widetilde{Y}$ of Y. However, as mentioned above, we have $h^{2,0}(\widetilde{Y}) = 0$. ☐

In order to finish the proof, we have to see what happens along the singular locus $Y_{\rm sing}$ of Y.

Lemma 3.3. *Any 2-dimensional component of $f^{-1}(Y_{\rm sing})$ is also Lagrangian for σ_X.*

Proof. Let Σ_2 be the union of the 2-dimensional components of $\Sigma := {\rm Sing}\, Y$ and let $y \in \Sigma_2$ be a general point. We claim that $f^{-1}(y)$ consists of a single point. By flattening, after blowing up Y to a smooth variety $\widetilde{Y}$, the exceptional fiber of $\tau : \widetilde{Y} \to Y$ has connected fiber over y, because Y is normal, and it parameterizes schemes z of finite length with support the fiber $f^{-1}(y)$. The local multiplicities of z at any of its points $x \in f^{-1}(\{y\})$ cannot be 1, as otherwise the local degree of f near the point x would be 1 and, by normality, f would be a local isomorphism, contradicting the fact that Y is singular at y. This implies that $f^{-1}(\{y\})$ contains at most one point, since the sum of the local degrees over $Y_{\rm reg}$ is 3. The argument above shows that points of $\widetilde{Y}$ over $y \in Y_{\rm sing}$ parameterize subschemes of length 3 supported at a single point $x \in X$ over y. It thus follows again by Mumford's theorem [26] that the restriction of σ_X to $f^{-1}(\Sigma_2)$ is the restriction of a 2-form on $\widetilde{Y}$, hence 0 by the argument already used. ☐

We now consider the Zariski closures $\overline{R}$ of R and $\overline{R'}$ of R'.

Corollary 3.4. *The intersection $\overline{R} \cap \overline{R'}$ is isotropic for σ_X. In particular, it has dimension 2, since there is no divisor in X which is isotropic for σ_X.*

Indeed, this is true away from $f^{-1}(Y_{\rm sing})$ by Lemma 3.2 and over $Y_{\rm sing}$ by Lemma 3.3.

The contradiction now comes from the fact that $\overline{R'}$ is a nonempty divisor in X, so that the restriction $\overline{\sigma}$ of σ_X to $\overline{R'}$, or rather its pull-back to a desingularization

$\tau : \widetilde{\overline{R'}} \to \overline{R'}$ of $\overline{R'}$, is nonzero. As the ramification divisor $\overline{R}$ is a big divisor on X since it is linearly equivalent to $f^*(-K_Y)$ over $Y_{\rm reg}$, it is an ample divisor by our assumptions, hence its pull-back $\tau'^*\overline{R}$ to $\widetilde{\overline{R'}}$ is big, where $\tau' : \widetilde{\overline{R'}} \to X$ is the composition of τ and the inclusion map $\overline{R'} \hookrightarrow X$. This contradicts the fact that the surface $\overline{R} \cap \overline{R'} \subset \overline{R'}$, hence also its inverse image in $\widetilde{\overline{R'}}$ is isotropic for the 2-form $\overline{\sigma}$ on $\widetilde{\overline{R'}}$. This concludes the proof of Proposition 3.1. □

4. Rational maps from hyper-Kähler fourfolds: a variant of a theorem of O'Grady

In the paper [27], O'Grady proves the following result.

Theorem 4.1. *Let X be a hyper-Kähler fourfold which is numerically equivalent to $\mathrm{K3}^{[2]}$. Assume that $\rho(X) = 1$ and $\mathrm{Pic}(X)$ is generated by one positive line bundle H with $q_X(H) = 2$, or equivalently, $H^4 = 12$. Then the rational map*

$$\phi_H \colon X \dashrightarrow \mathbb{P}^5$$

is either birational to a hypersurface of degree $12 \geq d \geq 6$, or of degree 2 over a hypersurface of degree 6 whose desingularization has $p_g \neq 0$.

Here, "numerically equivalent" means that the lattice $(H^2(X,\mathbb{Z}), q_X)$ is isomorphic to the corresponding lattice for $\mathrm{K3}^{[2]}$. As explained in [27], Theorem 4.1 is equivalent to excluding the possibilities where the image of ϕ_H is of dimension < 4 or a hypersurface of degree < 6. In these two cases, the image would be rationally connected by [22].

In this section, we are going to extend Theorem 4.1 to the situation studied in [9]. The hyper-Kähler fourfold X is only supposed to be very general with $\rho(X) = 2$ and to admit two line bundles L and M satisfying

$$L^4 = M^4 = 0, \quad L^2M^2 = 2, \tag{4.1}$$

which gives $(L+M)^4 = 12$ since this implies, by [5], that

$$L^3M = LM^3 = 0. \tag{4.2}$$

It is proved in [9, Theorem 1.7] that such an X has $b_2(X) = 23$ and the same Chern numbers and Fujiki constant as $\mathrm{K3}^{[2]}$, and that the Riemann–Roch polynomial $\chi(X, kL + k'M)$ coincides with the similarly defined polynomial on $\mathrm{K3}^{[2]}$ equiped with line bundles L, M satisfying (4.1); however, we do not know a priori that X is numerically equivalent to $\mathrm{K3}^{[2]}$. The following result is in fact needed in order to prove that X as above is deformation equivalent to $\mathrm{K3}^{[2]}$ so that, a posteriori, X is numerically equivalent to $\mathrm{K3}^{[2]}$ (see [9, Theorem 1.5]).

Theorem 4.2. *Assume that X, L, M are as above, with L, M nef and X very general with $\rho(X) = 2$, and that $h^0(X, L) = 0$. Then $\phi_{L+M} \colon X \dashrightarrow \mathbb{P}^5$ does not have rationally connected image.*

Remark 4.3. The conditions L, M nef and $h^0(X, L) = 0$ imply that no divisor in $|L + M|$ is reducible. Indeed, if L and M are nef, any effective divisor D on X

satisfies $q(L,D) \geq 0$ and $q(M,D) \geq 0$, so that D is a combination with integral nonnegative coefficients of L and M.

Note that [9, Proposition 6.3] proves that $\phi_{L+M}\colon X \dashrightarrow \mathbb{P}^5$ has rationally connected image, so that, in fact, an X as above, with L and M nef satisfying (4.1) and $h^0(X,L)=0$, does not exist.

The proof of Theorem 4.2 will be done in several steps. Although the statement is very similar to that of Theorem 4.1, we cannot use O'Grady's strategy, which proves first that any surface which is the complete intersection of two members of $|L+M|$ is reduced and irreducible, a statement that is a priori not true in our situation. Nevertheless, using the fact that $(L+M)^4 = 12$, and under the assumption that no divisor in $|L+M|$ is reducible, a number of his arguments go through in our situation where $\rho(X)=2$ and L, M are nef.

The following lemma will be very much used in the proof. We denote $l = c_1(L) \in \mathrm{Hdg}^2(X,\mathbb{Z})$, $m = c_1(M) \in \mathrm{Hdg}^2(X,\mathbb{Z})$.

Lemma 4.4. *Assume X is as above, very general with $\rho(X)=2$. Then there is no surface $\Sigma \subset X$ such that the class $(l+m)^2 - 3[\Sigma] \in \mathrm{Hdg}^4(X,\mathbb{Z})$ is pseudoeffective.*

Proof. We argue as in the proof of [9, Claim 6.2]. Any integral cohomology class $\eta \in H^4(X,\mathbb{Z})$ has an associated matrix

$$M_\eta = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

with $a = \langle \eta, l^2\rangle_X$, $b = \langle \eta, ml\rangle_X$, $c = \langle \eta, m^2\rangle_X$. If η is the class of a surface in X, this matrix is nonzero, since $L+M$ is ample and has nonnegative coefficients as L and M are nef. We follow some computations and arguments of [27], which we can do as we are in a very similar numerical situation, namely our X has by [9, Theorem 1.7] the same Chern numbers, Betti numbers and Fujiki constant as $\mathrm{Hilb}^2(\mathrm{K3})$. As $b_2(X)=23$, one has an isomorphism given by cup-product (see [5, 13])

$$\mathrm{Sym}^2 H^2(X,\mathbb{Q}) \cong H^4(X,\mathbb{Q}),$$

which induces a decomposition

$$H^4(X,\mathbb{Q}) = \mathrm{Sym}^2 H^2(X,\mathbb{Q})_{\mathrm{tr}} \oplus H^2(X,\mathbb{Q})_{\mathrm{tr}} \otimes \mathrm{NS}(X)_{\mathbb{Q}} \oplus \mathrm{Sym}^2 \mathrm{NS}(X)_{\mathbb{Q}}. \quad (4.3)$$

