

THE GREEN RING OF A FAMILY OF COPOINTED HOPF ALGEBRAS

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ABSTRACT. The copointed liftings of the Fomin–Kirillov algebra \mathcal{FK}_3 over the algebra of functions on the symmetric group \mathbb{S}_3 were classified by Andruskiewitsch and the author. We demonstrate here that those associated to a generic parameter are Morita equivalent to the regular blocks of well-known Hopf algebras: the Drinfeld doubles of the Taft algebras and the small quantum groups $u_q(\mathfrak{sl}_2)$. The indecomposable modules over these were classified independently by Chen, Chari–Premet and Suter. Consequently, we obtain the indecomposable modules over the generic liftings of \mathcal{FK}_3 . We decompose the tensor products between them into the direct sum of indecomposable modules. We then deduce a presentation by generators and relations of the Green ring.

1. INTRODUCTION

The distinctive feature of a Hopf algebra, and source of its greatest applications, is that its representations form a tensor category. Therefore it is natural to investigate the structure of the tensor products between its modules. This information is encoded in the Green ring, which was first considered in the context of finite groups by J. A. Green [18]. This is the ring generated by the isomorphism classes of modules with sum and product induced by the direct sum and tensor product of modules. A first problem which arises is to compute the indecomposable summands of the tensor product of two indecomposable modules.

In the present work, we address the above problem for the generic liftings of the Fomin–Kirillov algebra \mathcal{FK}_3 over the algebra on functions of the symmetric group \mathbb{S}_3 . This is a certain infinite subfamily of the copointed Hopf algebras over $\mathbb{k}^{\mathbb{S}_3}$ classified by Andruskiewitsch and the author in [3]. Moreover, we give a presentation by generators and relations of the Green ring.

Similar results are found in the literature, for instance, for: the small quantum groups $u_q(\mathfrak{sl}_2)$ [20]; the Taft algebras [10], their generalized versions [12, 21, 22] and

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their Drinfeld doubles [9, 13, 14, 25]; the (twisted) Drinfeld doubles of finite groups [29, 30]; the non-semisimple Hopf algebras of dimension 8 [28]; the basic Hopf algebras of finite representation type [19]; the Kac–Paljutkin type algebras [11]; and the Wu–Liu–Ding algebras [33, 31, 32]. The major obstacles for dealing with other Hopf algebras are that they usually are of infinite representation type and the explicit computations of actions are too involved, as we can appreciate in some partial results, e.g. [17, 23].

Let \mathcal{A} be a generic copointed Hopf algebra over $\mathbb{k}^{\mathbb{S}_3}$; we precisely define it in Section 2. In order to achieve our goals, we first demonstrate that \mathcal{A} is Morita equivalent to every regular (i.e., non-simple) block of the Drinfeld double $\mathcal{D}(T_n(\zeta))$ of a Taft algebra and also to those of the small quantum groups $u_q(\mathfrak{sl}_2)$ with $q^2 = \zeta$ a primitive n -th root of unity. As a direct consequence, we obtain the classification of the indecomposable \mathcal{A} -modules since the respective $\mathcal{D}(T_n(\zeta))$ -modules were classified in [8], and independently in [5, 26] for $u_q(\mathfrak{sl}_2)$.

Theorem 1.1. *The following list constitutes a complete set of non-isomorphic indecomposable \mathcal{A} -modules.*

- (1) *The simple modules ε and L .*
- (2) *The projective cover and injective hull P_ε and P_L of ε and L , respectively.*
- (3) *The syzygies and cosyzygy modules $\Omega^k(\varepsilon)$ and $\Omega^k(L)$, $k \in \mathbb{Z} \setminus \{0\}$.*
- (4) *The (k, k) -type modules $M_k(\varepsilon, \mathfrak{t})$ and $M_k(L, \mathfrak{t})$, $k \in \mathbb{N}$ and $0 \neq \mathfrak{t} \in \mathfrak{A}/\sim$.*

In particular, \mathcal{A} is of tame representation type.

We briefly describe the modules on the above list in Figure 1. Some of them were studied earlier in [4]. Here, ε denotes the trivial one-dimensional module given by the antipode. The reader can think of \mathfrak{A}/\sim as the projective line \mathbb{P}^1 .

P_L	P_ε	$M_k(\varepsilon, \mathfrak{t})$	$L \simeq L^*$
$\dots L \dots$	$\dots \varepsilon \dots$	$\dots kL \dots$	$\varepsilon \simeq \varepsilon^*$
$\dots \varepsilon \oplus \varepsilon \dots$	$\dots L \oplus L \dots$	$k\varepsilon$	$P_L \simeq P_L^*$
L	ε		$P_\varepsilon \simeq P_\varepsilon^*$
$\Omega^{2k+1}(\varepsilon)$	$\Omega^{2k}(\varepsilon)$	$\Omega^{2k+1}(L)$	$\Omega^{-k}(\varepsilon) \simeq \Omega^k(\varepsilon)^*$
$\dots 2(k+1)L \dots$	$\dots (2k+1)\varepsilon \dots$	$\dots 2(k+1)\varepsilon \dots$	$\Omega^{-k}(L) \simeq \Omega^k(L)^*$
$(2k+1)\varepsilon$	$2kL$	$(2k+1)L$	$M_k(L, \mathfrak{t}) \simeq M_k(\varepsilon, \mathfrak{t})^*$
		$2k\varepsilon$	

FIGURE 1. Loewy layers of some indecomposable modules; kM denotes the direct sum of $k \in \mathbb{N}$ copies of M . The remaining ones can be deduced using the isomorphisms on the right-hand side.

Next, we calculate the indecomposable summands of every tensor product between indecomposable modules imitating the strategy of [9]. This is a case-by-case analysis. We employ an inductive argument when tensoring with modules of the

infinite families. We also need to perform some explicit computations which we carry out in GAP [16] as we explain in Appendix A. These are very long and tedious calculations to do by hand; in the worst case we have to compute actions over the tensor product of two modules of dimension 18. Once we have these decompositions, we are ready to present the Green ring of \mathcal{A} by generators and relations using standard arguments. Explicitly, we demonstrate the following.

Theorem 1.2. *The Green ring of \mathcal{A} is isomorphic to the commutative \mathbb{Z} -algebra generated by the elements $\lambda, \rho, \omega, \bar{\omega}$ and $\mu_{k,\mathbf{t}}$, for all $k \in \mathbb{N}$ and $\mathbf{t} \in \mathfrak{A}/\sim$, subject to the relations in Table 1. Moreover, the set*

$$\mathcal{B} = \{ \lambda^i, \lambda^i \rho, \lambda^i \omega^s, \lambda^i \bar{\omega}^s, \lambda^i \mu_{k,\mathbf{t}} \mid i \in \{0, 1\}, s, k \in \mathbb{N}, \mathbf{t} \in \mathfrak{A}/\sim \}$$

is a \mathbb{Z} -basis of the Green ring of \mathcal{A} .

TABLE 1. Generators and relations of the Green ring of \mathcal{A}

$1 := [\varepsilon]$	$\lambda := [L]$	$\rho := [P_L]$	$\omega := [\Omega(\varepsilon)]$	$\bar{\omega} := [\Omega^{-1}(\varepsilon)]$	$\mu_{k,\mathbf{t}} := [M_k(\varepsilon, \mathbf{t})]$
					$\forall k \in \mathbb{N}, \mathbf{t} \in \mathfrak{A}/\sim$
(Rel. 1)		$\lambda^2 = 1 + 2\rho$			
(Rel. 2)		$\rho^2 = 2\rho + 2\lambda\rho$			
(Rel. 3)		$\omega\rho = \rho + 2\lambda\rho$			
(Rel. 4)		$\bar{\omega}\rho = \rho + 2\lambda\rho$			
(Rel. 5)		$\bar{\omega}\omega = 1 + 10\rho$			
(Rel. 6)		$\mu_{k,\mathbf{t}}\rho = k\rho + k\lambda\rho$			
(Rel. 7)		$\mu_{k,\mathbf{t}}\omega = k\lambda\rho + \mu_{k,\mathbf{t}}$			
(Rel. 8)		$\mu_{k,\mathbf{t}}\bar{\omega} = 3k\rho + \lambda\mu_{k,\mathbf{t}}$			
(Rel. 9)		$\mu_{k,\mathbf{t}}\mu_{j,\mathbf{s}} = -2jk\rho + jk\lambda\rho$			
(Rel. 10)		$\mu_{k,\mathbf{t}}\mu_{j,\mathbf{t}} = -2(j-1)k\rho + (j-1)k\lambda\rho + \mu_{k,\mathbf{t}} + \lambda\mu_{k,\mathbf{t}}$			
					$\forall \mathbf{s}, \mathbf{t} \in \mathfrak{A}, \mathbf{s} \not\sim \mathbf{t}, \forall j, k \in \mathbb{N}, k \leq j$

