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ON HYPONORMALITY AND A COMMUTING PROPERTY OF TOEPLITZ OPERATORS

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ABSTRACT. In this work we give sufficient conditions for hyponormality of Toeplitz operators on a weighted Bergman space when the analytic part of the symbol is a monomial and the conjugate part is a polynomial. We also extend a known commuting property of Toeplitz operators with a harmonic symbol on the Bergman space to weighted Bergman spaces.

1. Introduction

Let D denote the unit disk of radius in the complex plane, $d\nu_\alpha(z) = \frac{\alpha+1}{\pi}(1 - |z|^2)^\alpha dA(u)$, where $dA(z)$ is the Lebesgue measure on D and $\alpha > -1$. Denote by $L^2(D, d\nu_\alpha)$ the Hilbert space of complex valued functions on D that are square integrable with respect to ν_α. We write $\|f\|^2 = \int_D |f(z)|^2 \, d\nu_\alpha(z)$. When f is analytic on D, we have

$$f(u) = \sum_0^\infty c_m u^m, \quad \|f\|^2 = \sum_0^\infty \frac{m!\Gamma(\alpha+1)}{\Gamma(m+\alpha+2)} |c_m|^2.$$

Denote by $B^2_{a,\alpha}$ the space of analytic functions on D such that $\|f\|^2 < \infty$. It is known that $B^2_{a,\alpha}$ is a Hilbert space [7, 13] and an orthonormal basis is given by $e_m(z) = \frac{\sqrt{\Gamma(m+\alpha+2)}}{\sqrt{m!\Gamma(\alpha+1)}} z^m$. The Toeplitz operator with symbol f on $B^2_{a,\alpha}$ is defined by $T_f(k) = P(fk)$, where f is bounded and measurable on D, k is in $B^2_{a,\alpha}$ and P is the orthogonal projection of $L^2(D, d\nu_\alpha(z))$ onto $B^2_{a,\alpha}$. Hankel operators are defined by $H_f(k) = (I - P)(fk)$, f and k as before. Recall that a bounded operator A on a Hilbert space is hyponormal if $A^*A - AA^*$ is a positive operator. Hyponormality on the Hardy space was studied by C. Cowen in [3, 4]. Hyponormality of Toeplitz operators on the Bergman space of the unit disk ($\alpha = 0$) was first considered in [10]. An improvement of the necessary condition therein is due to P. Ahern and Ž. Čučković [1]. A new necessary condition, due to Ž. Čučković and R. Curto

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in a special case is found in [5]. Sufficient conditions when the analytic part is a monomial are given in [12]. Most results on hyponormality on weighted Bergman spaces treat very special cases of the symbol. We cite for example [9] and [8]. Recent results on hyponormality on weighted Bergman spaces with a general harmonic symbol can be found in [11]. Some results on hyponormality of Toeplitz operators with non-harmonic symbols are due to M. Fleeman and C. Liaw [6]. In this work we first give sufficient conditions for the hyponormality of Toeplitz operators with a symbol of the form $f+\overline{g}$, where f is a monomial and g is a polynomial and $\alpha = p$. In the second part we give a generalization of a commuting property of Toeplitz operators with a harmonic symbol on the Bergman space, due to S. Axler and Ž. Čučković [2], to weighted Bergman spaces.

2. Some general results

We assume f, g are in $L^\infty(D)$. Then we have:

(1) $T_{f+g} = T_f + T_g$;
(2) $T_f^* = T_{\overline{f}}$;
(3) $T_{\overline{f}} T_g = T_{\overline{f}g}$ if f or g are analytic on D.

The use of these properties leads to describing hyponormality in more than one form. These are known properties on the unweighted Bergman space [9] and hold also for weighted Bergman spaces.

