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PRIMITIVE DECOMPOSITIONS OF DOLBEAULT HARMONIC FORMS ON COMPACT ALMOST-KÄHLER MANIFOLDS

ANDREA CATTANEO, NICOLETTA TARDINI, AND ADRIANO TOMASSINI

Abstract. Let (X, J, g, ω) be a compact $2n$-dimensional almost-Kähler manifold. We prove primitive decompositions of ∂-, $\bar{\partial}$-harmonic forms on X in bidegree $(1, 1)$ and $(n-1, n-1)$ (such bidegrees appear to be optimal). We provide examples showing that in bidegree $(1, 1)$ the ∂- and $\bar{\partial}$-decompositions differ.

1. Introduction

In complex geometry, the Dolbeault cohomology plays a fundamental role in the study of complex manifolds, and a classical way to compute it on compact complex manifolds is through the use of the associated spaces of harmonic forms. More precisely, if X is a complex manifold, then the exterior derivative d splits as $\partial + \bar{\partial}$, and such operators satisfy $\bar{\partial}^2 = \partial^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. Hence, one can define the Dolbeault cohomology and its conjugate as

$$H^{\bullet,\bullet}_{\bar{\partial}}(X) := \frac{\operatorname{Ker}\bar{\partial}}{\operatorname{Im}\bar{\partial}}, \qquad H^{\bullet,\bullet}_{\partial}(X) := \frac{\operatorname{Ker}\partial}{\operatorname{Im}\partial}.$$

If X is compact and we fix an Hermitian metric, then it turns out that these spaces are isomorphic to the kernel of two suitable elliptic operators, $\Delta_{\bar{\partial}}$ and Δ_{∂}, respectively. More precisely, denoting with $\mathscr{H}^{\bullet,\bullet}_{\bar{\partial}}(X)$ and $\mathscr{H}^{\bullet,\bullet}_{\partial}(X)$ the spaces of harmonic forms, they have a cohomological meaning, namely

$$H^{\bullet,\bullet}_{\bar{\partial}}(X) \simeq \mathscr{H}^{\bullet,\bullet}_{\bar{\partial}}(X), \qquad H^{\bullet,\bullet}_{\partial}(X) \simeq \mathscr{H}^{\bullet,\bullet}_{\partial}(X),$$

and in particular their dimensions are holomorphic invariants.

Moreover, if the Hermitian metric is Kähler, then by the Kähler identities it turns out that $\Delta_{\bar{\partial}} = \Delta_{\partial}$ and in particular

$$\mathscr{H}^{\bullet,\bullet}_{\bar{\partial}}(X) = \mathscr{H}^{\bullet,\bullet}_{\partial}(X),$$

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therefore giving isomorphisms for the respective cohomologies, namely

$$H^{\bullet,\bullet}_{\bar\partial}(X) \simeq H^{\bullet,\bullet}_{\partial}(X).$$

The integrability assumption on the complex structure is crucial in the proof of all these results.

Furthermore, a remarkable feature of Kähler geometry is that the primitive decomposition of differential forms passes to cohomology and leads to a primitive decomposition of de Rham cohomology (see, e.g., [18]). Kähler geometry is at the crossroad of complex and symplectic geometry. From the symplectic point of view we recall that Tseng and Yau [17] introduced natural cohomologies on (compact) symplectic manifolds, involving the symplectic co-differential and the exterior derivative, proving a primitive decomposition for them.

If J is a non-integrable almost-complex structure on a $2n$-dimensional smooth manifold X, then the exterior derivative splits as $\mu + \partial + \bar\partial + \bar\mu$, and in particular $\bar\partial^2 \neq 0$. Hence, the standard Dolbeault cohomology and its conjugate are not well-defined. Recently, Cirici and Wilson [6] gave a definition for the Dolbeault cohomology in the non-integrable setting considering also the operator $\bar\mu$ together with $\bar\partial$. Such cohomology groups might be infinite-dimensional on compact almost-complex manifolds as shown in [7].

On the other hand, fixing an almost-Hermitian metric g on (X, J) one can develop a Hodge theory for harmonic forms on (X, J, g) without a cohomological counterpart. More precisely, setting, similarly to the integrable case,

$$\Delta_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial, \qquad \Delta_{\partial} := \partial\partial^* + \partial^*\partial,$$

it turns out that they are elliptic selfadjoint differential operators. Therefore, if X is compact, their kernels, denoted again with $\mathscr{H}^{\bullet,\bullet}_{\bar\partial}(X)$ and $\mathscr{H}^{\bullet,\bullet}_{\partial}(X)$, are finite dimensional complex vector spaces. Holt and Zhang [11] answered to a question of Kodaira and Spencer [9] showing that, contrarily to the complex case, the dimensions of the spaces of $\bar\partial$-harmonic $(0, 1)$-forms on a 4-dimensional manifold depend on the metric. Indeed they construct on the Kodaira–Thurston manifold an almost-complex structure that, with respect to different almost-Hermitian metrics, has varying $\dim \mathscr{H}^{0,1}_{\bar\partial}$. With different techniques, in [16] it was shown that also the dimension of the space of $\bar\partial$-harmonic $(1, 1)$-forms depends on the metric on 4-dimensional manifolds (for other results in this direction, see [13] and [10]).

We note that performing explicit computations of $\bar\partial$-harmonic forms is a difficult task and not much is known in higher dimensions (see [15], [3], [4] for some detailed computations).

In the present paper we study the validity of primitive decompositions on compact almost-Kähler manifolds in any dimension. More precisely, in Propositions 3.1 and 3.2, Theorem 3.4 and Corollary 3.5 we prove, on compact almost-Kähler $2n$-dimensional manifolds, primitive decompositions for $\bar\partial$- and ∂-harmonic forms in bidegrees $(p, 0)$, $(0, q)$, $(1, 1)$, $(n, n-p)$, $(n-q, n)$ and $(n-1, n-1)$, with $p, q \leq n$. One cannot hope to have such decompositions for any bidegree as shown in Example 5.3. For similar results in the case of Bott–Chern harmonic forms, we refer the reader to [12].