We emphasize that our results provide examples of non-quasitriangular Hopf algebras [4, Proposition 30] with commutative Green ring, and these form an infinite family of non-isomorphic Morita equivalent Hopf algebras with isomorphic Green ring (this does not necessarily imply that the representation categories are monoidally equivalent¹ but we were not able to see whether or not they are). Moreover, the categories of comodules of this family are monoidally equivalent to each

¹Even in the semisimple case, see e.g. [27].

other [4, Proposition 29], and hence these Hopf algebras are cocycle deformations of each other by [24, Corollary 5.9]. However, not all cocycle deformations of these Hopf algebras are Morita equivalent. Indeed, the non-generic liftings of \mathcal{FK}_3 over $\mathbb{k}^{\mathbb{S}_3}$ are also cocycle deformations of the generic liftings but their representation categories are quite different; for instance, their blocks have three or six simple modules, cf. [4]. We also observe that \mathcal{A} is a spherical Hopf algebra and the corresponding quotient category is monoidally equivalent to the category of $C_2 \times \mathbb{Z}$ -graded finite-dimensional vector spaces, see Corollary 4.4.

We conclude this introduction by comparing our work with Chen’s paper [9] which describes the Green ring of $\mathcal{D}(T_2(-1))$. This Drinfeld double has two singular (i.e., simple) blocks and only one which is regular. The latter turns out to be the representation category of a book Hopf algebra \mathbf{h} [2]. Therefore \mathbf{h} is Morita equivalent to \mathcal{A} and its indecomposable modules are classified as in Theorem 1.1. The two simple modules of \mathbf{h} are one-dimensional, say \tilde{L} and the trivial one $\tilde{\varepsilon}$. Then $\tilde{L} \otimes \tilde{L} \simeq \tilde{\varepsilon}$ and tensoring with \tilde{L} induces an involution on the family of indecomposable modules. Instead, our non-trivial simple module L is of dimension 5 and $L \otimes L \simeq \varepsilon \oplus 4P_L$. This projective summand is then propagated through the remaining tensor products. This makes the computations more difficult than in [9]. It is like $\text{Rep } \mathcal{A}$ is a deformation of $\text{Rep } \mathbf{h}$ over the projective modules. It would be interesting to develop a method to construct new tensor categories by deforming a known one as it occurs in the present situation.

The article is organized as follows. In Section 2 we introduce the generic co-pointed Hopf algebras over $\mathbb{k}^{\mathbb{S}_3}$. In Section 3, we prove that they are Morita equivalent to every regular block of $\mathcal{D}(T_n(\zeta))$ and $u_q(\mathfrak{sl}_2)$, and describe their indecomposable modules in detail. The bulk of our work is in Section 4, where we address the tensor products of every pair of indecomposable modules and prove Theorem 1.2. In Appendix A, we explain how we use GAP in our calculations.

2. THE FAMILY OF COPOINTED HOPF ALGEBRAS

We work over an algebraically closed field \mathbb{k} of characteristic zero. Let $\mathbb{k}^{\mathbb{S}_3}$ be the algebra of function on the symmetric group \mathbb{S}_3 . We denote by δ_g the characteristic function of $g \in \mathbb{S}_3$ and by e the neutral element of \mathbb{S}_3 . We recall that $\mathbb{k}^{\mathbb{S}_3}$ is a Hopf algebra where the comultiplication, antipode and counit of each δ_g are

$$\Delta(\delta_g) = \sum_{h \in \mathbb{S}_3} \delta_h \otimes \delta_{h^{-1}g}, \quad S(\delta_g) = \delta_{g^{-1}} \quad \text{and} \quad \varepsilon(\delta_g) = \delta_g(e).$$

We define

$$\mathfrak{A} = \{(a_{(23)}, a_{(12)}, a_{(13)}) \in \mathbb{k}^3 \mid a_{(23)} + a_{(12)} + a_{(13)} = 0\}. \tag{2.1}$$

We will consider in \mathfrak{A} the equivalence relation $\mathbf{t} \sim \mathbf{s} \Leftrightarrow \mathbf{t} = \lambda \mathbf{s}$ for some $\lambda \in \mathbb{k}^\times$.

We fix $\mathbf{a} \in \mathfrak{A}$ and, for each transposition (ij) , we set

$$f_{ij} = \sum_{g \in \mathbb{S}_3} (a_{(ij)} - a_{g^{-1}(ij)g}) \delta_g \in \mathbb{k}^{\mathbb{S}_3}.$$

The Hopf algebra $\mathcal{A}_{[\mathbf{a}]}$ defined in [3, Definition 3.4] is generated by $x_{(12)}$, $x_{(23)}$, $x_{(13)}$ and δ_g for all $g \in \mathbb{S}_3$, subject to the relations

$$\begin{aligned} x_{(ij)}^2 &= f_{ij}, & \delta_g x_{(ij)} &= x_{(ij)} \delta_{(ij)g}, & \delta_g \delta_h &= \delta_g(h) \delta_g, \\ x_{(23)} x_{(12)} + x_{(13)} x_{(23)} + x_{(12)} x_{(13)} &= 0, \\ x_{(12)} x_{(23)} + x_{(23)} x_{(13)} + x_{(13)} x_{(12)} &= 0 \end{aligned}$$

for all $(ij), g, h \in \mathbb{S}_3$. The comultiplication, the antipode and the counit of each generator $x_{(ij)}$ are

$$\begin{aligned} \Delta(x_{(ij)}) &= x_{(ij)} \otimes 1 + \sum_{h \in \mathbb{S}_3} \text{sgn}(h) \delta_h \otimes x_{h^{-1}(ij)h}, \\ S(x_{(ij)}) &= - \sum_{h \in \mathbb{S}_3} \text{sgn}(h) x_{h^{-1}(ij)h} \delta_{h^{-1}(ij)} \quad \text{and} \quad \varepsilon(x_{(ij)}) = 0. \end{aligned}$$

The elements $\delta_g, g \in \mathbb{S}_3$, generate a Hopf subalgebra isomorphic to $\mathbb{k}^{\mathbb{S}_3}$.

The dimension of $\mathcal{A}_{[\mathbf{a}]}$ is 72 and the elements $x \delta_g, g \in \mathbb{S}_3$ and $x \in \mathbb{B}$, with

$$\mathbb{B} := \left\{ \begin{array}{l} 1, \quad x_{(13)}, \quad x_{(13)} x_{(12)}, \quad x_{(13)} x_{(12)} x_{(13)}, \quad x_{(13)} x_{(12)} x_{(23)} x_{(12)}, \\ x_{(23)}, \quad x_{(12)} x_{(13)}, \quad x_{(12)} x_{(23)} x_{(12)}, \\ x_{(12)}, \quad x_{(23)} x_{(12)}, \quad x_{(13)} x_{(12)} x_{(23)}, \\ x_{(12)} x_{(23)} \end{array} \right\},$$

form a basis.

The following elements will play a distinguished role:

$$x_{\text{top}} := x_{(13)} x_{(12)} x_{(23)} x_{(12)}, \tag{2.2}$$

$$x_{\text{soc}} := (-1 - 2a)(1 - a) - x_{\text{top}}, \tag{2.3}$$

$$x_{L,\mathbf{t}} := -t_{(12)} x_{(13)} x_{(12)} + t_{(13)} x_{(12)} x_{(23)}, \tag{2.4}$$

$$x_{\varepsilon,\mathbf{t}} := -t_{(12)} x_{(12)} x_{(13)} + t_{(13)} x_{(23)} x_{(12)}, \tag{2.5}$$

where $\mathbf{t} = (t_{(23)}, t_{(12)}, t_{(13)}) \in \mathfrak{A}$.