As X is very general, the Mumford–Tate group of the Hodge structure on $H^2(X,\mathbb{Q})_{\mathrm{tr}}$ is the orthogonal group of the Beauville–Bogomolov form q_X, so that the only Hodge classes in $\mathrm{Sym}^2 H^2(X,\mathbb{Q})_{\mathrm{tr}} \subset H^4(X,\mathbb{Q})$ are multiples of the class c inducing the Beauville–Bogomolov form. By (4.1) and (4.2), the classes l^2 and m^2 satisfy

$$M_{l^2} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad M_{m^2} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

while the integral Hodge classes lm and $c_2(X)$ satisfy

$$M_{lm} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad M_{c_2(X)} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix},$$

with $\lambda = 30$ as for a hyper-Kähler fourfold of $\mathrm{K3}^{[2]}$ deformation type. It is indeed a general fact that the Beauville–Bogomolov form for hyper-Kähler fourfolds is a nonzero multiple of the quadratic form $q_{c_2(X)}(\alpha,\beta) = \langle \alpha\beta, c_2(X)\rangle_X$ on $H^2(X,\mathbb{Q})$. The computation of the coefficient λ is as in the case of $\mathrm{K3}^{[2]}$, since it is determined by the Riemann–Roch polynomial and the Fujiki constant. It follows from (4.3) that the space of rational Hodge classes on X is generated by $\mathrm{Sym}^2 \mathrm{NS}(X)_{\mathbb{Q}}$ and c, and the kernel of the map $\eta \to M_\eta$ on $\mathrm{Hdg}^4(X,\mathbb{Q})$ is of rank 1, generated by $c_2(X) - 15ml$.

Let $f = [\Sigma]$ and $e = (l+m)^2 - 3f \in H^4(X,\mathbb{Z})$ be the two pseudoeffective classes considered. The corresponding matrices M_e and M_f thus satisfy

$$3M_f + M_e = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix},$$

and as both matrices are nonzero, with integral nonnegative coefficients, we must have

$$M_f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_e = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{4.4}$$

Note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M_{\frac{1}{2}ml}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = M_{l^2+m^2+\frac{1}{2}ml}. \tag{4.5}$$

It follows from (4.4) and (4.5) that for some coefficient $\eta \in \mathbb{Q}$ we have

$$f = \frac{1}{2}ml + \eta(c_2(X) - 15ml), \quad e = l^2 + m^2 + \frac{1}{2}ml - 3\eta(c_2(X) - 15ml). \tag{4.6}$$

We now compute the self-intersection of these integral cohomology classes and conclude that

$$f^2 = \frac{1}{2} + \eta^2(c_2(X) - 15ml)^2 = \frac{1}{2} + 378\eta^2.$$

We thus conclude that $2 \cdot 378\eta^2$ is an integer, and as $378 = 27 \cdot 2 \cdot 7$, it follows that 6η is an integer. From the first equation in (4.6), with f effective, we now conclude that $\eta < 0$, since otherwise $\eta \geq \frac{1}{6}$ and $\frac{1}{2} - 15\eta < 0$, so

$$\eta c_2(X) = f + \left(15\eta - \frac{1}{2}\right) lm,$$

with all coefficients positive and f effective. This is equation (32) in [9, Proof of Claim 6.2], which is proved there to be impossible.

From the second equation in (4.6), we now deduce that

$$-3\eta c_2(X) = e - l^2 - m^2 + \left(-45\eta - \frac{1}{2}\right) ml. \tag{4.7}$$

We claim that this implies $\eta \geq -\frac{1}{18}$. This is proved by integrating against both terms of (4.7) a class α^2, where $\alpha \in H^{1,1}(X)_{\mathbb{R}}$ is in the boundary of the Kähler cone and satisfies $q(\alpha) = 0$. We get

$$0 = \langle e, \alpha^2\rangle_X - \langle l^2, \alpha^2\rangle_X - \langle m^2, \alpha^2\rangle_X + \left(-45\eta - \frac{1}{2}\right) \langle lm, \alpha^2\rangle_X. \tag{4.8}$$

Using the Fujiki relations (with Fujiki constant equal to 3), we have

$$\langle \beta\gamma, \alpha^2\rangle_X = 2q_X(\alpha,\gamma)q_X(\alpha,\beta)$$

for any α, β, $\gamma \in H^2(X,\mathbb{C})$ such that $q_X(\alpha)=0$. Thus (4.8) gives

$$0 = \langle e, \alpha^2\rangle_X - 2q_X(l,\alpha)^2 - 2q_X(m,\alpha)^2 + 2\left(-45\eta - \frac{1}{2}\right) q_X(l,\alpha)q_X(m,\alpha)$$

and, as e is pseudoeffective, $\langle e,\alpha^2\rangle_X \geq 0$ when α is in the boundary of the Kähler cone, which by [15, Proposition 3.2] is satisfied once $q_X(l,\alpha)\geq 0$, $q_X(m,\alpha)\geq 0$. In conclusion, we proved that

$$q_X(l,\alpha)^2 + q_X(m,\alpha)^2 + \left(45\eta + \frac{1}{2}\right) q_X(l,\alpha)q_X(m,\alpha) \geq 0$$

once $q_X(l,\alpha)\geq 0$, $q_X(m,\alpha)\geq 0$. It follows that $45\eta + \frac{1}{2} \geq -2$, which proves the claim.

As we know that 6η is an integer and $\eta < 0$, the claim gives a contradiction, proving the lemma. □

The proof of Theorem 4.2 will be obtained by a case-by-case study. Assuming ϕ_{L+M} has rationally connected image, we have, by adapting arguments of [27], the following three possibilities (the case where the image is a curve being directly excluded by the fact that no divisor in $|L+M|$ is reducible):

(1) $Y = \phi_{L+M}(X) \subset \mathbb{P}^5$ is a surface of degree $d \geq 4$.
(2) $Y = \phi_{L+M}(X)$ is a 3-fold of degree $3 \leq d \leq 6$. In the case of degree $d=6$, the indeterminacy locus of ϕ_{L+M} has dimension 0.
(3) $Y = \phi_{L+M}(X)$ is a 4-fold of degree $2\leq d\leq 4$ and the degree of $\phi_{L+M}\colon X \dashrightarrow Y$ is at least 3.

The bound on the degree d in (1) follows from the fact that the image Y is linearly nondegenerate in $\mathbb{P}^5$.

The bound on the degree d in (2) follows from the fact that the image Y is linearly nondegenerate in $\mathbb{P}^5$ and that the general fiber is a curve F such that $d[F]+e = (m+l)^3$ in $\mathrm{Hdg}^6(X,\mathbb{Z})$ for some pseudoeffective class e (we use the ampleness of $L+M$ here). Indeed, as we assumed $\rho(X)=2$, the group $\mathrm{Hdg}^6(X,\mathbb{Q})$ is generated by l^2m and m^2l. An integral pseudoeffective curve class in X can thus be written as $\alpha l^2 m + \beta m^2 l$. By intersecting with l and m, we find that 2α and 2β are integers, and furthermore, they are nonnegative since L and M are nef. Applying this argument to $[F]$ and e, and using $(m+l)^3 = 3m^2l + 3ml^2$, the equality $d[F]+e=(m+l)^3$, with $[F]$ and e pseudoeffective, implies that $d\leq 6$.

The bound on the degree d in (3) follows from ampleness of $L+M$ and the fact that $(l+m)^4 = 12$. Furthermore, as in [27] (see also Lemma 2.1), one uses the fact that the degree of X over Y is at least 3 since $p_g(\widetilde{Y})=0$. Here and in what follows we denote by $\widetilde{Y}$ a desingularization of Y and by $\tilde{\phi}\colon \widetilde{X}\to\widetilde{Y}$ a desingularization of $\phi\colon X \dashrightarrow \widetilde{Y}$.

We thus have to exclude each of these possibilities. Let us start by excluding a few easy cases.

Lemma 4.5. *The image $Y \subset \mathbb{P}^5$ of ϕ_{L+M} is not a surface of degree $d \geq 4$.*

Proof. Otherwise, the general fiber F would be a surface in X such that $(l+m)^2 - d[F] = e$, where e is the class of a surface (which is a union of irreducible components of the base-locus of $|L+M|$), and this is excluded by Lemma 4.4. $\square$

Lemma 4.6. *The image $Y \subset \mathbb{P}^5$ of ϕ_{L+M} is not a threefold of degree* 3.

