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ONE-SIDED EP ELEMENTS IN RINGS WITH INVOLUTION

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Abstract. This paper investigates the one-sided EP property of elements in rings with involution. Let R be a ring with involution $*$. Then $a \in R$ is said to be left (resp. right) EP if a is Moore–Penrose invertible and $aR \subseteq a^*R$ (resp. $a^*R \subseteq aR$). Many properties of EP elements are extended to one-sided versions. Some new characterizations of EP elements are presented in relation to the absorption law for Moore–Penrose inverses.

1. Introduction

The EP property was first discussed in 1950 by H. Schwerdtfeger [25], who defined a square complex matrix to be EP if it has the same range as its conjugate transpose. In the literature [7, 13], the notion of EP matrices was extended to EP elements in rings with involution by means of Moore–Penrose inverses: an element a in a ring R with involution $*$ is called EP if the Moore–Penrose inverse $a^\dagger$ of a exists and $aa^\dagger = a^\dagger a$, or, equivalently, if $a^\dagger$ exists and $aR = a^*R$ [7, Proposition 25]. The class of EP elements has very nice properties and important relations with some other classes of elements such as units and projections; it has been investigated by many authors (see, for example, [4, 15, 16, 18, 19, 20, 21, 22, 27]).

It is well known that an $n \times n$ complex matrix A is EP if and only if

$$A\mathcal{M}_n(\mathbb{C}) = A^\star \mathcal{M}_n(\mathbb{C}), \tag{1.1}$$

where $\mathcal{M}_n(\mathbb{C})$ denotes the $n \times n$ complex matrix ring and $A^\star$ denotes the conjugate transpose of A (see, for example, [2, p. 159, Exercise 17]). Since A and $A^\star$ have the same rank, the condition (1.1) is also equivalent to

$$A\mathcal{M}_n(\mathbb{C}) \subseteq A^\star \mathcal{M}_n(\mathbb{C}).$$

In [22], Patrício and Puystjens extended this fact to Dedekind-finite rings (i.e., rings for which every one-sided invertible element is two-sided invertible; $\mathcal{M}_n(\mathbb{C})$ is a typical example of such rings) by showing that an element a of a Dedekind-finite

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ring R with involution $*$ is EP if and only if $a^\dagger$ exists and $aR \subseteq a^*R$. But in general, this is not the case if the ring is not Dedekind-finite.

In this paper, for an arbitrary ring R with involution $*$, we investigate those elements $a \in R$ for which $a^\dagger$ exists and $aR \subseteq a^*R$ (resp. $a^*R \subseteq aR$), in which case such an a is said to be left (resp. right) EP. Many properties of EP elements are extended to one-sided versions. Various characterizations of one-sided EP elements are derived by making use of generalized inverses.

To begin with, we recall that an involution $*$ of a ring R is an anti-isomorphism with index two, that is, it satisfies $(r^*)^* = r$, $(rs)^* = s^*r^*$ and $(r+s)^* = r^* + s^*$ for each $r, s \in R$. An element $a \in R$ is said to be Moore–Penrose invertible (with respect to $*$) if there exists $x \in R$ satisfying the following Penrose equations [23, 12]:

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa.$$

Such an element x is unique when it exists, and is called the Moore–Penrose inverse of a and denoted by $a^\dagger$.

Throughout the paper, unless otherwise stated, R denotes a unital ring with involution $*$, and $R^\dagger$ denotes the set of all Moore–Penrose invertible elements of R.

2. The notion and basic properties of one-sided EP elements

In this section, we shall present the notion, examples and basic properties of one-sided EP elements. We begin with the following definition.

Definition 2.1. Let R be a ring with involution $*$. Then $a \in R$ is said to be *left EP* if a is Moore–Penrose invertible and $aR \subseteq a^*R$, and dually a is said to be *right EP* if a is Moore–Penrose invertible and $a^*R \subseteq aR$.

From the definition it follows directly that an element is EP if and only if it is both left and right EP. Moreover, since $a \in R^\dagger$ implies that $a^* \in R^\dagger$, we can see that a is left EP if and only if a^* is right EP. The following examples show that one-sided EP elements are, in general, not EP.