It holds that $\mathcal{A}_{[\mathbf{a}]} \simeq \mathcal{A}_{[\mathbf{b}]}$ if and only if $\mathbf{b} = \lambda(a_{\theta(23)}, a_{\theta(12)}, a_{\theta(13)})$ for a permutation θ of the transpositions and a non-zero scalar λ , and this gives a classification of the copointed Hopf algebras over $\mathbb{k}^{\mathbb{S}_3}$, see [3, Theorem 3.5]. The representation theory of $\mathcal{A}_{[\mathbf{a}]}$ depends on the number of scalars $a_{(ij)}$ which are equal, cf. [4]. In the present work, we will study the generic case, i.e., when the three scalars are different. Thus, without loss of generality, we will adopt the following convention.

Convention 2.1. We fix $a \in \mathbb{k} \setminus \{1, -\frac{1}{2}, -2\}$ and set $\mathcal{A} := \mathcal{A}_{[(1,a,-1-a)]}$. From now on, by ‘module’ we mean left \mathcal{A} -module.

3. THE CATEGORY OF \mathcal{A} -MODULES

Here, we first recall from [4] the simple and projective modules. Using them we prove the Morita equivalence announced in the introduction. Next, we describe in detail the remaining indecomposable modules.

3.1. \mathbb{S}_3 -degree. We will use the fact that any module M is by restriction a $\mathbb{k}^{\mathbb{S}_3}$ -module, or equivalently, an \mathbb{S}_3 -graded module. Explicitly, the homogeneous component $M[g]$ of degree $g \in \mathbb{S}_3$ is the subspace of M spanned by the elements $m \in M$ such that $\delta_h \cdot m = \delta_h(g) m$. We point out that $(M \otimes N)[g] = \bigoplus_{h \in \mathbb{S}_3} M[h] \otimes N[h^{-1}g]$. We denote by \mathbb{k}_g the one-dimensional $\mathbb{k}^{\mathbb{S}_3}$ -module concentrated in degree $g \in \mathbb{S}_3$.

3.2. Simple modules. There are only two simple modules [4, Theorem 1]. By abuse of notation, we denote by ε the simple module determined by the counit.

We denote by L the non-trivial simple module. It is five-dimensional with basis $\{v_g \mid e \neq g \in \mathbb{S}_3\}$. The action of \mathcal{A} is determined by

$$v_g \in L[g] \quad \text{and} \quad x_{(ij)} \cdot v_g = \begin{cases} v_{(ij)g} & \text{if } \text{sgn}(g) = 1, \\ f_{ij}(g) v_{(ij)g} & \text{if } \text{sgn}(g) = -1. \end{cases} \quad (3.1)$$

Clearly, the simple modules are self-dual:

$$\varepsilon^* \simeq \varepsilon \quad \text{and} \quad L^* \simeq L.$$

3.3. Projective modules. Given $g \in \mathbb{S}_3$, we consider the induced module

$$M_g := \text{Ind}_{\mathbb{k}^{\mathbb{S}_3}}^{\mathcal{A}}(\mathbb{k}_g) = \mathcal{A} \otimes_{\mathbb{k}^{\mathbb{S}_3}} \mathbb{k}_g.$$

Equivalently, M_g is the ideal $\mathcal{A}\delta_g$. Thus, $\{x\delta_g \mid x \in \mathbb{B}\}$ is a basis, $\dim M_g = 12$ and any morphism $f : M_g \rightarrow N$ is determined by its value on δ_g .

We can compute easily the \mathbb{S}_3 -degree of these basis elements. In fact, using the commutation relations we see that $x_{(i_1j_1)} \cdots x_{(i_lj_l)} \delta_g \in M_g[(i_1j_1) \cdots (i_lj_l)g]$. It follows that $\dim(M_g[h]) = 2$ for all $h \in \mathbb{S}_3$.

By [4, Lemma 7], $M_g \simeq M_h$ if $g \neq e \neq h$. Moreover, by [4, Theorem 1],

$$P_\varepsilon := M_e \quad \text{and} \quad P_L := M_{(132)}$$

are the projective covers and injective hulls of ε and L , respectively. In particular,

$$\begin{aligned} \text{soc}(P_\varepsilon) &= \mathcal{A} \cdot (x_{\text{top}}\delta_e) \simeq \varepsilon \simeq \text{top}(P_\varepsilon) \quad \text{and} \\ \text{soc}(P_L) &= \mathcal{A} \cdot (x_{\text{soc}}\delta_L) \simeq L \simeq \text{top}(P_L), \quad \text{where } \delta_L := \delta_{(132)}. \end{aligned}$$

The generators of the socles were given in [4, Lemmas 10 and 13]. It follows that the projective modules are self-dual:

$$P_\varepsilon^* \simeq P_\varepsilon \quad \text{and} \quad P_L^* \simeq P_L.$$

The following lemma will be useful.

Lemma 3.1. *Let M be a module and $m \in M[g]$.*

- (1) *If $g = (132)$ and $x_{\text{soc}} \cdot m \neq 0$, then $\mathcal{A} \cdot m \simeq P_L$ is a direct summand of M .*
- (2) *If $g = e$ and $x_{\text{top}} \cdot m \neq 0$, then $\mathcal{A} \cdot m \simeq P_\varepsilon$ is a direct summand of M .*

Proof. Since $P_L = M_{(132)}$ is induced from $\mathbb{k}_{(132)}$, there exists a morphism $F : P_L \rightarrow M$ such that $F(\delta_L) = m$. As $F(x_{\text{soc}}\delta_L) \neq 0$, F is not zero in $\text{soc}(P_L)$ which is simple. Then F is a monomorphism and (1) follows because P_L is injective. The proof of (2) is similar. □

3.4. Morita equivalence. Let $\mathcal{D}(T_n(\zeta))$ be the Drinfeld double of a Taft algebra $T_n(\zeta)$ where ζ is a primitive root of unity of order $n \geq 2$. Chen showed that the blocks of $\mathcal{D}(T_n(\zeta))$ are arranged in two Morita equivalence classes: the class of regular blocks and the class of singular ones [8, Proposition 3.1]. Moreover, each regular block is Morita equivalent to any regular block of $\mathcal{D}(T_m(\zeta'))$ for any $m \geq 2$ [8, Proposition 3.3]. The blocks of the Frobenius–Lusztig kernel $u_q(\mathfrak{sl}_2)$, with $q^2 = \zeta$, satisfy the same properties; see for instance [26, Section 5]. These facts are proved in *loc. cit.* by computing the basic algebra of each block. More precisely, the basic algebra corresponding to the regular blocks is the following.

Theorem 3.2 ([8, 26]). *The regular blocks of $\mathcal{D}(T_n(\zeta))$ and $u_q(\mathfrak{sl}_2)$ are Morita equivalent to the algebra B generated by e_i, u_i, w_i , with $i \in \{1, 2\}$, subject to the relations*

$$e_i e_j = \delta_{i,j} e_i, \quad u_1 w_1 = u_2 w_2, \quad w_1 u_1 = w_2 u_2, \\ e_2 u_i = u_i e_1 = u_i, \quad w_j e_2 = e_1 w_j = w_j,$$

$$u_i^2 = w_j^2 = u_i u_j = w_i w_j = u_i w_j = w_j u_i = u_i e_2 = e_1 u_i = e_2 w_j = w_j e_1 = 0$$

for all $i, j \in \{1, 2\}$ with $i \neq j$. It holds that $\dim B = 8$. □

The case $n = 2$ has a peculiarity. Explicitly, $\mathcal{D}(T_2(-1))$ and $u_q(\mathfrak{sl}_2)$ has only one regular block and it is closed under tensor products. This is a consequence of B being a quotient of $\mathcal{D}(T_2(-1))$ and $u_q(\mathfrak{sl}_2)$ by a central group-like element as we explain next. Instead, we point out that the regular blocks of $\mathcal{D}(T_n(\zeta))$ and $u_q(\mathfrak{sl}_2)$ for $n > 2$ are not closed under tensor products, cf. [6, 20].

