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ON CONFORMALLY COMPACT EINSTEIN MANIFOLDS

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Abstract. We survey some of the recent developments in the study of the compactness and uniqueness problems for some classes of conformally compact Einstein manifolds.

1. Introduction

Let X^d be a smooth manifold of dimension $d \geq 3$ with boundary $\partial X = M$. A smooth conformally compact metric g^+ on X is a Riemannian metric such that $g = r^2 g^+$ extends smoothly to the closure $\overline{X}$ for some defining function r to the boundary ∂X in X. A defining function r is a smooth non-negative function on the closure $\overline{X}$ such that $\partial X = \{r = 0\}$ and the differential $Dr \neq 0$ on ∂X. A conformally compact metric g^+ on X is said to be conformally compact Einstein (CCE) if, in addition,

$$\mathrm{Ric}[g^+] = -(d-1)g^+,$$

where Ric denotes the Ricci curvature. The most significant feature of a CCE manifold (X, g^+) is that the metric g^+ is canonically associated with the conformal structure $[\hat{g}]$ on the boundary at infinity ∂X, where $\hat{g} = g|_{T\partial X}$. $(\partial X, [\hat{g}])$ is called the *conformal infinity* of a conformally compact manifold (X, g^+). It is of great interest in both the mathematics and theoretical physics communities to understand the correspondences between conformally compact Einstein manifolds (X, g^+) and their conformal infinities $(\partial X, [\hat{g}])$, especially in the study of the AdS/CFT correspondence in theoretical physics (see Maldacena [31, 32, 33] and Witten [36]).

For a CCE manifold, given any conformal infinity h and for any defining function r, we always have $|\nabla_g r| \equiv 1$ on M. In fact, it is known that the full Riemann curvature tensor $\mathrm{Rm}[g^+]$ of the metric g^+ has the asymptotic expansion near the infinity, for all $1 \leq i, j, k, l \leq d$,

$$\mathrm{Rm}_{ijkl}[g_+] = -|\nabla r(x)|_g^2((g^+)_{ik}(g^+)_{jl} - (g^+)_{il}(g^+)_{jk}) + O(r^{-3}),$$

which yields the above claim. A conformally compact metric g^+ on X is called *asymptotically hyperbolic* (AH) if, in addition, $|\nabla_g r| \equiv 1$ on M; thus any CCE manifold is AH. Moreover, in any CCE manifold, given any conformal infinity h there exists a special defining function r, which we call *geodesic defining function*,

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such that $|\nabla_g r| \equiv 1$ in an asymptotic neighborhood $M \times [0, \epsilon)$ of M and $r^2 g^+|_{TM} = h$.

Applying properties of the geodesic defining functions, we have nice asymptotic expansions for the compactification metrics of CCE manifolds. It turns out the asymptotic behaviors of the metrics are slightly different when the dimension d is even or odd.

When d is even, the asymptotic behavior of the compactified metric g of a CCE manifold (X^d, g^+) with conformal infinity $(M^{d-1}, [h])$ [18, 17] takes the form

$$\begin{aligned} g &:= r^2 g^+ = dr^2 + g_r \\ &= dr^2 + h + g^{(2)} r^2 + \cdots \text{(even powers)} + g^{(d-1)} r^{d-1} + g^{(d)} r^d + \cdots \end{aligned} \tag{1.1}$$

on an asymptotic neighborhood $M \times (0, \epsilon)$, where r denotes the geodesic defining function of g. The $g^{(j)}$ are tensors on M, and $g^{(d-1)}$ is trace-free with respect to a metric in the conformal class on M. For j even and $0 \le j \le d-2$, the tensor $g^{(j)}$ is locally formally determined by the conformal representative, but $g^{(d-1)}$ is a non-local term which is not determined by the boundary metric h, subject to the trace-free condition.

When d is odd, the analogous expansion is

$$\begin{aligned} g &:= r^2 g^+ = dr^2 + g_r \\ &= dr^2 + h + g^{(2)} r^2 + \cdots \text{(even powers)} + g^{(d-1)} r^{d-1} + k r^{d-1} \log r + \cdots, \end{aligned} \tag{1.2}$$

where now the $g^{(j)}$ are locally determined for j even and $0 \le j \le d-2$, k is locally determined and trace-free, the trace of $g^{(d-1)}$ is locally determined, but the trace-free part of $g^{(d-1)}$ is formally undetermined.

We remark that h together with $g^{(d-1)}$ determine the asymptotic behavior of g [17, 2].

In this paper, we will first briefly survey some of the recent development in this research area. We will then describe a series of joint works: one by the autors of this paper [7], one by us and Jie Qing [8], and a third one by us, Jie Qing and X. Jin [9]; in them, we address the issues of compactness of sequences of CCE manifolds for some classes of such manifolds, and we also address the unique filling-in problems for the class of CCE manifolds constructed earlier by Lee and Graham [21].

2. Basics and a short survey

Some basic examples.

Example 1. A model case of a CCE manifold is the hyperbolic ball $\mathbb{B}^d$ with the Poincaré metric $g_{\mathbb{H}} := \frac{4}{(1-|x|^2)^2} \sum_{i=1}^d dx_i^2$, where $|x| := \sqrt{\sum_{i=1}^d x_i^2}$ is the usual euclidean norm of $x = (x_1, \ldots, x_d) \in \mathbb{B}^d = \{y \in \mathbb{R}^d : |y| < 1\}$. In this case, when the metric in the conformal infinity is $h = \frac{1}{4} g_c$, where g_c denotes the standard metric on the $(d-1)$-dimensional sphere S^{d-1}, the associated geodesic defining function is $r(x) = \frac{1-|x|}{1+|x|}$. The metric g_r in the expansion (1.1) above is $g_r(x) = (1 - r(x)^2)^2 h$.

Example 2. Another class of examples of CCE manifolds was constructed by Graham and Lee in 1991 [21], where they proved that metrics in a small $C^{2,\alpha}$ neighborhood of the standard metric g_c on S^{d-1} are allowed as the conformal infinity of some CCE metrics on the unit ball B^d when $d \geq 4$.