Proof. By [12], a linearly normal 3-fold Y of degree 3 in $\mathbb{P}^5$ is a cone over a rational normal scroll. Such a Y is fibered into linear spaces over $\mathbb{P}^1$ and has many reducible hyperplane sections, in the sense that it is swept-out by reducible hyperplane sections, with at least two mobile irreducible components. In that case, X would thus have, by taking pull-back under ϕ_{L+M}, reducible divisors in $|L+M|$, contradicting our assumption that $h^0(X, L) = 0$ (see Remark 4.3). $\square$

Lemma 4.7. *The image $Y \subset \mathbb{P}^5$ of ϕ_{L+M} is not a fourfold of degree* 4.

Proof. By item (3) above, the rational map $\phi_{L+M} \colon X \dashrightarrow Y$ has degree ≥ 3. As $(L+M)^4 = 12$ and $L+M$ is ample, the case where $\dim Y = \deg Y = 4$ is possible only if ϕ_{L+M} is a morphism of degree 3 (see [27, Corollary 4.7]). As $L+M$ is ample, the morphism ϕ_{L+M} is quasifinite to its image and the same is true for the induced morphism $\phi_{L+M} \colon X \to Y_n$, where Y_n is the normalization of Y. The big divisors are ample on X since, by Remark 4.3, the pseudoeffective cone of X is generated by two nef line bundles, and the regular locus of Y_n has a big anticanonical bundle, hence this would contradict Proposition 3.1. $\square$

Lemma 4.8. *If the image $Y \subset \mathbb{P}^5$ of ϕ_{L+M} is a hypersurface of degree* 3*, the degree of $\phi_{L+M} \colon X \dashrightarrow Y$ is* 3.

Proof. The rational map ϕ_{L+M} is of degree ≥ 3 by item (3) above, and it cannot be of degree ≥ 5 since $(L+M)^4 = 12 \geq \deg Y \deg \phi_{L+M}$, because $L+M$ is ample (see [27]). So we have to exclude the case where $\deg \phi_{L+M} = 4$ and $\deg Y = 3$, where the equality $(L+M)^4 = 12 = \deg Y \deg \phi_{L+M}$ holds, implying that ϕ_{L+M} is a morphism (see [27, Corollary 4.7]). Let $C \subset Y$ be a general plane section and let $C_X \subset X$ be its inverse image in X. We observe that Y cannot be singular in codimension 1, as otherwise it would have reducible hyperplane sections; hence, by taking the inverse images under the morphism ϕ_{L+M}, X has reducible members in $|L+M|$. It follows that the curve C is a smooth elliptic curve. We use now the results proved in the course of the proof of Proposition 6.4 and in [9, Lemma 6.8]. They imply that, under our assumptions on X, L, M, the rational map $\phi_{2L+M|C_X}$ factors through the degree 4 rational map $\phi_{L+M|C_X} \colon C_X \to C$. Note that the linear systems $|L+M|$ and $|2L+M|$ on X have no fixed components. Indeed, this is clear for the first one as $|L+M|$ has no reducible divisors; for the second one, as we assumed $h^0(X, L) = 0$, and we have $h^0(X, 2L+M) = 10$, $h^0(X, L+M) = 6$, the only fixed component could be in $|2L|$ and we would then have $h^0(X, M) = 10$, or it could be in $|M|$ and we would then have $h^0(X, 2L) = 10$. Both possibilities are easily ruled out, using [9, Lemma 5.1] and [16] (see [9, Subsection 5.1] for the complete argument). As the curve C_X is mobile, it follows that the linear systems

$|L+M|$ and $|2L+M|$ have no base points along C_X, hence the factorization of the morphisms mentioned above shows that the linear systems $H^0(X, L+M)_{|C_X}$ and $H^0(X, 2L+M)_{|C_X}$ are pulled back from linear systems on C. A fortiori, we get that the line bundle $(2L+M)_{|C_X}$ is pulled back from a line bundle on C, hence the degree of $(2L+M)_{|C_X}$ is divisible by 4. This contradicts the fact that

$$\deg(2L+M)_{|C_X} = (2L+M)(L+M)^3 = 3(2L+M)(L^2M+ML^2) = 18,$$

which is obtained by using the equalities $L^2M^2 = 2$, $L^3M = 0$, $LM^3 = 0$ of (4.1) and (4.2). ☐

Lemma 4.9. *The image $Y \subset \mathbb{P}^5$ of ϕ_{L+M} is not a fourfold of degree* 2*.*

Proof. If Y is a quadric, the general plane section C of Y, defined by a 3-dimensional vector subspace $W_3 \subset H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)) = H^0(X, L+M)$, is a smooth conic, as otherwise Y would be singular in codimension 1, hence reducible. We thus have $C \cong \mathbb{P}^1$ and denote by $\mathcal{O}_{\mathbb{P}^1}(1)$ the degree 1 line bundle on C. We recall from [9, Proof of Proposition 6.4] that, under our assumptions on X, L, M, assuming that Y is a fourfold, and given a general plane section C of Y, the mobile part X_C of $\phi_{L+M}^{-1}(C)$, or equivalently the Zariski closure of the locus in $X \setminus \mathrm{BL}(L+M)$ which is defined by W_3, is an irreducible curve with the following properties (we denote below by $\phi_{L+M,C}\colon X_C \to C$ the restriction of ϕ_{L+M} to X_C):

(1) $\dim H^0(X, L+M)_{|X_C} = 3$.
(2) $\dim H^0(X, 2L+M)_{|X_C} = 5$ or 4, and in the second case, $\phi_{2L+M}(X_C)$ is a rational cubic curve in $\mathbb{P}^3$.
(3) $\dim H^0(X, 3L+2M)_{|X_C} \leq 8$.

(a) If $\dim H^0(X, 2L+M)_{|X_C} = 5$, denoting by W_5 the space $H^0(X, 2L+M)_{|X_C}$ and by $W_3 \cong \mathrm{Sym}^2 W_2$ the space $H^0(X, L+M)_{|X_C}$, with $W_2 := H^0(C, \mathcal{O}_{\mathbb{P}^1}(1))$, we study the multiplication maps

$$\mu\colon W_2 \otimes W_5 \to H^0(X_C, (2L+M)_{|X_C} \otimes \phi_{L+M,C}^*\mathcal{O}_{\mathbb{P}^1}(1))$$

with image W', and

$$\mu'\colon W_2 \otimes W' \to H^0(X, 3L+2M)_{|X_C}$$

with $\operatorname{rank} \mu' \leq 8$. We get by the Hopf lemma applied to both multiplication maps that $\dim W' = 6$ or $\dim W' = 7$. In the first case, the equality in the Hopf lemma is satisfied by μ, and in the second case, the equality in the Hopf lemma is satisfied by μ'. In both cases, we conclude that

$$W_5 = \phi_{L+M}^* H^0(C, \mathcal{O}_{\mathbb{P}^1}(4)) = \phi_{L+M}^* H^0(C, \mathcal{O}_C(2)). \tag{4.9}$$

It follows that the rational morphism $\phi_{2L+M}\colon X \dashrightarrow \mathbb{P}^9$ factors rationally through Y. Furthermore, the linear system $|2L+M|$ has no fixed component, as we already explained in the previous proof. We also observe that the quadric Y must be of rank at least 5, as otherwise it would have many reducible hyperplane sections, and X would contain reducible divisors in $|L+M|$. It follows that $\mathrm{Pic}(Y \setminus \mathrm{Sing}\, Y) =$

$\mathbb{Z}\mathcal{O}_Y(1)$. These facts, together with the equality (4.9), imply that we have an equality of divisors in X,

$$2L + M = 2(L + M) - E,$$

where E is an effective divisor in X contracted by ϕ_{L+M}. Thus E belongs to $|M|$ and must be irreducible and contracted by ϕ_{L+M} to an irreducible subvariety W of Y. Furthermore, this equality induces an equality of spaces of sections,

$$H^0(X, 2L + M) = H^0(Y, \mathcal{O}_Y(2) \otimes \mathcal{I}_W).$$

As $H^0(X, 2L+M)$ is of dimension 10 (see [9]), W imposes at most 11 conditions to the quadrics. On the other hand, W must generate linearly at least a $\mathbb{P}^4$. Otherwise, $W \subset \mathbb{P}^3$ and Y is swept-out by linear sections containing W. Thus there would be reducible divisors in $|L+M|$, namely inverse images of general hyperplane sections of Y containing W, which contain E and a mobile component, contradicting our assumptions. Finally W cannot be a curve. Otherwise this curve would have degree at least 4 and the map $E \to W$ would have 2-dimensional fibers of class F; choosing three general points on the curve W and two general hyperplanes in $\mathbb{P}^5$ containing these three points, we would conclude that $(l + m)^2 - 3F$ is effective in X, which is excluded by Lemma 4.4. An irreducible linearly nondegenerate surface W in $\mathbb{P}^4$ or $\mathbb{P}^5$ imposes at least 12 conditions to quadrics (as the only irreducible linearly nondegenerate surface in $\mathbb{P}^4$ contained in three quadrics is a cubic scroll, residual of a plane in the complete intersection of two quadrics, see [14, p. 50]), and this is a contradiction.