We notice that, even though the metric is almost-Kähler, the decompositions of $\bar\partial$- and ∂-harmonic forms might differ. Indeed, in Section 4 we show explicitly that, differently from the Kähler case, one can have $\Delta_{\bar\partial} \neq \Delta_{\partial}$, and also

$$\mathscr{H}^{1,1}_{\bar\partial}(X) \neq \mathscr{H}^{1,1}_{\partial}(X).$$

We observe that a key ingredient in the proof of the results in [16] (see also [11]) is indeed the primitive decomposition of $\bar\partial$-harmonic $(1,1)$-forms on 4-dimensional manifolds. In fact, in this dimension in Proposition 4.1 we prove the general equality $\mathscr{H}^{1,1}_{\bar\partial}(X) = \mathscr{H}^{1,1}_{\partial}(X)$.

All the examples we present are nilmanifolds, of dimensions 6 and 8, endowed with possibly non-left-invariant almost-Kähler structures.

We recall that if one wants to mimic and recover all the Kähler identities, the proper operator to consider is $\bar\delta := \bar\partial + \mu$ (see [5], [14]). However, considering just the operator $\bar\partial$ on almost-Kähler manifolds we are able to see how genuinely almost-Kähler manifolds differ from Kähler ones. More precisely, the study of the kernel of $\Delta_{\bar\partial}$ illuminates the purely almost-complex properties.

2. Preliminaries

In this section we recall some basic facts about almost-complex and almost-Hermitian manifolds and fix some notations. Let X be a smooth manifold of dimension $2n$ and let J be an almost-complex structure on X, namely a $(1,1)$-tensor on X such that $J^2 = -\operatorname{id}$. Then J induces on the space of forms $A^\bullet(X)$ a natural bigrading, namely

$$A^\bullet(X) = \bigoplus_{p+q=\bullet} A^{p,q}(X).$$

Accordingly, the exterior derivative d splits into four operators:

$$\begin{gathered} d : A^{p,q}(X) \to A^{p+2,q-1}(X) \oplus A^{p+1,q}(X) \oplus A^{p,q+1}(X) \oplus A^{p-1,q+2}(X), \\ d = \mu + \partial + \bar\partial + \bar\mu, \end{gathered}$$

where μ and $\bar\mu$ are differential operators that are linear over functions. In particular, they are related to the Nijenhuis tensor N_J by

$$(\mu\alpha + \bar\mu\alpha)(u,v) = \frac{1}{4}\alpha\left(N_J(u,v)\right),$$

where $\alpha \in A^1(X)$. Hence, J is integrable, that is, J induces a complex structure on X if and only if $\mu = \bar\mu = 0$.

In general, since $d^2 = 0$, one has

$$\begin{cases} \mu^2 = 0 \\ \mu\partial + \partial\mu = 0 \\ \partial^2 + \mu\bar{\partial} + \bar{\partial}\mu = 0 \\ \partial\bar{\partial} + \bar{\partial}\partial + \mu\bar{\mu} + \bar{\mu}\mu = 0 \\ \bar{\partial}^2 + \bar{\mu}\partial + \partial\bar{\mu} = 0 \\ \bar{\mu}\bar{\partial} + \bar{\partial}\bar{\mu} = 0 \\ \bar{\mu}^2 = 0. \end{cases}$$

In particular, $\bar{\partial}^2 \neq 0$, and so the Dolbeault cohomology of X

$$H^{\bullet,\bullet}_{\bar{\partial}}(X) := \frac{\operatorname{Ker}\bar{\partial}}{\operatorname{Im}\bar{\partial}}$$

is well defined if and only if J is integrable. The same holds for the operator ∂.

If g is an Hermitian metric on (X, J) with fundamental form ω and $*$ is the associated $\mathbb{C}$-linear Hodge-$*$ operator, one can consider the adjoint operators

$$d^* = - * d *, \quad \mu^* = - * \bar{\mu} *, \quad \partial^* = - * \bar{\partial} *, \quad \bar{\partial}^* = - * \partial *, \quad \bar{\mu}^* = - * \mu *,$$

and, for $D \in \left\{d, \partial, \bar{\partial}, \mu, \bar{\mu}\right\}$, one defines the associated Laplacians

$$\Delta_D := DD^* + D^*D,$$

and we will denote the kernel by

$$\mathscr{H}^{p,q}_D(X) := \operatorname{Ker}\Delta_{D_{|A^{p,q}(X)}}.$$

These spaces will be called the *spaces of D-harmonic forms*. The operators $\Delta_{\bar{\partial}}$ and Δ_{∂} are second-order, elliptic, differential operators; in particular, if X is compact, the associated spaces of harmonic forms are finite-dimensional, and their dimensions will be denoted by $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{\partial}$.

If X is compact, then we easily deduce the following relations for a (p,q)-form α:

$$\begin{cases} \Delta_{\partial}\,\alpha = 0 \iff \partial\alpha = 0,\ \bar{\partial} * \alpha = 0, \\ \Delta_{\bar{\partial}}\,\alpha = 0 \iff \bar{\partial}\alpha = 0,\ \partial * \alpha = 0, \end{cases}$$

which characterize the spaces of harmonic forms.

3. Primitive decompositions of Dolbeault harmonic forms

Let (X, J, g, ω) be a $2n$-dimensional almost-Hermitian manifold. We denote with

$$L : \Lambda^k X \to \Lambda^{k+2} X, \quad \alpha \mapsto \omega \wedge \alpha$$

the Lefschetz operator and with

$$\Lambda : \Lambda^k X \to \Lambda^{k-2} X, \quad \Lambda = - * L *$$

its dual. A k-form α_k on X, for $k \leq n$, is said to be *primitive* if $\Lambda\alpha_k = 0$, or equivalently, $L^{n-k+1}\alpha_k = 0$. Then, the following vector bundle decomposition holds (see, e.g., [18]):

$$\Lambda^k X = \bigoplus_{r\geq \max(k-n,0)} L^r(P^{k-2r}X),$$

where

$$P^s X := \ker\left(\Lambda : \Lambda^s X \to \Lambda^{s-2}X\right)$$

is the bundle of s-primitive forms. Accordingly, given any k-form α_k on X, we can write

$$\alpha_k = \sum_{r\geq \max(k-n,0)} \frac{1}{r!} L^r \beta_{k-2r}, \tag{3.1}$$

where $\beta_{k-2r} \in \Gamma(P^{k-2r}X)$, that is,

$$\Lambda\beta_{k-2r} = 0,$$

or equivalently

$$L^{n-k+2r+1}\beta_{k-2r} = 0.$$

Furthermore, the decomposition above is compatible with the bidegree decomposition on the bundle of complex k-forms $\Lambda^k_{\mathbb{C}}X$ induced by J, that is,

$$P^k_{\mathbb{C}}X = \bigoplus_{p+q=k} P^{p,q}X,$$

where

$$P^{p,q}X = P^k_{\mathbb{C}}X \cap \Lambda^{p,q}X.$$

For any given $\beta_k \in P^kX$, we have the following formula (cf. [18, p. 23, Théorème 2]):

$$*L^r\beta_k = (-1)^{\frac{k(k+1)}{2}} \frac{r!}{(n-k-r)!} L^{n-k-r} J\beta_k. \tag{3.2}$$

In what follows we will write $P^\bullet = P^\bullet X$ and so on.