Example 3. The AdS-Schwarzschild space $(R^2 \times S^2, g_m^+)$, where m is any positive number and

$$g_m^+ = V dt^2 + V^{-1} dr^2 + r^2 g_c,$$

with $V = 1 + r^2 - \frac{2m}{r}$, $t \in S^1(\lambda)$, g_c the surface measure on S^2, $r \in [r_h, +\infty)$, and r_h the positive root of $1 + r^2 - \frac{2m}{r} = 0$. It turns out that in this case there are two different values of m such that both g_m^+ are conformal compact Einstein fillings for the same boundary metric $S^1(\lambda) \times S^2$. This is the famous non-unique filling-in example of Hawking and Page [26].

Existence and non-existence results.

The most important existence result is the *ambient metric* construction by Fefferman and Graham [15, 17]. As a consequence of their construction, for any given compact manifold (M^{d-1}, h) with an analytic metric h, some CCE metric exists on some tubular neighborhood $M^n \times (0, \epsilon)$ of M. This result was later extended to manifolds with smooth metrics by Gursky and Székelyhidi [25].

As we have mentioned before, a perturbation result of Graham and Lee [21] asserts that in a neighborhood of the standard metric g_c on S^{d-1}, there exists a conformal compact Einstein metric on B^d with any given conformal infinity h.

Recent results of Gursky and Han [23] and of Gursky, Han and Stolz [24] showed that when X is spin and of dimension $4k \geq 8$, and when the Yamabe invariant $Y(M, [h]) > 0$, there are topological obstructions to the existence of a CCE metric g^+ defined in the interior of X with conformal infinity given by $[h]$. The basic idea is to adapt the classical Lichnerowicz result on the vanishing of the $\widehat{A}$-genus for spin manifolds of positive scalar curvature. Indeed, suppose g^+ is a CCE filling-in of $[h]$; then one can use the compactification of Lee to obtain a metric $g = r^2 g^+$ with positive scalar curvature which is smooth up to the boundary, and such that M is totally geodesic with respect to g. It follows that the index of the Dirac operator (with respect to APS boundary conditions) is zero. However, using well-known properties of the index, it is possible to construct examples of spin manifolds with boundary M and conformal classes $[h]$ of positive Yamabe invariant on M such that the index of the Dirac operator (with respect to any extension of any metric in $[h]$) has non-vanishing index. For example, on the round sphere S^{4k-1} with $k \geq 2$, there are infinitely many such conformal classes.

The results in [23] and [24] were based on a key fact pointed out earlier by J. Qing [35], which in turn relies on previous work by J. Lee [28].

Lemma 2.1. *On a CCE manifold (X^d, g^+), assuming $Y(\partial X, [h]) > 0$, there exists a compactification of g^+ with positive scalar curvature; hence the relative Yamabe invariant $Y(X, \partial X, [r^2 g^+]) > 0$.*

Uniqueness and non-uniqueness results.

Under the assumption of the positive mass theorem, J. Qing [35] has established (B^d, g_H) as the unique CCE manifold with $(S^{d-1}, [g_c])$ as its conformal infinity. The proof of this result was later refined and established without using the positive mass theorem by Li, Qing and Shi [30] (see also Dutta and Javaheri [14]). Later in Sections 4 and 5 we will also prove the uniqueness of the CCE extension of the metrics constructed by Graham and Lee [21] for all $d \geq 4$.

As we mentioned in Example 3 above, when the conformal infinity is $S^1(\lambda) \times S^2$ with the product metric, Hawking and Page [26] constructed non-unique CCE filling-ins.

In a recent series of joint works [7, 8, 9], we address the compactness issue of sequences of metrics on CCE manifolds. The question is as follows. Given a sequence of CCE manifolds (X^d, g^+), with $M = \partial X$ and $\{g_i\} = \{r_i^2 g_i^+\}$ a sequence of compactified metrics, set $h_i = g_i|_{TM}$; assuming that $\{h_i\}$ forms a compact family of metrics in M, when is it true that some representatives $\bar{g}_i \in [g_i]$ with $\{\bar{g}_i|_M = h_i\}$ also form a compact family of metrics in $\bar{X}$? One main difficulty in addressing the compactness problem is the existence of some non-local term in the asymptotic expansion of the metric near the conformal infinity. For example, in the case $d = 4$ the $g^{(3)}$ term in the asymptotic expansion of $g = r^2 g^+$ in (1.1) is non-local, as it depends on both $h = g|_M$ and g^+.

One application of compactness is the uniqueness result of the CCE extension of Graham and Lee for the metrics on $\mathbb{S}^{d-1}$ close to the standard canonical metric on $\mathbb{S}^{d-1}$. As we have mentioned before, in the model case—the hyperbolic space form—it was proved in [35] (see also [14] and a later different proof in [30]) that $(\mathbb{B}^d, g_{\mathbb{H}})$ is the unique CCE manifold with the standard canonical metric on $\mathbb{S}^{d-1}$ as its conformal infinity. The compactness result permits us to generalize the global uniqueness in the above setting. Such result could be considered also as a stability result for the hyperbolic space.

In this work, if there is no confusion, we drop the argument g for the various curvature tensors Ric, Rm, etc.

3. Compactness result in high dimensions, $d \geq 5$

In this section, we consider a general d-dimensional CCE manifold (X^d, g^+), with $d \geq 5$. A general consideration is what is a "good" choice of the compactification of g^+ one should use. A most natural consideration is the compactication of the Yamabe metric (i.e., the metric which minimizes the L^1 norm of the scalar curvature in the compactified conformal class of metrics $[g^+]$ with fixed volume, which we know exists). The problem with that choice is that we do not see how to control the corresponding boundary metric of the Yamabe metric. Instead, in [9] and in the earlier works [7] and [8], we consider a special choice of compactification with some given boundary metric to start with. In the case when $d \geq 5$, the metric we choose and denote by g^* is the metric that was considered in a paper by Case and Chang [6] and called adapted metric.

Given (X^d, g^+) a CCE manifold and a representative metric h in its conformal infinity, we can solve the PDE

$$-\Delta_{g^+} v - \frac{(d-1)^2 - 9}{4} v = 0 \quad \text{on } X^d$$

and define our adapted metric g^* as $g^* := v^{\frac{4}{d-4}} g^+ = \rho^2 g^+$, with $g^*|_M = h$, the fixed metric on the conformal infinity of (X^d, g^+). We now describe some special properties of the metric g^*.