(b) The other case, where $\dim H^0(X, 2L + M)_{|X_C} = 4$, and $\phi_{2L+M}(X_C)$ is a rational cubic curve in $\mathbb{P}^3$, is still easier. Indeed, we prove as above (see [9]) that the rational map ϕ_{2L+M} factors through ϕ_{L+M}, and thus there is a linear system on Y which is of degree 3 on the plane sections C of Y. This is impossible under our assumptions since, as we argued above, the quadric Y has rank at least 5, hence Y has cyclic Picard group generated by $\mathcal{O}_Y(1)$. ☐

Lemma 4.10. *The image $Y \subset \mathbb{P}^5$ of ϕ_{L+M} is not a threefold of degree* 6*.*

Proof. First of all we prove

Claim 4.11. *Assume* $\operatorname{Pic} X$ *is generated by L and M, with L and M nef isotropic, and the image $Y \subset \mathbb{P}^5$ of ϕ_{L+M} is a threefold of degree $d = 4$, 5 or 6. Then the general fiber F of ϕ_{L+M} is a curve of class $\frac{1}{2}(L^2M + M^2L)$, and it is of genus* 3 *when $d = 6$.*

As before, by "general fiber of ϕ_{L+M}" we mean "general fiber of a desingularization $\tilde{\phi}_{L+M} \colon \widetilde{X} \to Y$ of ϕ_{L+M}".

Proof of Claim 4.11. Using the fact that $L + M$ is ample, and arguing as in [27], the image f of F in the group of 1-cycles of X modulo numerical equivalence (or in $H_2(X, \mathbb{Z})$) satisfies

$$df = (l + m)^3 - e = 3(l^2m + m^2l) - e, \tag{4.10}$$

where the class e is the class of a pseudoeffective 1-cycle. Under our assumptions on L, M, the group of pseudoeffective 1-cycles is contained in the cone generated over $\mathbb{Q}$ by l^2m and lm^2. Furthermore, the classes $\frac{1}{2}l^2m$ and $\frac{1}{2}lm^2$ are integral and any integral cohomology class in $\langle l^2m, m^2l\rangle_{\mathbb{Q}}$ is an integral combination of $\frac{1}{2}l^2m$ and $\frac{1}{2}lm^2$ as one sees by intersecting them with L and M. It now follows from (4.10) with $d \geq 4$ that one of the following possibilities holds:

$$f = \frac{1}{2}\big(l^2m + m^2l\big), \quad f = \frac{1}{2}l^2m, \quad f = \frac{1}{2}m^2l.$$

Next we observe that the image of F in X is an irreducible component of the intersection of three members of $|L + M|$, and it follows by adjunction that

$$\deg K_F \leq 3(l + m)f. \tag{4.11}$$

If $f = \frac{1}{2}l^2m$ or $f = \frac{1}{2}m^2l$, we get from (4.11) that $\deg K_F \leq 3$, that is, F is of genus 0, 1 or 2, which contradicts Proposition 1.3. Hence we conclude that $f = \frac{1}{2}(l^2m+m^2l)$, which proves the first statement. When $d = 6$, we have $e = 0$ by the inequality (4.10) and it follows as in [27] that the base locus of $|L+M|$ consists of isolated points. The equality in (4.11) would imply that the holomorphic Euler–Poincaré characteristic $\chi(\widetilde{Z}, \mathcal{O}_{\widetilde{Z}})$ equals the holomorphic Euler–Poincaré characteristic $\chi(Z, \mathcal{O}_Z)$, where Z is the complete intersection of three members of $|L + M|$ and $\widetilde{Z}$ is its normalization. This is not possible, since the normalization map $\widetilde{Z} \to Z$ is nontrivial because Z is connected and $\widetilde{Z}$ is not connected; hence we conclude that the inequality is strict in (4.11). Therefore we get in this case $\deg K_F < 6$ and so the genus of F is at most 3, hence equal to 3 by Proposition 1.3. ☐

We now concentrate on the case of degree $d = 6$. As noted above, the indeterminacy locus of ϕ_{L+M} consists of isolated points. We first prove

Claim 4.12. *There is a single indeterminacy point $x \in X$.*

Proof. Let $\tau\colon \widetilde{X} \to X$, $\tilde{\phi}_{L+M}\colon \widetilde{X} \to Y$ be a resolution of indeterminacies of ϕ_{L+M}. As we know that τ has rank at most 1 over the indeterminacy points $x_1, \dots, x_N$, each irreducible component of the canonical divisor $K_{\widetilde{X}}$ of $\widetilde{X}$, defined as the zero-locus of the form $\tau^*\sigma_X^4$, appears with multiplicity at least 3. If $F \subset \widetilde{X}$ is a general fiber, we know by Claim 4.11 that $K_{\widetilde{X}} \cdot F = 4$, hence it follows that F meets a single irreducible component of $K_{\widetilde{X}}$. This implies that $N = 1$, as the image of F in X passes through all indeterminacy points x_i. ☐

We now examine the order of vanishing of sections of $L + M$ at x.

Claim 4.13. *(i) There is no section of $L + M$ vanishing at x to order* 3 *or more.*

(ii) There exists a section of $L + M$ whose zero set is nonsingular at x. The rank of the evaluation map $e_x\colon H^0(X, L + M) \to \Omega_{X,x} \otimes (L + M)$ is exactly 1.

(iii) Let $V_x \subset T_{X,x}$ be the hyperplane defined by any linear form in $\operatorname{Im} e_x$. Then the rank of the evaluation map $H^0(X, L + M) \to \operatorname{Sym}^2 V_x^* \otimes (L + M)$ is 5.

Proof. (i) We have $\tau^*(L+M) \cdot F = 2$ by Claim 4.11. If a section of $L+M$ vanishes to order ≥ 3, it thus vanishes on all the curves $\tau(F)$, hence on X.

(ii) If all sections of $L+M$ vanish to order ≥ 2 at x, the local intersection number at x of 4 sections of $L+M$ forming a regular sequence is at least 16, contradicting the fact that $(L+M)^4 = 12$. Suppose now that there are two sections s, s' of $L+M$ with independent differentials at x. Choosing them general, they define a smooth surface $S \subset X$ passing through x. This surface is swept out by curves $\tau(F)$ contained in it, hence it follows by the same argument as before that a nonzero section in $H^0(X, L+M)_{|S}$ cannot vanish at order ≥ 3 at x. There cannot be a complete intersection of three sections of $L+M$ which is smooth at x (since there are at least 6 curves F_i passing through x in such complete intersection), hence any section in $H^0(X, L+M)_{|S}$ vanishes to order ≥ 2 at x. The space $H^0(X, L+M)_{|S}$ is 4-dimensional and the evaluation map

$$e_{x,S} \colon H^0(X, L+M)_{|S} \to \operatorname{Sym}^2 \Omega_{S,x} \otimes (L+M)$$

has rank at most 3. Hence $e_{x,S}$ has a nontrivial kernel providing a section whose restriction to S is nonzero and vanishes to order 3 at x. This contradiction proves (ii).

