REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 68, No. 1, 2024, Pages 49–54
Published online: December 20, 2024
https://doi.org/10.33044/revuma.3574

HAAR WAVELET CHARACTERIZATION OF DYADIC LIPSCHITZ REGULARITY

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To Pola Harboure and Roberto Macías

Abstract. We obtain a necessary and sufficient condition on the Haar coefficients of a real function f defined on $\mathbb{R}^+$ for the Lipschitz α regularity of f with respect to the ultrametric $\delta(x,y) = \inf\{|I| : x, y \in I; I \in \mathcal{D}\}$, where $\mathcal{D}$ is the family of all dyadic intervals in $\mathbb{R}^+$ and α is positive. Precisely, $f \in \mathrm{Lip}_\delta(\alpha)$ if and only if $|\langle f h^j_k \rangle| \leq C2^{-(\alpha+1/2)j}$ for some constant C, every $j \in \mathbb{Z}$ and every $k = 0, 1, 2, \ldots$ Here, as usual, $h^j_k(x) = 2^{j/2}h(2^j x - k)$ and $h(x) = \mathcal{X}_{[0,1/2)}(x) - \mathcal{X}_{[1/2,1)}(x)$.

Arde de abejas el aguaribay, arde.
Ríen los ojos, los labios, hacia las islas azules
a través de la cortina
de los racimos
pálidos.

Juan L. Ortiz

1. Introduction

In [4] and [3] (see also [2]), M. Holschneider and Ph. Tchamitchian provide characterizations of the Lipschitz α regularity of a function in $L^2(\mathbb{R})$ for $0 < \alpha < 1$ in terms of the behaviour of the continuous wavelet transform. The result is that a given function is Lipschitz α if and only if its continuous wavelet transform satisfies a power law in the absolute value of the scale parameter. Here Lipschitz α refers to the classical definition with respect to the usual metric in $\mathbb{R}$, i.e., $|f(x) - f(y)| \leq C\,|x-y|^\alpha$ for some constant $C > 0$ and every x and y in $\mathbb{R}$. In [1] these results are extended to more general moduli of regularity of functions when the basic wavelet is the Haar wavelet. The method used in [1] provides the tool for the analysis of

2020 *Mathematics Subject Classification.* 42C15.

Key words and phrases. wavelets, Lipschitz regularity.

This work was supported by the Ministerio de Ciencia, Tecnología e Innovación (MINCYT) in Argentina: Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) grant PIP-2021-2023-11220200101940CO, and Universidad Nacional del Litoral grant CAI+D 50620190100070LI.

pointwise regularity through the discrete wavelet transform associated to dyadic scaling and integer translations of the Haar wavelet. The natural Lipschitz α class, in our setting, is defined through the dyadic distance instead of the usual one.

The result of this paper is contained in the next statement.

Theorem 1.1. *Let f be a real valued function in $L^1_{\text{loc}}(\mathbb{R}^+)$. Let $h^j_k(x) = 2^{j/2} \times h(2^j x - k)$, where $h(x) = \mathcal{X}_{[0,1/2)}(x) - \mathcal{X}_{[1/2,1)}(x)$, $j \in \mathbb{Z}$, $k = 0, 1, 2, \ldots$, and $\langle f, h^j_k \rangle = \int_{\mathbb{R}^+} f(x) h^j_k(x)\, dx$. Let α be any positive number. Then, the boundedness of the sequence*

$$\left\{ 2^{\left(\alpha+\frac{1}{2}\right)j} \left| \langle f, h^j_k \rangle \right| : j \in \mathbb{Z},\, k = 0, 1, 2, \ldots \right\}$$

is equivalent to the essential boundedness of the quotients

$$\frac{|f(x) - f(y)|}{\delta^\alpha(x, y)}, \quad x \neq y,$$

where $\delta(x, y) = \inf\{|I| : x, y \in I; I \in \mathcal{D}\}$ with $\mathcal{D}$ the family of all dyadic intervals in $\mathbb{R}^+$.

In Section 2 we introduce the basic facts and notation, and Section 3 is devoted to the proof of Theorem 1.1.

