FAMILIES OF CONVEX TILINGS

RICHARD KENYON

ABSTRACT. We study tilings of polygons R with arbitrary convex polygonal tiles. Such tilings come in continuous families obtained by moving tile edges parallel to themselves (keeping edge directions fixed). We study how the tile shapes and areas change in these families. In particular we show that if R is convex, the tile shapes can be arbitrarily prescribed (up to homothety). We also show that the tile areas and tile "orientations" determine the tiling. We associate to a tiling an underlying bipartite planar graph $\mathcal{G}$ and its corresponding Kasteleyn matrix K. If $\mathcal{G}$ has quadrilateral faces, we show that K is the differential of the map from edge intercepts to tile areas, and extract some geometric and probabilistic consequences.

1. INTRODUCTION

In 1903 Dehn [2] showed that an $a \times b$ rectangle can be tiled with squares (of not necessarily the same size) if and only if a/b is rational. Tutte [12] gave an analogous result for tilings of convex regions with equilateral triangles. These results were generalized in [7] where we gave a correspondence between tilings with "horizontal trapezoids" (trapezoids with two horizontal edges) and harmonic functions on planar Markov chains. In a further generalization, with Scott Sheffield [8] we discussed tilings with general convex polygons, relating them to Kasteleyn theory and the Matrix-Tree Theorem.

In this paper we study *families* of tilings of polygonal regions with convex polygons having fixed edge directions. We obtain new results regarding the space of possible shapes of tiles and their possible areas. We also explore the connections between tilings and the dimer model on the underlying bipartite graph.

1.1. **Results.** Our first result is illustrated in Figure 1. Given a convex tiling of a convex polygon R and, for each tile t, a new copy t' of t with the same edge directions but of possibly different shape, we show (Theorem 3.1) that there is a unique, up to homothety, "combinatorially equivalent" tiling (see below for the
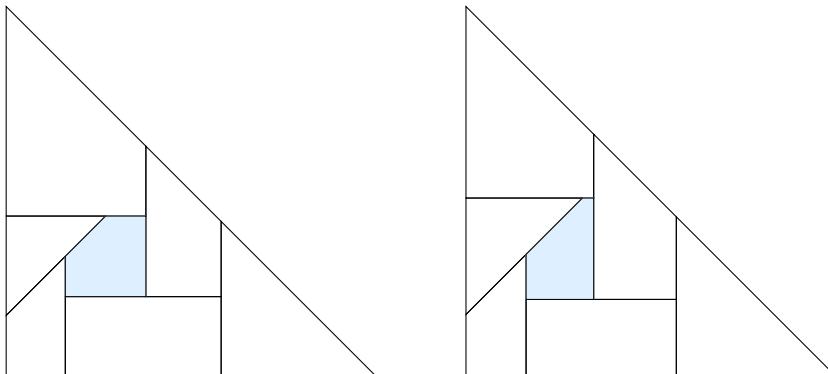
FIGURE 1. Combinatorially equivalent tilings in which one of the
tiles (shaded) has changed shape. The other tiles have changed
only by homothety. One can similarly change the shape of several,
or all, tiles, the remaining ones retaining their shape up to homo-
thety.

definition) of a new convex polygon R', with homothetic[1] copies of the new tiles.
A similar statement holds when R is not convex. This result is analogous to a
theorem about packings of strictly convex bodies due to Schramm [10]: see the
remarks after Corollary 3.3.

Another set of results involves the study of the tile areas and orientations for a
given tiling family. (We define "tile orientation" below—a tile has two orientations
and changes orientation if it undergoes a 180° rotation in the plane.) Figure 2
shows two combinatorially equivalent tilings with corresponding tiles having the
same areas. Notice that the small triangle has its orientation flipped from one
tiling to the next.

It is a remarkable fact about *rectangle tilings of a rectangle* that one can find for
any tiling a combinatorially equivalent rectangle tiling with *any* tuple of positive
areas [13]. Moreover, for prescribed areas the set of equivalent rectangle tilings is in
bijection with a set of st-orientations (acyclic orientations with a unique source and
sink) of a related graph; see [1]. These statements seem to hold only for rectangle
tilings; for other families of convex tilings, like the one illustrated in Figure 2, the
situation is more complicated. We show (Theorem 4.1) that for a given tuple of
areas and orientations there is at most one tiling. Furthermore, if the underlying
bipartite graph (see below) has quadrilateral faces, the set of feasible areas for a
given choice of orientations is either empty or topologically a ball of full dimension
$d = (\# \text{ of tiles}) - 1$ (see Corollary 4.3).

_____

[1]Here and in the rest of the paper by *homothety* we mean scaling by a real number, allowing
positive and negative homotheties (which are positive homotheties followed by rotation by π),
and also, in degenerate cases, zero homotheties which scale a tile down to a point.
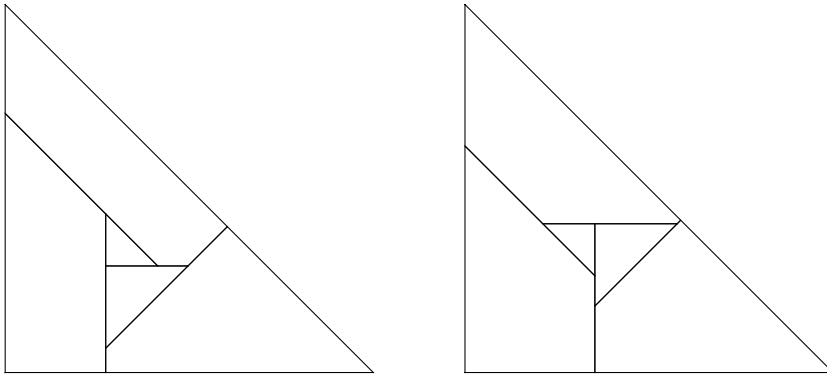
FIGURE 2. Combinatorially equivalent tilings with corresponding tiles of the same area.