We recall that by [5, Corollary 5.4] such decompositions in primitive forms pass to the spaces of d-harmonic forms whenever there exists an almost-Kähler metric. More precisely, if (X, J, ω) is a compact $2n$-dimensional almost-Kähler manifold, then, for every p, q,

$$\mathscr{H}^{p,q}_d(X) = \bigoplus_{r\geq \max(k-n,0)} L^r\left(\mathscr{H}^{p-r,q-r}_d(X) \cap P^{p-r,q-r}\right).$$

In fact, this holds also for the spaces of $\bar\delta$- and δ-harmonic forms introduced in [14], where $\bar\delta := \bar\partial + \mu$ and $\delta := \partial + \bar\mu$. Indeed, by [14, Proposition 6.2 and Theorem 6.7], one has

$$\mathscr{H}^{p,q}_d(X) = \mathscr{H}^{p,q}_{\bar\delta}(X) = \mathscr{H}^{p,q}_{\delta}(X).$$

Next, we are going to study such decompositions for $\bar\partial$-harmonic forms. First, notice that, since $(p, 0)$-forms and $(0, q)$-forms are trivially primitive, we immediately derive the following results.

Proposition 3.1. *Let (X, J, g, ω) be a compact $2n$-dimensional almost-Hermitian manifold (with $n \geq 2$). Then the following decompositions hold for every $p, q \leq n$:*

$$\mathscr{H}^{p,0}_{\bar\partial} = \mathscr{H}^{p,0}_{\bar\partial} \cap P^{p,0}, \qquad \mathscr{H}^{0,q}_{\bar\partial} = \mathscr{H}^{0,q}_{\bar\partial} \cap P^{0,q},$$
$$\mathscr{H}^{p,0}_{\partial} = \mathscr{H}^{p,0}_{\partial} \cap P^{p,0}, \qquad \mathscr{H}^{0,q}_{\partial} = \mathscr{H}^{0,q}_{\partial} \cap P^{0,q}.$$

By applying to such decompositions the Hodge-$*$ operator and formula (3.2), we obtain the following result.

Proposition 3.2. *Let (X, J, g, ω) be a compact $2n$-dimensional almost-Hermitian manifold (with $n \geq 2$). Then the following decompositions hold for every $p, q \leq n$:*

$$\mathscr{H}^{n,n-p}_{\bar\partial} = L^{n-p}\big(\mathscr{H}^{p,0}_{\partial} \cap P^{p,0}\big), \qquad \mathscr{H}^{n-q,n}_{\bar\partial} = L^{n-q}\big(\mathscr{H}^{0,q}_{\partial} \cap P^{0,q}\big),$$
$$\mathscr{H}^{n,n-p}_{\partial} = L^{n-p}\big(\mathscr{H}^{p,0}_{\bar\partial} \cap P^{p,0}\big), \qquad \mathscr{H}^{n-q,n}_{\partial} = L^{n-q}\big(\mathscr{H}^{0,q}_{\bar\partial} \cap P^{0,q}\big).$$

As a consequence, we derive the following corollary.

Corollary 3.3. *Let (X, J, g, ω) be a compact $2n$-dimensional almost-Hermitian manifold (with $n \geq 2$). Then,*

$$\mathscr{H}^{n,0}_{\bar\partial} = \mathscr{H}^{n,0}_{\partial} \qquad \textit{and} \qquad \mathscr{H}^{0,n}_{\bar\partial} = \mathscr{H}^{0,n}_{\partial}.$$

Proof. This follows taking $p = n$ and $q = n$ in Proposition 3.2. Otherwise, it can be proved directly. Indeed, let α be an $(n, 0)$-form (the case $(0, n)$ is similar); then α is primitive, and by Formula (3.2), $*\alpha = c_n\, \alpha$, with $c_n \neq 0$ a constant depending only on the dimension of X. Therefore, for bidegree reasons,

$$\alpha \in \mathscr{H}^{n,0}_{\bar\partial} \iff \bar\partial\alpha = 0 \iff \bar\partial * \alpha = 0 \iff \alpha \in \mathscr{H}^{n,0}_{\partial}. \qquad \square$$

We show now that primitive decompositions hold also in other suitable degrees as soon as we assume the existence of an almost-Kähler metric.

Theorem 3.4. *Let (X, J, g, ω) be a compact $2n$-dimensional almost-Kähler manifold (with $n \geq 2$). Then the following decomposition holds:*

$$\mathscr{H}^{1,1}_{\bar\partial} = \mathbb{C} \cdot \omega \oplus \big(\mathscr{H}^{1,1}_{\bar\partial} \cap P^{1,1}\big).$$

Proof. Let $\alpha_{1,1} \in A^{1,1}(X)$. Then the primitive decomposition (3.1) reads as

$$\alpha_{1,1} = \beta_{1,1} + \beta\omega, \tag{3.3}$$

where

$$\beta_{1,1} \in A^{1,1}(X), \qquad \beta_{1,1} \wedge \omega^{n-1} = 0, \qquad \beta \in \mathscr{C}^\infty(X; \mathbb{C}).$$

The form $\alpha_{1,1}$ belongs to $\mathscr{H}^{1,1}_{\bar\partial}$ if and only if $\alpha_{1,1}$ satisfies the equations

$$\bar\partial\alpha_{1,1} = 0, \quad \partial * \alpha_{1,1} = 0. \tag{3.4}$$

By (3.2) we compute

$$*\alpha_{1,1} = -\frac{1}{(n-2)!}\beta_{1,1} \wedge \omega^{n-2} + \beta\frac{1}{(n-1)!}\omega^{n-1}. \tag{3.5}$$

Therefore, by (3.3), (3.5), taking into account that g is almost-Kähler, equations (3.4) are equivalent to

$$\begin{cases} \bar\partial\beta_{1,1} + \bar\partial\beta\wedge\omega = 0 \\ -\dfrac{1}{(n-2)!}\partial\beta_{1,1}\wedge\omega^{n-2} + \partial\beta\wedge\dfrac{1}{(n-1)!}\omega^{n-1} = 0. \end{cases} \tag{3.6}$$