(iii) The argument is the same as before, since, denoting by $X_s \subset X$ the zero locus of a general section s of $L+M$ (so that $V_x = T_{X_s,x}$), any element of $H^0(X, L+M)_{|X_s}$ has 0 differential at x but cannot vanish to order ≥ 3 at x. The conclusion thus follows from the fact that $H^0(X, L+M)_{|X_s}$ has dimension 5. $\square$

A contradiction arises as follows: the 5-dimensional space of quadrics on $\mathbb{P}(V_x)$ given by Claim 4.13 (iii) either has no base point, or is the space of quadrics vanishing at a point $u \in \mathbb{P}(V_x)$. In both cases, if we take three general sections of $H^0(X, L+M)_{|X_s}$, they provide a rational map $X_s \dashrightarrow \mathbb{P}^2$ that is undefined only at x, at which the three sections vanish at order 2. Blowing up x in X_s, and denoting by $E_{x,s}$ the exceptional divisor over x, we get sections of $(L+M)_{|X_s}(-2E_{x,s})$. The restricted rational map $\phi_{L+M|E_{x,s}} \colon E_{x,s} \dashrightarrow \mathbb{P}^2$ is given by a general linear system of quadrics vanishing at one point in the second case, or by a linear system of quadrics without base points in the first case. It is thus generically finite of degree ≤ 4. However, this contradicts the fact that it factors as the composition of the dominant rational map

$$E_{x,s} \dashrightarrow Y_s \to \mathbb{P}^2,$$

where Y_s is the hyperplane section of Y defined by s and the second map is a general linear projection, hence has degree 6. $\square$

Combining Lemmas 4.5, 4.6, 4.7, 4.9 and 4.10 we find that, in order to prove Theorem 4.2, we only have to prove the following Proposition 4.14, which eliminates the case where the image is a cubic hypersurface, and Proposition 4.20, which excludes the cases where Y is a 3-fold of degree 4 or 5 in $\mathbb{P}^5$.

Proposition 4.14. *The image $Y = \phi_{L+M}(X) \subset \mathbb{P}^5$ cannot be a cubic hypersurface.*

We establish a few lemmas in order to prove Proposition 4.14. We first prove

Lemma 4.15. *If Y is a cubic hypersurface, it cannot be singular in codimension* 1.

Proof. If the singular locus of Y has dimension 3, it must be a $\mathbb{P}^3$, and either Y is a cone over a cubic surface or the equation of Y takes the form

$$f_Y = x_0^2 x_2 + x_0 x_1 x_3 + x_1^2 x_4, \tag{4.12}$$

for an adequate choice of coordinates x_i, $x_0 = x_1 = 0$ being the equations defining the $\mathbb{P}^3$ contained in $\operatorname{Sing} Y$. The first case is excluded as follows: If Y is a cone over a cubic surface S, the linear projection $\pi\colon Y \dashrightarrow S$ from the vertex composes with ϕ_{L+M} to give a dominant rational map

$$\psi = \pi \circ \phi_{L+M}\colon X \dashrightarrow S$$

with general fiber F_x, $x \in S$. For any general set $\{x_1, x_2, x_3\}$ of three collinear points in S, the three surfaces F_{x_i} are homologous in X and satisfy $[F_{x_1}] + [F_{x_2}] + [F_{x_3}] + e = (l+m)^2$ in $H^4(X, \mathbb{Z})$, where e is the class of an effective surface in X, which contradicts Lemma 4.4. In the second case, where Y is defined by an equation f_Y as in (4.12), Y has many reducible hyperplane sections. Indeed, in the above coordinates the hyperplane section $\{x_2 = 0\}$ is the union of the two components $\{x_1 = x_2 = 0\} \subset Y$ and $\{x_0x_3 + x_1x_4 = x_2 = 0\} \subset Y$. Using the natural $\mathrm{SO}(3)$ (or $\mathrm{SL}(2)$) action on Y, it is easy to see that both components are mobile. Thus X would have reducible members in $|L+M|$, which is excluded by assumption. □

By Lemma 4.15, if Y is a cubic hypersurface, the general plane sections $C := P \cap Y$ are smooth elliptic plane curves. We now choose a desingularization $\widetilde{Y}$ of Y, so that $Y_{\mathrm{reg}} \subset \widetilde{Y}$, and prove

Lemma 4.16. *If Y is a cubic hypersurface, there exists a line bundle $\mathcal{L}$ on $\widetilde{Y}$ such that $\deg \mathcal{L}_{|C} = 5$ and the pull-back $\phi_{L+M}^*\mathcal{L}$ of $\mathcal{L}$ to X satisfies*

$$2L + M = \phi_{L+M}^*\mathcal{L}(-E) \tag{4.13}$$

for some effective divisor E in X which is contracted by $\phi_{L+M}\colon X \dashrightarrow \widetilde{Y}$. Furthermore, the sections of $2L+M$ are pulled back from sections of $\mathcal{L}$ on $\widetilde{Y}$. In particular,

$$h^0(Y_{\mathrm{reg}}, \mathcal{L}_{\mathrm{reg}}) \geq h^0(X, 2L+M) = 10, \tag{4.14}$$

where $\mathcal{L}_{\mathrm{reg}} := \mathcal{L}_{|Y_{\mathrm{reg}}}$.

Proof. Denote by $D \subset X$ the curve $\phi_{L+M}^{-1}(C)$ (that is, the mobile part of the closed algebraic subset defined by the three equations α, β, γ of $L+M$ on X corresponding to the three sections of $\mathcal{O}_Y(1)$ defining C). We recall from [9, Proof of Proposition 6.4] that, under our assumptions, the linear systems

$$W_3 := H^0(X, L+M)_{|D}, \quad W_5 := H^0(X, 2L+M)_{|D}, \quad W_8 := H^0(X, 3L+2M)_{|D}$$

are of respective dimension 3, ≥ 5, ≤ 8. Then [9, Lemma 6.8] proves, using the multiplication map

$$\mu\colon W_3 \otimes W_5 \to W_8,$$

that these three linear systems are pulled back from linear systems W_3', W_5', W_8' on the curve C. By removing the base points, we may assume that the linear systems W_3', W_5' and W_8' have no base points on C. This defines a line bundle $\mathcal{L}_C$ on C such that $W'_5 \subset H^0(C, \mathcal{L}_C)$ and has no base points. Note that W'_3 gives the embedding of C as a plane curve.

Claim 4.17. *One has $W'_5 = H^0(C, \mathcal{L}_C)$; equivalently, the line bundle $\mathcal{L}_C$ on C has degree* 5*.*

Proof. We have a base-point-free not necessarily complete linear system $W'_5 \subset H^0(C, \mathcal{L}_C)$ of dimension ≥ 5 on C such that the image of the multiplication map

$$W'_3 \otimes W'_5 \to H^0(C, \mathcal{L}_C(1))$$

has rank ≤ 8. Up to taking a general vector subspace, we can assume $\dim W'_5 = 5$. Let x, y, z be three general points of C. Then the linear system $W'_{2,x,y,z}$ of elements of W'_5 vanishing on x, y and z has dimension 2 and the rank of the multiplication map

$$W'_3 \otimes W'_{2,x,y,z} \to H^0(C, \mathcal{L}_C(1)(-x-y-z))$$

is at most 5, hence has a nontrivial kernel. By the base-point-free pencil trick, one has $H^0(C, \mathcal{L}_C^{-1}(x+y+z)(1)) \neq 0$, hence $\deg \mathcal{L}_C^{-1}(x+y+z)(1) > 0$ since x, y, z are arbitrary. It follows that $\deg \mathcal{L}_C < 6$, hence $\deg \mathcal{L}_C = 5$ and the linear system W'_5 is complete. ☐

We now conclude the proof of Lemma 4.16. As the rational map ϕ_{2L+M} on each curve $D \subset X$ as above factors through the corresponding curve $C \subset Y$, there exists a line bundle $\mathcal{L}$ on the chosen desingularization $\widetilde{Y}$ of Y such that $|\mathcal{L}|$ has no fixed components and $\phi_{\mathcal{L}} \circ \phi_{L+M} = \phi_{2L+M}$. As we already explained in the proof of Lemma 4.8, the 10-dimensional linear system $H^0(X, 2L+M)$ has no fixed component. This implies formula (4.13); the divisor E appears because a divisor contracted by ϕ_{L+M} can appear in the fixed part of the linear system $\phi^*_{L+M}|\mathcal{L}|$. Equality (4.14) follows. Finally, as $H^0(\widetilde{Y}, \mathcal{L})$ has no fixed component and $C \subset \widetilde{Y}$ is in general position, $H^0(\widetilde{Y}, \mathcal{L})_{|C}$ has no base point, hence $\mathcal{L}_{|C} = \mathcal{L}_C$, where $\mathcal{L}_C$ appears in Claim 4.17. It thus follows from Claim 4.17 that $\deg \mathcal{L}_{|C} = 5$. ☐

Lemma 4.16 indicates that if Y is a cubic hypersurface, it has a singular locus which is of dimension at least 1. Indeed, if $\operatorname{Sing} Y$ is isolated, the general hyperplane section Y' of Y is smooth, hence has Picard number 1 and thus any line bundle on Y_{reg} has degree divisible by 3 on the plane sections of Y'. Going farther, we now prove