If however we generalize the notion of tiling to "homology tiling" (which is defined via a winding number condition), we have a cleaner statement: given any tiling of a convex polygon R and a new tuple of areas, summing to the area of R, there is a unique combinatorially equivalent homology tiling of R with the new areas and the same tile orientations; see Theorem 4.4. Since for rectangles any homology tiling is an actual tiling, this generalizes the above. We furthermore conjecture that if no two interior edge directions are parallel, there is a unique homology tiling of R for any choice of areas and orientations; see Conjecture 4.5 and Figure 8.

1.2. **Definitions.** By a *tiling* of a region $R \subset \mathbb{R}^2$ we mean a covering of R with convex polygons, with nonintersecting interiors, as in Figure 1. We do not require the tiles to line up edge-to-edge. In fact in this figure there are no interior vertices where every tile which meets that vertex, meets at one of its corners: we will consider such a vertex to be *degenerate*. We take the point of view that such a vertex (if in the interior) is a degeneration of a tiling with one more tile, located at that vertex, which has shrunk to a point, as in Figure 3. Similarly, if multiple tiles meet at a corner of R, each tile meeting at one of its corners, then we consider such a tiling to be a degeneration of a tiling of a polygon R' having an extra edge or edges at that corner, whose lengths have gone to zero, as in Figure 4. This point of view on degeneracies is slightly unusual, but as we vary tilings in families such degeneracies are natural. In a nondegenerate tiling, each vertex of a tile is at a 'T' (or a T with several legs, as for the lower middle vertex in the right panel of Figure 6), except for the convex corners of R. These tilings are called "t-graphs" in [8].

Associated to a (nondegenerate) tiling is a bipartite planar graph $\mathcal{G}$. It has a white vertex for each tile, a black vertex for each maximal line segment in the tile boundaries, and an edge for an adjacency; see Figure 5. Note that some black vertices correspond to edges of the boundary of the tiled region; we'll call these
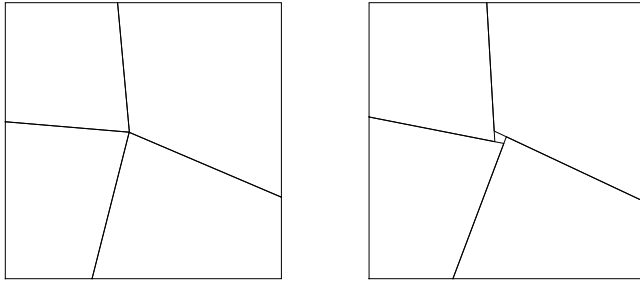
FIGURE 3. A tiling with a degenerate tile, and a nearby nonde-
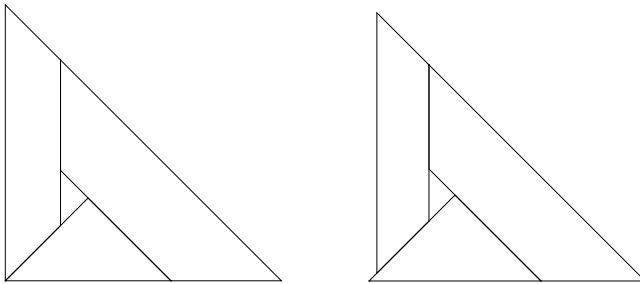generate tiling.

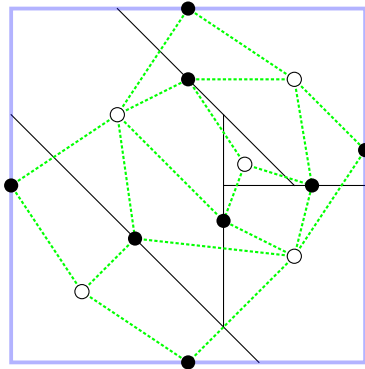FIGURE 4. A degenerate tiling, and a nearby nondegenerate tiling.

FIGURE 5. Bipartite graph (green) associated to a convex tiling.
Two tilings of R are combinatorially equivalent if they have the
same underlying bipartite graph with the same boundary identifi-
cations.

boundary vertices. Each quad face of $\mathcal{G}$ corresponds to a 'T' in the tiling: two segments meet, one of which ends in the interior of another. Bounded faces of higher degree in $\mathcal{G}$ correspond to T's with multiple legs: two or more segments end at an interior point of another segment.

We call two tilings *combinatorially equivalent* if their graphs are isomorphic respecting boundary vertices, that is, there is an isomorphism fixing the boundary vertices pointwise, by which we mean fixing which edge of the boundary polygon they correspond to.

To encode the tiling completely (up to translation), we can record, for each edge $e = wb$ of the bipartite graph $\mathcal{G}$, the edge vector in $\mathbb{C}$ giving the counterclockwise boundary edge of tile w as a subset of segment b. This turns $\mathcal{G}$ into a bipartite *network*, that is, an edge-weighted bipartite graph, with complex weights.