Let \mathfrak{h} be the book Hopf algebra [2] generated by G, X and Y subject to the relations $G^2 = 1, X^2 = Y^2 = 0, GX = -XG, GY = -YG$ and $XY = YX$, with comultiplication

$$\Delta(G) = G \otimes G, \quad \Delta(X) = X \otimes G + 1 \otimes X, \quad \Delta(Y) = Y \otimes 1 + G \otimes Y.$$

It turns out that B is isomorphic to \mathfrak{h} as an algebra by letting $G = e_1 - e_2, X = u_1 - w_2$ and $Y = w_1 - u_2$. Thus, B inherits a Hopf algebra structure and we have the following.

Proposition 3.3. *Let q be a primitive root of unity of order 2 and C_2 the cyclic group of order 2. Then there exist central extensions of Hopf algebras*

$$\mathbb{k} \rightarrow \mathbb{k}C_2 \rightarrow \mathcal{D}(T_2(-1)) \rightarrow \mathfrak{h} \rightarrow \mathbb{k} \quad \text{and} \quad \mathbb{k} \rightarrow \mathbb{k}C_2 \rightarrow u_q(\mathfrak{sl}_2) \rightarrow \mathfrak{h} \rightarrow \mathbb{k}.$$

Moreover, the category of representations of \mathfrak{h} is monoidally equivalent to the regular block of both $\mathcal{D}(T_2(-1))$ and $u_q(\mathfrak{sl}_2)$.

Proof. We recall that $\mathcal{D}(T_2(-1))$ is generated by a, b, c, d subject to the relations $ba = -ab, bd = -db, ca = -ac, dc = -cd, bc = cb, a^2 = d^2 = 0, b^2 = c^2 = 1$ and $da + ad = 1 - bc$, where b and c are group-like elements and a and d are skew-primitive elements; see for instance [9, p. 1460]. Then the Hopf subalgebra generated by bc gives the desired extension.

In the case of $u_q(\mathfrak{sl}_2)$, this is generated by E, K, F subject to the relations $K^4 = 1, KE = q^2 EK, KF = q^{-2} FK, K^2 = 0 = F^2$ and $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$,

where K is a group-like element and E and F are skew-primitive elements; see for instance [26]. Then the desired extension is given by considering the element K^2 .

In consequence, these extensions give rise to monoidal functors from $\text{Rep } \mathbf{h}$ to $\text{Rep } \mathcal{D}(T_2(-1))$ and to $\text{Rep } u_q(\mathfrak{sl}_2)$, respectively. It is not difficult to realize that these functors are equivalences with the respective regular blocks. \square

We can appreciate from the description of its simple and projective modules that \mathcal{A} has only one block. We next show that the corresponding basic algebra is isomorphic to B .

Proposition 3.4. *\mathcal{A} is Morita equivalent to \mathbf{h} and to the regular blocks of $\mathcal{D}(T_n(\zeta))$ and $u_q(\mathfrak{sl}_2)$ for any $n \geq 2$. In particular, the indecomposable \mathcal{A} -modules are classified by Theorem 1.1.*

Proof. Let $P = P_\epsilon \oplus P_L$ and let $E := \text{End}_{\mathcal{A}}(P)$ be the basic algebra of \mathcal{A} . We denote by

$$\begin{pmatrix} e_1, p_1 & g_1, g_2 \\ f_1, f_2 & e_2, p_2 \end{pmatrix} \in \begin{pmatrix} \text{Hom}_{\mathcal{A}}(M_e, M_e) & \text{Hom}_{\mathcal{A}}(M_e, M_{(132)}) \\ \text{Hom}_{\mathcal{A}}(M_{(132)}, M_e) & \text{Hom}_{\mathcal{A}}(M_{(132)}, M_{(132)}) \end{pmatrix}$$

the morphisms determined by

- $e_1 = \text{id}_{M_e}$ and $e_2 = \text{id}_{M_{(132)}}$.
- $f_1(\delta_L) = x_{(12)}x_{(13)}\delta_e$ and $f_2(\delta_L) = x_{(23)}x_{(12)}\delta_e$.
- $g_1(\delta_e) = x_{(13)}x_{(12)}\delta_L$ and $g_2(\delta_e) = x_{(12)}x_{(23)}\delta_L$.
- $p_1(\delta_e) = \delta_e$ and $p_2(\delta_L) = x_{\text{soc}}\delta_L$.

These form a basis of E because

$$E \simeq \begin{pmatrix} \text{Hom}_{\mathbb{k}^{\mathfrak{S}_3}}(\mathbb{k}_e, M_e) & \text{Hom}_{\mathbb{k}^{\mathfrak{S}_3}}(\mathbb{k}_e, M_{(132)}) \\ \text{Hom}_{\mathbb{k}^{\mathfrak{S}_3}}(\mathbb{k}_{(132)}, M_e) & \text{Hom}_{\mathbb{k}^{\mathfrak{S}_3}}(\mathbb{k}_{(132)}, M_{(132)}) \end{pmatrix}$$

and these Hom spaces are of dimension two, cf. Subsection 3.3.

By a computation in GAP, we see that

$$\begin{aligned} e_1 + e_2 &= \text{id}_P, & e_i e_j &= \delta_{i,j} e_i, & 1 \leq i, j \leq 2, \\ f_i e_2 &= e_1 f_i = f_i, & g_i e_1 &= e_2 g_i = g_i, & g_i f_i = 0, & f_i g_i = 0, & 1 \leq i \leq 2, \\ g_2 f_1 &= p_2, & g_1 f_2 &= -p_2, & f_2 g_1 &= p_1, & f_1 g_2 &= -p_1; \end{aligned}$$

it is enough to verify these equalities after evaluating in δ_e and δ_L .

These are the same relations defining the basic algebra of $\mathcal{D}(T_2(-1))$ and hence $E \simeq B$, see [8, p. 2818]; notice that our convention for the composition of morphisms is opposite to that in *loc. cit.* This proves the first part of the proposition. Therefore the indecomposable modules over \mathcal{A} are given by translating [8, Theorem 3.12]. This is Theorem 1.1. \square

We will describe in detail the non-simple non-projective indecomposable modules listed on Theorem 1.1 in the following subsections. We notice that every indecomposable module M satisfies $\text{top}(M) \simeq M/\text{soc}(M)$. According to Chen, we say that M is of (m, n) -type if $\text{soc}(M)$ is the direct sum of n simple modules and $M/\text{soc}(M)$ is the direct sum of m simple modules. Given a module M and

$k \in \mathbb{N}$, we will denote by kM the direct sum of k copies of M . We recall that a Morita equivalence also preserves simple, indecomposable, injective and projective modules, exact sequences, injective and projective resolutions.

3.5. The syzygy modules. We recall that the syzygy Ω is an endofunctor of the stable category of modules. It is computed by taking the kernels of projective covers. Here, $\Omega^k(\varepsilon)$ and $\Omega^k(L)$ shall denote certain representatives of the syzygy applied iteratively to the simple modules ε and L . They are defined inductively using the minimal projective resolutions in [7, p. 772] as follows.

We set $\Omega^0(\varepsilon) := \varepsilon$. For $k \geq 1$, $\Omega^{k+1}(\varepsilon)$ is determined by the exact sequence

$$0 \longrightarrow \Omega^{k+1}(\varepsilon) \longrightarrow (k+1)P_\varepsilon \longrightarrow \Omega^k(\varepsilon) \longrightarrow 0 \quad \text{for } k \text{ even,} \tag{3.2}$$

$$0 \longrightarrow \Omega^{k+1}(\varepsilon) \longrightarrow (k+1)P_L \longrightarrow \Omega^k(\varepsilon) \longrightarrow 0 \quad \text{for } k \text{ odd.} \tag{3.3}$$

Equally, $\Omega^0(L) := L$ and, for $k \geq 1$, $\Omega^{k+1}(L)$ is determined by the exact sequence

$$0 \longrightarrow \Omega^{k+1}(L) \longrightarrow (k+1)P_L \longrightarrow \Omega^k(L) \longrightarrow 0 \quad \text{for } k \text{ even,} \tag{3.4}$$

$$0 \longrightarrow \Omega^{k+1}(L) \longrightarrow (k+1)P_\varepsilon \longrightarrow \Omega^k(L) \longrightarrow 0 \quad \text{for } k \text{ odd.} \tag{3.5}$$