Example 2.2. We employ the construction of Jacobson [9]. Let R be the ring of all row and column finite matrices over a field, and let $*$ be the transpose map of matrices. Let $a = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix} \in R$. A routine calculation shows that $a^*a = 1_R$ and $aa^* = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \neq 1_R$, from which we can see that $a \in R^\dagger$ and $a^\dagger = a^*$. Moreover, since $a^*a = 1$, it follows that $a^*R = R$; and since a is not right invertible, it follows that $aR \subsetneq R$. Therefore, we have $aR \subsetneq a^*R$, which, together with $a \in R^\dagger$, imply that a is left EP but not (right) EP.

Example 2.3. Let K be a field, and let

$$R = K\langle x, y : x^2y = x,\ xy^2 = y,\ xyx = x,\ yxy = y\rangle$$

be the free algebra over K in the noncommuting variables x, y satisfying $x^2y = x = xyx$ and $xy^2 = y = yxy$. Observe that the set $\mathcal{B} = \{xy, y^m x^n : m, n \geq 0\}$ forms a basis of R, and so any element $r \in R$ can be uniquely written in the form $r = k_0 xy + \sum_{i=1}^{p} k_i y^{m_i} x^{n_i}$ for some $k_0, k_i \in K$, $m_i, n_i, p \geq 0$. Define

$$*:\ R \to R, \quad r \longmapsto r^* = k_0 xy + \sum_{i=1}^{p} k_i y^{n_i} x^{m_i}.$$

Then, by [26, Example 4.2], $*$ is an involution of R. Now we claim that

(i) x is a partial isometry (i.e., $xx^*x = x$, or, equivalently, $x \in R^\dagger$ and $x^\dagger = x^*$);
(ii) x is right EP but not (left) EP.

Indeed, since $x^* = (x^1y^0)^* = x^0y^1 = y$ and $xyx = x$, it follows that x is a partial isometry. Since $x^*R = yR = (xy^2)R \subseteq xR$, it follows that x is right EP. Moreover, if $xR \subseteq x^*R$, then $x = x^*s = ys$ for some $s \in R$, and so $x = (yxy)s = yx^2$, contradicting the assumption on x, y; thus, x is not (left) EP.

By [18], if a is a partial isometry (or, more generally, a is star-dagger, i.e., $a^\dagger a^* = a^* a^\dagger$) and is EP, then it is normal (i.e., $aa^* = a^*a$). Here, we notice from the above two examples that, in general, a partial isometry being left or right EP does not imply that it is normal.

The next result characterizes the one-sided EP property by making use of Moore–Penrose inverses.

Proposition 2.4. *Let $a \in R^\dagger$. Then the following statements are equivalent:*

(1) *a is left EP.*
(2) *$a^\dagger a^2 = a$.*
(3) *$(a^\dagger)^2 a = a^\dagger$.*

Proof. (1)⇒(2). By (1), there exists $r \in R$ such that $a = a^*r$, so

$$a^\dagger a^2 = a^\dagger a(a^*r) = [(a^\dagger a)^* a^*]r = (aa^\dagger a)^* r = a^*r = a.$$

(2)⇒(3). Since

$$\begin{aligned}(a^\dagger)^2 a = (a^\dagger a a^\dagger) a^\dagger a &= a^\dagger (aa^\dagger)^* (a^\dagger a)^* \\ &= a^\dagger [(a^\dagger a)(aa^\dagger)]^* = a^\dagger [(a^\dagger a^2) a^\dagger]^*,\end{aligned}$$

it follows from (2) that $(a^\dagger)^2 a = a^\dagger [(a^\dagger a^2)a^\dagger]^* = a^\dagger (aa^\dagger)^* = a^\dagger$.

(3)⇒(1). If $(a^\dagger)^2 a = a^\dagger$, then

$$a = (a^\dagger)^* a^* a = [(a^\dagger)^2 a]^* a^* a = a^* [(a^\dagger)^2]^* a^* a \in a^* R,$$

which implies that a is left EP. □

Proposition 2.5. *Let $a \in R^\dagger$. Then the following statements are equivalent:*

(1) *a is right EP.*
(2) $a^2a^\dagger = a$.
(3) $a(a^\dagger)^2 = a^\dagger$.