1.3. Connections with dimers and the Kasteleyn matrix.
As we vary the shapes of tiles in the family of combinatorially equivalent tilings, we can record the areas as functions of the edge intercepts. If bounded faces of $\mathcal{G}$ are quadrilateral, we show (see Section 4.2) that the map Ψ from edge intercepts to tile areas has differential $D\Psi = K$, where K is a *Kasteleyn matrix*, a matrix whose maximal minors count perfect matchings of the underlying graph $\mathcal{G}$. This observation, along with the reconstruction of a tiling from a graph given in [8] (Theorem 3.2 below) allows us to realize the Kasteleyn matrix of a bipartite planar graph (with quad faces), up to gauge equivalence, as the differential of a mapping. Some geometric and probabilistic implications of this fact are given in Section 4.2 and Theorem 4.2.

2. BACKGROUND ON DIMERS

We recall here background on the dimer model; see further details in [6]. A dimer cover, or perfect matching, of a graph $\mathcal{G}$ is a collection of edges with the property that each vertex is the endpoint of a unique edge in the collection. If $\nu = \{\nu_e\}_{e \in E}$ is a positive edge weight function on $\mathcal{G}$, we can associate to each dimer cover m a weight $\nu(m) = \prod_{e \in m} \nu_e$.

Kasteleyn [4] showed how to compute the weighted sum of dimer covers of any planar graph. When $\mathcal{G}$ is bipartite and planar, this sum is the determinant of a *Kasteleyn matrix* for $\mathcal{G}$, defined as follows. Let $K = (K_{w,b})_{w \in W, b \in B}$ be a matrix with rows indexing white vertices and columns indexing black vertices and

$$K_{w,b} = \begin{cases} c_{wb}\nu_{wb} & \text{if } w \sim b, \\ 0 & \text{otherwise,} \end{cases}$$

where c_{wb} is a complex number of modulus 1 with the property that the alternating product of c's around a bounded face of length 2ℓ is $(-1)^{\ell+1}$. Here by alternating

product around a face $w_1, b_1, w_2, \ldots, w_\ell, b_\ell$ we mean the first, divided by the second, times the third, and so on:

$$\frac{c_{w_1 b_1} c_{w_2 b_2} \ldots c_{w_\ell, b_\ell}}{c_{w_2 b_1} \ldots c_{w_1 b_\ell}} = (-1)^{\ell+1}.$$

Theorem 2.1 (Kasteleyn [4]). *Let K be a Kasteleyn matrix for a planar bipartite network $\mathcal{G}$. Then*

$$Z := \sum_{dimer\ covers\ m} \nu(m) = |\det K|.$$

Any two choices K, K' of Kasteleyn matrix for $\mathcal{G}$ are diagonally equivalent: $K' = D_W K D_B$, where D_B, D_W are diagonal matrices with diagonal entries of modulus 1. In Kasteleyn's original formulation the complex signs were just ± 1; any Kasteleyn matrix with complex signs is diagonally equivalent to one with real signs.

Given two edge weight functions ν, ν' on $\mathcal{G}$ we say they are *gauge equivalent* if there is a (nonzero) function f on vertices such that $\nu'_{wb} = \nu_{wb} f(w) f(b)$. Gauge equivalent weights give rise to the same probability measure on dimer covers, where a dimer cover has probability proportional to its weight (under a gauge equivalence, the constant of proportionality will change but not the probability).

Edge weight functions ν, ν' are gauge equivalent if their Kasteleyn matrices are diagonally equivalent (with diagonal matrices D_B, D_W with diagonal entries $f(b), f(w)$ respectively). Edge weight functions ν, ν' are gauge equivalent if and only if they have the same *face weights*, where the face weight X_f of a (bounded) face f with vertices $w_1, b_1, \ldots, b_\ell$ is

$$X_f = \frac{\nu_{w_1 b_1} \nu_{w_2 b_2} \ldots \nu_{w_\ell, b_\ell}}{\nu_{w_2 b_1} \ldots \nu_{w_1 b_\ell}}.$$

Given an arbitrary set of positive face weights X_f, it is easy to construct a corresponding edge weight function giving rise to those face weights. So the set of equivalence classes of edge weights is globally parameterized by $(\mathbb{R}_+)^F$, where F is the set of bounded faces of $\mathcal{G}$.

We note that the gauge transformations for the edge weights and edge signs are essentially the same operation: multiplying on the left and right by either a positive real diagonal matrix or a diagonal matrix of complex numbers of modulus 1. Therefore we can combine these two operations into a single one. We say two Kasteleyn matrices K, K' are *gauge equivalent* if they are diagonally equivalent via (nonsingular) complex-valued diagonal matrices. Gauge equivalent Kasteleyn matrices correspond to the same probability measure on dimer covers.

3. All shapes are possible

3.1. **Space of shapes of a tile.** Given a convex polygon t, let C_t be the space of convex polygons with the same number of edges and the same edge directions (in the same cyclic order), up to a global sign, and up to translation. The space C_t naturally has a linear structure: if t has edges $z_1, \ldots, z_n$ in counterclockwise order then any other tile in C_t will have edges $a_1 z_1, \ldots, a_n z_n$ for reals a_i, all of the same

sign: all positive or all negative, subject to the linear condition $\sum a_i z_i = 0$. If we parameterize C_t with the n-tuple $(a_1, \ldots, a_n)$, it is naturally the interior of a cone in an $(n-2)$-dimensional real vector space V_t. More precisely, C_t is the union of two open convex cones which are negatives of each other: C_t^+ consists of those tiles in which $a_i > 0$, and C_t^- of those in which $a_i < 0$. These two cones are called the two *orientations* of t. For convenience we also add the origin to C_t since as we vary tiles in families we sometimes have to deal with the case where a tile shrinks to a point and then moves into the other orientation.