Recall that the fourth-order Paneitz operator is given by (see [34, 5, 20])

$$P_4 = (-\Delta)^2 + \delta \left(4A - \frac{d-2}{2(d-1)} R \right) \nabla + \frac{d-4}{2} Q_4, \tag{3.1}$$

where $A = \frac{1}{d-2}\big(\operatorname{Ric} - \frac{R}{2(d-1)} g\big)$ is the Schouten tensor, δ is the dual operator of the differential ∇, R denotes the scalar curvature and Q_4 is a fourth-order Q-curvature. More precisely, let $\sigma_k(A)$ denote the k-th symmetric function of the eigenvalues of A and set $Q_4 := -\Delta\sigma_1(A) + 4\sigma_2(A) + \frac{d-4}{2}\sigma_1(A)^2$. For an Einstein metric with $\operatorname{Ric}_{g^+} = -(d-1)g^+$, we have

$$P_4[g^+] = \left(-\Delta_{g^+} - \frac{(d-1)^2 - 1}{4} \right) \circ \left(-\Delta_{g^+} - \frac{(d-1)^2 - 9}{4} \right).$$

Therefore,

$$Q_4[g^*] = \frac{2}{d-4} P_4[g^*] 1 = \frac{2}{d-4} v^{\frac{d+4}{d-4}} P_4[g^+] v = 0.$$

Moreover, g^* is totally geodesic on the boundary (see [9, Lemma 2.6]).

We now recall some basic calculations for curvatures under conformal changes. Write $g^+ = r^{-2} g$ for some defining function r and calculate

$$\operatorname{Ric}[g^+] = \operatorname{Ric}[g] + (d-2) r^{-1} \nabla^2 r + (r^{-1}\Delta r - (d-1) r^{-2} |\nabla r|^2) g,$$

so that

$$R[g^+] = r^2 \left(R[g] + \frac{2d-2}{r} \Delta r - \frac{d(d-1)}{r^2} |\nabla r|^2 \right).$$

Here the covariant derivatives are calculated with respect to the metric g (or adapted metrics g^* in the following). Therefore, for adapted metrics g^* of a conformally compact Einstein metric g^+, one has

$$R[g^*] = 2(d-1)\rho^{-2}(1 - |\nabla\rho|^2),$$

which in turn gives

$$\operatorname{Ric}[g^*] = -(d-2)\rho^{-1}\nabla^2\rho + \frac{4-d}{4(d-1)} R[g^*] g^*$$

and

$$R[g^*] = -\frac{4(d-1)}{d+2} \rho^{-1} \Delta\rho.$$

Given (X^d, g^+) a CCE manifold with the conformal infinity $(\partial X, [h])$ of non-negative Yamabe type, an important property of the g^* metric (proved in the earlier work of Case and Chang [6, Lemma 4.2]) is that $g^* = \rho^2 g^+$, the adapted metrics associated with the metric h with positive scalar curvature in the conformal infinity, have positive scalar curvature $R[g^*] > 0$ on X, which implies, in particular, that

$$\|\nabla\rho\|[g^*] \leq 1.$$

This property is one of the main ingredients in our blow-up analysis.

Another important property in the blow-up analysis is the non-collapsing result for adapted metrics g^* when the conformal infinity $(\partial X, [h])$ is of positive Yamabe type (see [8, Lemma 3.3] and [9, Lemma 2.11]). That is, the volume of any geodesic ball with radius equal to 1 is uniformly bounded below by some positive constant when the curvature tensor is bounded.

We recall that the Yamabe invariant of the conformal infinity $(\partial X, [h])$ is defined as

$$Y(\partial X, [h]) = \inf_{\widetilde{h}\in[h]} \frac{\int_{\partial X} R[\widetilde{h}]\, d\mathrm{vol}[\widetilde{h}]}{\mathrm{Vol}(\partial X, \widetilde{h})^{(d-3)/(d-1)}}.$$

We now split the discussion into two cases.

Case I, when d is even

We first consider the case when d is even. In this case, due to the vanishing obstruction tensor [17, 19] for CCE manifolds, the curvature tensor satisfies an elliptic system. More precisely, let R_{ikjl}, R_{ij} and W_{ikjl} be the Riemann, Ricci and Weyl curvature tensors, respectively. We recall the definition of the fourth-order Bach tensor B on d-dimensional manifolds (X^d, g) as

$$B_{ij} := \frac{1}{d-3}\nabla^k\nabla^l W_{kijl} + \frac{1}{d-2} W_{kijl}R^{kl}. \tag{3.2}$$

Recall also that the Cotten tensor $\mathcal{C}$ is defined as

$$\mathcal{C}_{ijk} = A_{ij,k} - A_{ik,j},$$

where A is the Schouten tensor. It turns out there is a relation between the divergence of the Weyl tensor and the Cotton tensor, namely

$$\nabla^l W_{ijkl} = (d-3)\mathcal{C}_{kij}.$$

Applying this relation, we can write the Bach tensor into the following equations:

$$(d-2)B_{ij} = \Delta R_{ij} - \frac{d-2}{2(d-1)}\nabla_i\nabla_j R - \frac{1}{2(d-1)}\Delta R g_{ij} + Q_1(\mathrm{Rm}), \tag{3.3}$$

where $Q_1(\mathrm{Rm})$ is some quadratic term on the Riemann curvature tensor,

$$\begin{aligned} Q_1(\mathrm{Rm}) := 2W_{ikjl}R^{kl} - \frac{d}{d-2}R_i{}^k R_{jk} + \frac{d}{(d-1)(d-2)}RR_{ij} \\ + \left(\frac{1}{d-2}R_{kl}R^{kl} - \frac{R^2}{(d-1)(d-2)}\right) g_{ij}. \end{aligned}$$

We recall that the adapted metric g^* has flat Q_4-curvature, i.e., $Q_4[g^*] = 0$, which can be rewritten into the following form:

$$-\Delta R = -\frac{d^3 - 4d^2 + 16d - 16}{4(d-2)^2(d-1)}R^2 + \frac{4(d-1)}{(d-2)^2}|\,\mathrm{Ric}\,|^2. \tag{3.4}$$

We will now incorporate the Q_4-flat property of g^* to the Bach equation of g^* to derive estimates of the curvature of g^*.