Notice that $\Omega(\Omega^k(\varepsilon)) \simeq \Omega^{k+1}(\varepsilon)$ and $\Omega(\Omega^k(L)) \simeq \Omega^{k+1}(L)$. These are the unique $(k+1, k)$ -type modules. Moreover, the modules $\Omega^k(\varepsilon)$ and $\Omega^k(L)$ are characterized as the unique (up to isomorphism) indecomposable modules fitting in the exact sequences

$$0 \longrightarrow k\varepsilon \longrightarrow \Omega^k(\varepsilon) \longrightarrow (k+1)L \longrightarrow 0 \quad \text{for } k \text{ odd,} \tag{3.6}$$

$$0 \longrightarrow kL \longrightarrow \Omega^k(\varepsilon) \longrightarrow (k+1)\varepsilon \longrightarrow 0 \quad \text{for } k \text{ even,} \tag{3.7}$$

$$0 \longrightarrow kL \longrightarrow \Omega^k(L) \longrightarrow (k+1)\varepsilon \longrightarrow 0 \quad \text{for } k \text{ odd,} \tag{3.8}$$

$$0 \longrightarrow k\varepsilon \longrightarrow \Omega^k(L) \longrightarrow (k+1)L \longrightarrow 0 \quad \text{for } k \text{ even} \tag{3.9}$$

(see [7, Theorem 3.14]). We remark that the modules on the left- and right-hand sides of the above sequences coincide with the socle and the top of the middle terms by [7, Corollary 3.16].

3.6. The cosyzygy modules. The cosyzygy Ω^{-1} is also an endofunctor of the stable category but it is calculated by taking the cokernels of injective hulls. By [7, Theorem 3.14], $\Omega^{-k}(\varepsilon)$ and $\Omega^{-k}(L)$ are the unique $(k, k+1)$ -type modules. Moreover, by dualizing the exact sequences (3.6)–(3.9), we have that

$$\Omega^{-k}(\varepsilon) \simeq (\Omega^k(\varepsilon))^* \quad \text{and} \quad \Omega^{-k}(L) \simeq (\Omega^k(L))^*$$

for all $k \in \mathbb{N}$. We point out that $\Omega^{-k}(\varepsilon)$ and $\Omega^{-k}(L)$ can be constructed using the dual exact sequences of (3.2)–(3.5).

3.7. The (k, k) -type indecomposable modules. Let $k \in \mathbb{N}$ and $0 \neq \mathbf{t} \in \mathfrak{A}$. We choose $\widehat{\mathbf{t}} \in \mathfrak{A}$ such that $(\widehat{t}_{(12)}, \widehat{t}_{(13)})$ and $(t_{(12)}, t_{(13)})$ are linearly independent. Inspired in [4, Definition 11] and [9, Lemmas 3.29–30], we define the module $M_k(\varepsilon, \mathbf{t})$ as the vector space with basis $\{w_g^\ell \mid g \in \mathbb{S}_3, 1 \leq \ell \leq k\}$ and action given by

$$w_g^\ell \in M_k(\varepsilon, \mathbf{t})[g] \quad \text{and} \tag{3.10}$$

$$x_{(ij)} \cdot w_g^\ell = \begin{cases} 0 & \text{if } g = e, \\ w_{(ij)g}^\ell & \text{if } g \neq e \text{ and } \text{sgn}(g) = 1, \\ f_{(ij)}(g) w_{(ij)g}^\ell & \text{if } g \neq (ij) \text{ and } \text{sgn}(g) = -1, \\ t_{(ij)} w_e^\ell + \widehat{t}_{(ij)} w_e^{\ell-1} & \text{if } g = (ij) \end{cases}$$

for all $g \in \mathbb{S}_3$ and $1 \leq \ell \leq k$. It is an straightforward computation to verify that this definition respects the defining relations of \mathcal{A} .

Proposition 3.5. *The following statements hold:*

- (1) $M_k(\varepsilon, \mathbf{t})$ is a (k, k) -type indecomposable module fitting in the exact sequence

$$0 \longrightarrow k\varepsilon \longrightarrow M_k(\varepsilon, \mathbf{t}) \longrightarrow kL \longrightarrow 0 \tag{3.11}$$

for all $k \in \mathbb{N}$ and $0 \neq \mathbf{t} \in \mathfrak{A}$.

- (2) Any (k, k) -type indecomposable module with $k\varepsilon$ as socle is isomorphic to $M_k(\varepsilon, \mathbf{t})$ for some $0 \neq \mathbf{t} \in \mathfrak{A}$.

- (3) $M_k(\varepsilon, \mathbf{t}) \simeq M_k(\varepsilon, \widetilde{\mathbf{t}})$ if and only if $\mathbf{t} \sim \widetilde{\mathbf{t}}$.

- (4) The definition of $M_k(\varepsilon, \mathbf{t})$ does not depend on the election of $\widehat{\mathbf{t}}$.

- (5) For all $k \in \mathbb{N}$ and $0 \neq \mathbf{t} \in \mathfrak{A}$, there exists an exact sequence

$$0 \longrightarrow M_1(\varepsilon, \mathbf{t}) \longrightarrow M_{k+1}(\varepsilon, \mathbf{t}) \longrightarrow M_k(\varepsilon, \mathbf{t}) \longrightarrow 0. \tag{3.12}$$

Moreover, any module fitting in such an exact sequence is isomorphic to either $M_{k+1}(\varepsilon, \mathbf{t})$ or $M_1(\varepsilon, \mathbf{t}) \oplus M_k(\varepsilon, \mathbf{t})$.

Proof. The modules $M_1(\varepsilon, \mathbf{t})$ are exactly the modules introduced in [4, Definition 11] and hence, for $k = 1$, this proposition is [4, Lemmas 12 and 21].

We now consider the case $k > 1$. We begin by proving that $M_k(\varepsilon, \mathbf{t})$ is of (k, k) -type. We can see from the very definition that $k\varepsilon$ is a submodule of $\text{soc } M_k(\varepsilon, \mathbf{t})$ and $[M_k(\varepsilon, \mathbf{t}) : \varepsilon] = [M_k(\varepsilon, \mathbf{t}) : L] = k$. Suppose there is a copy of L in the socle. Thus, there would exist $0 \neq w \in M_k(\varepsilon, \mathbf{t})[(132)]$ such that $x_{(12)}x_{(23)} \cdot w = 0$ by the definition of L . Let us say $w = \sum_{\ell=1}^k a_\ell w_{(132)}^\ell$ with $a_\ell \in \mathbb{k}$. Then

$$\begin{aligned} x_{(12)}x_{(23)} \cdot w &= t_{(12)} \sum_{\ell=1}^k a_\ell w_e^\ell + \widehat{t}_{(12)} \sum_{\ell=1}^{k-1} a_{\ell+1} w_e^\ell \\ &= t_{(12)} a_k w_e^k + \sum_{\ell=1}^{k-1} (t_{(12)} a_\ell + \widehat{t}_{(12)} a_{\ell+1}) w_e^\ell. \end{aligned}$$

If $t_{(12)} \neq 0$, we see that $a_k = 0$ and then deduce inductively that $a_\ell = 0$ for all $1 \leq \ell < k$. Hence $w = 0$, a contradiction. If $t_{(12)} = 0$, then $t_{(13)}$ should be non-zero

and we get the same conclusion by considering $x_{(13)}x_{(12)} \cdot w$. Therefore $M_k(\varepsilon, \mathbf{t})$ is of (k, k) -type.