The vector space V_t in which C_t sits has another natural linear coordinate system, using the intercepts of the lines defining the edges of t; see Section 4.2 below.

3.2. **Space of shapes of a tiling.** Given a convex tiling $T = \cup_{w \in W} t_w$ of a polygon R, let us consider the set $C_T = C_T(R)$ of combinatorially equivalent tilings, in which each tile has the same edge directions, up to sign, as the corresponding tile of T. Note that C_T also has a linear structure: an element of C_T is determined by the shapes of its individual tiles, so C_T lies inside the product $\prod_w C_{t_w} \subset \prod_w V_{t_w}$, and is defined by the linear conditions that certain sums of edge vectors are zero: around each interior black vertex of $\mathcal{G}$ there is one such condition, and for each boundary edge the sum of tile edge vectors equals the vector of the corresponding edge of R.

The space C_T has a natural subdivision into regions $C_T(\sigma)$ where the individual tiles have fixed orientations: $\sigma \in \{-1, 1\}^k$ corresponds to a choice of orientation for each tile. Each subset $C_T(\sigma)$ is either empty or a convex polytope: given any two tilings $T_1, T_2 \in C_T(\sigma)$ then any convex combination of T_1 and T_2 is also a tiling of $C_T(\sigma)$. The full space C_T is not typically convex, however. These polytopes $C_T(\sigma)$ are glued in C_T along parts where one or more tiles degenerate to points.

3.3. **Kasteleyn matrix.** Let $T = \cup_{w \in W} t_w$ be a tiling of a convex polygon R, and let $\mathcal{G}$ be the associated bipartite network. We note that $\mathcal{G}$ has $n - 1$ more black vertices than white vertices, where n is the number of edges of R. This fact follows from an Euler characteristic argument: the tiling divides R into $|W|$ open 2-cells, $|B|$ open line segments and n vertices. By the Euler characteristic,

$$\chi(R) = 1 = |W| - |B| + n.$$

We let B_{int} be the interior black vertices, and B_∂ the boundary vertices. Thus $|W| = |B_{\text{int}}| + 1$.

Let K be the weighted bipartite adjacency matrix of $\mathcal{G}$; K has rows indexing the white vertices (tiles of T) and columns indexing the black vertices (maximal segments in the tile boundaries). The entry $K_{w,b}$ is the complex number giving the vector of the edge of tile w in direction b, oriented counterclockwise around the tile boundary.

Note that K is a *Kasteleyn matrix* for $\mathcal{G}$, in the sense that the alternating product of edge weights around any face of $\mathcal{G}$ is real with sign $(-1)^{\ell+1}$, where 2ℓ is the degree of the face (the number of edges on its boundary). See Figure 6 for the proof of this fact.
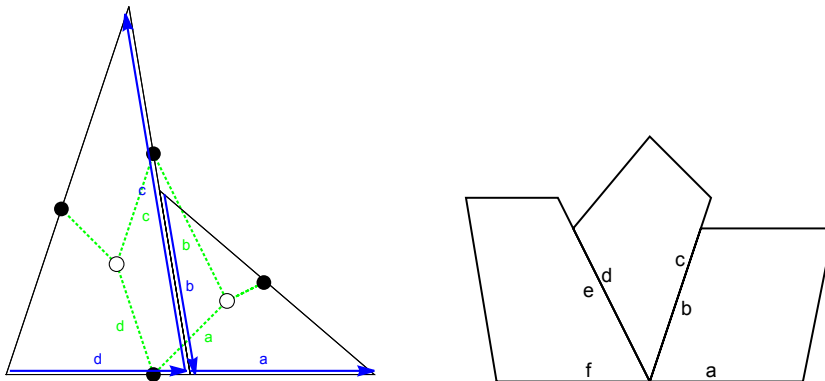
FIGURE 6. In the first panel, the lower central vertex in blue corresponds to the green quadrilateral face of $\mathcal{G}$, with edge weights a, b, c, d. We have $\frac{ac}{bd} < 0$, since a, d point in the same direction and b, c in opposite directions. Similarly, in the second panel the lower central vertex corresponds to a hexagonal face; we have $\frac{ace}{bdf} > 0$.

It is a bit harder to see that K has full rank $|W|$, but this is important. That K has full rank follows from Kasteleyn's theorem, Theorem 2.1. We simply need to show that, once we remove all but one of the black boundary vertices from $\mathcal{G}$, the new graph $\mathcal{G}'$ has at least one dimer cover. This is a surprising fact since it relates a geometric property (of being a convex tiling) to a combinatorial one. We can prove this is as follows. Rotate R so that one boundary edge b_0 is horizontal. Rotate further by a very small positive amount so that each tile w has a unique vertex v_w with lowest y-value. Since v_w is a 'T', exactly one of the two edges of face w meeting at v_w has v_w as its lower endpoint; match w to the black vertex corresponding to this edge. The leftmost face containing b_0 along one of its edges is matched to b_0. This concludes the proof that K has maximal rank $|W|$.