Proposition 2.1. *Let f, g be bounded and analytic on D. Then the following are equivalent:*

(i) *$T_{f+\overline{g}}$ is hyponormal.*
(ii) $H_{\overline{g}}^* H_{\overline{g}} \leq H_{\overline{f}}^* H_{\overline{f}}$.
(iii) $\|(I-P)(\overline{g}k)\| \leq \|(I-P)(\overline{f}k)\|$ *for any k in $B^2_{a,\alpha}$.*
(iv) $\|\overline{g}k\|^2 - \|P(\overline{g}k)\|^2 \leq \|\overline{f}k\|^2 - \|P(\overline{f}k)\|^2$ *for any k in $B^2_{a,\alpha}$.*
(v) *$H_{\overline{g}} = KH_{\overline{f}}$, where K is of norm less than or equal to one.*

We also need the following lemmas.

Lemma 2.2. *For s and t integers, we have $P(\overline{z}^t z^s) = \frac{s!\Gamma(s-t+\alpha+2)}{\Gamma(s+\alpha+2)(s-t)!} z^{s-t}$ if $s \geq t$ and $P(\overline{z}^t z^s) = 0$ if $s < t$.*

Lemma 2.3. *If $\alpha = p$ is an integer, then the matrix of $H_{\overline{z^m}}^* H_{\overline{z^m}}$ with respect to the orthonormal basis $\{e_m\}_{m=0}^\infty$ is given by*

$$d_i = \frac{(m+i)!(i+p+1)!}{i!(m+i+p+1)!} \quad \text{if } i < m$$

and

$$d_i = \frac{(m+i)!(i+p+1)!}{i!(m+i+p+1)!} - \frac{i!(i-m+p+1)!}{(i-m)!(i+p+1)!} \quad \text{if } i \geq m.$$

For the sake of simplification set $Q_r = (r+1)(r+2)\dots(r+p+1) = \frac{\Gamma(r+p+2)}{\Gamma(r+1)}$ for any nonnegative integer r. We have $d_i = \frac{Q_i}{Q_{m+i}}$ if $i < m$ and $d_i = \frac{Q_i^2 - Q_{m+i}Q_{i-m}}{Q_i Q_{m+i}}$ if $i \geq m$. We then have the following results.

3. The sufficient condition

Proposition 3.1. *Let n and m be integers with $n > m \geq 1$. Then there exists N_m such that if $n \geq N_m$, then $T_{z^m+\lambda\overline{z^n}}$ is hyponormal on $B^2_{a,\alpha}$ if and only if*

$$|\lambda| \leq \inf\left\{\sqrt{\frac{Q_{n+i}}{Q_{m+i}}\frac{Q_i^2 - Q_{m+i}Q_{i-m}}{Q_i^2 - Q_{n+i}Q_{i-n}}},\ i \geq n\right\}.$$

Proof. Hyponormality is equivalent to $|\lambda|^2 H^*_{\overline{z^n}} H_{\overline{z^n}} \leq H^*_{\overline{z^m}} H_{\overline{z^m}}$, which is equivalent to the three inequalities

$$|\lambda|^2 \frac{Q_i}{Q_{n+i}} \leq \frac{Q_i}{Q_{m+i}} \qquad \text{if } i < m, \tag{3.1}$$

$$|\lambda|^2 \frac{Q_i}{Q_{n+i}} \leq \frac{Q_i^2 - Q_{m+i}Q_{i-m}}{Q_{i+m}Q_i} \qquad \text{if } m \leq i < n, \tag{3.2}$$

$$|\lambda|^2 \frac{Q_i^2 - Q_{n+i}Q_{i-n}}{Q_iQ_{n+i}} \leq \frac{Q_i^2 - Q_{m+i}Q_{i-m}}{Q_iQ_{m+i}} \qquad \text{if } n \leq i. \tag{3.3}$$

Inequality (3.1) is equivalent to

$$|\lambda| \leq \min\left\{\sqrt{\frac{Q_{n+i}}{Q_{m+i}}},\ i < m\right\} = \Delta^1_{m,n}.$$

Inequality (3.2) is equivalent to

$$|\lambda| \leq \min\left\{\sqrt{\frac{Q_{n+i}}{Q_{m+i}}\frac{(Q_i^2 - Q_{m+i}Q_{i-m})}{Q_i^2}},\ m \leq i < n\right\} = \Delta^2_{m,n}.$$