Recall that it follows from [17, 19, 27] that when d is even, the metrics conformal to Einstein ones have the vanishing obstruction tensor $\mathcal{O}_{ij}$. That is,

$$\begin{aligned}\mathcal{O}_{ij} &= (\Delta)^{(d-4)/2}\frac{1}{d-3}\nabla^j\nabla^l W_{ijkl} + \text{lower order terms}\\ &= (\Delta)^{(d-4)/2}B_{ij} + \text{lower order terms}\\ &= 0.\end{aligned} \tag{3.5}$$

For example, when $d = 6$, we can rewrite (3.5) as

$$\begin{aligned}\Delta B_{ij} = B_{ij,k}{}^{k} = 2W_{kijl}B^{kl} + 4A_k{}^kB_{ij} - 8A^{kl}\mathcal{C}_{(ij)k,l}\\ + 4\mathcal{C}_{ki}{}^{l}\mathcal{C}_{lj}{}^{k} - 2\mathcal{C}_i{}^{kl}\mathcal{C}_{jkl} - 4A^k{}_{k,l}\mathcal{C}_{ij}{}^{l} + 4W_{kijl}A^k{}_mA^{ml},\end{aligned}$$

where $2\mathcal{C}_{(ij)k} = \mathcal{C}_{ijk} + \mathcal{C}_{jik}$. Gathering (3.3), (3.4) and (3.5), the Ricci tensor satisfies a $(d-2)$th-order elliptic system. This allows us to apply some standard elliptic PDE techniques, to obtain an ε-regularity result for the Ricci tensor, and then for the metrics g^*. This is the key step which permits us to do various blow-up analyses and derive the following compactness result (see [9, Theorem 1.1]).

Theorem 3.1. *Let $d \geq 6$ be even. Let X be a smooth oriented d-dimensional manifold with boundary $\partial X = \mathbb{S}^{d-1}$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. Assume that the set $\{h_i\}$ of metrics on the boundary with non-negative scalar curvature that represent the conformal infinities lies in a given set $\mathcal{C}$ of metrics that is of positive Yamabe type and compact in the $C^{k,\alpha}$ Cheeger–Gromov topology, with $k \geq d-2$. Moreover, assume that there exists some positive constant $C_0 > 0$ such that the Yamabe invariant of the conformal infinities is uniformly bounded below by C_0. Then there exists a sufficiently small $\delta_0 > 0$ such that if either*

(1′) $\int_{X^d}(|W|^{d/2}\,\mathrm{dvol})[g_i^+] < \delta_0$ *or*

(1″) $Y(\partial X, [h_i]) \geq Y(\mathbb{S}^{d-1}, [g_{\mathbb{S}}]) - \delta_0$,

then the set $\{g_i^*\}$ of the adapted metrics (after diffeomorphisms that fix the boundary) is compact in the $C^{k,\alpha'}$ Cheeger–Gromov topology for all $0 < \alpha' < \alpha$.

Case II, when d is odd

When the dimension d of the manifold X is odd, in general we would not expect the strong estimate C^{d-1} as in the case when d is even, due to the occurrence of the term of $kr^{d-1}\log r$ in the expansion of the metric g as in (1.2). The coefficient k of this term happens to be the obstruction tensor [17, 19] on the boundary of X and in general it may not vanish. It turns out we can apply a different strategy to reach a similar compactness result as in Theorem 3.1 under a stronger, namely C^6, regularity assumption. This strategy actually works for all dimensions d. Instead of exploring the vanishing of the obstruction tensor as in the case when d is even, we explore the regularity property of the Einstein metric g^+. To do so, we will apply the gauge fixing techniques for Einstein metrics developed in [9, Lemma 4.6

and Lemma 4.7]. We first obtain some regularity property of the metrics near the neighborhood of the boundary; we then introduce some suitable weighted spaces and apply them to avoid the degeneration of the metric and reach the ε-regularity property in the conformal infinity of the CCE metrics.

Let us introduce some notation. We firstly choose smooth local coordinates $\theta = (\theta^2, \theta^2, \dots, \theta^d)$ on an open set $U \subset \partial X$. We then extend θ to $(\theta^1, \theta) = (\rho, \theta^2, \theta^2, \dots, \theta^d)$ on the open subset $\Omega = [0, \epsilon) \times U \subset \overline{X}$, where ρ is the above defining function and $\epsilon > 0$ is some small positive number.

For any fixed point $p \in \partial X$, let Ω be a neighborhood and let (ρ, θ) be the background coordinates such that $\theta(p) = 0$. For each $R > 0$ sufficiently small, we define $Z_R(p) \subset \Omega \subset \overline{X}$:

$$Z_R(p) = \{(\rho, \theta) \in \Omega : |\theta| < R,\ 0 < \rho < R\}.$$

In [13], Chruściel, Delay, Lee and Skinner used the gauged Einstein equation to study the regularity problem and, later on, Biquard and Herzlich [4] proved a local version. Let us consider the non-linear functional on the d-dimensional open set $Z_R(p)$, with $p \in \partial X$, introduced by Biquard [3]: for two asymptotically hyperbolic metrics g^+ and k^+,

$$F(g^+, k^+) \coloneqq \mathrm{Ric}[g^+] + (d-1)g^+ - \delta^*_{g^+}(B_{k^+}(g^+)),$$

where $B_{k^+}(g^+)$ is a linear condition, essentially the infinitesimal version of the harmonicity condition

$$B_{k^+}(g^+) \coloneqq \delta_{k^+} g^+ + \frac{1}{2} d\, \mathrm{tr}_{k^+}(g^+).$$

We have, for any asymptotically hyperbolic metrics k^+,

$$D_1 F(k^+, k^+) = \frac{1}{2}(\Delta_L + 2(d-1)),$$

where D_1 denotes the partial differentiation of F with respect to its first variable, and the Lichnerowicz Laplacian Δ_L on symmetric 2-tensors is given by

$$\Delta_L \coloneqq \nabla^* \nabla[k^+] + 2\overset{\circ}{\mathrm{Ric}}[k^+] - 2\overset{\circ}{\mathrm{Rm}}[k^+],$$

where

$$\overset{\circ}{\mathrm{Ric}}[k^+](u)_{ij} = \frac{1}{2}(R_{im}[g^+]{u_j}^m + R_{jm}[k^+]{u_i}^m)$$

and

$$\overset{\circ}{\mathrm{Rm}}[k^+](u)_{ij} = R_{imjl}[k^+]u^{ml}.$$

It is clear that for any CCE metrics g^+ we have

$$F(g^+, g^+) = 0.$$

Suppose $(X^d, \partial X, g^+)$ is conformally compact Einstein with positive conformal infinity $(\partial X, [h])$ and with dimension $d \geq 5$. Assume that, under the adapted metrics g^*, we have

(1) $\| \mathrm{Rm}_{g^*} \|_{C^0} \leq 1$;
(2) $\|h\|_{C^6} \leq N$ for some positive constant $N > 0$.