3.4. **From tiling to tiling.**

Theorem 3.1. *Given a tiling $T = \cup t_w$ of a convex polygon R and, for each tile t_w, a new tile t'_w in C_{t_w}, there is a unique (up to homothety) combinatorially equivalent tiling $S = \cup s_w$ of a convex polygon $R' \in C_R$, where $s_w = a_w t'_w$ is a homothetic copy of t'_w, and where each $a_w \in \mathbb{R}$ can be positive, zero or negative.*

Proof. In order to make a tiling out of the tiles $a_w t'_w$, the reals $\{a_w\}$ must satisfy a linear relation around each interior segment, that is, at each non-boundary black vertex $b \in B_{\mathrm{int}}$. More precisely, let K be the Kasteleyn matrix associated to T, and let K' be the matrix obtained from K by replacing the row for t_w by the corresponding row for t'_w (that is, the row with the same nonzero entries, but these corresponding to the edge vectors of t'_w instead of t_w). The equation on the $\{a_w\}$ is $\sum_w a_w K'_{wb} = 0$ for non-boundary vertices b. Since K' still has full rank $|W|$, and

$|B_{\mathrm{int}}| = |W| - 1$, there is a unique solution vector $(a_w)_{w \in W}$ up to scale. Since each column of K' has a constant argument, the unique solution can be chosen real.

It remains to see that this solution defines a tiling. Let $\mathcal{G}^*$ be the dual graph of $\mathcal{G}$, without the external vertex. We glue in a disk for each bounded face of $\mathcal{G}^*$ (each non-boundary vertex of $\mathcal{G}$) to make a topological disk D in which $\mathcal{G}^*$ is embedded. The initial tiling defines a continuous map from D to R which collapses each black face of $\mathcal{G}^*$ to a segment, and maps each white face to the corresponding tile in an orientation-preserving fashion. We can deform the tiles t_w continuously from their initial values (since the space of shapes $\bar{C}_t$ of each tile is connected); along the deformation the map from D remains orientation-preserving, sending the boundary of D homeomorphically to the boundary of R' and preserving the boundary orientation. The orientation of the mapping is preserved even if the individual tiles change their tile-orientation (which rotates a tile by π, but preserves its orientation as a map), and the mapping is surjective to $\mathbb{R}'$, so defines a tiling. $\square$

If R is initially not convex, the same argument works, but since each boundary edge of R will have its length adjusted to make R', this might result in R' being self-intersecting; nonetheless there is at least a locally injective tiling of R'.

3.5. **From a graph to a tiling.** Suppose we are just given the planar bipartite graph $\mathcal{G}$ with positive edge weights $\{\nu_{wb}\}$, and a convex polygon R, along with a cyclic identification of the black boundary vertices of $\mathcal{G}$ with the edges of R. Then under a certain nondegeneracy condition there is an associated tiling. We say $\mathcal{G}$ is 2-*nondegenerate* if every pair of edges having distinct endpoints (at most one of the four of which is a boundary vertex) can be completed to a dimer cover of $\mathcal{G}$ using exactly one of the boundary vertices. For example, if $\mathcal{G}$ has a degree-2 vertex it is *not* 2-nondegenerate. See Figure 7 for another 2-degenerate example. There exists a tiling under these conditions:

Theorem 3.2 ([8]). *Suppose $\mathcal{G}, R$ are as above and $\mathcal{G}$ is 2-nondegenerate. Then there is a unique gauge equivalent choice of edge weights and a unique tiling of R with those edge weights.*

Proof. See [8] for the complete proof. We'll give the construction (and prove uniqueness) here. Let $q_1, \ldots, q_n \in \mathbb{C}$ be the counterclockwise boundary edges of R. Let K be a Kasteleyn matrix for $\mathcal{G}$. We need to find diagonal matrices D_W, D_B so that $K' = D_W K D_B$ defines the tiling, that is, $K' 1_B = 0$ and $1_W K'$ is the vector which is zero on internal black vertices and q_i on the ith boundary black vertex. (Here 1_B and 1_W are the "all-1s" vectors.) The solution is given in three linear algebraic steps.

First, find a nonzero vector v_W so that $v_W K$ is zero on B_{int}. Since K restricted to B_{int} has full rank and has one more row than column, there is a unique solution up to scale. Define D_W to be the diagonal matrix with entries of v_W.

Second, for $b \in B_\partial$, the sum of the bth column of $D_W K D_B$ needs to add to q_i; this determines $(D_B)_{b,b}$.
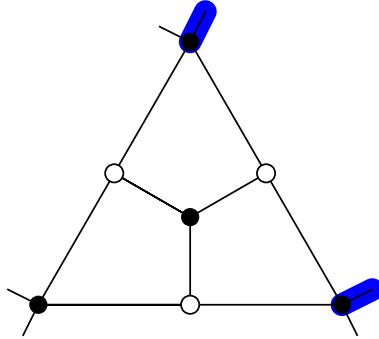
FIGURE 7.   A graph which contains a subgraph as shown (which is
connected to the rest of the graph only via the three black corner
vertices) is 2-degenerate; for example, no dimer cover contains
both blue edges, since the three black vertices have among them
only two remaining neighbors. In any tiling with this graph the
tiles for the three white vertices will be reduced to segments. If
we swap the colors $W \leftrightarrow B$, however, then a nondegenerate tiling
may exist, but in any such tiling the tiles for the four white vertices
will have convex union.

Finally, the remaining $|B_{\mathrm{int}}|$ entries of D_B are determined by the linear equations
$K D_B 1_B = 0$. (Here there is one more equation than variable, but since the sum of
each column is zero the last equation is a consequence of the previous ones).

The fact that K' determines an actual tiling is a consequence of the maximum
principle: we refer the reader to [8]. The 2-nondegeneracy implies that white tiles
do not degenerate to segments.   □

Since the edge weights modulo gauge are described by the face weights $\{X_f\}$,
we have the following.