Lemma 4.18. *If Y is a cubic hypersurface, the singular locus of Y has dimension at least* 2*.*

Proof. Assume by contradiction that the singular locus of the cubic hypersurface Y has dimension ≤ 1. The notation $\mathcal{L}_{\mathrm{reg}} \in \operatorname{Pic} Y_{\mathrm{reg}}$ being as in Lemma 4.16, we prove

Claim 4.19. *There exists a divisor $D \subset Y$ which is a linear $\mathbb{P}^3 \subset Y \subset \mathbb{P}^5$ such that*

$$\mathcal{L}_{\mathrm{reg}} = \mathcal{O}_{Y_{\mathrm{reg}}}(2)(-D).$$

Proof. Let $S \subset Y$ and $H \subset Y$ be general linear sections of Y, with $\dim S = 2$, $\dim H = 3$ and $S \subset H$. The surface S is thus smooth by our assumption and contained in Y_{reg}. Assume $c_1(\mathcal{L}_{|S})^2 \geq 7$. Then, denoting $\mathcal{L}' := \mathcal{O}_{Y_{\mathrm{reg}}}(2) \otimes \mathcal{L}_{\mathrm{reg}}^{-1}$, the line bundle $\mathcal{L}'_S := \mathcal{L}'_{|S}$ satisfies

$$c_1(\mathcal{L}'_S) \cdot K_S = -1, \quad c_1(\mathcal{L}'_S)^2 \geq 7 + 12 - 20 = -1; \tag{4.15}$$

hence we have $\chi(S, \mathcal{L}'_S) \geq 1$ and it follows from the first equality in (4.15) that $h^2(S, \mathcal{L}'_S) = 0$, hence $h^0(S, \mathcal{L}'_S) \neq 0$. Thus $\mathcal{L}'_S = \mathcal{O}_S(\Delta_S)$ for some line $\Delta_S \subset S$. We now show that the lines Δ_S for all $S \subset Y_{\mathrm{reg}}$ fill-in only a divisor D in Y_{reg}. To see this, we observe that if $C \subset S$ is a smooth plane section, the intersection $\Delta_S \cap C$ is a point $x \in C$ that satisfies

$$\mathcal{O}_C(x) = \mathcal{L}'_{|C}.$$

It follows that x does not depend on S containing C. Fixing x and moving C containing x, we finally conclude that, if a point $x \in \Delta_S$ for some S, then $x \in \Delta_S$ for any S containing x. This proves the existence of the divisor D. This divisor is then a $\mathbb{P}^3$, since it contains at least a 4-dimensional family of lines (indeed, a given line is contained in a 4-dimensional family of surfaces S and there is an 8-dimensional family of surfaces S). Finally, we found that the divisor $D \cong \mathbb{P}^3 \subset Y$ satisfies

$$\mathcal{L}_{|S} = \mathcal{I}_D(2)_{|S}$$

for any smooth surface $S \subset Y_{\mathrm{reg}}$. It easily follows that $\mathcal{L}_{\mathrm{reg}} = \mathcal{I}_D(2)$.

We next assume that $c_1(\mathcal{L}_{|S})^2 \leq 5$. Then, as $|\mathcal{L}_{|S}|$ has no fixed part, it is nef and we have $h^1(S, \mathcal{L}_{|S}) = 0$, $h^2(S, \mathcal{L}_{|S}) = 0$, since $-K_S$ is ample. Hence we have in this case

$$h^0(S, \mathcal{L}_{|S}) = 1 + \frac{c_1(\mathcal{L}_{|S})^2 - c_1(\mathcal{L}_{|S}) \cdot K_S}{2} \leq 6.$$

Comparing with (4.14), we get $h^0(Y_{\mathrm{reg}}, \mathcal{L}_{\mathrm{reg}} \otimes \mathcal{I}_S) \geq 4$, and considering as above a general pair $S \subset H \subset Y$, we conclude that either

(1) $h^0(Y_{\mathrm{reg}}, \mathcal{L}_{\mathrm{reg}}(-1)) \geq 2$, or
(2) $h^0(H_{\mathrm{reg}}, \mathcal{L}_{\mathrm{reg}}(-1)_{|H_{\mathrm{reg}}}) \geq 3$.

In the case (1), the divisor of a general section of $\mathcal{L}_{\mathrm{reg}}(-1)$ has degree 2, hence it is reduced by Bertini and there are two possibilities:

(a) The divisor of this section is the intersection $Q_3 \cap Y_{\mathrm{reg}}$ for some 3-dimensional quadric $Q_3 \subset Y$ such that $Q_3 \cap Y_{\mathrm{reg}} \in |\mathcal{L}_{\mathrm{reg}}(-1)|$. There is then a residual $\mathbb{P}^3 \subset Y$ such that $\mathbb{P}^3 + Q_3$ is a hyperplane section of Y and the lemma is also proved in this case.

(b) The divisor of this section is the intersection with Y_{reg} of the union $D_1 \cup D_2$ of two $\mathbb{P}^3$'s contained in Y such that $(D_1 \cup D_2) \cap Y_{\mathrm{reg}} \in |\mathcal{L}(-1)|$ and $\dim D_1 \cap D_2 = 1$. In fact, this case is impossible because the intersection of $D_1 \cup D_2$ with a general cubic surface $S \subset Y$ as above is the disjoint union of two lines, hence is a rigid divisor. It follows immediately that $h^0(Y_{\mathrm{reg}}, \mathcal{L}_{\mathrm{reg}}(-1)) \leq 1$, contradicting the inequality (1).

The case (2) is excluded as follows. We get that the divisor of a general section of $\mathcal{L}_{\rm reg}(-1)_{|H_{\rm reg}}$ has degree 2, hence it is reduced by Bertini and there are two possibilities:

(a) There is a 2-dimensional quadric $Q_2 \subset H$ such that $Q_2 \cap H_{\rm reg} \in |\mathcal{L}_{|H_{\rm reg}}(-1)|$. But then $h^0(H_{\rm reg}, \mathcal{L}_{\rm reg}(-1)_{|H_{\rm reg}}) \leq 2$, contradicting the inequality (2).

(b) There are two planes P, $P' \subset H$ such that $(P+P') \cap H_{\rm reg} \in |\mathcal{L}_{|H_{\rm reg}}(-1)|$. But then we find as above $h^0(H_{\rm reg}, \mathcal{L}(-1)_{|H_{\rm reg}}) \leq 1$, contradicting the inequality (2).

The claim is thus proved. $\square$

We now conclude the proof of Lemma 4.18. Let $D \cong \mathbb{P}^3 \subset Y$ be as in Claim 4.19. We have $h^0(Y_{\rm reg}, \mathcal{I}_D(2)) = 11$, while $h^0(X, 2L+M) = 10$. The inclusion $H^0(X, 2L+M) \subset H^0(Y_{\rm reg}, \mathcal{I}_D(2))$ given by Lemmas 4.16 and Claim 4.19 is thus the inclusion of a hyperplane. Let $H_X \subset X$ be a general member of $|L+M|$, that is, the inverse image $\phi_{L+M}^{-1}(H_Y)$, where $H_Y \subset Y$ is a general hyperplane section. Then $H^0(Y_{\rm reg}, \mathcal{I}_{H_Y} \otimes \mathcal{I}_D(2)) = H^0(Y_{\rm reg}, \mathcal{I}_D(1))$ has dimension 2, hence it intersects nontrivially the hyperplane $H^0(X, 2L+M) \subset H^0(Y_{\rm reg}, \mathcal{I}_D(2))$, providing a nonzero section of the line bundle

$$(2L+M) \otimes \mathcal{I}_{H_X} = L$$

on X, which is excluded by the hypotheses of Theorem 4.2. The lemma is thus proved. $\square$

Proof of Proposition 4.14. Using Lemmas 4.18 and 4.15, the singular locus of a cubic hypersurface $Y = \operatorname{Im} \phi$ has dimension 2. We observe now that the arguments in Lemma 4.18 involving smooth cubic surfaces appearing as general linear sections of Y when $\dim(\operatorname{Sing} Y) \leq 1$ extend in a straightforward way if the general cubic surface section has Duval singularities, which happens if the order of vanishing of the defining equation f_Y of Y along any 2-dimensional component of its singular locus is not 3 (see [7]). Indeed, we can work in that case with a crepant resolution of singularities of these surfaces. However, if $\dim(\operatorname{Sing} Y) = 2$ and f_Y vanishes to order 3 along a component of $\operatorname{Sing} Y$, Y is a cone over an elliptic curve in $\mathbb{P}^2$. This case is excluded since Y would then have many reducible hyperplane sections. This concludes the proof of Proposition 4.14. $\square$

Proposition 4.20. *Let X, L, M be as above. Then the image $Y = \phi_{L+M}(X)$ cannot be a linearly nondegenerate threefold of degree 4 or 5 in $\mathbb{P}^5$.*

Assuming by contradiction that Y is a threefold of degree 4 or 5, we first prove the following lemmas.