2. Dyadic distance in $\mathbb{R}^+$ and the Haar system

The set of nonnegative real numbers is denoted here by $\mathbb{R}^+$. The family of all dyadic intervals in $\mathbb{R}^+$ is the disjoint union of the classes $\mathcal{D}^j$, $j \in \mathbb{Z}$, where $\mathcal{D}^j = \{I^j_k = [k2^{-j}, (k+1)2^{-j}) : k = 0, 1, 2, \ldots\}$ are the dyadic intervals of level j. Notice that with this notation, when j grows, the partitions of $\mathbb{R}^+$ get refined and the intervals smaller. Since given two points x and y of $\mathbb{R}^+$ there exists some $j_0 \in \mathbb{Z}$ such that $0 \leq \max\{x, y\} < 2^{-j_0}$, we have that $x, y \in I^{j_0}_0$. Hence, the class of all dyadic intervals $I \in \mathcal{D}$ such that both x and y belong to I is non-empty. Therefore, if $|E|$ denotes the Lebesgue length of the measurable set E, we have that

$$\delta(x, y) = \inf \{|I| : x, y \in I; I \in \mathcal{D}\}$$

is a well-defined nonnegative real number. Even more, δ is an ultrametric in $\mathbb{R}^+$. In other words,

(i) $\delta(x, y) = 0$ if and only if $x = y$;
(ii) $\delta(x, y) = \delta(y, x)$ for every x, y in $\mathbb{R}^+$;
(iii) $\delta(x, z) \leq \max\{\delta(x, y), \delta(y, z)\}$ for every x, y, z in $\mathbb{R}^+$.

The triangle inequality follows from the properties of the family $\mathcal{D}$. In fact, given x, y and z in $\mathbb{R}^+$, let $I(x, y)$ and $I(y, z)$ denote the smallest dyadic intervals containing x, y and y, z, respectively; then, one of these intervals contains the other because $y \in I(x, y) \cap I(y, z) \neq \emptyset$. Assume $I(x, y) \supseteq I(y, z)$; then $\delta(x, z) \leq |I(x, y)| = \max\{|I(y, z)|, |I(x, y)|\} = \max\{\delta(y, z), \delta(x, y)\}$. In particular, δ is a metric in $\mathbb{R}^+$. Notice that $|x - y| \leq \delta(x, y)$, but $\frac{\delta(x,y)}{|x-y|}$ is unbounded.

Hence every Lipschitz α function f in the usual sense ($|f(x) - f(y)| \leq C\,|x-y|^{\alpha}$) is also a $\mathrm{Lip}_\delta(\alpha)$ function, i.e.,

$$|f(x) - f(y)| \leq C\delta^{\alpha}(x,y)$$

for some constant C and every x and y in $\mathbb{R}^+$. On the other hand, there are $\mathrm{Lip}_\delta(\alpha)$ functions which are not Lipschitz α in the classical sense. In fact, $\mathcal{X}_I$, $I \in \mathcal{D}$, is in the class $\mathrm{Lip}_\delta(1)$. We also observe that in contrast with the class Lipschitz α for every $\alpha > 1$, which is trivial, there exist nonconstant $\mathrm{Lip}_\delta(\alpha)$ functions for every $\alpha > 0$.

Let us now review the basic properties of the Haar system. Set $h_0^0(x) = \mathcal{X}_{[0,1/2)}(x) - \mathcal{X}_{[1/2,1)}(x)$ and $h_k^j(x) = 2^{j/2}h_0^0(2^j x - k)$ for $j \in \mathbb{Z}$ and $k = 0, 1, 2, \ldots$ The family $\mathscr{H} = \{h_k^j : j \in \mathbb{Z}, k = 0, 1, 2, \ldots\}$ is the Haar system in $\mathbb{R}^+$. It is well known that $\mathscr{H}$ is an orthonormal basis for $L^2(\mathbb{R}^+)$. Since for each $I \in \mathcal{D}$ there is one and only one $h \in \mathscr{H}$ supported in I, we write sometimes h_I to denote the $h \in \mathscr{H}$ supported in $I \in \mathcal{D}$ and sometimes I_h to denote the dyadic support of $h \in \mathscr{H}$. From the basic character of $\mathscr{H}$ in $L^2(\mathbb{R}^+)$ we have that, given $f \in L^2(\mathbb{R}^+)$,

$$f = \sum_{h \in \mathscr{H}} \langle f, h\rangle h,$$

in the $L^2(\mathbb{R}^+)$-sense, with $\langle f, h\rangle = \int_{\mathbb{R}^+} f(x)h(x)\,dx$. The sequence of coefficients $\{\langle f, h\rangle : h \in \mathscr{H}\}$ is well defined even for functions in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$, since the Haar functions are bounded and have bounded support.

3. Proof of Theorem 1.1

The easy part of Theorem 1.1 follows as usual from the vanishing of the mean of the Haar functions. Let us state and prove it.