Corollary 3.3. *For fixed convex R and 2-nondegenerate $\mathcal{G}$, the space of $(\mathcal{G}, R)$-
tilings is homeomorphic to $\mathbb{R}_+^F$, with global coordinates provided by the face weights.*

A somewhat stronger variant of Theorem 3.1 above is as follows. Let $\mathcal{G}$ be a bi-
partite graph satisfying the conditions of Theorem 3.2, and for each white vertex w
choose a convex k-gon t_w (where k is the degree of w), under the constraint that if
two white vertices have a black neighbor in common then the corresponding edges
(corresponding to the adjacencies with the black vertex) have the same direction
up to sign. Suppose moreover that the directions of the boundary black vertices
can be oriented so that as one moves counterclockwise around the boundary of $\mathcal{G}$,
they turn once around, that is, there is a convex polygon with those edge directions
in cyclic order. Then there is a tiling of a convex polygon with the combinatorics
of $\mathcal{G}$ in which for each white vertex w the corresponding tile is homothetic to t_w.
This is a tiling analog of the theorem of Schramm [10]: Schramm's result gives

a packing of given strictly convex sets (up to homothety) with a given adjacency graph; our theorem gives a tiling with given convex polygons (up to homothety) and given underlying graph. The analogy is not perfect, however, since the underlying adjacency graphs play different roles.

4. Areas and the Kasteleyn matrix

4.1. **The area form.** Recall the cone $C_t \subset V_t$ of convex polygons with edge directions in the directions of a convex polygon t. On C_t there is a quadratic form q_t giving the area of the polygon. It is a homogeneous quadratic function of the edge lengths, which can be defined on all of V_t. By a theorem of Thurston [11], it has signature $(1, n-3)$ when t is a (convex) n-gon. In particular, if we consider the subset of C_t of polygons of fixed area $A > 0$, it consists of two disjoint hyperboloid components H_+ and H_-, one for each of the two different orientations of t.

Theorem 4.1. *Let T be a nondegenerate convex tiling of a convex polygon R and consider the space of combinatorially equivalent tilings of R with the same edge directions and the same tile orientations. Then for a given set of tile areas there is at most one tiling.*

Proof. Suppose there are two combinatorially equivalent tilings T_1, T_2 of R having the same tile orientations σ and the same tile areas. Recall that $C_T(\sigma)$ is convex. Consider the tiling $sT_1 + (1 - s)T_2$, where $T_1, T_2 \in C_T(\sigma)$ and $s \in [0, 1]$. If tiles $t_1, t_2 \in C_t$ have the same area and orientation then a convex combination $st_1 + (1 - s)t_2$ has area which is a strictly concave function of s (the line segment joining t_1 and t_2 lies on one side of the hyperboloid component H). In particular, the area for $s \in (0, 1)$ is larger than the area at the endpoints. It is impossible that all tile areas increase, since the boundary is fixed. Therefore the areas of corresponding tiles of T_1 and T_2 cannot be all equal, unless $T_1 = T_2$. Thus the areas are uniquely determined by R and σ. □

4.2. **The Kasteleyn matrix as differential.** For a line $ax + by = c$ let us define the *intercept* to be the signed distance from the line to the origin: $\pm c/\sqrt{a^2 + b^2}$, with the sign depending on which side the origin is on (chosen arbitrarily).

Let T be a tiling of a convex polygon R, whose underlying graph has quad faces. The space C_T can be uniquely parameterized by the intercepts $\{i_b\}$ for $b \in B_{\text{int}}$, that is, $C_T \subset \mathbb{R}^{B_{\text{int}}}$.

Let us consider a slightly larger space: fix the outer polygon R except for one edge b_0 of R, whose intercept we allow to vary. Let $B_0 = B_{\text{int}} \cup \{b_0\}$. As noted earlier, $|B_0| = |W|$. We can redefine $C_T \subset \mathbb{R}^{B_0}$ to be this larger space. Let $C_T(\sigma) \subset C_T$ be the corresponding subset with fixed tile orientations.

Let $\Psi \colon C_T \to \mathbb{R}^W$ be the map from intercepts $\{i_b\}_{b \in B_0}$ to tile areas $\{A_w\}_{w \in W}$. It is a quadratic function, that is, each area is a quadratic function of the intercepts. From our definition of intercepts, the differential $D\Psi$ satisfies $\partial A_w/\partial i_b = \pm|K_{w,b}|$, that is, plus or minus the length of the edge of face w in direction b. The sign here depends on which side of the edge the face is on. This matrix $D\Psi = (\pm|K_{w,b}|)$ is

in fact gauge equivalent to K: $K_{w,b} = \pm e^{i\theta_b}(D\Psi)_{w,b}$, where θ_b is the direction of edge b and the sign depends only on b.

Thus we see that $K_\Psi := D\Psi$ is a Kasteleyn matrix for $\mathcal{G}$ which is also the differential of the map Ψ.

The inverse K_Ψ^{-1} is the differential of the map Ψ^{-1} from areas to intercepts. Let $\mathcal{G}_0$ be the graph obtained from $\mathcal{G}$ by removing all black boundary vertices except b_0. Dimer covers of $\mathcal{G}_0$ are counted by $|\det K_\Psi|$. We can use K_Ψ^{-1} to relate the tiling geometry to the edge probabilities of a random dimer covering of $\mathcal{G}_0$, as follows. The probability of edge wb is $K(w,b)K^{-1}(b,w)$ (and this quantity is independent of choice of gauge); see [5]. The fact that the probability is positive means that $\frac{\partial A_w}{\partial i_b}$ and $\frac{\partial i_b}{\partial A_w}$ have the same sign. This implies that if we increase the area of a tile t by a small amount, keeping the other tile areas fixed, then each edge of t moves *outward*, as if we were blowing air into the tile. The probability p_{wb} is the relative proportion of area along b which swept out as A_w increases (and i_b moves outwards).