After multiplying the first equation by ω^{n-2} and the second by $(n-2)!$, we obtain

$$\begin{cases} \bar\partial\beta_{1,1}\wedge\omega^{n-2} + \bar\partial\beta\wedge\omega^{n-1} = 0 \\ -\partial\beta_{1,1}\wedge\omega^{n-2} + \dfrac{1}{n-1}\partial\beta\wedge\omega^{n-1} = 0, \end{cases}$$

and taking the sum of the last two equations we obtain

$$(\bar\partial\beta_{1,1} - \partial\beta_{1,1})\wedge\omega^{n-2} + \left(\bar\partial\beta + \frac{1}{n-1}\partial\beta\right)\wedge\omega^{n-1} = 0.$$

By definition, we have

$$d^c = i(\bar\partial - \partial + \mu - \overline{\mu}),$$

where $|\mu| = (2,-1)$, $|\overline{\mu}| = (-1,2)$. Consequently, the last equation can be written as

$$\left(\bar\partial\beta + \frac{1}{n-1}\partial\beta\right)\wedge\omega^{n-1} = id^c\beta_{1,1}\wedge\omega^{n-2}.$$

Applying $-id^c$ to both sides of the above equation, we obtain

$$\left[(\bar\partial - \partial + \mu - \bar\mu)\left(\bar\partial\beta + \frac{1}{n-1}\partial\beta\right)\right]\wedge\omega^{n-1} = 0,$$

which yields

$$\left(\frac{1}{n-1} + 1\right)\partial\bar\partial\beta\wedge\omega^{n-1} = 0,$$

since $\partial\bar\partial + \bar\partial\partial = 0$ on functions and the other contributions vanish by bidegree reasons when we take the wedge product with ω^{n-1}. Therefore,

$$\partial\bar\partial\left(\beta\cdot\omega^{n-1}\right) = 0,$$

from which we derive that $\beta \equiv \beta_0 \in \mathbb{C}$ is constant (see, for instance [8], [1, Theorem 10] or [16, Proposition 3.4] for the 4-dimensional case). Hence

$$\alpha_{1,1} = \beta_{1,1} + \beta_0\omega,$$

so from (3.6) or from

$$\begin{aligned} \bar\partial\beta_{1,1} &= \bar\partial\alpha_{1,1} - \bar\partial(\beta_0\omega) \qquad = 0 \\ \partial * \beta_{1,1} &= \partial * \alpha_{1,1} - \partial * (\beta_0\omega) = 0, \end{aligned}$$

we have that $\beta \in \mathscr{H}^{1,1}_{\bar\partial}$ and $\beta_{1,1}$ is primitive. This proves that

$$\mathscr{H}^{1,1}_{\bar\partial} \subset \mathbb{C}\cdot\omega \oplus \left(\mathscr{H}^{1,1}_{\bar\partial} \cap P^{1,1}\right).$$

Conversely, if $\alpha_{1,1} = \beta_0\omega + \beta_{1,1}$, with $\beta_0 \in \mathbb{C}$ and $\beta_{1,1} \in \mathscr{H}^{1,1}_{\bar\partial} \cap P^{1,1}$, we easily conclude that $\partial * \alpha_{1,1} = 0$ and $\bar\partial\alpha_{1,1} = 0$. The decomposition is thus proved. □

As a consequence we obtain the following primitive decompositions.

Corollary 3.5. *Let (X, J, g, ω) be a compact $2n$-dimensional almost-Kähler manifold (with $n \geq 2$). Then the following decompositions hold:*

(i) $\mathscr{H}^{1,1}_{\partial} = \mathbb{C} \cdot \omega \oplus \left(\mathscr{H}^{1,1}_{\partial} \cap P^{1,1}\right)$,
(ii) $\mathscr{H}^{n-1,n-1}_{\bar\partial} = \mathbb{C}\omega^{n-1} \oplus L^{n-2}\left(\mathscr{H}^{1,1}_{\partial} \cap P^{1,1}\right)$,
(iii) $\mathscr{H}^{n-1,n-1}_{\partial} = \mathbb{C}\omega^{n-1} \oplus L^{n-2}\left(\mathscr{H}^{1,1}_{\bar\partial} \cap P^{1,1}\right)$.

Proof. The first decomposition follows from the one proved in Theorem 3.4 by conjugation.

To prove the second, observe that the Hodge-$*$ operator induces an isomorphism $\mathscr{H}^{1,1}_{\partial} \simeq \mathscr{H}^{n-1,n-1}_{\bar\partial}$. Via this isomorphism, ω corresponds to ω^{n-1}, while by (3.2) on primitive $(1,1)$-forms we have $* = -\frac{1}{(n-2)!}L^{n-2}$. So we just have to apply $*$ to the decomposition of the previous point.

Finally, the last point follows from the second by conjugation. $\square$

Recall that by [5] (see also [14]) on compact almost-Kähler manifolds we have

$$\Delta_{\bar\partial} + \Delta_{\mu} = \Delta_{\partial} + \Delta_{\bar\mu},$$

and so, for every p, q,

$$\mathscr{H}^{p,q}_{\bar\partial} \cap \mathscr{H}^{p,q}_{\mu} = \mathscr{H}^{p,q}_{\partial} \cap \mathscr{H}^{p,q}_{\bar\mu}.$$

In particular, if J is integrable, namely (X, J, g, ω) is a compact Kähler manifold, one recovers the well-known identities

$$\Delta_{\bar\partial} = \Delta_{\partial}$$

and

$$\mathscr{H}^{p,q}_{\bar\partial} = \mathscr{H}^{p,q}_{\partial}.$$

Therefore, one could wonder if this last identity holds true also in the non-integrable case for some special bidegrees. More precisely, we want to show that the two primitive decompositions we obtained in Theorem 3.4 and Corollary 3.5 for $\mathscr{H}^{1,1}_{\bar\partial}$ and $\mathscr{H}^{1,1}_{\partial}$ are not the same.

4. Relations between $\Delta_{\bar\partial}$ and Δ_{∂}

Let us start by considering the 4-dimensional case. Let $\alpha_{1,1}$ be a primitive $(1,1)$-form on an almost-Kähler 4-dimensional manifold X. It follows from (3.2) that $*\alpha_{1,1} = -\alpha_{1,1}$. As a consequence we have the following result.