Proof. The proof is similar to that of Proposition 2.4. □

Corollary 2.6. *Let $a \in R^\dagger$. Then a is left EP if and only if $a^\dagger$ is right EP.*

Corollary 2.7. *For $a \in R$, the following statements are equivalent:*

(1) *a is EP.*
(2) *a is left EP and $aR = a^2R$.*
(3) *a is right EP and $Ra = Ra^2$.*

Proof. (1)⇒(2). If a is EP, then it is automatically left and right EP, and by Proposition 2.5 we obtain $a = a^2a^\dagger$, which implies that $aR = a^2R$.

(2)⇒(1). Suppose that a is left EP and $aR = a^2R$. Then we have $a = a^\dagger a^2$ and $a^\dagger = (a^\dagger)^2a$ by Proposition 2.4, and $a = a^2r$ for some $r \in R$. Therefore, we can get

$$\begin{aligned} aa^\dagger &= a[(a^\dagger)^2a] = a[(a^\dagger)^2a^2r] \\ &= a[a^\dagger(a^\dagger a^2)r] = aa^\dagger ar \\ &= ar = (a^\dagger a^2)r = a^\dagger(a^2r) = a^\dagger a, \end{aligned}$$

as desired.

(1)⇔(3). It can be proved similarly. □

Recall that an element r is called Hermitian (or self-adjoint) if $r^* = r$, and that an Hermitian idempotent is called a projection. As is well known, an element $a \in R$ is Moore–Penrose invertible if and only if there exist two projections $p, q \in R$ such that $aR = pR$ and $Ra = Rq$, in which case p and q are uniquely determined by $p = aa^\dagger$ and $q = a^\dagger a$ (see, for example, [24, Theorem 2.12]). Following Kaplansky [11], such projections p and q are called the left and right projections of a, respectively. Clearly, a is EP if and only if, in addition, $p = q$. Now, for one-sided EP elements, we have the following.

Theorem 2.8. *Let $a \in R^\dagger$, and let p and q be the left and right projections of a, respectively. Then the following statements are equivalent:*

(1) *a is left EP.*
(2) *$a = uq$ for some left invertible element u commuting with q.*
(3) $qa = aq$.
(4) $qp = p$.
(5) $pq = p$.

Proof. (1)⇒(2). Suppose that a is left EP. Let $u = a+1-a^\dagger a$. A direct calculation shows that

$$uq = (a + 1 - a^\dagger a)(a^\dagger a) = aa^\dagger a = a \qquad \text{and} \qquad qu = a^\dagger a^2,$$

so we have $uq = a = qu$ by Proposition 2.4. Moreover, letting $u_l^{-1} = a^\dagger + 1 - a^\dagger a$, we see that u is left invertible as

$$\begin{aligned} u_l^{-1}u &= (a^\dagger + 1 - a^\dagger a)(a + 1 - a^\dagger a) \\ &= a^\dagger a + [a^\dagger - (a^\dagger)^2 a] + (a - a^\dagger a^2) + (1 - a^\dagger a) \\ &= a^\dagger a + (1 - a^\dagger a) \qquad \text{(by Proposition 2.4)} \\ &= 1. \end{aligned}$$

(2)⇒(3). It is clear.

(3)⇒(4). Right multiplying $qa = aq$ by $a^\dagger$ and applying $qa^\dagger = a^\dagger$, we can get

$$qp = qaa^\dagger = aa^\dagger = p.$$

(4)⇒(5). Involuting the equation $qp = p$ gives $p^*q^* = p^*$. Since p, q are Hermitian, it follows that $pq = p$.

(5)⇒(1). Left multiplying $pq = p$ by $a^\dagger$ and applying $a^\dagger p = a^\dagger$, we can get $a^\dagger(a^\dagger a) = a^\dagger$. Thus, a is left EP by Proposition 2.4. □

Remark 2.9.

(i) By interchanging p and q in (2), (3), (4), (5), and replacing left invertibility of u in (2) with right invertibility, we are led to characterizations of the right EP property.