We continue by showing that $M_k(\varepsilon, \mathbf{t})$ is indecomposable by induction on k . Let us assume the contrary, i.e. that $M_k(\varepsilon, \mathbf{t}) = N \oplus \bar{N}$ is a direct sum of submodules. We claim that $w_{(132)}^1$ belongs to either N or \bar{N} . In fact, let us assume that $w_{(132)}^1 = n + \bar{n}$. Thus, there are scalars a_1, \dots, a_k such that

$$n = \sum_{\ell=1}^k a_\ell w_{(132)}^\ell \in N \quad \text{and} \quad \bar{n} = (1 - a_1) w_{(132)}^1 - \sum_{\ell=2}^k a_\ell w_{(132)}^\ell \in \bar{N}.$$

Hence

$$\begin{aligned} x_{(12)}x_{(23)} \cdot n &= t_{(12)} \sum_{\ell=1}^k a_\ell w_e^\ell + \widehat{t}_{(12)} \sum_{\ell=1}^{k-1} a_{\ell+1} w_e^\ell \in N \quad \text{and} \\ x_{(13)}x_{(12)} \cdot n &= t_{(13)} \sum_{\ell=1}^k a_\ell w_e^\ell + \widehat{t}_{(13)} \sum_{\ell=1}^{k-1} a_{\ell+1} w_e^\ell \in N. \end{aligned}$$

If $\sum_{\ell=1}^{k-1} a_{\ell+1} w_e^\ell \neq 0$, then $\sum_{\ell=1}^k a_\ell w_e^\ell$ and $\sum_{\ell=1}^{k-1} a_{\ell+1} w_e^\ell$ are linearly independent. Since $(\widehat{t}_{(12)}, \widehat{t}_{(13)})$ and $(t_{(12)}, t_{(13)})$ are also linearly independent, we deduce that $x_{(12)}x_{(23)} \cdot n$ and $x_{(13)}x_{(12)} \cdot n$ are linearly independent as well. In particular, we can infer that $\sum_{\ell=1}^{k-1} a_{\ell+1} w_e^\ell \in N$. However, if we follow the same reasoning with \bar{n} instead of n , we would also conclude that $\sum_{\ell=1}^{k-1} a_{\ell+1} w_e^\ell \in \bar{N}$. As $N \cap \bar{N} = 0$, it follows that $\sum_{\ell=1}^{k-1} a_{\ell+1} w_e^\ell = 0$ and hence either $n = w_{(132)}^1$ or $\bar{n} = w_{(132)}^1$, as claimed. This completes the proof of our claim.

Without loss of generality we can assume that $w_{(132)}^1 \in N$. Hence the submodule generated by $w_{(132)}^1$ is contained in N , i.e., $w_g^1 \in N$ for all $g \in \mathbb{S}_3$. In consequence, we have the following isomorphisms:

$$M_k(\varepsilon, \mathbf{t})/(\mathcal{A} \cdot w_{(132)}^1) \simeq M_{k-1}(\varepsilon, \mathbf{t}) \simeq N/(\mathcal{A} \cdot w_{(132)}^1) \oplus \bar{N}.$$

By induction, it should hold either $\bar{N} = 0$ or $N = \mathcal{A} \cdot w_{(132)}^1$. If $\bar{N} = 0$, the proof is complete. If $N = \mathcal{A} \cdot w_{(132)}^1$, there is $b \in \mathbb{k}$ such that $w_{(132)}^2 - bw_{(132)}^1 \in \bar{N}$. Then

$$\begin{aligned} x_{(12)}x_{(23)} \cdot \left(w_{(132)}^2 - bw_{(132)}^1 \right) &= t_{(12)}w_e^2 + (\widehat{t}_{(12)} - t_{(12)}b) w_e^1 \in \bar{N} \quad \text{and} \\ x_{(13)}x_{(12)} \cdot \left(w_{(132)}^2 - bw_{(132)}^1 \right) &= t_{(13)}w_e^2 + (\widehat{t}_{(13)} - t_{(13)}b) w_e^1 \in \bar{N}. \end{aligned}$$

Thus, the same reasoning of the above paragraph allows us to infer that $w_e^1 \in \bar{N}$. Therefore $w_e^1 \in N \cap \bar{N}$ which contradicts our assumption $M_k(\varepsilon, \mathbf{t}) = N \oplus \bar{N}$. In conclusion, $M_k(\varepsilon, \mathbf{t})$ is indecomposable and this finishes the proof of item (1) of the proposition.

In order to prove the remaining items we first observe that the existence of the exact sequence (3.12) is immediate from the very definition. On the other hand, we know by [7, p. 783] that there is a unique (up to isomorphism) indecomposable module fitting in such an exact sequence. Moreover, [7, Theorem 4.16] states that

any $(k + 1, k + 1)$ -type indecomposable module with socle $(k + 1)\varepsilon$ is constructed in this way. Thus, we see by induction that item (2) holds for all $k > 1$. Equally, as item (3) holds for $k = 1$, it does for $k > 1$ as well as item (4) by *loc. cit.* \square

We now study the dual module $M_k(L, \mathbf{t}) := M_k(\varepsilon, \mathbf{t})^*$. Let $\{\bar{w}_g^\ell \mid g \in \mathbb{S}_3, 1 \leq \ell \leq k\}$ be the basis of $M_k(L, \mathbf{t})$ such that

$$\langle \bar{w}_g^\ell, w_h^l \rangle = (-1)^{\delta_e(g)} \delta_g(h^{-1}) \delta_{\ell, k-l+1} \quad \forall g, h \in \mathbb{S}_3, 1 \leq \ell, l \leq k.$$

The action of \mathcal{A} on this basis is given by

$$\bar{w}_g^\ell \in M_k(L, \mathbf{t})[g] \quad \text{and} \tag{3.13}$$

$$x_{(ij)} \cdot \bar{w}_g^\ell = \begin{cases} t_{(ij)} \bar{w}_{(ij)}^\ell + \hat{t}_{(ij)} \bar{w}_{(ij)}^{\ell-1} & \text{if } g = e, \\ f_{(ij)}(g) \bar{w}_{(ij)g}^\ell & \text{if } g \neq e \text{ and } \text{sgn}(g) = 1, \\ \bar{w}_{(ij)g}^\ell & \text{if } g \neq (ij) \text{ and } \text{sgn}(g) = -1, \\ 0 & \text{if } g = (ij) \end{cases}$$

for all $g \in \mathbb{S}_3$ and $1 \leq \ell \leq k$. In fact, we have that

$$\begin{aligned} \langle x_{(ij)} \cdot \bar{w}_g^\ell, w_h^l \rangle &= \langle \bar{w}_g^\ell, S(x_{(ij)}) \cdot w_h^l \rangle = \langle \bar{w}_g^\ell, - \sum_{\sigma \in \mathbb{S}_3} \text{sgn}(\sigma) x_{\sigma^{-1}(ij)\sigma} \delta_{\sigma^{-1}(ij)} \cdot w_h^l \rangle \\ &= \langle \bar{w}_g^\ell, \text{sgn}(h) p_{h,(ij)} w_{h(ij)}^l + \text{sgn}(h) q_{h,(ij)} w_{h(ij)}^{l-1} \rangle \end{aligned}$$

for certain scalars $p_{(ij),h}$ and $q_{(ij),h}$ given by (3.10). This is not zero only for $(ij)g = h^{-1}$ and either $\ell = k - l + 1$ or $\ell = k - (l - 1) + 1$. Thus, (3.13) follows by a case-by-case analysis which we leave to the reader.

We notice that $M_1(L, \mathbf{t})$ is the module $W_{\mathbf{t}}(\mathbb{k}_e, L)$ of [4, Definition 14], although there is a typo in *loc. cit.*

Proposition 3.6. *The following statements hold:*

- (1) $M_k(L, \mathbf{t})$ is a (k, k) -type indecomposable module fitting in the exact sequence

$$0 \longrightarrow kL \longrightarrow M_k(L, \mathbf{t}) \longrightarrow k\varepsilon \longrightarrow 0 \tag{3.14}$$

for all $k \in \mathbb{N}$ and $0 \neq \mathbf{t} \in \mathfrak{A}$.

- (2) Any (k, k) -type indecomposable module with kL as socle is isomorphic to $M_k(L, \mathbf{t})$ for some $0 \neq \mathbf{t} \in \mathfrak{A}$.
- (3) $M_k(L, \mathbf{t}) \simeq M_k(L, \tilde{\mathbf{t}})$ if and only if $\mathbf{t} \sim \tilde{\mathbf{t}}$.
- (4) The definition of $M_k(L, \mathbf{t})$ does not depend on the election of $\hat{\mathbf{t}}$.
- (5) For all $k \in \mathbb{N}$ and $0 \neq \mathbf{t} \in \mathfrak{A}$, there exists an exact sequence

$$0 \longrightarrow M_1(L, \mathbf{t}) \longrightarrow M_{k+1}(L, \mathbf{t}) \longrightarrow M_k(L, \mathbf{t}) \longrightarrow 0. \tag{3.15}$$

Moreover, any module fitting in such an exact sequence is isomorphic to either $M_{k+1}(L, \mathbf{t})$ or $M_1(L, \mathbf{t}) \oplus M_k(L, \mathbf{t})$.