Proposition 4.1. *Let X be an almost-Kähler 4-dimensional manifold. Then, on $(1,1)$-forms we have*

$$\Delta_{\bar\partial_{|A^{1,1}}} = \Delta_{\partial_{|A^{1,1}}},$$

and in particular their kernels coincide:

$$\mathscr{H}^{1,1}_{\bar\partial} = \mathscr{H}^{1,1}_{\partial}.$$

Notice that this follows also from [5], since on almost-Kähler manifolds we have $\Delta_{\bar\partial}+\Delta_{\mu}=\Delta_{\partial}+\Delta_{\bar\mu}$, and on $(1,1)$-forms on 4-dimensional almost-Kähler manifolds, $\Delta_{\mu}=\Delta_{\bar\mu}=0$.

We show now that in higher dimension the equality

$$\Delta_{\bar\partial_{|A^{1,1}}}=\Delta_{\partial_{|A^{1,1}}}$$

does not hold in general.

Example 4.2. Let $\mathbb{T}^6=\mathbb{Z}^6\backslash\mathbb{R}^6$ be the 6-dimensional torus with coordinates $(x_1,x_2,x_3,y_1,y_2,y_3)$ on $\mathbb{R}^6$. Let $f=f(x_2)$ be a non-constant $\mathbb{Z}$-periodic function, and we define the following non-left-invariant almost-complex structure J on $\mathbb{T}^6$:

$$J\partial_{x_1}:=e^{-f}\partial_{y_1},\quad J\partial_{x_2}:=\partial_{y_2},\quad J\partial_{x_3}:=\partial_{y_3}.$$

A global co-frame of $(1,0)$-forms is given by

$$\Phi^1:=dx_1+i\,e^f dy_1,\quad \Phi^2:=dx_2+i\,dy_2,\quad \Phi^3:=dx_3+i\,dy_3.$$

The structure equations are

$$d\Phi^1=-\frac{1}{4}f'(x_2)\Phi^{12}-\frac{1}{4}f'(x_2)\Phi^{2\bar1}-\frac{1}{4}f'(x_2)\Phi^{1\bar2}+\frac{1}{4}f'(x_2)\Phi^{\bar1\bar2}$$

and $d\Phi^2=d\Phi^3=0$. Then, the $(1,1)$-form

$$\omega:=\frac{i}{2}e^{-f}\Phi^{1\bar1}+\frac{i}{2}\Phi^{2\bar2}+\frac{i}{2}\Phi^{3\bar3}$$

is a compatible symplectic structure, namely (J,ω) is an almost-Kähler structure on $\mathbb{T}^6$.

Notice now that by a direct computation

$$\bar\mu\Phi^{1\bar3}=\frac{1}{4}f'(x_2)\Phi^{\bar1\bar2\bar3}\neq0$$

and

$$\mu\Phi^{1\bar3}=0.$$

Therefore, from [5], we have

$$(\Delta_{\bar\partial}-\Delta_{\partial})\Phi^{1\bar3}=-\bar\mu^*\bar\mu\Phi^{1\bar3}\neq0.$$

The last point follows either by direct computation or by noticing that

$$\bar\mu^*\bar\mu\Phi^{1\bar3}\neq0\quad\iff\quad\|\bar\mu\Phi^{1\bar3}\|^2\neq0\quad\iff\quad\bar\mu\Phi^{1\bar3}\neq0.$$

Another example is provided by the following 8-dimensional nilmanifold with a left-invariant almost-Kähler structure.

Example 4.3. We recall the following construction contained in [2]. Set

$$\mathbb{H}(1,2):=\left\{\begin{bmatrix}1&0&x_1&z_1\\0&1&x_2&z_2\\0&0&1&y\\0&0&0&1\end{bmatrix}\mid x_1,x_2,y,z_1,z_2\in\mathbb{R}\right\}.$$

Let Γ be the subgroup of matrices with integral entries. Let $X := \Gamma\backslash\mathbb{H}(1,2)$ and define

$$M := X \times \mathbb{T}^3.$$

Denoting with u, v, w coordinates on $\mathbb{T}^3$ we consider the following left-invariant 1-forms:

$$e^1 := dx_2, \quad e^2 := dx_1, \quad e^3 := dy, \quad e^4 := du,$$
$$e^5 := dz_1 - x_1 dy, \quad e^6 := dz_2 - x_2 dy, \quad e^7 := dv, \quad e^8 := dw,$$

and the structure equations become

$$de^1 = de^2 = de^3 = de^4 = de^7 = de^8 = 0, \quad de^5 = -e^{23}, \quad de^6 = -e^{13}.$$

We define the symplectic structure

$$\omega := e^{15} + e^{26} + e^{37} + e^{48},$$

and we take the compatible almost-complex structure defined by the following coframe of $(1,0)$-forms:

$$\psi^1 := e^1 + i\,e^5, \quad \psi^2 := e^2 + i\,e^6, \quad \psi^3 := e^3 + i\,e^7, \quad \psi^4 := e^4 + i\,e^8.$$

By direct computation we get

$$d\psi^{1\bar{4}} = -\frac{i}{4}\psi^{23\bar{4}} - \frac{i}{4}\psi^{2\bar{3}\bar{4}} + \frac{i}{4}\psi^{3\bar{2}\bar{4}} - \frac{i}{4}\psi^{\bar{2}\bar{3}\bar{4}};$$

hence

$$\mu\psi^{1\bar{4}} = 0, \quad \bar{\mu}\psi^{1\bar{4}} = -\frac{i}{4}\psi^{\bar{2}\bar{3}\bar{4}}.$$

Therefore,

$$(\Delta_{\bar{\partial}} - \Delta_{\partial})\psi^{1\bar{4}} = (\Delta_{\bar{\mu}} - \Delta_{\mu})\psi^{1\bar{4}} = \bar{\mu}^*\bar{\mu}\psi^{1\bar{4}} \neq 0,$$

proving that

$$\Delta_{\bar{\partial}} \neq \Delta_{\partial}$$

on $(1,1)$-forms. However, one can show that their kernels coincide, namely $\mathscr{H}^{1,1}_{\bar{\partial}} = \mathscr{H}^{1,1}_{\partial}$.

Remark 4.4. We want to point out that finding explicit examples of almost-Kähler manifolds with $\Delta_{\bar{\partial}} \neq \Delta_{\partial}$ seems to be not so obvious. In fact, we couldn't find any left-invariant example in dimension 6.