(ii) Recall from [8] that an element $a \in R^\dagger$ is called bi-EP if $a(a^\dagger)^2 a = a^\dagger a^2 a^\dagger$, i.e., if the two projections of a commute. From the equivalence of (1), (4) and (5) in Theorem 2.8 and from (i) it follows that every left or right EP element is bi-EP.

(iii) Given any $a \in R^\dagger$, consider the multiplicative semigroup S generated by a, p and q, where p and q are the left and right projections of a, respectively. If a is left EP, then by Theorem 2.8, we have $qa = aq = a$ and $qp = pq = p$, whence it follows that S becomes a monoid with q as the identity. Conversely, if S has q as the identity, then $qa = aq$, and so by Theorem 2.8 again, a is left EP. From a similar argument, it follows that a is right EP if and only if S becomes a monoid with p as the identity.

According to [27, Theorem 4.4], an element $a \in R^\dagger$ is EP if and only if $a^\dagger = ua$ for some unit u (see [3] for the operator version). Now for left EP elements we have the following result.

Theorem 2.10. *If $a \in R$ is left EP, then $a = a^\dagger v$ for some left invertible element $v \in R$ and $a^\dagger = wa$ for some right invertible element $w \in R$. Conversely, if $a \in R^\dagger$, and it satisfies $a \in a^\dagger R$ or $a^\dagger \in Ra$, then a is left EP.*

Proof. If a is left EP, then by Proposition 2.4, $a^\dagger a^2 = a$ and $(a^\dagger)^2 a = a^\dagger$. Write

$$v = a^2 + 1 - a^\dagger a, \quad w = (a^\dagger)^2 + 1 - a^\dagger a.$$

Then we see that

$$a^\dagger v = a^\dagger a^2 + [a^\dagger - (a^\dagger)^2 a] = a, \quad wa = (a^\dagger)^2 a + (a - a^\dagger a^2) = a^\dagger,$$

and v is left invertible and w right invertible since

$$\begin{aligned} wv &= wa^2 + w(1 - a^\dagger a) \\ &= a^\dagger a + [(a^\dagger)^2 - (a^\dagger)^3 a + (1 - a^\dagger a)^2] \qquad (\text{by } wa = a^\dagger) \\ &= a^\dagger a + 1 - a^\dagger a = 1. \end{aligned}$$

Conversely, let $a \in R^\dagger$. Since $a^\dagger = a^*(a^\dagger)^* a^\dagger$, it follows from $a \in a^\dagger R$ that $aR \subseteq a^*R$, so a is left EP. Similarly, since $a = (a^\dagger)^* a^* a$, and $a^\dagger \in Ra$ implies $(a^\dagger)^* \in a^*R$, it follows from $a^\dagger \in Ra$ that $a \in (a^\dagger)^* R \subseteq a^*R$, and thus a is left EP. □

In [22], it was proved that if R is a Dedekind-finite ring, then $a \in R^\dagger$ and $aR \subseteq a^*R$ imply that $aR = a^*R$ (i.e., left EP elements in a Dedekind-finite ring are EP). Here, we use Theorem 2.10 to give another proof.

Corollary 2.11 (cf. [22]). *Let R be a Dedekind-finite ring. Then $a \in R$ is EP if and only if it is left or right EP.*

Proof. It suffices to prove the "if" part. If a is left EP, then by Theorem 2.10 there exists a left invertible element v such that $a = a^\dagger v$. Since R is a Dedekind-finite ring, it follows that v is invertible, and so $a^\dagger = av^{-1}$. Thus, $a^* = a^\dagger a a^* = (av^{-1})aa^* \in aR$, which implies that a is also right EP. So a is EP. If a is right EP, then $a^\dagger$ is left EP by Corollary 2.6. So it can be seen from the previous steps that $a^\dagger$ is EP. Again by Corollary 2.6, a is EP. □

3. Further characterizations of one-sided EP elements

Given any $a \in R^\dagger$, consider elements of the four types

$$aa^* \cdots aa^*, \quad a^*a \cdots a^*a, \quad (aa^* \cdots aa^*)a, \quad (a^*a \cdots a^*a)a^*.$$

For them, write the following two sets:

$$\begin{aligned} \Delta_a &= \{(aa^*)^m, (a^*a)^m : m > 0\}, \\ \Gamma_a &= \{(aa^*)^n a, (a^*a)^n a^* : n \geq 0\}. \end{aligned}$$