Proof. It follows from Proposition 3.5 by taking duals. \square

We can infer the projective covers, the injective hulls, the syzygies and the cosyzygies of the (k, k) -type indecomposable modules from the next result.

Proposition 3.7. *For all $k \in \mathbb{N}$ and $0 \neq \mathbf{t} \in \mathfrak{A}$, we have the exact sequences*

$$0 \longrightarrow M_k(L, \mathbf{t}) \longrightarrow kP_L \longrightarrow M_k(\varepsilon, \mathbf{t}) \longrightarrow 0 \quad \text{and} \quad (3.16)$$

$$0 \longrightarrow M_k(\varepsilon, \mathbf{t}) \longrightarrow kP_\varepsilon \longrightarrow M_k(L, \mathbf{t}) \longrightarrow 0. \quad (3.17)$$

Moreover, for $k = 1$, the image of the inclusions in the above exact sequences satisfy

$$M_1(L, \mathbf{t}) \simeq \mathcal{A} \cdot (x_{L, \mathbf{t}} \delta_L) \subset P_L \quad \text{and} \quad M_1(\varepsilon, \mathbf{t}) \simeq \mathcal{A} \cdot (x_{\varepsilon, \mathbf{t}} \delta_\varepsilon) \subset P_\varepsilon.$$

Proof. The first part is proved in [9, p. 1462]. We next prove the claims for $k = 1$.

By a computation in GAP, we see that the morphism $P_L \rightarrow M_1(\varepsilon, \mathbf{t})$ induced by the assignment $\delta_L \mapsto w_{(132)}$ is an epimorphism and the element $x_{L, \mathbf{t}} \delta_L$ belongs to the kernel. By the proof of [4, Lemma 15], we know that $x_{L, \mathbf{t}} \delta_L$ generates a submodule isomorphic to $M_1(L, \mathbf{t})$. This proves $M_1(L, \mathbf{t}) \simeq \mathcal{A} \cdot (x_{L, \mathbf{t}} \delta_L) \subset P_L$.

Similarly, one can show the other claim. We only note that the kernel of the epimorphism $P_\varepsilon \rightarrow M_1(\varepsilon, \mathbf{t})$ induced by $\delta_\varepsilon \mapsto \bar{w}_\varepsilon$ contains $x_{\varepsilon, \mathbf{t}} \delta_\varepsilon$ and this element generates a submodule isomorphic to $M_1(\varepsilon, \mathbf{t})$ by the proof of [4, Lemma 12]. \square

4. TENSOR PRODUCTS BETWEEN INDECOMPOSABLE MODULES

In the following subsections, we decompose the tensor product of every pair of indecomposable modules into the direct sum of indecomposable modules. We then prove our main result over the Green ring of \mathcal{A} .

4.1. Preliminaries. We recall some well-known facts which are useful in the calculus of tensor products. We will use them without an explicit mention.

First, tensoring and dualizing are exact functors. The tensor product with a projective module is projective; see for instance [15].

Second, let M, N, P, Q be modules with P and Q projective, and hence injective, forming the exact sequence

$$0 \longrightarrow M \oplus P \longrightarrow E \longrightarrow N \oplus Q \longrightarrow 0.$$

Then, using the Krull–Schmidt theorem, it follows that $E \simeq \tilde{E} \oplus P \oplus Q$ for some module \tilde{E} fitting in the exact sequence

$$0 \longrightarrow M \longrightarrow \tilde{E} \longrightarrow N \longrightarrow 0.$$

Lemma 4.1 ([9, Lemma 3.12]). *Let $\{S_1, S_2\} = \{\varepsilon, L\}$ and M be an indecomposable module such that $M/\text{soc } M \simeq kS_1$ for some $k \geq 1$. If $f : sP_{S_1} \oplus tP_{S_2} \rightarrow M$ is an epimorphism with $s, t \geq 1$, then $s \geq k$ and $\text{Ker}(f) \simeq \Omega(M) \oplus (s - k)P_{S_1} \oplus tP_{S_2}$.*

Proof. It is exactly [9, Lemma 3.12] except for the claim that $s \geq k$, which follows using that kP_{S_1} is the projective cover of M . \square

In Appendix A we explain how we implemented our computations in GAP.

4.2. The tensor product of L with itself. We have the following isomorphism:

$$L \otimes L \simeq \varepsilon \oplus 2P_L. \tag{4.1}$$

Indeed, the projective direct summand is generated by $n_1 := v_{(13)} \otimes v_{(23)}$ and $n_2 := v_{(12)} \otimes v_{(13)}$. To prove this, according to Lemma 3.1, we must calculate the action of x_{soc} on them. By a computation in GAP, we see that

$$\begin{aligned} x_{\text{soc}} \cdot n_1 &= (2a^2 - a - 1) (v_{(23)} \otimes v_{(12)} - v_{(12)} \otimes v_{(13)} - (a + 2)v_{(123)} \otimes v_{(123)}) \quad \text{and} \\ x_{\text{soc}} \cdot n_2 &= (2a^2 + a - 2) (v_{(23)} \otimes v_{(12)} - v_{(13)} \otimes v_{(23)} - (2a + 1)v_{(123)} \otimes v_{(123)}). \end{aligned}$$

These elements are non-zero as $a \neq 1, -\frac{1}{2}, -2$. Then $N_1 := \mathcal{A} \cdot n_1$ and $N_2 := \mathcal{A} \cdot n_2$ are direct summands of $L \otimes L$ isomorphic to P_L by Lemma 3.1. Also, $x_{\text{soc}} \cdot n_1$ and $x_{\text{soc}} \cdot n_2$ are clearly linearly independent and hence $N_1 \cap N_2 = 0$ because they generate the socle of N_1 and N_2 . Therefore $L \otimes L = N_1 \oplus N_2 \oplus N \simeq 2P_L \oplus N$ for some submodule N . Since $\dim(L \otimes L) = \dim(2P_L) + 1$, N must be isomorphic to ε and (4.1) is proved. We point out that N is spanned by

$$\begin{aligned} \frac{1}{2a + 1} v_{(23)} \otimes v_{(23)} - \frac{1}{a + 2} v_{(12)} \otimes v_{(12)} - \frac{1}{a - 1} v_{(13)} \otimes v_{(13)} \\ - v_{(123)} \otimes v_{(132)} + v_{(132)} \otimes v_{(123)}. \end{aligned}$$

4.3. Tensoring by projectives. We will prove that

$$P_L \otimes L \simeq 4P_L \oplus P_\varepsilon \simeq L \otimes P_L \quad \text{and} \quad P_\varepsilon \otimes L \simeq 5P_L \simeq L \otimes P_\varepsilon. \tag{4.2}$$

We begin by proving the first isomorphism. Let us apply $-\otimes L$ to the exact sequence (3.8) for $k = 1$:

$$0 \longrightarrow L \otimes L \xrightarrow{(4.1)} \varepsilon \oplus 2P_L \longrightarrow \Omega(L) \otimes L \longrightarrow 2\varepsilon \otimes L \simeq 2L \longrightarrow 0.$$

Then $\Omega(L) \otimes L \simeq 2P_L \oplus N$ for some module N of dimension 11. On the other hand, we apply $-\otimes L$ to the exact sequence (3.4) for $k = 0$:

$$0 \longrightarrow \Omega(L) \otimes L \simeq 2P_L \oplus N \longrightarrow P_L \otimes L \longrightarrow L \otimes L \xrightarrow{(4.1)} \varepsilon \oplus 2P_L \longrightarrow 0.$$

Then $P_L \otimes L \simeq 4P_L \oplus Q$ for some projective module Q such that

$$0 \longrightarrow N \longrightarrow Q \longrightarrow \varepsilon \longrightarrow 0.$$

Therefore $Q \simeq P_\varepsilon$ and the first isomorphism of (4.2) follows. Notice that the same proof runs for $L \otimes P_L$.