Proposition 3.1. *Let $f \in \mathrm{Lip}_\delta(\alpha)$, $\alpha > 0$. Set $[f]_{\mathrm{Lip}_\delta(\alpha)}$ to denote the infimum of the constants $C > 0$ such that $|f(x) - f(y)| \leq C\delta^{\alpha}(x,y)$, $x, y \in \mathbb{R}^+$. Then $|\langle f, h_I\rangle| \leq [f]_{\mathrm{Lip}_\delta(\alpha)}\,|I|^{\alpha+\frac{1}{2}}$ for every $I \in \mathcal{D}$.*

Proof. For $I = [a_I, b_I) \in \mathcal{D}$ we have $\int_{\mathbb{R}^+} h_I(x)\,dx = 0$; hence

$$\begin{aligned}
|\langle f, h_I\rangle| &= \left|\int_{\mathbb{R}^+} f(x)h_I(x)\,dx\right| \\
&= \left|\int_{\mathbb{R}^+} (f(x) - f(a_I))h_I(x)\,dx\right| \\
&\leq \int_I |f(x) - f(a_I)|\,|h_I(x)|\,dx \\
&\leq [f]_{\mathrm{Lip}_\delta(\alpha)} \int_I \delta^{\alpha}(x, a_I)\,|I|^{-\frac{1}{2}}\,dx \\
&\leq [f]_{\mathrm{Lip}_\delta(\alpha)}\,|I|^{\alpha-\frac{1}{2}} \int_I dx \\
&= [f]_{\mathrm{Lip}_\delta(\alpha)}\,|I|^{\alpha-\frac{1}{2}+1} = [f]_{\mathrm{Lip}_\delta(\alpha)}\,|I|^{\alpha+\frac{1}{2}}.
\end{aligned}$$

□

In order to prove that the size of the coefficients guarantee the regularity of f, we start by stating and proving a lemma. Given an interval $I \in \mathcal{D}$, we denote by I^- and I^+ its left and right halves, respectively. Notice that when $I \in \mathcal{D}^j$, both I^- and I^+ belong to $\mathcal{D}^{j+1}$. Given a locally integrable function f, we write $m_I(f)$ to denote the mean value of f on $I \in \mathcal{D}$. In other words, $m_I(f) = \frac{1}{|I|}\int_I f(x)\,dx$.

Lemma 3.2. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$. Then, for every $I \in \mathcal{D}$, we have*

$$|m_{I^-}(f) - m_{I^+}(f)| = 2\,|I|^{-\frac{1}{2}}\,|\langle f, h_I\rangle|\,.$$

Proof. Let $I \in \mathcal{D}$ be given. Then

$$\begin{aligned}
|m_{I^-}(f) - m_{I^+}(f)| &= \left|\frac{2}{|I|}\int_{I^-} f(x)\,dx - \frac{2}{|I|}\int_{I^+} f(x)\,dx\right| \\
&= 2\,|I|^{-\frac{1}{2}}\left|\int_I |I|^{-\frac{1}{2}}\left(\mathcal{X}_{I^-}(x) - \mathcal{X}_{I^+}(x)\right) f(x)\,dx\right| \\
&= 2\,|I|^{-\frac{1}{2}}\left(\int_{\mathbb{R}^+} h_I(x) f(x)\,dx\right) \\
&= 2\,|I|^{-\frac{1}{2}}\,|\langle f, h_I\rangle|\,.
\end{aligned}$$

□

Proposition 3.3. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ be such that, for some constant $A > 0$, we have*

$$|\langle f, h_I\rangle| \le A\,|I|^{\alpha+\frac{1}{2}}$$

for every $I \in \mathcal{D}$. Then $f \in \mathrm{Lip}_\delta(\alpha)$ and $[f]_{\mathrm{Lip}_\delta(\alpha)} \le C_\alpha A$ with $C_\alpha = \sup\{2, \frac{1}{2^\alpha - 1}\}$.

Proof. Let $x < y$ be two points in $\mathbb{R}^+$. Let $I \in \mathcal{D}$ be the smallest dyadic interval containing x and y. In other words, $|I| = \delta(x,y)$. Since $x < y$, necessarily $x \in I^-$ and $y \in I^+$. Set $I^x_1 = I^-$ and $I^y_1 = I^+$. Now let I^x_2 be the half of I^x_1 to which x belongs, and I^y_2 the half of I^y_1 with $y \in I^y_2$. In general, once I^x_l and I^y_l are defined, we select I^x_{l+1} as the only half of I^x_l with $x \in I^x_{l+1}$ and I^y_{l+1} as the only half of I^y_l with $y \in I^y_{l+1}$. In this way, for a fixed positive integer k, we have

$$I^x_k \subset I^x_{k-1} \subset \cdots \subset I^x_2 \subset I^x_1 \subset I$$

and

$$I^y_k \subset I^y_{k-1} \subset \cdots \subset I^y_2 \subset I^y_1 \subset I.$$