Lemma 3.1. *If $a \in R^\dagger$, then $\Delta_a \cup \Gamma_a \subseteq R^\dagger$; moreover,*

$$[(aa^*)^m]^\dagger = [(a^\dagger)^* a^\dagger]^m, \tag{3.1}$$

$$[(a^*a)^m]^\dagger = [a^\dagger (a^\dagger)^*]^m,$$

$$[(aa^*)^n a]^\dagger = a^\dagger [(a^\dagger)^* a^\dagger]^n, \tag{3.2}$$

$$[(a^*a)^n a^*]^\dagger = (a^\dagger)^* [a^\dagger (a^\dagger)^*]^n, \tag{3.3}$$

and

$$p_{(aa^*)^m} = q_{(aa^*)^m} = p_{(aa^*)^n a} = q_{(a^*a)^n a^*} = aa^\dagger, \tag{3.4}$$

$$p_{(a^*a)^m} = q_{(a^*a)^m} = p_{(a^*a)^n a^*} = q_{(aa^*)^n a} = a^\dagger a, \tag{3.5}$$

where $p_{(\cdot)}$ and $q_{(\cdot)}$ denote the left and right projections of $(\cdot)$, respectively.

Proof. It can be checked directly. □

It is clear that every element in Δ_a is Hermitian, and hence EP. But elements in Γ_a need not be EP. The next two results reveal the relationship between EP properties of a and elements in Γ_a.

Proposition 3.2. *Let $a \in R^\dagger$ and $n \geq 0$. Then the following statements are equivalent:*

(1) *a is left EP.*
(2) *$(aa^*)^n a$ is left EP.*
(3) *$(a^*a)^n a^*$ is right EP.*

Proof. (1)⇔(2). Write $b = (aa^*)^n a$. By (3.4) and (3.5), we obtain $bb^\dagger = aa^\dagger$ and $b^\dagger b = a^\dagger a$. Therefore, by Theorem 2.8,

$$(1) \;\Leftrightarrow\; (aa^\dagger)(a^\dagger a) = aa^\dagger \;\Leftrightarrow\; (bb^\dagger)(b^\dagger b) = bb^\dagger \;\Leftrightarrow\; (2).$$

(2)⇔(3). Since $(a^*a)^n a^* = [(aa^*)^n a]^*$, the result follows directly. □

Proposition 3.3. *Let $a \in R^\dagger$ and $n \geq 0$. Then the following statements are equivalent:*

(1) *a is right EP.*
(2) *$(aa^*)^n a$ is right EP.*
(3) *$(a^*a)^n a^*$ is left EP.*

Proof. It is dual to Proposition 3.2. □

Given two invertible elements $a, b \in R$, one can easily verify that

$$a^{-1}(a + b)b^{-1} = a^{-1} + b^{-1}.$$

This fact is usually known as the absorption law for ordinary inverses [1, 10, 14]. For Moore–Penrose inverses, we first see

Proposition 3.4. *Let $a \in R^\dagger$, $n \geq 0$ and $d = (aa^*)^n a$. Then $a^\dagger(a+d)d^\dagger = a^\dagger + d^\dagger$ and $d^\dagger(d + a)a^\dagger = d^\dagger + a^\dagger$.*

Proof. By (3.2), (3.4) and (3.5), we first get $d^\dagger = a^\dagger[(a^\dagger)^* a^\dagger]^n$, $dd^\dagger = aa^\dagger$ and $d^\dagger d = a^\dagger a$. Since $a^\dagger a d^\dagger = d^\dagger$, it follows that

$$a^\dagger(a + d)d^\dagger = a^\dagger a d^\dagger + a^\dagger d d^\dagger = d^\dagger + a^\dagger a a^\dagger = d^\dagger + a^\dagger.$$

Since $d^\dagger a a^\dagger = d^\dagger$, it follows that

$$d^\dagger(d + a)a^\dagger = d^\dagger d a^\dagger + d^\dagger a a^\dagger = a^\dagger a a^\dagger + d^\dagger = a^\dagger + d^\dagger.$$ □

However, in general, for two elements $a, b \in R^\dagger$, $a^\dagger(a + b)b^\dagger$ and $a^\dagger + b^\dagger$ are not equal. We next consider the relations between one-sided EP properties and the absorption law for Moore–Penrose inverses.