For multiple edges we can also draw a geometric conclusion. Given two edges $w_1 b_1$ and $w_2 b_2$ of $\mathcal{G}$, suppose the signs of the intercepts are chosen so that K_{w_1,b_1}, $K_{w_2,b_2} > 0$. The probability of the pair of edges $w_1 b_1$ and $w_2 b_2$ is (see [5])

$$K(w_1, b_1)K(w_2, b_2) \det \begin{pmatrix} K^{-1}(b_1, w_1) & K^{-1}(b_2, w_1) \\ K^{-1}(b_1, w_2) & K^{-1}(b_2, w_2) \end{pmatrix}.$$

The fact that this is positive implies that if we vary only the areas A_{w_1}, A_{w_2}, keeping other areas fixed, the map from those areas to the intercepts i_{b_1}, i_{b_2} is orientation-preserving. This is because the above determinant is the Jacobian of the map from areas A_{w_1}, A_{w_2} to intercepts i_{w_1}, i_{w_2}. Consequently, for example, if $dA_{w_1} + \beta dA_{w_2}$ is a perturbation of areas which fixes i_{b_1}, and $\beta > 0$, then it increases i_{b_2} (and if $\beta < 0$, it decreases i_{b_2}).

Here is another application.

Theorem 4.2. Ψ *is a diffeomorphism from* $C_T(\sigma)$ *onto its image.*

Proof. The differential of the map from intercepts to areas is the Kasteleyn matrix K_Ψ and thus has nonzero determinant. Therefore Ψ is locally a diffeomorphism. Injectivity follows from Theorem 4.1 above. $\square$

Since $C_T(\sigma)$ is an open polytope, we have:

Corollary 4.3. *For a family of tilings with fixed tile orientations* σ, *and whose underlying graph* $\mathcal{G}$ *has quad faces, the set of possible areas* $\Psi(C_T(\sigma))$ *is homeomorphic to a ball of dimension* $d = \# \text{tiles} - 1$.

Without the hypothesis on quad faces the dimension of C_T is smaller, so it is not clear what the corresponding statement should be.

4.3. **Homology tilings.** For a convex tile t recall that $V_t \cong \mathbb{R}^{n-2}$ is the vector space of closed polygonal paths with edges in the direction (up to sign) of the edges of t. We call such closed paths "homology tiles"; they may be self-intersecting. On V_t let q_t be the area form: the integral of $x\,dy$ over the closed path. Recall that

q_t has signature $(1, n - 3)$. We let V_t^+, V_t^- be the two disjoint subsets of V_t where $q_t > 0$. These are interiors of cones; each $V_t^{\pm}$ contains one of the two components of C_t.

Let T be a convex tiling of a convex polygon R, whose underlying graph has quad faces. We can parameterize $C_T = C_T(R)$ using the intercepts $i_b \in \mathbb{R}$. In order for a tiling to exist the i_b must satisfy certain inequalities. If the i_b fail to satisfy those inequalities we say we have a *homology* tiling. That is, the space of homology tilings is just the space $\mathbb{R}^{B_{\text{int}}}$ allowing the intercepts to take arbitrary real values.

A homology tiling can be characterized in terms of winding numbers: for a point p in general position inside R, the sum of winding numbers of the homology tiles t around p is 1, and for a point p in general position outside R, the sum of winding numbers is 0. (This can be proved by showing that the sum of winding numbers for a point remains constant when an intercept line passes across the point.)

We define $V_T \subset \mathbb{R}^{B_{\text{int}}}$ to be those homology tilings whose tiles have positive area. On $\mathbb{R}^{B_{\text{int}}}$, a choice of orientation σ for the tiles cuts out a convex region $V_T(\sigma) \subset V_T$, where individual homology tiles have fixed orientation.

Theorem 4.4. *Let T be a convex tiling of a convex polygon R, whose underlying graph has quad faces. Let $(A_1, \dots, A_{|W|})$ be a collection of positive reals summing to the area of R. Then there is a unique combinatorially equivalent homology tiling of R such that tile w has area A_w and the same tile orientation as the original tile.*

Proof. $V_T(\sigma)$ is a convex set cut out by a finite number of cones. If $V_T(\sigma)$ is nonempty, it is bounded: if it were unbounded then by going to ∞ along a line parallel to the boundary, some tile area would tend to ∞, contradicting the property that the sum of the areas equals the area of R.

For fixed positive $\{A_w\}$, the function $Q(\{i_b\}) = \sum_w A_w \log q_w$ is a concave and analytic function on $V_T(\sigma)$ and tends to $-\infty$ on the boundary, and so has a unique critical point. At the critical point,

$$\sum_w \frac{A_w}{q_w} \frac{\partial q_w}{\partial i_b} = 0,$$

that is

$$\sum_w \frac{A_w}{q_w} K_{w,b} = 0$$

or

$$\left(\frac{A_w}{q_w}\right) \in \ker K^*,$$

the left kernel of K restricted to B_{int}. But recalling that $\ker K^*$ contains 1_W, if the kernel is of dimension 1 then $q_w = sA_w$ (for a global scale factor s) and we are done.