Inequality (3.3) is equivalent to

$$|\lambda| \leq \inf\left\{\sqrt{\frac{Q_{n+i}}{Q_{m+i}}\frac{Q_i^2 - Q_{m+i}Q_{i-m}}{Q_i^2 - Q_{n+i}Q_{i-n}}},\ i \geq n\right\} = \Delta^3_{m,n}.$$

For the first inequality (3.1), if we set

$$R(i) = \frac{Q_{n+i}}{Q_{m+i}} = \frac{(n+i+1)(n+i+2)\ldots(n+i+p+1)}{(m+i+1)(m+i+2)\ldots(m+i+p+1)},$$

using logarithmic differentiation we can see that $R(i)$ decreases with i, so (3.1) is equivalent to

$$\Delta^1_{m,n} = \sqrt{\frac{Q_{n+m-1}}{Q_{2m-1}}}.$$

For the second inequality (3.2), since $\frac{Q_{m+i}Q_{i-m}}{Q_i^2}$ increases with i, we get that $\frac{Q_{n+i}}{Q_{m+i}}\frac{(Q_i^2-Q_{m+i}Q_{i-m})}{Q_i^2}$ decreases with i and that

$$\Delta^2_{m,n} = \sqrt{\frac{Q_{2n-1}}{Q_{m+n-1}}\frac{(Q_{n-1}^2 - Q_{m+n-1}Q_{n-1-m})}{Q_{n-1}^2}}.$$

It is clear that $\Delta^2_{m,n} \leq \Delta^1_{m,n}$. We also have

$$\Delta^3_{m,n} \leq \lim_{i\to\infty}\sqrt{\frac{Q_{n+i}}{Q_{m+i}}\frac{Q_i^2 - Q_{m+i}Q_{i-m}}{Q_i^2 - Q_{n+i}Q_{i-n}}} = \frac{m}{n}.$$

Set $R_1(i) = \frac{Q_{m+i}Q_{i-m}}{Q_{n+i}Q_{i-n}}$. Using logarithmic differentiation we verify that $R_1(i)$ decreases with i. Since $\lim\limits_{i\to\infty}\dfrac{Q_{m+i}Q_{i-m}}{Q_{n+i}Q_{i-n}} = 1$, we get $Q_{m+i}Q_{i-m} \geq Q_{n+i}Q_{i-n}$ and $\frac{Q_i^2 - Q_{m+i}Q_{i-m}}{Q_i^2 - Q_{n+i}Q_{i-n}} \leq 1$. Thus $\Delta^3_{m,n} \leq \Delta^1_{m,n}$. Let us verify that $\Delta^3_{m,n} \leq \Delta^2_{m,n}$ for large n. It is enough to verify that the following inequality holds for n large:

$$\frac{m^2}{n^2} \leq \frac{Q_{2n-1}}{Q_{m+n-1}}\frac{(Q^2_{n-1} - Q_{m+n-1}Q_{n-1-m})}{Q^2_{n-1}}.$$

Setting

$$S_{m,n} = \frac{n^2 Q_{2n-1}(Q^2_{n-1} - Q_{m+n-1}Q_{n-1-m})}{m^2 Q_{m+n-1}Q^2_{n-1}},$$

a computation shows that

$$\begin{aligned}\frac{(Q^2_{n-1} - Q_{m+n-1}Q_{n-1-m})}{Q^2_{n-1}} &= 1 - \frac{(n^2-m^2)\dots((n+p)^2-m^2)}{n^2\dots(n+p)^2}\\ &= m^2A^1_n - m^4A^2_n + \dots + (-1)^p m^{2(p+1)}A^{p+1}_n,\end{aligned}$$

where

$$A^1_n = \frac{1}{n^2} + \dots + \frac{1}{(n+p)^2}, \quad A^2_n = \sum_{0\leq s\neq t}^{p}\frac{1}{(n+s)^2(n+t)^2}, \dots,$$

$$A^{p+1}_n = \frac{1}{n^2\dots(n+p)^2}.$$

We have

$$m^2A^1_n - m^4A^2_n \geq \frac{m^2}{n^2} + \frac{pm^2}{(n+p)^2} - \frac{p(p+1)m^4}{2n^2(n+1)^2}.$$