We will prove the ε-regularity. Namely, Rm_{g^*} is in the Hölder space $C^{1,\alpha}$ for all $\alpha \in (0,1)$ (or, equivalently, the adapted metric g^* is in the Hölder space $C^{3,\alpha}$) near the boundary ∂X.

We can identify $\{p \in \bar{X} : \rho(p) \leq r_1\} = [0, r_1] \times \partial X$ for some $r_1 > 0$ as a submanifold with the boundary. We consider a C^4 compactified AH manifold on $[0, r_1/2] \times \partial X$,

$$t = d\rho^2 + h + \rho^2 h^{(2)}, \quad t^+ = \rho^{-2} t,$$

where $h^{(2)} = g^{(2)}$ is the Fefferman–Graham expansion term and intrinsically determined by the boundary metric h ($g^{(2)}$ is the Schouten tensor of h for the adapted metric). Given $2R < r_1/2$, we look for a local diffeomorphism $\Phi : Z_R(p) \to \Phi(Z_R(p)) \subset Z_{2R}(p)$ such that $\Phi^* g^+$ solves the gauged Einstein equation in $Z_{R/2}(p)$,

$$F(\Phi^* g^+, t^+) = 0. \tag{3.6}$$

We divide the boundary $\partial Z_R(p) \coloneqq \partial^\infty Z_R(p) \cup \partial^{\mathrm{int}} Z_R(p) = (\{\rho = 0\} \cap \partial Z_R(p)) \cup (\{\rho > 0\} \cap \partial Z_R(p))$. Recall that CCE g^+ and regular AH t^+ have the same conformal infinity h on ∂X. We try to find a $C^{2,\alpha}$ (with $\alpha \in (0,1)$) local diffeomorphism $\Phi : Z_R(p) \to Z_{2R}(p)$ fixing the boundary $\partial^\infty Z_R(p)$ such that the gauged condition is satisfied in $Z_{R/2}(p)$ up to the diffeomorphism Φ, that is,

$$B_{t^+}(\Phi^* g^+) = 0 \quad \text{ in } Z_{R/2}(p).$$

Thus, the gauged Einstein equation (3.6) is satisfied in $Z_{R/2}(p)$. Such equation permits us to establish that $\rho^2(\Phi^* g^+ - t^+)$ is in the Hölder space $C^{3,\alpha}$ for all $\alpha \in (0,1)$, which in turn implies that $\rho^2 \Phi^* g^+$ is in $C^{3,\alpha}$. Using the fact that g^+ is CCE, we derive the regularity result for the Cotton tensor in the Hölder space $C^{0,\alpha}$. Hence, it follows from (3.2) and (3.3) that the Ricci tensor Ric is in the Hölder space $C^{1,\alpha}$ in $Z_{R/2}(p)$, which yields the desired ε-regularity in $Z_{R/2}(p)$. Once the ε-regularity is established, the rest of the proof is as in the even-dimensional case. Finally, we prove the following compactness result. For more details, see [9, Theorem 1.2].

Theorem 3.2. *Let $d \geq 4$ be even. Let X be a smooth oriented d-dimensional manifold with boundary $\partial X = \mathbb{S}^{d-1}$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. Assume the set $\{h_i\}$ of metrics on the boundary with non-negative scalar curvature that represent the conformal infinities lies in a given set $\mathcal{C}$ of metrics that is of positive Yamabe type and compact in the C^6 Cheeger–Gromov topology. Moreover, assume there exists some positive constant $C > 0$ such that the Yamabe invariant of the conformal infinities is uniformly bounded below by C. Then under the above assumptions $(1')$ or $(1'')$, the set $\{g_i^*\}$ of the adapted metrics (after diffeomorphisms that fix the boundary) is compact in the $C^{3,\alpha}$ Cheeger–Gromov topology for all $0 < \alpha < 1$.*

4. Uniqueness of Graham–Lee metrics in high dimensions, $d \geq 5$

As an application of Theorem 3.2, we are able to establish the global uniqueness for the CCE metrics on X^d with prescribed conformal infinities that are very close to the conformal round $(d-1)$-sphere as in the work [21] (see also [29]).

Theorem 4.1 ([9, Theorem 1.3])**.** *For a given conformal $(d-1)$-sphere $(\mathbb{S}^{d-1},[h])$, with $d \geq 5$, that is sufficiently close to the round one in the C^6 topology, there is exactly one conformally compact Einstein metric g^+ on X^d whose conformal infinity is the prescribed conformal $(d-1)$-sphere $(\mathbb{S}^{d-1},[h])$. Moreover, the topology of X is that of the ball $\mathbb{B}^d$.*

We remark that there exists a unique CCE filling-in metric when the conformal infinity is the standard sphere [35] (see also [14, 30]). We remark that the uniqueness result in the above theorem is the stability property of the uniqueness result of the model case—hyperbolic space.

Theorem 4.1 could be proved by contradiction. Assume that there is a sequence of conformal $(d-1)$-dimensional spheres $(\mathbb{S}^{d-1},[h_i])$ that converges to the round sphere such that, for each i, there exist two non-isometric conformally compact Einstein metrics g_i^+ and $\widetilde{g}_i^+$, and g_i^* and $\widetilde{g}_i^*$ are the corresponding adapted metrics.