Proposition 3.5. *Let $a \in R^\dagger$. Then the following statements are equivalent:*

(1) *a is left EP.*
(2) *$a^\dagger(a + b)b^\dagger = a^\dagger + b^\dagger$ for every $b \in \Delta_a \cup \Gamma_a$.*
(3) *$a^\dagger(a + b)b^\dagger = a^\dagger + b^\dagger$ for some $b \in \Delta_a \cup \Gamma_a - \{(aa^*)^n a : n \geq 0\}$.*

Proof. (1)⇒(2). Assume (1). In view of Proposition 3.4, it is enough to show that $a^\dagger(a+b)b^\dagger = a^\dagger + b^\dagger$ holds for every $b \in \Delta_a \cup \Gamma_a - \{(aa^*)^n a : n \geq 0\}$. For such a b, we claim that

$$a^\dagger ab^\dagger = b^\dagger \quad \text{and} \quad a^\dagger bb^\dagger = a^\dagger. \tag{3.6}$$

If this is the case, then $a^\dagger(a+b)b^\dagger = a^\dagger ab^\dagger + a^\dagger bb^\dagger = a^\dagger + b^\dagger$. To verify (3.6), we see:

Case (i): When $b = (aa^*)^m$, we have $b^\dagger = [(a^\dagger)^* a^\dagger]^m$ and $bb^\dagger = aa^\dagger$ by (3.1), (3.4); so $a^\dagger bb^\dagger = a^\dagger aa^\dagger = a^\dagger$, $a^\dagger ab^\dagger = a^\dagger a[(a^\dagger)^* a^\dagger]^m$. Since a being left EP gives

$$a^\dagger a(a^\dagger)^* = (a^\dagger a)^*(a^\dagger)^* = [(a^\dagger)^2 a]^* = (a^\dagger)^*,$$

we can get $a^\dagger ab^\dagger = [a^\dagger a(a^\dagger)^*]a^\dagger[(a^\dagger)^* a^\dagger]^{m-1} = [(a^\dagger)^* a^\dagger]^m = b^\dagger$, as desired.

Case (ii): When $b = (a^*a)^m$, we have $a^\dagger ab^\dagger = a^\dagger a[a^\dagger(a^\dagger)^*]^m = [a^\dagger(a^\dagger)^*]^m = b^\dagger$ immediately. Moreover, by (3.5), $bb^\dagger = a^\dagger a$; since a is left EP, it follows that $a^\dagger bb^\dagger = (a^\dagger)^2 a = a^\dagger$.

Case (iii): When $b = (a^*a)^n a^*$, we have $b^\dagger = (a^\dagger)^*[a^\dagger(a^\dagger)^*]^n$ and $bb^\dagger = a^\dagger a$ by (3.3), (3.5). Hence, $a^\dagger ab^\dagger = (a^\dagger a)^* b^\dagger = [(a^\dagger)^2 a]^*[a^\dagger(a^\dagger)^*]^n$, $a^\dagger bb^\dagger = (a^\dagger)^2 a$. Since a is left EP, we have $(a^\dagger)^2 a = a^\dagger$, and so $a^\dagger ab^\dagger = (a^\dagger)^*[a^\dagger(a^\dagger)^*]^n = b^\dagger$, $a^\dagger bb^\dagger = a^\dagger$.

Therefore, (1)⇒(2) is completed.

(2)⇒(3) is clear.