Lemma 4.21. *The threefold Y cannot be a cone $\pi\colon Y \dashrightarrow S$ over a surface S in $\mathbb{P}^4$.*

Proof. As in the proof of Lemma 4.5, this would indeed contradict Lemma 4.4 by considering the composite map $\pi \circ \phi_{L+M}\colon X \dashrightarrow S$. $\square$

Lemma 4.22. *The threefold $Y \subset \mathbb{P}^5$ is not contained in a quadric of rank ≤ 4.*

Proof. Indeed, Y would have otherwise many reducible hyperplane sections, contradicting the fact that all members of $|L+M|$ are irreducible. ☐

We now exclude the case of degree 4.

Lemma 4.23. *The threefold Y cannot be of degree* 4.

Proof. As Y is not a cone and is linearly nondegenerate, linearly normal in $\mathbb{P}^5$, the Swinnerton-Dyer classification [28] tells us that Y is the complete intersection of two quadrics in $\mathbb{P}^5$. In particular, Y contains a line through any of its points. Furthermore, if Y is smooth, its family of lines is smooth and connected and Y contains four lines through a general point $y \in Y$. Let $\Delta_1, \dots, \Delta_4$ be the four lines through y. Then the Δ_i are contained in the projectivized tangent space $\mathbb{P}^3_y$ of Y at y, and $\mathbb{P}^3_y \cap Y \supseteqq \cup_i \Delta_i$, while by smoothness of Y, $\mathbb{P}^3_y \cap Y$ has dimension 1; hence we have in fact $\mathbb{P}^3_y \cap Y = \cup_i \Delta_i$. The inverse images $S_i := \phi^{-1}_{L+M}(\Delta_i)$ are then cohomologous in X and their common class f satisfies

$$4f + e = (l+m)^2$$

for some pseudoeffective class in X. This contradicts again Lemma 4.4. We now consider the case where Y is singular and try to extend the argument above. We still know that Y is swept-out by lines and that there exist at least four lines passing through a general point of Y. Unfortunately, we do not know that the lines are homologous or algebraically equivalent in Y, so the above argument fails. However, we have the following

Sublemma 4.24. *If $Y = \operatorname{Im} \phi_{L+M}$ is the intersection of two quadrics in $\mathbb{P}^5$, there are at most two algebraic equivalence classes of mobile lines in Y.*

Proof. We first claim that the family of conics in Y has at most two irreducible 4-dimensional components whose general point parameterizes a conic passing through the general point of Y. Indeed, Y is not swept-out by planes, as otherwise it would have many reducible hyperplane sections, which is excluded. Hence we can consider only the family of conics in Y which are not contained in a plane contained in Y. But these conics are in bijection with planes contained in one quadric Q_t containing Y and not contained in Y. By Lemma 4.22, Y is not contained in any quadric of rank ≤ 4. The family of planes in Q_t thus has two irreducible components of dimension 3 if Q_t is smooth and only one, also of dimension 3, if Q_t is singular of rank 5. Thus the family of planes contained in one of the Q_t has one or two components, according to whether the double cover of the projective line parameterizing the quadrics Q_t containing Y determined by the choice of a ruling is reducible or not. This proves the claim. Let now $y \in Y$ be a general point. There are (at least) four lines $l_1, \dots, l_4$ in Y passing through y, and the union of any two of these lines is a conic in Y passing through y. Hence the cycles $l_i + l_j$ belong to only two algebraic equivalence classes of 1-cycles in Y, and it follows immediately that these four lines belong to at most two algebraic equivalence classes of 1-cycles in Y. ☐

Corollary 4.25. *(i) There exists an algebraic equivalence class $\mathcal{C}$ of 1-cycles on Y such that, through a general point $y \in Y$, there pass at least two lines of the class $\mathcal{C}$.*

(ii) There exist chains of three lines $\Delta_1, \Delta_2, \Delta_3 \subset Y$, $\Delta_1 \cap \Delta_2 \neq \emptyset$, $\Delta_2 \cap \Delta_3 \neq \emptyset$, such that the Δ_i pass through the general point of Y and the three lines Δ_i are in the class $\mathcal{C}$.

Proof. Statement (i) immediately follows from Sublemma 4.24, since there pass four lines through a general point of Y.

(ii) Given a general point y of Y, there are two lines Δ_1, Δ_2 in the algebraic equivalence class $\mathcal{C}$ and passing through y. Choosing another point $y' \in \Delta_2$, we can choose a deformation Δ_3 of Δ_1 (hence also in the class $\mathcal{C}$) passing through y'. This gives the desired chain. □

Corollary 4.25 leads to a contradiction as follows: indeed the three lines Δ_i forming a connected chain are all contained in a mobile $\mathbb{P}^3_{\Delta_\bullet}$. Assume first that $\mathbb{P}^3_{\Delta_\bullet} \cap Y$ is 1-dimensional; then we get, by taking inverse images in X, an equality of codimension 2 cycles in X,

$$\phi^*_{L+M}(\mathbb{P}^3_{\Delta_\bullet} \cap Y) = T_1 + T_2 + T_3 + T \quad \text{in } A^2(X), \tag{4.16}$$

where T is the class of an effective surface in X and $T_i = \phi^{-1}_{L+M}(\Delta_i)$. As the three lines Δ_i are algebraically equivalent in Y, the three surfaces T_i are numerically equivalent in X, and thus (4.16) contradicts Lemma 4.4. It remains to analyze the case where $\mathbb{P}^3_{\Delta_\bullet} \cap Y$ has a 2-dimensional component for general $\Delta_\bullet$. If this component is mobile, then Y has many reducible hyperplane sections, which is excluded. If this component is fixed, it must be a plane $P \subset Y$ since it is contained in the intersection of at least two $\mathbb{P}^3$'s, and this plane has the property that any mobile line in Y in the algebraic equivalence class $\mathcal{C}$ intersects P. In that case, under the linear projection $\pi_P \colon Y \dashrightarrow \mathbb{P}^2$ from P, Y maps to a curve of degree > 1 in $\mathbb{P}^2$, since the fiber of the projection π_P passing through the general point y contains at least two lines and hence must be of dimension at least 2. Therefore Y has many reducible hyperplane sections, which gives again a contradiction. Lemma 4.23 is thus proved, hence also Proposition 4.20 in the case of degree 4. □

Proof of Proposition 4.20. By Lemma 4.23, we only have to exclude the case where Y is a threefold of degree 5.

Claim 4.26. *Y is not contained in a quadric.*

Proof. If Y is contained in a quadric $Q \subset \mathbb{P}^5$, Q must have rank at least 5 by Lemma 4.22. The general hyperplane section $Q_H := Q \cap H$ of Q is then a smooth quadric of dimension 3 which contains a surface of degree 5, contradicting the fact that $\operatorname{Pic} Q_H$ is generated by $\mathcal{O}_Q(1)$. □

Denote by $n \colon Y_n \to Y$ the normalization of Y. Thus Y_n is smooth in codimension 1. For a general $\mathbb{P}^3 \subset \mathbb{P}^5$, the general section $C_n := n^{-1}(C)$, $C := Y \cap \mathbb{P}^3$, of Y_n is a smooth connected curve. We denote by S a general hyperplane section of Y, $S_n \subset Y_n$ its inverse image in Y_n, and consider the inclusions $C_n \subset S_n \subset Y_n$.