Clearly, there exists N^1_m such that, for $n \geq N^1_m$, $m^2A^1_n - m^4A^2_n \geq \frac{m^2}{n^2}$. A similar argument shows that, for k odd, there exists N^k_m such that for $n \geq N^k_m$, $m^{2k}A^k_n - m^{2(k+1)}A^{k+1}_n \geq 0$ for $3 \leq k \leq p$ ($3 \leq k \leq p-1$ if p is even). If we put $\max\{N^k_m,\ k \text{ odd},\ 1 \leq k \leq p\} = N_m$, and noticing that $\frac{Q_{2n-1}}{Q_{m+n-1}} \geq 1$, we get, for $n \geq N_m$,

$$\frac{n^2Q_{2n-1}(Q^2_{n-1} - Q_{m+n-1}Q_{n-1-m})}{m^2Q_{m+n-1}Q^2_{n-1}} \geq 1.$$

For $n \geq N_m$, we get that $T_{u^m+\lambda\overline{u^n}}$ is hyponormal if and only if $|\lambda| \leq \Delta^3_{m,n}$. □

Note that when $m = n$, hyponormality is equivalent to $|\lambda| \leq 1$. We now consider the case $m > n$.

Proposition 3.2. *Let n and m be integers with $m > n \geq 1$. Then $T_{z^m+\lambda\overline{z^n}}$ is hyponormal on $B^2_{a,2}$ if and only if $|\lambda| \leq \sqrt{\frac{Q_{2n-1}}{Q_{m+n-1}}}$.*

Proof. Hyponormality is equivalent to $|\lambda|^2 H^*_{\overline{z^n}} H_{\overline{z^n}} \leq H^*_{\overline{z^m}} H_{\overline{z^m}}$, which is equivalent to the three inequalities

$$|\lambda|^2 \frac{Q_i}{Q_{n+i}} \leq \frac{Q_i}{Q_{m+i}} \qquad \text{if } i < n, \tag{3.4}$$

$$|\lambda|^2 \frac{Q_i^2 - Q_{n+i}Q_{i-n}}{Q_iQ_{n+i}} \leq \frac{Q_i}{Q_{m+i}} \qquad \text{if } n \leq i \leq m-1, \tag{3.5}$$

$$|\lambda|^2 \frac{Q_i^2 - Q_{n+i}Q_{i-n}}{Q_iQ_{n+i}} \leq \frac{Q_i^2 - Q_{m+i}Q_{i-m}}{Q_iQ_{m+i}} \qquad \text{if } m \leq i. \tag{3.6}$$

Inequality (3.4) is equivalent to

$$|\lambda| \leq \min\left\{\sqrt{\frac{Q_{n+i}}{Q_{m+i}}},\ i < n\right\}.$$

The ratio $\frac{Q_{n+i}}{Q_{m+i}}$ increases with i, so the inequality (3.4) is equivalent to

$$|\lambda| \leq \sqrt{\frac{Q_{2n-1}}{Q_{m+n-1}}} = \Gamma^1_{m,n}.$$

Inequality (3.5) is equivalent to

$$|\lambda| \leq \min\left\{\sqrt{\frac{Q_{n+i}}{Q_{m+i}} \frac{Q_i^2}{(Q_i^2 - Q_{n+i}Q_{i-n})}},\ n \leq i < m\right\}.$$

Again both $\frac{Q_{n+i}}{Q_{m+i}}$ and $\frac{Q_{i+n}Q_{i-n}}{Q_i^2}$ increase with i, so we get that $\frac{Q_{n+i}}{Q_{m+i}} \frac{Q_i^2}{(Q_i^2 - Q_{n+i}Q_{i-n})}$ increases with i, which leads to

$$|\lambda| \leq \sqrt{\frac{Q_{2n}}{Q_{m+n}} \frac{Q_n^2}{(Q_n^2 - (p+1)!Q_{2n})}} = \Gamma^2_{m,n}.$$