Generically in $\mathbb{R}^{B_{\text{int}}}$, $\ker K^*$ is of dimension 1: this is true for the original tiling and nearby, so true on a Zariski dense set. So $\ker K^*$ is of dimension strictly larger than 1 only on a hypersurface X defined by $\det K_{w_0} = 0$, the determinant of K
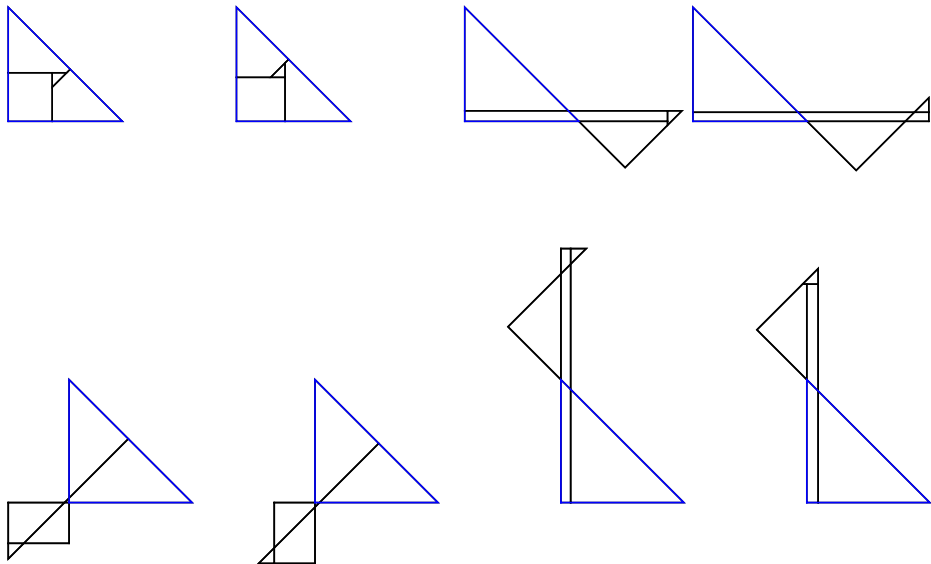
FIGURE 8. The 8 homology tilings of a triangle R (the triangle in blue) with areas equal to those of the first tiling.

when we remove row w_0 (since 1_W is in the left kernel of K, all maximal minors are equal, so X has no dependence on the choice of w_0.) For generic areas ker K^* is thus one-dimensional. The proof now follows by compactness of V_T, taking a limit of solutions with generic areas. $\square$

Conjecture 4.5. *Let T be a convex tiling of a convex polygon R, with the property that no two internal edges are parallel, and the underlying graph has quad faces. Then for any choice of tile orientations $\sigma \in \{1, -1\}^W$ there exists a unique homology tiling of R or $-R$ such that tile w has positive area and orientation σ_w.*

In Figure 8 we can see 8 out of the 16 possible solutions for a given set of areas (the other 8 are obtained by rotation of these by π).

If some edges are parallel, then apparently some of these solutions "go to infinity" and we have fewer than 2^W tilings. For example in the rectangle tiling case, we get significantly fewer, as discussed previously and in [1].

If all tiles are quadrilaterals, V_T is defined by a hyperplane arrangement; in this case bounded regions in the complement of the hyperplanes correspond bijectively to orientations for which there is a homology tiling.

References

[1] A. Abrams and R. Kenyon, Fixed-energy harmonic functions, *Discrete Anal.* **2017**, Paper No. 18, 21 pp. MR 3734203.

[2] M. Dehn, Über Zerlegung von Rechtecken in Rechtecke, *Math. Ann.* **57** (1903), no. 3, 314–332. MR 1511212.

[3] A. B. Goncharov and R. Kenyon, Dimers and cluster integrable systems, *Ann. Sci. Éc. Norm. Supér. (4)* **46** (2013), no. 5, 747–813. MR 3185352.

[4] P. W. Kasteleyn, Graph theory and crystal physics, in *Graph Theory and Theoretical Physics*, 43–110, Academic Press, London, 1967. MR 0253689.

[5] R. Kenyon, Local statistics of lattice dimers, *Ann. Inst. H. Poincaré Probab. Statist.* **33** (1997), no. 5, 591–618. MR 1473567.

[6] R. Kenyon, Lectures on dimers, in *Statistical Mechanics*, 191–230, IAS/Park City Math. Ser., 16, Amer. Math. Soc., Providence, RI, 2009. MR 2523460.

[7] R. Kenyon, Tilings and discrete Dirichlet problems, *Israel J. Math.* **105** (1998), 61–84. MR 1639727.

[8] R. W. Kenyon and S. Sheffield, Dimers, tilings and trees, *J. Combin. Theory Ser. B* **92** (2004), no. 2, 295–317. MR 2099145.

[9] A. Postnikov, Total positivity, Grassmannians, and networks, https://arxiv.org/abs/math/0609764 [math.CO], 2006.

[10] O. Schramm, Existence and uniqueness of packings with specified combinatorics, *Israel J. Math.* **73** (1991), no. 3, 321–341. MR 1135221.

[11] W. P. Thurston, Shapes of polyhedra and triangulations of the sphere, in *The Epstein Birthday Schrift*, 511–549, Geom. Topol. Monogr., 1, Geom. Topol. Publ., Coventry, 1998. MR 1668340.

[12] W. T. Tutte, Dissections into equilateral triangles, in *The Mathematical Gardner*, 127–139, Prindle, Weber & Schmidt, Boston, 1981. MR 593154.

[13] S. Wimer, I. Koren and I. Cederbaum, Floorplans, planar graphs, and layouts, *IEEE Trans. Circuits and Systems* **35** (1988), no. 3, 267–278. MR 0931855.

Richard Kenyon
Department of Mathematics, Yale University, New Haven, CT 06520, USA
richard.kenyon@yale.edu