Even though $\Delta_{\bar{\partial}_{|A^{1,1}}} \neq \Delta_{\partial_{|A^{1,1}}}$ in general, we wonder whether their kernels coincide. Before showing that this is not the case we notice that the equality $\mathscr{H}^{1,1}_{\bar{\partial}} = \mathscr{H}^{1,1}_{\partial}$ is equivalent to $\mathscr{H}^{1,1}_{\bar{\partial}} \cap P^{1,1} = \mathscr{H}^{1,1}_{\partial} \cap P^{1,1}$.

Lemma 4.5. *Let (X, J, g, ω) be an almost-Kähler manifold. Then $\mathscr{H}^{1,1}_{\bar\partial} = \mathscr{H}^{1,1}_{\partial}$ if and only if $\mathscr{H}^{1,1}_{\bar\partial} \cap P^{1,1} = \mathscr{H}^{1,1}_{\partial} \cap P^{1,1}$.*

Proof. We prove only the non-trivial implication. Let $\alpha_{1,1} \in \mathscr{H}^{1,1}_{\bar\partial}$; then we can decompose it as $\alpha_{1,1} = c\,\omega + \beta_{1,1}$ with $c \in \mathbb{C}$ and $\beta_{1,1} \in \mathscr{H}^{1,1}_{\bar\partial} \cap P^{1,1}$. Now,

$$\Delta_\partial \alpha_{1,1} = c \cdot \Delta_\partial \omega + \Delta_\partial \beta_{1,1} = 0 + 0 = 0,$$

so $\alpha_{1,1} \in \mathscr{H}^{1,1}_{\partial}$. The other inclusion is similar. □

We observe the following:

Lemma 4.6. *Let (X^{2n}, J, g, ω) be a $2n$-dimensional almost-Kähler manifold. Let $k := p + q \leq n$ and let $\alpha \in P^{p,q}$. Then,*

$$\bar\partial \alpha = 0 \quad \Longrightarrow \quad \partial^* \alpha = 0.$$

Similarly,

$$\partial \alpha = 0 \quad \Longrightarrow \quad \bar\partial^* \alpha = 0.$$

Proof. By (3.2) we have

$$*\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{i^{p-q}}{(n-k)!} \alpha \wedge \omega^{n-k}.$$

Since ω is closed, this readily implies that $\bar\partial * \alpha = 0$. The same holds switching $\bar\partial$ and ∂. □

Lemma 4.7. *Let (X, J, g, ω) be an almost-Kähler manifold. Let $\alpha_{1,1} \in \mathscr{H}^{1,1}_{\bar\partial} \cap P^{1,1}$. Then $d^* \alpha_{1,1} = 0$.*

Proof. Since $*\alpha_{1,1}$ is an $(n-1, n-1)$-form, by the previous lemma we have

$$d * \alpha_{1,1} = (\partial + \bar\partial) * \alpha_{1,1} = \partial * \alpha_{1,1} + \bar\partial * \alpha_{1,1} = 0.$$ □

Lemma 4.8. *Let (X, J, g, ω) be an almost-Kähler manifold. Let $\alpha_{1,1} \in \mathscr{H}^{1,1}_{\bar\partial} \cap P^{1,1}$. Then $d\alpha_{1,1}$, $\mu\alpha_{1,1}$, $\partial\alpha_{1,1}$, $\bar\partial\alpha_{1,1}$ and $\bar\mu\alpha_{1,1}$ are primitive.*

Proof. From the previous lemma and (3.2) we deduce that

$$0 = d * \alpha_{1,1} = -\frac{1}{(n-2)!} d(\alpha \wedge \omega^{n-2}) = -\frac{1}{(n-2)!} d\alpha \wedge \omega^{n-2}.$$

So $d\alpha_{1,1}$ is primitive, and by decomposition in types we deduce that also $\mu\alpha_{1,1}$, $\partial\alpha_{1,1}$, $\bar\partial\alpha_{1,1}$ and $\bar\mu\alpha_{1,1}$ are primitive. □

We finally show that, in general, on compact almost-Kähler manifolds we have

$$\mathscr{H}^{1,1}_{\bar\partial} \neq \mathscr{H}^{1,1}_{\partial}.$$

By Lemma (4.5) this will be done using primitive forms.

Example 4.9. Using the same notations as in Example 4.2 we consider $\mathbb{T}^6 = \mathbb{Z}^6\backslash\mathbb{R}^6$. Let $g = g(x_3, y_3)$ be a function on $\mathbb{T}^6$. We define an almost-complex structure J setting as global co-frame of $(1,0)$-forms

$$\varphi^1 := e^g dx_1 + i\, e^{-g} dy_1, \quad \varphi^2 := dx_2 + i\, dy_2, \quad \varphi^3 := dx_3 + i\, dy_3.$$

The structure equations are

$$d\varphi^1 = V_3(g)\varphi^{3\bar{1}} - \bar{V}_3(g)\varphi^{\bar{1}\bar{3}},$$

where $\{V_1, V_2, V_3\}$ is the global frame of vector fields dual to $\left\{\varphi^1, \varphi^2, \varphi^3\right\}$, and $d\varphi^2 = d\varphi^3 = 0$. Assume finally that g satisfies $V_3(g) \neq 0$.

Then, the $(1,1)$-form

$$\omega := \frac{i}{2}\varphi^{1\bar{1}} + \frac{i}{2}\varphi^{2\bar{2}} + \frac{i}{2}\varphi^{3\bar{3}}$$

is a compatible symplectic structure, namely (J, ω) is an almost-Kähler structure on $\mathbb{T}^6$.

Notice now that

$$\bar{\partial}\varphi^{1\bar{2}} = V_3(g)\varphi^{3\bar{1}\bar{2}} \neq 0,$$

namely, $\varphi^{1\bar{2}} \notin \mathscr{H}^{1,1}_{\bar{\partial}}$ but $\varphi^{1\bar{2}} \in \mathscr{H}^{1,1}_{\partial}$. Indeed, $\partial\varphi^{1\bar{2}} = 0$, and since $\varphi^{1\bar{2}}$ is primitive and ω is closed,

$$\bar{\partial} * \varphi^{1\bar{2}} = \bar{\partial}(-\omega \wedge \varphi^{1\bar{2}}) = -\omega \wedge \bar{\partial}\varphi^{1\bar{2}} = -\omega \wedge \left(V_3(g)\varphi^{3\bar{1}\bar{2}}\right) = 0.$$

Hence, $\partial^*\varphi^{1\bar{2}} = - * \bar{\partial} * \varphi^{1\bar{2}} = 0$.