Since $\frac{Q_{n+i}}{Q_{m+i}}$ increases with i and $\frac{Q_i^2}{Q_i^2 - Q_{n+i}Q_{i-n}} \geq 1$, we have $\Gamma^1_{m,n} \leq \Gamma^2_{m,n}$. Inequality (3.6) is equivalent to

$$|\gamma| \leq \min\left\{\sqrt{\frac{Q_{n+i}}{Q_{m+i}} \frac{Q_i^2 - Q_{m+i}Q_{i-m}}{Q_i^2 - Q_{n+i}Q_{i-n}}},\ i \geq m\right\} = \Gamma^3_{m,n}.$$

Using logarithmic differentiation we can verify that $\frac{Q_{m+i}Q_{i-m}}{Q_{n+i}Q_{i-n}}$ increases with i.

Since $\lim_{i\to\infty} \frac{Q_{m+i}Q_{i-m}}{Q_{n+i}Q_{i-n}} = 1$, we deduce that $Q_{m+i}Q_{i-m} \leq Q_{n+i}Q_{i-n}$ and $Q_i^2 - Q_{m+i}Q_{i-m} \geq Q_i^2 - Q_{n+i}Q_{i-n}$. From the fact that $\frac{Q_{n+i}}{Q_{m+i}}$ increases with i, it follows that $\Gamma^3_{m,n} \geq \Gamma^1_{m,n}$. This proves the result. □

Note that the result holds also when $m = n$.

Denote by U_1 the unit ball of $B^{2\perp}_{a,\alpha}$, the orthogonal of $B^2_{a,\alpha}$ in $L^2(D, d\nu_\alpha(z))$.

Definition 3.3. For $f \in B^2_{a,\alpha}$, define the set Ω_f by

$$\Omega_f = \left\{ g \in B^2_{a,\alpha} : \sup_{l \in U_1} |\langle \overline{g}, \overline{k}l \rangle| \leq \sup_{l \in U_1} |\langle \overline{f}, \overline{k}l \rangle| \text{ for any } k \in H^\infty \right\}.$$

We see, from the density of H^∞ in $B^2_{a,\alpha}$ and Proposition 3.2, that when g and f are in H^∞, $g \in \Gamma_f$ is equivalent to $T_{f+\overline{g}}$ being hyponormal. The following proposition lists some properties of Ω_f.

Proposition 3.4. *For $f \in B^2_{a,\alpha}$, the following holds:*

(1) *Ω_f is convex and balanced.*
(2) *If $g \in \Omega_f$, then $g + \lambda$ is in Ω_f for any complex number λ.*
(3) *$f \in \Omega_f$.*
(4) *Ω_f is closed in the weak topology of $L^2(D, d\nu_\alpha(w))$.*

The proof of these properties is similar to the case $\alpha = 0$ in [3] and is therefore omitted. Using this proposition we get our first main result when $\alpha = p$.

Theorem 3.5. *Let $(\gamma_i)_{i \geq 1}$ be complex numbers such that $\sum_{i \geq 1} |\gamma_i| \leq 1$, and let $m \geq 1$ be an integer. Then $T_{z^m + \sum_{1 \leq n \leq m} \gamma_n \Gamma^1_{m,n} \overline{z^n} + \sum_{N_m \leq n} \gamma_n \Delta^3_{m,n} \overline{z^n}}$ is hyponormal.*

4. The commuting property

We continue to use the notations of the previous sections: D denotes the unit disk in the complex plane and $\alpha > -1$ a real number. $B^2_{a,\alpha}$ is the Hilbert space of analytic functions f on D such that $\|f\|^2 = \int_D |f(z)|^2 \, d\nu_\alpha(z) < \infty$, where $d\nu_\alpha(z) = \frac{(\alpha+1)}{\pi}(1 - |z|^2)^\alpha dA(z)$ and $dA(z) = r dr d\theta$ is the Lebesgue measure on D. For h bounded measurable on D, the Toeplitz operator T_h is defined on $B^2_{a,\alpha}$ by $T_h(f) = P(hf)$, where P is the orthogonal projection of $L^2(D, d\nu_\alpha)$ on $B^2_{a,\alpha}$. When $\alpha = 0$, S. Axler and Ž. Čučković [2] showed the following theorem:

Theorem 4.1. *Suppose g and h are bounded harmonic functions on D. Then $T_g T_h = T_h T_g$ if and only if one of the following holds:*

(i) *g and h are both analytic on D.*
(ii) *$\overline{g}$ and $\overline{h}$ are both analytic on D.*
(iii) *There exist constants a and b, not both zero, such that $ag + bh$ is constant on D.*

In what follows we will show that the above result holds on $B^2_{a,\alpha}$ for any $\alpha > -1$.

4.1. **The second main result.** We begin by recalling some definitions from [2].

Definition 4.2. A function $u \in C(D) \cap L^1(D, d\nu_\alpha)$ is said to have *the area version of the invariant mean value property* if $\int_D u \circ \varphi \, d\nu_\alpha = u(\varphi(0))$ for any $\varphi \in \mathrm{Aut}(D)$.

Definition 4.3. If $u \in C(D)$, the *radialization* of u is given by $R(u)(w) = \frac{1}{2\pi}\int\limits_0^{2\pi} u(we^{i\theta})\,d\theta$.

We can state the result that is used in the generalization as follows.

Lemma 4.4. *Suppose $u \in C(D) \cap L^1(D, d\nu_\alpha)$. Then u is harmonic on D if and only if $\int\limits_D u \circ \varphi\, d\nu_\alpha = u(\varphi(0))$ and $R(u \circ \varphi) \in C(\overline{D})$ for all $\varphi \in \mathrm{Aut}(D)$.*

Proof. If u is harmonic, then $u \circ \varphi$ is also harmonic, and it is easy to see that $\int\limits_D u \circ \varphi\, d\nu_\alpha = u(\varphi(0))$. Since $R(u \circ \varphi)$ is constant by the mean value property, we have that $R(u \circ \varphi) \in C(\overline{D})$. Assume now that $\int\limits_D u \circ \varphi\, d\nu_\alpha = u(\varphi(0))$ and $R(u \circ \varphi) \in C(\overline{D})$. Let ψ be an automorphism of the disk. We have

$$\int\limits_D R(u \circ \varphi)(\psi(w))\, d\nu_\alpha(w) = \int\limits_D \int_0^{2\pi} u\left(\varphi(\psi(w)e^{i\theta})\right) \frac{d\theta}{2\pi}\, d\nu_\alpha(w).$$

Set $\varphi\left(\psi(w)e^{i\theta}\right) = f_\theta(w)$ as in [2]. Then f_θ is an automorphism of the disk, and we can easily verify (see [2]) that $|(f_\theta^{-1})'(z)| \le C$ for all $z \in D$ and $\theta \in [0, 2\pi]$. If we write $f_\theta(z) = \zeta \frac{\lambda - z}{1 - \overline{\lambda} z}$ with $|\zeta| = 1$ and $|\lambda| < 1$, then we have $1 - |f_\theta^{-1}(z)|^2 = \frac{(1-|z|^2)(1-|\lambda|^2)}{|1-\overline{\gamma} z|^2}$, where $\gamma = e^{i\mu}\lambda$ for some real μ. Thus, noting that $1 - |f_\theta^{-1}(z)|^2 \le C_1(1 - |z|^2)$ and changing variables, we get

$$\begin{aligned}
&\int_0^{2\pi} \int_D \left|u\left(\varphi(\psi(w)e^{i\theta})\right)\right| d\nu_\alpha(w) \frac{d\theta}{2\pi} \\
&\qquad = \frac{\alpha+1}{\pi} \int_0^{2\pi} \int_D |u(z)| |(f_\theta^{-1})'(z)|^2 \left(1 - |f_\theta^{-1}(z)|^2\right)^\alpha dA(z) \frac{d\theta}{2\pi} \\
&\qquad \le C_2 \int_D |u(z)|\, d\nu_\alpha(z).
\end{aligned}$$