Up to a subsequence, both g_i^* and $\widetilde{g}_i^*$ converge to the adapted metric $g_{\mathbb{H}}^*$ of hyperbolic space in the $C^{3,\alpha}$ Cheeger–Gromov sense due to Theorem 3.2. On the other hand, there exists a diffeomorphism φ_i of class $C^{2,\alpha}$ for all $\alpha \in (0,1)$ (equal to the identity on the boundary), such that

$$F(\varphi_i^*\widetilde{g}_i^+, g_i^+) = 0.$$

Moreover, $\|\varphi_i(x) - x\|_{C^{2,\alpha}} \to 0$ and $\|\varphi_i^*\widetilde{g}_i^* - g_i^*\|_{C^{1,\alpha}} \to 0$ when $i \to \infty$. By the implicit function theorem, we have local uniqueness around each g_i^+, which implies that, for large i,

$$g_i^+ = \varphi_i^*\widetilde{g}_i^+.$$

5. Compactness and uniqueness in dimension $d = 4$

In this section, we report results in dimension 4 established in [7, 8]. On a 4-dimensional CCE manifold (X^4, g^+), we will consider a special choice of compactification $g^* = g_{FG} = \rho^2 g^+$, called *Fefferman–Graham's compactification* (also called *FG metric* or *FG compactification*). The FG metric was first studied by Fefferman and Graham [16], who introduced the PDE

$$-\Delta w = -(d-1) \quad \text{on } X^d \tag{5.1}$$

and showed the connection between the integral of some coefficient of the asymptotic expansion of w and that of the renormalized volume of CCE manifolds (X^d, g^+). Their result was later put into the geometric setting by Chang, Qing and Yang [37], who considered the metric $g^* := e^{2w} g^+$ and related the behavior of the renormalized volume to that of the Q-curvature of g^*.

We remark that in the special case when $d = 4$, the FG metric $g^* = e^{2w} g^+$ on a CCE 4-manifold (X^4, g^+) is a natural dimensional continuation of the adapted metrics on a CCE d-manifold (X^d, g^+) when $d \geq 5$, in the following sense: Fixed a boundary metric h, let us call v_s the solution of the Poisson equation

$$-\Delta_{g^+} v - s(d-1-s)v = 0 \qquad \text{on } X^d \tag{5.2}$$

when we choose $s = \frac{d}{2} + 1$. Then, for $d \geq 5$, the adapted metric on X^d which we introduced earlier in Section 4 is defined as $g^* = v_s^{\frac{d-4}{2}} g^+ = \rho_s^2 g^+$ with $g^*|_M = h$. When $d = 4$, we have $s = \frac{d}{2} + 1 = 3 = d - 1$; then the solution w of (5.1) satisfies

$$w = - \frac{d}{ds}\Big|_{s=d-1} v_s,$$

and the FG metric is defined as the compactified metric $g^* = e^{2w} g^+ = \rho^2 g^+$. Note that when $s = d - 1$, the natural solution of the Poisson equation (5.2) is $v_s \equiv 1$, and therefore ρ is the limiting function of ρ_s when s tends to $d - 1$. We refer the reader to the expository article [12] for further explanation of the relationship between the adapted metric and the FG metric, and of the connection between the FG metric and the notion of renormalized volume and other integral conformal invariants in the CCE setting.

Thus the FG metric on X^4 satisfies properties similar to the adapted metrics defined on X^d when $d \geq 5$. The most important among them are: (a) the FG metric g^* has free Q_4-curvature and positive scalar curvature, and (b) its restriction to the boundary M is totally geodesic. For simplicity, we choose the boundary metric h be the Yamabe metric representative of the conformal infinity. With the same arguments as in Theorem 3.2, we obtain the same result in dimension 4 (see [8, Theorem 1.3]).

Theorem 5.1. *Suppose that X is a smooth oriented 4-manifold with boundary $\partial X = \mathbb{S}^3$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. Assume the following conditions:*

1. *the set $\{h_i\}$ of Yamabe metrics that represent the conformal infinities lies in a given set $\mathcal{C}$ of metrics that is of positive Yamabe type and compact in the $C^{k,\alpha}$ Cheeger–Gromov topology with $k \geq 3$ and with some $\alpha \in (0,1)$;*
2. *there exists some $\delta_0 > 0$ such that either*
   (2a) $\int_{X^4} (|W|^2 \, d\mathrm{vol})[g_i^+] < \delta_0$ *or*
   (2b) $Y(\partial X, [h_i]) \geq Y(\mathbb{S}^3, [g_{\mathbb{S}}]) - \delta_0$
   holds.

Then, the set $\{g_i^*\}$ of the FG compactifications (after diffeomorphisms that fix the boundary) is compact in the $C^{k,\alpha'}$ Cheeger–Gromov topology for all $\alpha' \in (0, \alpha)$.

We now present some general compactness results on X^4 without the assumptions that the Weyl tensor is small in L^2 norm or that the Yamabe invariant of the conformal infinity is close to that of the standard sphere (see [7, 8]).

We first introduce some geometric quantities. In [7, Lemma 2.1], for a CCE manifold (X^4, g^+) with any compactification g, we introduce the notion of 2-tensor S, which on a 3-manifold M^3 is defined as

$$(S[g])_{\alpha,\beta} \coloneqq \nabla^i (W[g])_{i\alpha\mathbf{n}\beta} + \nabla^i (W[g])_{i\beta\mathbf{n}\alpha} - \nabla^{\mathbf{n}} (W[g])_{\mathbf{n}\alpha\mathbf{n}\beta} - \frac{4}{3} H[g] (W[g])_{\alpha\mathbf{n}\beta}{}^{\mathbf{n}}$$

where $W[g]$ denotes the Weyl tensor, $H[g]$ the mean curvature on the boundary M, i is the full index, α, β represent the tangential indices, and $\mathbf{n}$ is the outward unit normal of the boundary under the metric g. When the compactified metric g has totally geodesic boundary, it takes the form

$$(S[g])_{\alpha,\beta} = \frac{1}{2}\partial_{\mathbf{n}} \operatorname{Ric}[g]_{\alpha,\beta} - \frac{1}{12}\partial_{\mathbf{n}} R[g]\, h_{\alpha,\beta}.$$

The 2-tensor S is conformally invariant, in the sense that

$$S[r^2 g] = r^{-1} S[g].$$

The connection of the S tensor to that of $g^{(3)}$ in (1.1) is the following (see [7, Remark 2.2, (2.7)]): Under any compactification by a geodesic defining function r, $g = r^2 g^+$ has $\partial_{\mathbf{n}} R[g] = 0$ on M, thus

$$(S[g])_{\alpha,\beta} = -\frac{3}{2} g^{(3)}_{\alpha,\beta}.$$

This shows that $g^{(3)}$ is also a local conformal invariant, which has been stated by Graham [18].