(3)⇒(1). If $a^\dagger(a+b)b^\dagger = a^\dagger + b^\dagger$ for some $b = (aa^*)^m$, left multiplying this equation by $1 - a^\dagger a$, we get $0 = (1 - a^\dagger a)b^\dagger$, and so $a^\dagger ab^\dagger = b^\dagger$. Right multiplying $a^\dagger ab^\dagger = b^\dagger$ by b and using $b^\dagger b = aa^\dagger$, we get $(a^\dagger a)(aa^\dagger) = aa^\dagger$. Therefore, a is left EP by Theorem 2.8. Or else, if $a^\dagger(a+b)b^\dagger = a^\dagger + b^\dagger$ for some $b = (a^*a)^m$ or $b = (a^*a)^n a^*$, right multiplying this equation by $1 - bb^\dagger$, we then obtain $0 = a^\dagger(1 - bb^\dagger)$, and so $a^\dagger = a^\dagger bb^\dagger$. Since $bb^\dagger = a^\dagger a$, it follows that $a^\dagger = (a^\dagger)^2 a$. Therefore, a is left EP by Proposition 2.4. □

Proposition 3.6. *Let $a \in R^\dagger$. Then the following statements are equivalent:*

(1) *a is right EP.*
(2) *$b^\dagger(b+a)a^\dagger = b^\dagger + a^\dagger$ for every $b \in \Delta_a \cup \Gamma_a$.*
(3) *$b^\dagger(b+a)a^\dagger = b^\dagger + a^\dagger$ for some $b \in \Delta_a \cup \Gamma_a - \{(aa^*)^n a : n \geq 0\}$.*

Proof. It is dual to Proposition 3.5. □

In addition to the Moore–Penrose inverse, there exist also some other generalized inverses that are closely related to EP properties. Recall that $a \in R$ is group invertible if there exists $x \in R$ such that

$$axa = a, \quad xax = x, \quad ax = xa,$$

in which case such an x is unique, denoted by $a^\#$, and called the group inverse of a; a is core invertible if there exists $x \in R$ such that

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad xa^2 = a, \quad ax^2 = x,$$

in which case such an x is unique, denoted by $a^{\oplus}$, and called the core inverse of a; a is dual core invertible if there exists $x \in R$ such that

$$axa = a, \quad xax = x, \quad (xa)^* = xa, \quad a^2x = a, \quad x^2a = x,$$

in which case such an x is unique, denoted by $a_{\textcircled{\#}}$, and called the dual core inverse of a.

It was proved in [24, Theorem 3.1] that if a is EP then the four generalized inverses $a^{\dagger}$, $a^{\#}$, $a^{\textcircled{\#}}$, $a_{\textcircled{\#}}$ exist and are equal; and conversely if any two of $a^{\dagger}$, $a^{\#}$, $a^{\textcircled{\#}}$, $a_{\textcircled{\#}}$ exist and are equal then a is EP. However, when a is merely left or right EP, it can be seen from Example 2.2 that in general, the three generalized inverses $a^{\#}$, $a^{\textcircled{\#}}$ and $a_{\textcircled{\#}}$ do not exist while $a^{\dagger}$ is always assumed to exist.

In the rest of this paper, we shall derive the corresponding one-sided version of [24, Theorem 3.1] by taking advantage of some one-sided generalized inverses. For our purpose, first recall from [5, 6] that, given two elements $b, c \in R$, $a \in R$ is said to be (b, c)-invertible if there exists $x \in R$ such that

$$x \in bR \cap Rc, \quad xab = b, \quad cax = c,$$

in which case such an x is unique and called the (b, c)-inverse of a. Moreover, a is said to be left (b, c)-invertible if there exists $x \in Rc$ satisfying $xab = b$, in which case any such x is called a left (b, c)-inverse of a; dually, a is said to be right (b, c)-invertible if there exists $x \in bR$ satisfying $cax = c$, in which case any such x is called a right (b, c)-inverse of a.

According to [17, 5, 24], by choosing specific elements b and c, the Moore–Penrose inverse, group inverse, core inverse and dual core inverse can all be expressed in terms of (b, c)-inverses:

- a is Moore–Penrose invertible if and only if a is (a^*, a^*)-invertible, in which case $a^{\dagger}$ coincides with the (a^*, a^*)-inverse of a;
- a is group invertible if and only if a is (a, a)-invertible, in which case $a^{\#}$ coincides with the (a, a)-inverse of a;
- a is core invertible if and only if a is (a, a^*)-invertible, in which case $a^{\textcircled{\#}}$ coincides with the (a, a^*)-inverse of a;
- a is dual core invertible if and only if a is (a^*, a)-invertible, in which case $a_{\textcircled{\#}}$ coincides with the (a^*, a)-inverse of a.