So Fubini's theorem leads to

$$\begin{aligned}
\int\limits_D \int_0^{2\pi} u\left(\varphi(\psi(w)e^{i\theta})\right) \frac{d\theta}{2\pi}\, d\nu_\alpha(w) &= \int_0^{2\pi} \int\limits_D u\left(\varphi(\psi(w)e^{i\theta})\right) d\nu_\alpha(w) \\
&= \int_0^{2\pi} \int_D u \circ f_\theta(w)\, d\nu_\alpha(w) \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} u\left(f_\theta(0)\right) \frac{d\theta}{2\pi},
\end{aligned}$$

i.e.,

$$\int\limits_D R(u \circ \varphi)(\psi(w))\, d\mu_\alpha(w) = \int_0^{2\pi} u\left(\varphi(\psi(0)e^{i\theta})\right) \frac{d\theta}{2\pi} = R(u \circ \varphi)(\psi(0)).$$

Thus $R(u \circ \varphi)$ is continuous and has the area version of the invariant mean value theorem. By [2] it is harmonic on D. Since $R(u \circ \varphi)$ is radial, we deduce that it is constant and equal to $R(u \circ \varphi)(0)$. So $\int_0^{2\pi} u \circ \varphi(re^{i\theta}) \frac{d\theta}{2\pi} = u \circ \varphi(0)$. This holds for any φ automorphism of the unit disk. As in [2], we deduce that u is harmonic on D. □

For φ an automorphism of the unit disk, define the operator on $B^2_{a,\alpha}$ given by $V_\varphi f = f \circ \varphi . (\varphi')^{1+\frac{\alpha}{2}}$.

Lemma 4.5. *The operator V_φ is unitary.*

Proof.

$$(\alpha+1) \int_D |f \circ \varphi(w)|^2 |\varphi'(w)|^2 |\varphi'(w)|^\alpha \left(1-|w|^2\right)^\alpha \frac{dA(w)}{\pi}$$
$$= (\alpha+1) \int_D |f(z)|^2 |\varphi'\left(\varphi^{-1}(z)\right)|^\alpha \left(1-|\varphi^{-1}(z)|^2\right)^\alpha \frac{dA(z)}{\pi}.$$

Since $\left(1-|\varphi^{-1}(z)|^2\right)^\alpha = \left(1-|z|^2\right)^\alpha |(\varphi^{-1})'(z)|^\alpha$ and $(\varphi^{-1})'(z) = \frac{1}{\varphi'(\varphi^{-1}(z))}$, the result follows. □

Since $V_\varphi = T_{(\varphi')^{1+\alpha/2}} C_\varphi$ and $V_\varphi^* = V_\varphi^{-1}$, the adjoint is given by

$$V_\varphi^* f = \left(\varphi' \circ \varphi^{-1}\right)^{-1-\alpha/2} f \circ \varphi^{-1}.$$

The proof of the following lemma is straightforward and is therefore omitted.

Lemma 4.6. *For φ an automorphism of the unit disk and h bounded measurable on D, we have $V_\varphi T_h V_\varphi^* = T_{h\circ\varphi}$.*

We can now state the main result, which is a generalization of Theorem 1 in [2]. The proof is similar and thus omitted.

Theorem 4.7. *Let g and h be bounded and harmonic on the unit disk D. Then $T_g T_h = T_h T_g$ on $B^2_{a,\alpha}$ if and only if one of the following holds:*

(i) *g and h are analytic on D.*
(ii) *$\overline{g}$ and $\overline{h}$ are analytic on D.*
(iii) *There exist constants a and b in $\mathbb{C}$, not both zero, such that $ag + bh$ is constant on D.*

As in [2], we obtain a characterization of normality of Toeplitz operators, with a harmonic symbol, on $B^2_{a,\alpha}$.

Corollary 4.8. *Let f be bounded harmonic on D. Then T_f is normal if and only if $f(D)$ lies on some line in $\mathbb{C}$.*

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