The compactness result in the general case can be stated as follows (see [8, Theorem 1.1] and also [7, Theorem 1.1]):

Theorem 5.2. *Suppose that X is a smooth oriented 4-manifold with boundary $\partial X = \mathbb{S}^3$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. Assume that the condition (1) from Theorem 5.1 holds, and assume also the following conditions:*

(2″) *the FG compactifications $\{g_i^* = \rho_i^2 g_i^+\}$ associated with the Yamabe representatives $\{h_i\}$ on the boundary satisfy*

$$\lim_{r\to 0} \sup_i \sup_{x\in\partial X} \oint_{B(x,r)} |S_i|[g_i^*]\, d\mathrm{vol}[h_i] = 0;$$

(3) $H_2(X, \mathbb{Z}) = 0$.

Then, the set $\{g_i^*\}$ of FG compactifications (after diffeomorphisms that fix the boundary) forms a compact family in the $C^{k,\alpha'}$ Cheeger–Gromov topology for all $\alpha' \in (0, \alpha)$.

We remark that we are aware that in the paper [1] M. Anderson asserted similar compactness results in the CCE setting under no assumptions on the (analogue of the) non-local tensor S. We have difficulty understanding some key estimates in his arguments.

The key points for the compactness result in the general case on 4-dimensional CCE manifolds are the following: on the one hand, the condition (2″) in Theorem 5.2 rules out the boundary blow-up; on the other hand, the topological condition (3) in Theorem 5.2 rules out the interior blow-up.

We now explain the connection of the S tensor to other scalar curvature invariants for the metric g^*, which plays a key role in the results [7, Theorem 1.7] and [8, Theorem 1.2].

Recall that on a 4-manifold (X^4, g), a fourth-order Q_4-curvature is given by

$$Q_4[g] \coloneqq -\frac{1}{6}\Delta R - \frac{1}{2}|\operatorname{Ric}|^2 + \frac{1}{6}R^2.$$

Q_4-curvature is naturally associated with a fourth-order Paneitz operator (3.1). The relation of the pair $\{Q_4, P_4\}$ in 4 dimensions is like that of the well-known pair $\{K, -\Delta\}$ in 2 dimensions, where K denotes the Gaussian curvature:

$$\begin{aligned} -\Delta[g] + K[g] &= K[e^{2w}g]e^{2w} \quad \text{on } X^2, \\ P_4[g]w + Q_4[g] &= Q_4[e^{2w}g]e^{4w} \quad \text{on } X^4, \end{aligned}$$

for conformal changes of the metric. For a 4-manifold (X^4, g) with boundary, in the earlier works of Chang and Qing [10, 11], in connection with the fourth-order Q-curvature, a third-order non-local boundary curvature T was introduced on ∂X to study the boundary behavior of g. The relation of the pair (Q_4, T) is a generalization of that of the Dirichlet–Neumann pair $(-\Delta, \partial_{\mathbf{n}})$. The expression of T-curvature is in general complicated, but in the special case when g is totally geodesic, the expression T takes the simple form

$$T[g] \coloneqq \frac{1}{12}\partial_{\mathbf{n}} R.$$

We can state another compactness result (see [8, Theorem 1.2] and also [7, Theorem 1.7]).

Theorem 5.3. *Suppose that X is a smooth oriented 4-manifold with boundary $\partial X = \mathbb{S}^3$. Let $\{g_i^+\}$ be a set of conformally compact Einstein metrics on X. Assume the conditions (1) from Theorem 5.1, (3) from Theorem 5.2, and*

(2‴) *For the associated Fefferman–Graham's compactifications $\{g_i^* = \rho_i^2 g_i^+\}$ with the Yamabe representatives $\{h_i\}$ on the boundary,*

$$\liminf_{r\to 0} \inf_i \inf_{x\in\partial X} \oint_{B(x,r)} T[g_i^*] \, d\mathrm{vol}[h_i] \geq 0.$$

Then, the set $\{g_i^*\}$ is compact in the $C^{k,\alpha'}$ Cheeger–Gromov topology for all $\alpha' \in (0, \alpha)$ up to diffeomorphisms that fix the boundary, provided $k \geq 7$.

We remark that with the same arguments as in high dimensions (see Theorem 4.1), we also reach a global uniqueness result in dimension 4 (see [8, Theorem 1.9]). Namely,

Theorem 5.4. *For a given conformal 3-sphere $(\mathbb{S}^3, [h])$ that is sufficiently close to the round one in the $C^{3,\alpha}$ Cheeger–Gromov topology with some $\alpha \in (0, 1)$, there is exactly one conformally compact Einstein metric g^+ on $\mathbb{B}^4$ whose conformal infinity is the prescribed conformal 3-sphere $(\mathbb{S}^3, [h])$.*

References

[1] M. T. Anderson, Einstein metrics with prescribed conformal infinity on 4-manifolds, *Geom. Funct. Anal.* **18** (2008), no. 2, 305–366. MR 2421542.

[2] O. Biquard, Continuation unique à partir de l'infini conforme pour les métriques d'Einstein, *Math. Res. Lett.* **15** (2008), no. 6, 1091–1099. MR 2470386.

[3] O. Biquard, Einstein deformations of hyperbolic metrics, in *Surveys in Differential Geometry: Essays on Einstein Manifolds*, 235–246, Surv. Differ. Geom., 6, Int. Press, Boston, MA, 1999. MR 1798612.

[4] O. Biquard and M. Herzlich, Analyse sur un demi-espace hyperbolique et poly-homogénéité locale, *Calc. Var. Partial Differential Equations* **51** (2014), no. 3-4, 813–848. MR 3268872.

[5] T. P. Branson, S.-Y. A. Chang and P. C. Yang, Estimates and extremals for zeta function determinants on four-manifolds, *Comm. Math. Phys.* **149** (1992), no. 2, 241–262. MR 1186028.

[6] J. S. Case and S.-Y. A. Chang, On fractional GJMS operators, *Comm. Pure Appl. Math.* **69** (2016), no. 6, 1017–1061. MR 3493624.