Hence

$$\begin{aligned}
f(x) - f(y) = &\left(f(x) - m_{I^x_k}(f)\right) \\
&+ \left(m_{I^x_k}(f) - m_{I^x_{k-1}}(f)\right) + \cdots + \left(m_{I^x_2}(f) - m_{I^x_1}(f)\right) \\
&+ \left(m_{I^x_1}(f) - m_{I^y_1}(f)\right) \\
&+ \left(m_{I^y_1}(f) - m_{I^y_2}(f)\right) + \cdots + \left(m_{I^y_{k-1}}(f) - m_{I^y_k}(f)\right) \\
&+ \left(m_{I^y_k}(f) - f(y)\right).
\end{aligned}$$

Then

$$\begin{aligned}|f(x)-f(y)| \le &\left|f(x)-m_{I_k^x}(f)\right| \\ &+\sum_{l=2}^{k}\left|m_{I_l^x}(f)-m_{I_{l-1}^x}(f)\right| \\ &+\left|m_{I_1^x}(f)-m_{I_1^y}(f)\right| \\ &+\sum_{l=1}^{k-1}\left|m_{I_l^y}(f)-m_{I_{l+1}^y}(f)\right| \\ &+\left|m_{I_k^y}(f)-f(x)\right| \\ =&\ \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.\end{aligned}$$

Let us start by bounding the central term III. Notice that $I_1^x = I^-$ and $I_1^y = I^+$, with $|I| = \delta(x,y)$. Then by Lemma 3.2, we have

$$\begin{aligned}\mathrm{III} &= \left|m_{I_1^x}(f)-m_{I_1^y}(f)\right| \\ &= |m_{I^-}(f)-m_{I^+}(f)| \\ &= 2\,|I|^{-\frac{1}{2}}\,|\langle f,h_I\rangle| \\ &\le 2A\,|I|^{-\frac{1}{2}}\,|I|^{\alpha+\frac{1}{2}} \\ &= 2A\,|I|^{\alpha} \\ &= 2A\delta^{\alpha}(x,y),\end{aligned}$$

which has the desired form. The terms II and IV can be handled in the same way; let us deal with II. Take a generic term of the sum II, and use again Lemma 3.2:

$$\begin{aligned}\left|m_{I_l^x}(f)-m_{I_{l-1}^x}(f)\right| &= \left|\frac{1}{|I_l^x|}\int_{I_l^x} f - \frac{1}{|I_{l-1}^x|}\left(\int_{I_l^x} f + \int_{I_{l-1}^x\setminus I_l^x} f\right)\right| \\ &= \left|\frac{1}{2}\frac{1}{|I_l^x|}\int_{I_l^x} f - \frac{1}{2}\frac{1}{\left|I_{l-1}^x\setminus I_l^x\right|}\int_{I_{l-1}^x\setminus I_l^x} f\right| \\ &= \frac{1}{2}\left|m_{I_l^x}(f)-m_{I_{l-1}^x\setminus I_l^x}(f)\right| \\ &= \frac{1}{2}2\left|I_{l-1}^x\right|^{-\frac{1}{2}}\left|\langle f,h_{I_{l-1}^x}\rangle\right| \\ &\le A\left|I_{l-1}^x\right|^{-\frac{1}{2}}\left|I_{l-1}^x\right|^{\alpha+\frac{1}{2}} \\ &= A\left|I_{l-1}^x\right|^{\alpha} \\ &= A\frac{2^{\alpha}}{2^{\alpha l}}\,|I|^{\alpha}.\end{aligned}$$

Then

$$\begin{aligned}\mathrm{II} &= \sum_{l=2}^{k} \left| m_{I_l^x}(f) - m_{I_{l-1}^x}(f) \right| \\ &\leq A 2^{\alpha} |I|^{\alpha} \sum_{l \geq 2} \frac{1}{2^{\alpha l}} \\ &= \frac{A}{2^{\alpha} - 1} \delta^{\alpha}(x, y).\end{aligned}$$

The same estimate holds for IV. Let $C_\alpha = \sup\{2, \frac{1}{2^\alpha - 1}\}$. Then

$$|f(x) - f(y)| \leq \left| f(x) - m_{I_k^x}(f) \right| + A C_\alpha \delta^\alpha(x, y) + \left| f(y) - m_{I_k^y}(f) \right|$$

uniformly in k. Now, from the differentiation theorem, we have for almost all x and almost all y that $m_{I_k^x}(f) \to f(x)$ as $k \to \infty$ and $m_{I_k^y}(f) \to f(y)$ as $k \to \infty$. Hence, for those values of x and y in $\mathbb{R}^+$ we get the result

$$|f(x) - f(y)| \leq A C_\alpha \delta^\alpha(x, y). \qquad \square$$

Propositions 3.1 and 3.3 prove Theorem 1.1.

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Received: September 14, 2022
Accepted: March 7, 2023
Early view: August 22, 2024