Meanwhile, left (a, a)-inverses, left (a, a^*)-inverses and left (a^*, a)-inverses can be regarded as left versions of group inverses, core inverses and dual core inverses, respectively. By using the language of these one-sided generalized inverses, the next two results generalize [24, Theorem 3.1] to one-sided versions.

Proposition 3.7. *For $a \in R$, the following statements are equivalent:*

(1) *a is left EP.*
(2) *$a^{\dagger}$ exists and $a^{\dagger}$ is a left (a, a)-inverse of a.*
(3) *$a^{\dagger}$ exists and $a^{\dagger}$ is a left (a, a^*)-inverse of a.*
(4) *$a^{\dagger}$ exists and $a^{\dagger}$ is a left (a^*, a)-inverse of a.*

Proof. (1)⇒(2). If a is left EP, then $a^{\dagger}$ exists, and by Proposition 2.4 we have $a^{\dagger} = (a^{\dagger})^2 a \in Ra$ and $a^{\dagger}a^2 = a$. So by the definition, $a^{\dagger}$ is a left (a, a)-inverse of a.

(2)⇒(3). Assume (2). For (3), it is enough to show $a^{\dagger} \in Ra^*$. This follows naturally by $a^{\dagger} = a^{\dagger}(aa^{\dagger})^* = a^{\dagger}(a^{\dagger})^* a^* \in Ra^*$.

(3)⇒(1). Assume (3). Since $a^{\dagger}$ is a left (a, a^*)-inverse of a, it follows that $a^{\dagger}a^2 = a$. Thus, a is left EP by Proposition 2.4.

(1)⇔(4). If a is left EP, then $a^\dagger$ exists and satisfies $a^\dagger aa^* = a^*$; moreover, by Proposition 2.4, $a^\dagger = (a^\dagger)^2 a \in Ra$. Thus, $a^\dagger$ is a left (a^*, a)-inverse of a. Conversely, assume (4). Then there exists $r \in R$ such that $a^\dagger = ra$, and so $a^\dagger = r(aa^\dagger a) = (ra)a^\dagger a = (a^\dagger)^2 a$. Thus, a is left EP by Proposition 2.4. □

Proposition 3.8. *For $a \in R$, the following statements are equivalent:*

(1) *a is left EP.*
(2) *There exists $x \in R$ which is both a left (a, a^*)-inverse and a left (a^*, a)-inverse of a.*
(3) *There exists $x \in R$ which is both a left (a^*, a^*)-inverse and a left (a, a)-inverse of a.*
(4) *There exists $x \in R$ which is both a left (a^*, a^*)-inverse and a left (a, a^*)-inverse of a.*
(5) *There exists $x \in R$ which is both a left (a^*, a^*)-inverse and a left (a^*, a)-inverse of a.*

Proof. (1)⇔(2). Assume that a is left EP. Then $a^\dagger$ exists, and by Proposition 3.7, $x = a^\dagger$ is both a left (a, a^*)-inverse and a left (a^*, a)-inverse of a. Conversely, assume that such an x exists. Since x is a left (a^*, a)-inverse of a, we have $xaa^* = a^*$, so $(xa)^* = a^*x^* = xaa^*x^* = xa(xa)^*$. It follows that

$$\text{(i) } (xa)^* = xa, \quad \text{(ii) } a = (xaa^*)^* = axa.$$

Since x is also a left (a, a^*)-inverse of a, we have $x = ra^*$ for some $r \in R$ and $xa^2 = a$. Now, by $x = ra^*$ and $axa = a$, we obtain

$$x = r(axa)^* = ra^*(ax)^* = x(ax)^* \quad \text{and} \quad ax = ax(ax)^*,$$

which implies that

$$\text{(iii) } (ax)^* = ax, \quad \text{(iv) } x = x(ax)^* = xax.$$

Therefore, by the definition, $x = a^\dagger$. Since $xa^2 = a$, it follows by Proposition 2.4 that a is left EP.

(1)⇔(3)⇔(4)⇔(5). According to [28, Theorem 2.16], x being a left (a^*, a^*)-inverse of a amounts to $x = a^\dagger$. Therefore, the equivalence of (1), (3), (4), and (5) follows by Proposition 3.7. □

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