[7] S.-Y. A. Chang and Y. Ge, Compactness of conformally compact Einstein manifolds in dimension 4, *Adv. Math.* **340** (2018), 588–652. MR 3886176.

[8] S.-Y. A. Chang, Y. Ge and J. Qing, Compactness of conformally compact Einstein 4-manifolds II, *Adv. Math.* **373** (2020), 107325, 33 pp. MR 4130465.

[9] S.-Y. A. Chang, Y. Ge, X. Jin and J. Qing, On compactness of conformally compact Einstein manifolds and uniqueness of Graham–Lee metrics, III (in preparation).

[10] S.-Y. A. Chang and J. Qing, The zeta functional determinants on manifolds with boundary. I. The formula, *J. Funct. Anal.* **147** (1997), no. 2, 327–362. MR 1454485.

[11] S.-Y. A. Chang and J. Qing, The zeta functional determinants on manifolds with boundary. II. Extremal metrics and compactness of isospectral set, *J. Funct. Anal.* **147** (1997), no. 2, 363–399. MR 1454486.

[12] S. Y. A. Chang and R. A. Yang, On a class of non-local operators in conformal geometry, *Chin. Ann. Math. Ser. B* **38** (2017), no. 1, 215–234. MR 3592161.

[13] P. T. Chruściel, E. Delay, J. M. Lee and D. N. Skinner, Boundary regularity of conformally compact Einstein metrics, *J. Differential Geom.* **69** (2005), no. 1, 111–136. MR 2169584.

[14] S. Dutta and M. Javaheri, Rigidity of conformally compact manifolds with the round sphere as the conformal infinity, *Adv. Math.* **224** (2010), no. 2, 525–538. MR 2609014.

[15] C. Fefferman and C. R. Graham, Conformal invariants, *Astérisque*, no. S131 (1985), 95–116. MR 0837196.

[16] C. Fefferman and C. R. Graham, Q-curvature and Poincaré metrics, *Math. Res. Lett.* **9** (2002), no. 2-3, 139–151. MR 1909634.

[17] C. Fefferman and C. R. Graham, *The Ambient Metric*, Annals of Mathematics Studies, 178, Princeton University Press, Princeton, NJ, 2012. MR 2858236.

[18] C. R. Graham, Volume and area renormalizations for conformally compact Einstein metrics, *Rend. Circ. Mat. Palermo (2)* Suppl. No. 63 (2000), 31–42. MR 1758076.

[19] C. R. Graham and K. Hirachi, The ambient obstruction tensor and Q-curvature, in *AdS/CFT Correspondence: Einstein Metrics and Their Conformal Boundaries*, 59–71, IRMA Lect. Math. Theor. Phys., 8, Eur. Math. Soc., Zürich, 2005. MR 2160867.

[20] C. R. Graham, R. Jenne, L. J. Mason and G. A. J. Sparling, Conformally invariant powers of the Laplacian. I. Existence, *J. London Math. Soc. (2)* **46** (1992), no. 3, 557–565. MR 1190438.

[21] C. R. Graham and J. M. Lee, Einstein metrics with prescribed conformal infinity on the ball, *Adv. Math.* **87** (1991), no. 2, 186–225. MR 1112625.

[22] C. R. Graham and M. Zworski, Scattering matrix in conformal geometry, *Invent. Math.* **152** (2003), no. 1, 89–118. MR 1965361.

[23] M. J. Gursky and Q. Han, Non-existence of Poincaré–Einstein manifolds with prescribed conformal infinity, *Geom. Funct. Anal.* **27** (2017), no. 4, 863–879. MR 3678503.

[24] M. J. Gursky, Q. Han and S. Stolz, An invariant related to the existence of conformally compact Einstein fillings, *Trans. Amer. Math. Soc.* **374** (2021), no. 6, 4185–4205. MR 4251226.

[25] M. J. Gursky and G. Székelyhidi, A local existence result for Poincaré–Einstein metrics, *Adv. Math.* **361** (2020), 106912, 50 pp. MR 4041200.

[26] S. W. Hawking and D. N. Page, Thermodynamics of black holes in anti-de Sitter space, *Comm. Math. Phys.* **87** (1982/83), no. 4, 577–588. MR 0691045.

[27] D. W. Helliwell, Boundary regularity for conformally compact Einstein metrics in even dimensions, *Comm. Partial Differential Equations* **33** (2008), no. 4-6, 842–880. MR 2424380.

[28] J. M. Lee, The spectrum of an asymptotically hyperbolic Einstein manifold, *Comm. Anal. Geom.* **3** (1995), no. 1-2, 253–271. MR 1362652.

[29] J. M. Lee, Fredholm operators and Einstein metrics on conformally compact manifolds, *Mem. Amer. Math. Soc.* **183** (2006), no. 864, vi+83 pp. MR 2252687.

[30] G. Li, J. Qing and Y. Shi, Gap phenomena and curvature estimates for conformally compact Einstein manifolds, *Trans. Amer. Math. Soc.* **369** (2017), no. 6, 4385–4413. MR 3624414.

[31] J. Maldacena, The large N limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2** (1998), no. 2, 231–252. MR 1633016.

[32] J. M. Maldacena, TASI 2003 Lectures on AdS/CFT, https://arxiv.org/abs/hep-th/0309246, 2003.

[33] J. Maldacena, Einstein gravity from conformal gravity, https://arxiv.org/abs/1105.5632 [hep-th], 2011.

[34] S. M. Paneitz, A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds (summary), *SIGMA Symmetry Integrability Geom. Methods Appl.* **4** (2008), Paper 036, 3 pp. MR 2393291.

[35] J. Qing, On the rigidity for conformally compact Einstein manifolds, *Int. Math. Res. Not.* **2003**, no. 21, 1141–1153. MR 1962123.

[36] E. Witten, Anti de Sitter space and holography, *Adv. Theor. Math. Phys.* **2** (1998), no. 2, 253–291. MR 1633012.

[37] P. Yang, S.-Yu. A. Chang and J. Qing, On the renormalized volumes for conformally compact Einstein manifolds, *J. Math. Sci. (N.Y.)* **149** (2008), no. 6, 1755–1769; MR 2336463.

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