5. Primitive decompositions in dimension 6

Notice that in view of Propositions 3.1, 3.2, Theorem 3.4 and Corollary 3.5 we have a full primitive description of all $\bar{\partial}$-harmonic forms on compact 4-dimensional almost-Kähler manifolds. It is therefore natural to ask what happens for bidegrees different from $(p,0)$, $(0,q)$, (n,q), (p,n), $(1,1)$ and $(n-1,n-1)$ in higher dimension. The first interesting dimension to consider is 6, and in this case the only bidegrees left are $(2,1)$ and $(1,2)$. Let us focus, for instance, on bidegree $(2,1)$. The primitive decomposition of forms is

$$A^{2,1}(X) = P^{2,1} \oplus L\left(A^{1,0}(X)\right).$$

Passing to $\bar{\partial}$-harmonic forms, it follows that

$$\mathscr{H}^{2,1}_{\bar{\partial}} \supseteq \left(\mathscr{H}^{2,1}_{\bar{\partial}} \cap P^{2,1}\right) \oplus L\left(\mathscr{H}^{1,0}_{\bar{\partial}}\right);$$

indeed, on compact almost-Kähler manifolds, for bidegree reasons and [5] one has

$$\mathscr{H}^{1,0}_{\bar{\partial}} = \mathscr{H}^{1,0}_{\bar{\partial}} \cap \mathscr{H}^{1,0}_{\mu} = \mathscr{H}^{1,0}_{\partial} \cap \mathscr{H}^{1,0}_{\bar{\mu}}.$$

Therefore, it is natural to wonder whether such inclusion is indeed an identity. In fact, this is not the case in general, as shown by the following proposition.

Proposition 5.1. *There exists a compact almost-Kähler* 6*-dimensional manifold* (X,J,ω) *such that*

$$\mathscr{H}^{2,1}_{\bar\partial} \neq \left(\mathscr{H}^{2,1}_{\bar\partial} \cap P^{2,1}\right) \oplus L\left(\mathscr{H}^{1,0}_{\bar\partial}\right).$$

Proof. We refer the reader to Example 5.3 for the proof. □

First we need the following lemma, which will allow us to work only with left-invariant forms.

Lemma 5.2. *Let* $X^6 = \Gamma\backslash G$ *be the compact quotient of a* 6*-dimensional, connected, simply-connected Lie group by a lattice and let* (J,ω) *be a left-invariant almost-Kähler structure on* X*. Let* $\eta \in A^{2,1}(X)$ *be a left-invariant* $(2,1)$*-form on* X *with primitive decomposition*

$$\eta = \alpha + L\beta.$$

Then, α *and* β *are left-invariant.*

Proof. Let $\eta \in A^{2,1}(X)$ be a left-invariant $(2,1)$-form on X. Its primitive decomposition is

$$\eta = \alpha + L\beta,$$

with $\alpha \in A^{2,1}(X)$ primitive, i.e., $L\alpha = 0$ and $\beta \in A^{1,0}(X)$. Notice that β is indeed primitive for bidegree reasons. We apply L to the decomposition and obtain

$$L\eta = L^2\beta.$$

Since ω is left-invariant, we have that $L\eta$, and so $L^2\beta$, are left-invariant. Now, since $L^2 : \Lambda^1 \to \Lambda^5$ is an isomorphism at the level of the exterior algebra, it follows that also β is left-invariant. As a consequence, since $L\beta$ and η are left-invariant, we get that also α is left-invariant. □

Example 5.3. Let X be the Iwasawa manifold defined as the quotient $X := \Gamma\backslash\mathbb{H}_3$, where

$$\mathbb{H}_3 := \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}$$

and

$$\Gamma := \left\{ \begin{bmatrix} 1 & \gamma_1 & \gamma_3 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix} \mid \gamma_1, \gamma_2, \gamma_3 \in \mathbb{Z}[\,i\,] \right\}.$$

Then, setting $z_j = x_j + iy_j$, there exists a basis of left-invariant 1-forms $\{e_i\}$ on X given by

$$\begin{cases} e^1 = dx_1 \\ e^2 = dy_1 \\ e^3 = dx_2 \\ e^4 = dy_2 \\ e^5 = dx_3 - x_1 dx_2 + y_1 dy_2 \\ e^6 = dy_3 - x_1 dy_2 - y_1 dx_2. \end{cases}$$

The following structure equations hold:

$$\begin{cases} de^1 = 0 \\ de^2 = 0 \\ de^3 = 0 \\ de^4 = 0 \\ de^5 = -e^{13} + e^{24} \\ de^6 = -e^{14} - e^{23}. \end{cases}$$

Let us consider the non integrable left-invariant almost-complex structure J given by

$$\phi^1 = e^1 + ie^6, \quad \phi^2 = e^2 + ie^5, \quad \phi^3 = e^3 + ie^4$$

being a global coframe of $(1,0)$-forms. By a direct computation the structure equations become (see also [15])

$$\begin{aligned} 4\, d\phi^1 &= -\phi^{13} - i\phi^{23} + \phi^{1\bar{3}} + \phi^{3\bar{1}} - i\phi^{2\bar{3}} + i\phi^{3\bar{2}} + \phi^{\bar{1}\bar{3}} - i\phi^{\bar{2}\bar{3}}, \\ 4\, d\phi^2 &= -i\phi^{13} + \phi^{23} - i\phi^{1\bar{3}} + i\phi^{3\bar{1}} - \phi^{2\bar{3}} - \phi^{3\bar{2}} - i\phi^{\bar{1}\bar{3}} - \phi^{\bar{2}\bar{3}}, \\ d\phi^3 &= 0. \end{aligned}$$

Endow (X, J) with the left-invariant almost-Kähler structure given by

$$\omega = 2(e^{16} + e^{25} + e^{34}) = i(\phi^{1\bar{1}} + \phi^{2\bar{2}} + \phi^{3\bar{3}}).$$

We want to find an element $\eta \in A^{2,1}(X)$ which is contained in $\mathscr{H}^{2,1}_{\bar{\partial}}$ but is not contained in

$$\big(\mathscr{H}^{2,1}_{\bar{\partial}} \cap P^{2,1}\big) \oplus L\big(\mathscr{H}^{1,0}_{\bar{\partial}}\big).$$

Thanks to Lemma 5.2 it is sufficient to work with left-invariant forms. Indeed if we find $\eta \in \mathscr{H}^{2,1}_{\bar{\partial}}$ left-invariant that cannot be decomposed as $\eta = \alpha + L\beta$, with $\alpha \in \mathscr{H}^{2,1}_{\bar{\partial}} \cap P^{2,1}$ and $\beta \in \mathscr{H}^{1,0}_{\bar{\partial}}$, both left-invariant forms, then $\eta \notin \big(\mathscr{H}^{2,1}_{\bar{\partial}} \cap P^{2,1}\big) \oplus L\big(\mathscr{H}^{1,0}_{\bar{\partial}}\big)$.