Claim 4.27. *The curve $C \subset \mathbb{P}^3$ is a smooth genus 2 curve of degree 5 (in particular it is isomorphic to C_n). It is thus contained in a quadric.*

Proof. The second statement follows from the first one, since it implies that $h^0(C, \mathcal{O}_C(2)) = 9$. Let $\tau\colon \widetilde{Y} \to Y_n$ be a desingularization, and let $\tau' := n \circ \tau$. We have

$$H^0(\widetilde{Y}, {\tau'}^* \mathcal{O}_Y(1)) \cong H^0(Y_n, n^* \mathcal{O}_Y(1)) \cong H^0(Y, \mathcal{O}_Y(1)). \tag{4.17}$$

Indeed, via the dominant rational map

$$ {\tau'}^{-1} \circ \phi_{L+M}\colon X \dashrightarrow \widetilde{Y},$$

sections of ${\tau'}^*\mathcal{O}_Y(1)$ on $\widetilde{Y}$ pull back to sections of $L + M$ on X, while by construction, the pull-back map $\phi^*_{L+M}\colon H^0(Y, \mathcal{O}_Y(1)) \to H^0(X, L + M)$ is an isomorphism. We have

$$H^1(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}) = 0, \tag{4.18}$$

since $\widetilde{Y}$ is rationally dominated by X. Let $\widetilde{S} := {\tau'}^{-1}(S) \subset \widetilde{Y}$, so that $\widetilde{S} \in |{\tau'}^*\mathcal{O}_Y(1)|$. By the vanishing (4.18), and using the relation (4.17), we conclude that $h^0(\widetilde{S}, {\tau'}^*\mathcal{O}_S(1)) = 5$. Furthermore, we also have $H^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = 0$, using (4.18) and the fact that $\widetilde{S} \subset \widetilde{Y}$ is big and nef. As $C_n \subset \widetilde{S}$ belongs to $|{\tau'}^*\mathcal{O}_S(1)|$, we conclude as before that, denoting $\mathcal{O}_{C_n}(1) := {\tau'}^*\mathcal{O}_C(1)$,

$$h^0(C_n, \mathcal{O}_{C_n}(1)) = h^0(\widetilde{S}, {\tau'}^*\mathcal{O}_S(1)) - 1 = 4,$$

which implies that $h^1(C_n, \mathcal{O}_{C_n}(1)) = 0$, hence $g(C_n) = 2$, since we know that the line bundle $\mathcal{O}_{C_n}(1)$ on C_n has degree 5. It follows that $\mathcal{O}_{C_n}(1)$ is very ample on C_n, hence C_n is isomorphic to C. ∎

We get a contradiction from Claims 4.26 and 4.27 by observing that both restriction maps

$$H^0(\mathbb{P}^5, \mathcal{I}_Y(2)) \to H^0(\mathbb{P}^4, \mathcal{I}_S(2)),$$
$$H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^3, \mathcal{I}_C(2))$$

are surjective. Indeed, the surjectivity in both cases is implied by the respective vanishings $H^1(\mathbb{P}^5, \mathcal{I}_Y(1)) = 0$ and $H^1(\mathbb{P}^4, \mathcal{I}_S(1)) = 0$, that come from the fact that both $Y \subset \mathbb{P}^5$ and $S \subset \mathbb{P}^4$ are linearly normal (for the surface S, this follows indeed from the arguments given in the previous proof). This concludes the proof of Proposition 4.20. ∎

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References

[1] N. Addington and M. Lehn, On the symplectic eightfold associated to a Pfaffian cubic fourfold, *J. Reine Angew. Math.* **731** (2017), 129–137. MR 3709062.

[2] A. Alzati and G. P. Pirola, On the holomorphic length of a complex projective variety, *Arch. Math. (Basel)* **59** (1992), no. 4, 398–402. MR 1179468.

[3] C. Bai. On Abel–Jacobi maps of Lagrangian families, https://arxiv.org/abs/2203.06242 [math.AG], 2022.

[4] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld and B. Ullery, Measures of irrationality for hypersurfaces of large degree, *Compos. Math.* **153** (2017), no. 11, 2368–2393. MR 3705293.

[5] F. A. Bogomolov, On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky), *Geom. Funct. Anal.* **6** (1996), no. 4, 612–618. MR 1406665.

[6] F. Charles, Two results on the Hodge structure of complex tori, *Math. Z.* **300** (2022), no. 4, 3623–3643. MR 4395105.

[7] I. A. Chel′tsov, Birationally rigid Fano varieties, *Russian Math. Surveys* **60** (2005), no. 5, 875–965. MR 2195677.

[8] C. H. Clemens and P. A. Griffiths, The intermediate Jacobian of the cubic threefold, *Ann. of Math.* (2) **95** (1972), 281–356. MR 0302652.

[9] O. Debarre, D. Huybrechts, E. Macrì and C. Voisin. Computing Riemann–Roch polynomials and classifying hyper-Kähler fourfolds, https://arxiv.org/abs/2201.08152 [math.AG], 2022.

[10] R. Donagi and E. Markman, Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles, in *Integrable Systems and Quantum Groups (Montecatini Terme, 1993)*, 1–119, Lecture Notes in Math., 1620, Fond. CIME/CIME Found. Subser, Springer, Berlin, 1998. MR 1397273.

[11] L. Ein and R. Lazarsfeld, The Konno invariant of some algebraic varieties, *Eur. J. Math.* **6** (2020), no. 2, 420–429. MR 4104556.

[12] D. Eisenbud and J. Harris, On varieties of minimal degree (a centennial account), in *Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, 3–13, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987. MR 0927946.

[13] D. Guan, On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four, *Math. Res. Lett.* **8** (2001), no. 5-6, 663–669. MR 1879810.

[14] J. Harris. *Geometry of Algebraic Curves*, Notes by Akhil Mathew, Harvard, 2011. http://math.uchicago.edu/~amathew/287y.pdf

[15] D. Huybrechts, The Kähler cone of a compact hyperkähler manifold, *Math. Ann.* **326** (2003), no. 3, 499–513. MR 1992275.

[16] D. Huybrechts and C. Xu, Lagrangian fibrations of hyperkähler fourfolds, *J. Inst. Math. Jussieu* **21** (2022), no. 3, 921–932. MR 4404129.

[17] P. Ionescu, Embedded projective varieties of small invariants, in *Algebraic Geometry, Bucharest 1982*, 142–186, Lecture Notes in Math., 1056, Springer, Berlin, 1984. MR 0749942.

[18] J. Kollár, Nonrational hypersurfaces, *J. Amer. Math. Soc.* **8** (1995), no. 1, 241–249. MR 1273416.

[19] K. Konno, Minimal pencils on smooth surfaces in $\mathbb{P}^3$, *Osaka J. Math.* **45** (2008), no. 3, 789–805. MR 2468594.

[20] R. Laza, The moduli space of cubic fourfolds, *J. Algebraic Geom.* **18** (2009), no. 3, 511–545. MR 2496456.

[21] C. Lehn, M. Lehn, Ch. Sorger and D. van Straten, Twisted cubics on cubic fourfolds, *J. Reine Angew. Math.* **731** (2017), 87–128. MR 3709061.

[22] H.-Y. Lin, Lagrangian constant cycle subvarieties in Lagrangian fibrations, *Int. Math. Res. Not. IMRN* **2020**, no. 1, 14–24. MR 4050560.

[23] O. Martin, The degree of irrationality of most Abelian surfaces is 4, *Ann. Sci. Éc. Norm. Supér. (4)* **55** (2022), no. 2, 569–574. MR 4411875.

[24] D. Matsushita, On fibre space structures of a projective irreducible symplectic manifold, *Topology* **38** (1999), no. 1, 79–83. MR 1644091.

[25] D. Matsushita, Addendum: “On fibre space structures of a projective irreducible symplectic manifold” [*Topology* **38** (1999), no. 1, 79–83], *Topology* **40** (2001), no. 2, 431–432. MR 1808227.

[26] D. Mumford, Rational equivalence of 0-cycles on surfaces, *J. Math. Kyoto Univ.* **9** (1968), 195–204. MR 0249428.

[27] K. G. O'Grady, Irreducible symplectic 4-folds numerically equivalent to $(K3)^{[2]}$, *Commun. Contemp. Math.* **10** (2008), no. 4, 553–608. MR 2444848.

[28] H. P. F. Swinnerton-Dyer, An enumeration of all varieties of degree 4, *Amer. J. Math.* **95** (1973), 403–418. MR 0340253.

[29] B. van Geemen and C. Voisin, On a conjecture of Matsushita, *Int. Math. Res. Not. IMRN* **2016**, no. 10, 3111–3123. MR 3551831.

[30] C. Voisin, Triangle varieties and surface decomposition of hyper-Kähler manifolds, to appear in *Recent Developments in Algebraic Geometry: To Miles Reid for His 70th Birthday*, London Mathematical Society Lecture Note Series, Cambridge University Press.

[31] C. Voisin, Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties, in *K3 Surfaces and Their Moduli*, 365–399, Progr. Math., 315, Birkhäuser/Springer, 2016. MR 3524175.

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