A long but direct and straightforward computation shows that the space of left-invariant $\bar{\partial}$-harmonic $(2,1)$-forms is

$$\mathbb{C}\langle \phi^{13\bar{1}} + \phi^{23\bar{2}},\, \phi^{13\bar{2}} + \phi^{23\bar{1}} - 2i\phi^{23\bar{2}},\, \phi^{13\bar{3}} + \phi^{23\bar{3}} \rangle,$$

while it is immediate to see that the space of left-invariant forms which are contained in $L\big(\mathscr{H}^{1,0}_{\bar{\partial}}\big)$ is

$$\mathbb{C}\langle \phi^{13\bar{1}} + \phi^{23\bar{2}} \rangle.$$

Since, for instance, $L(\phi^{13\bar{2}} + \phi^{23\bar{1}} - 2i\phi^{23\bar{2}}) = -2iL(\phi^{23\bar{2}}) \neq 0$, it means that $\phi^{13\bar{2}} + \phi^{23\bar{1}} - 2i\phi^{23\bar{2}}$ is not primitive. Therefore, $\phi^{13\bar{2}} + \phi^{23\bar{1}} - 2i\phi^{23\bar{2}}$ is a left-invariant, $\bar{\partial}$-harmonic $(2,1)$-form, but it is not contained in

$$\big(\mathscr{H}^{2,1}_{\bar{\partial}} \cap P^{2,1}\big) \oplus L\big(\mathscr{H}^{1,0}_{\bar{\partial}}\big).$$

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References

[1] D. Angella, N. Istrati, A. Otiman, and N. Tardini, Variational problems in conformal geometry, *J. Geom. Anal.* **31** no. 3 (2021), 3230–3251. DOI MR Zbl

[2] P. de Bartolomeis and A. Tomassini, On formality of some symplectic manifolds, *Internat. Math. Res. Notices* no. 24 (2001), 1287–1314. DOI MR Zbl

[3] A. Cattaneo, A. Nannicini, and A. Tomassini, Kodaira dimension of almost Kähler manifolds and curvature of the canonical connection, *Ann. Mat. Pura Appl. (4)* **199** no. 5 (2020), 1815–1842. DOI MR Zbl

[4] A. Cattaneo, A. Nannicini, and A. Tomassini, On Kodaira dimension of almost complex 4-dimensional solvmanifolds without complex structures, *Internat. J. Math.* **32** no. 10 (2021), Paper No. 2150075, 41 pp. DOI MR Zbl

[5] J. Cirici and S. O. Wilson, Topological and geometric aspects of almost Kähler manifolds via harmonic theory, *Selecta Math. (N.S.)* **26** no. 3 (2020), Paper No. 35, 27 pp. DOI MR Zbl

[6] J. Cirici and S. O. Wilson, Dolbeault cohomology for almost complex manifolds, *Adv. Math.* **391** (2021), Paper No. 107970, 52 pp. DOI MR Zbl

[7] R. Coelho, G. Placini, and J. Stelzig, Maximally non-integrable almost complex structures: an h-principle and cohomological properties, *Selecta Math. (N.S.)* **28** no. 5 (2022), Paper No. 83, 25 pp. DOI MR Zbl

[8] P. Gauduchon, La 1-forme de torsion d'une variété hermitienne compacte, *Math. Ann.* **267** no. 4 (1984), 495–518. DOI MR Zbl

[9] F. Hirzebruch, Some problems on differentiable and complex manifolds, *Ann. of Math. (2)* **60** (1954), 213–236. DOI MR Zbl

[10] T. Holt, Bott-Chern and $\bar{\partial}$ harmonic forms on almost Hermitian 4-manifolds, *Math. Z.* **302** no. 1 (2022), 47–72. DOI MR Zbl

[11] T. Holt and W. Zhang, Harmonic forms on the Kodaira-Thurston manifold, *Adv. Math.* **400** (2022), Paper No. 108277, 30 pp. DOI MR Zbl

[12] R. Piovani and N. Tardini, Bott-Chern harmonic forms and primitive decompositions on compact almost Kähler manifolds, *Ann. Mat. Pura Appl. (4)* **202** no. 6 (2023), 2749–2765. DOI MR Zbl

[13] R. Piovani and A. Tomassini, Bott-Chern Laplacian on almost Hermitian manifolds, *Math. Z.* **301** no. 3 (2022), 2685–2707. DOI MR Zbl

[14] N. Tardini and A. Tomassini, Differential operators on almost-Hermitian manifolds and harmonic forms, *Complex Manifolds* **7** no. 1 (2020), 106–128. DOI MR Zbl

[15] N. Tardini and A. Tomassini, Almost-complex invariants of families of six-dimensional solvmanifolds, *Complex Manifolds* **9** no. 1 (2022), 238–260. DOI MR Zbl

[16] N. Tardini and A. Tomassini, $\bar{\partial}$-harmonic forms on 4-dimensional almost-Hermitian manifolds, *Math. Res. Lett.* **30** no. 5 (2023), 1617–1637. DOI MR Zbl

[17] L.-S. Tseng and S.-T. Yau, Cohomology and Hodge theory on symplectic manifolds: I, *J. Differential Geom.* **91** no. 3 (2012), 383–416. MR Zbl Available at http://projecteuclid.org/euclid.jdg/1349292670.

[18] A. Weil, *Introduction à l'étude des variétés kählériennes*, Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Actualités Sci. Ind., no. 1267, Hermann, Paris, 1958. MR Zbl

Andrea Cattaneo
Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Unità di Matematica e Informatica, Università degli Studi di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy
andrea.cattaneo@unipr.it

Nicoletta Tardini
Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Unità di Matematica e Informatica, Università degli Studi di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy
nicoletta.tardini@unipr.it

Adriano Tomassini✉
Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Unità di Matematica e Informatica, Università degli Studi di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy
adriano.tomassini@unipr.it

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