The proof of the third isomorphism of (4.2) is similar. We first apply $-\otimes L$ to the exact sequence (3.6) for $k = 1$:

$$0 \longrightarrow \varepsilon \otimes L \simeq L \longrightarrow \Omega(\varepsilon) \otimes L \simeq 2L \otimes L \xrightarrow{(4.1)} 2\varepsilon \oplus 4P_L \longrightarrow 0.$$

Then $\Omega(\varepsilon) \otimes L \simeq 4P_L \oplus N$ for some module N of dimension 7. Second, we apply $-\otimes L$ to the exact sequence (3.2) for $k = 0$:

$$0 \longrightarrow \Omega(\varepsilon) \otimes L \simeq 4P_L \oplus N \longrightarrow P_\varepsilon \otimes L \longrightarrow \varepsilon \otimes L \simeq L \longrightarrow 0.$$

Then $P_\varepsilon \otimes L \simeq 4P_L \oplus Q$ for some projective module Q such that

$$0 \longrightarrow N \longrightarrow Q \longrightarrow L \longrightarrow 0.$$

Therefore $Q \simeq P_L$, which proves the third isomorphism of (4.2). Notice that the same proof runs for $L \otimes P_\varepsilon$.

Remark 4.2. Let $[M : L]$ and $[M : \varepsilon]$ denote the number of composition factors isomorphic to L and ε , respectively, of a module M . By induction on the length of M , it is easy to see that

$$\begin{aligned} M \otimes P_L &\simeq P_L \otimes M \simeq [M : L]P_\varepsilon \oplus (4[M : L] + [M : \varepsilon])P_L \quad \text{and} \\ M \otimes P_\varepsilon &\simeq P_\varepsilon \otimes M \simeq [M : \varepsilon]P_\varepsilon \oplus 5[M : L]P_L. \end{aligned}$$

4.4. Tensor products between syzygy modules. Let $s, t \in \mathbb{Z}_{\geq 0}$. Then

$$\Omega^s(L) \otimes \Omega^t(L) \simeq \Omega^t(L) \otimes \Omega^s(L) \tag{4.3}$$

$$\simeq \Omega^{s+t}(\varepsilon) \oplus \begin{cases} s(3t+2)P_L & \text{for } s \text{ odd and } t \text{ even,} \\ stP_\varepsilon \oplus 2stP_L & \text{for } s \text{ odd and } t \text{ odd,} \\ stP_\varepsilon \oplus 2(s+1)(t+1)P_L & \text{for } s \text{ even and } t \text{ even,} \\ (3s+2)tP_L & \text{for } s \text{ even and } t \text{ odd;} \end{cases}$$

$$\Omega^s(L) \otimes \Omega^t(\varepsilon) \simeq \Omega^t(L) \otimes \Omega^s(\varepsilon) \tag{4.4}$$

$$\simeq \Omega^{s+t}(L) \oplus \begin{cases} stP_\varepsilon \oplus 2stP_L & \text{for } s \text{ odd and } t \text{ even,} \\ s(3t+2)P_L & \text{for } s \text{ odd and } t \text{ odd,} \\ (3s+2)tP_L & \text{for } s \text{ even and } t \text{ even,} \\ stP_\varepsilon \oplus 2(t+1)(s+1)P_L & \text{for } s \text{ even and } t \text{ odd;} \end{cases}$$

$$\Omega^s(\varepsilon) \otimes \Omega^t(\varepsilon) \simeq \Omega^t(\varepsilon) \otimes \Omega^s(\varepsilon) \tag{4.5}$$

$$\simeq \Omega^{s+t}(\varepsilon) \oplus \begin{cases} (3s+2)tP_L & \text{for } s \text{ odd and } t \text{ even,} \\ stP_\varepsilon \oplus 2(s+1)(t+1)P_L & \text{for } s \text{ odd and } t \text{ odd,} \\ stP_\varepsilon \oplus 2stP_L & \text{for } s \text{ even and } t \text{ even,} \\ (3t+2)sP_L & \text{for } s \text{ even and } t \text{ odd.} \end{cases}$$

The demonstrations of these isomorphisms are by induction on s and t . We only prove (4.3) since the remaining proofs are similar. We will compute $\Omega^s(L) \otimes \Omega^t(L)$. We observe that our reasoning will rely on the isomorphisms in (4.2) and therefore $\Omega^s(L) \otimes \Omega^t(L) \simeq \Omega^t(L) \otimes \Omega^s(L)$.

For $s = t = 0$, this is (4.1). Next, we show the isomorphism for $s + 1$ and $t = 0$. If s is even, we apply $-\otimes L$ to (3.4) with $k = s$. We obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow \Omega^{s+1}(L) \otimes L &\longrightarrow (s+1)P_L \otimes L \stackrel{(4.2)}{\simeq} 4(s+1)P_L \oplus (s+1)P_\varepsilon \longrightarrow \\ &\longrightarrow \Omega^s(L) \otimes L \stackrel{\text{IH}}{\simeq} \Omega^s(\varepsilon) \oplus 2(s+1)P_L \longrightarrow 0. \end{aligned}$$

Then, by Lemma 4.1, $\Omega^{s+1}(L) \otimes L \simeq \Omega^{s+1}(\varepsilon) \oplus 2(s+1)P_L$ as claimed. If s is odd, we apply $-\otimes L$ to (3.5) with $k = s$ and obtain the exact sequence

$$0 \rightarrow \Omega^{s+1}(L) \otimes L \rightarrow (s+1)P_\varepsilon \otimes L \stackrel{(4.2)}{\simeq} 5(s+1)P_L \rightarrow \Omega^s(L) \otimes L \stackrel{\text{IH}}{\simeq} \Omega^s(\varepsilon) \oplus s2P_L \rightarrow 0.$$

Again, by Lemma 4.1, $\Omega^{s+1}(L) \otimes L \simeq \Omega^{s+1}(\varepsilon) \oplus 2(s+2)P_L$ as claimed.

Now, with a similar strategy, we show the isomorphism for $s \in \mathbb{Z}_{\geq 0}$ and $t + 1$. We apply $\Omega^s(L) \otimes -$ to the exact sequences (3.4) or (3.5), depending on the parity of $k = t$, and use Lemma 4.1. We must analyze four cases. First, if s is odd and t is even, we have the exact sequence

$$\begin{aligned} 0 &\longrightarrow \Omega^s(L) \otimes \Omega^{t+1}(L) \longrightarrow \\ &\longrightarrow \Omega^s(L) \otimes (t+1)P_L \stackrel{\text{Remark 4.2 and (3.8)}}{\simeq} (5s+1)(t+1)P_L \oplus s(t+1)P_\varepsilon \longrightarrow \\ &\longrightarrow \Omega^s(L) \otimes \Omega^t(L) \stackrel{\text{IH}}{\simeq} \Omega^{s+t}(\varepsilon) \oplus s(3t+2)P_L \longrightarrow 0. \end{aligned}$$

Then $\Omega^s(L) \otimes \Omega^{t+1}(L) \simeq \Omega^{s+t+1}(\varepsilon) \oplus s(t+1)P_\varepsilon \oplus 2s(t+1)P_L$ as claimed. Second, if s is even and t is even, we have

$$\begin{aligned} 0 &\longrightarrow \Omega^s(L) \otimes \Omega^{t+1}(L) \longrightarrow \\ &\longrightarrow \Omega^s(L) \otimes (t+1)P_L \stackrel{\text{Remark 4.2 and (3.9)}}{\simeq} (5s+4)(t+1)P_L \oplus (s+1)(t+1)P_\varepsilon \longrightarrow \\ &\longrightarrow \Omega^s(L) \otimes \Omega^t(L) \stackrel{\text{IH}}{\simeq} \Omega^{s+t}(\varepsilon) \oplus stP_\varepsilon \oplus 2(s+1)(t+1)P_L \longrightarrow 0. \end{aligned}$$

Then $\Omega^s(L) \otimes \Omega^{t+1}(L) \simeq \Omega^{s+t+1}(\varepsilon) \oplus (3s+2)(t+1)P_L$ as claimed. Third, if s is odd and t is odd, we have

$$\begin{aligned} 0 &\longrightarrow \Omega^s(L) \otimes \Omega^{t+1}(L) \longrightarrow \\ &\longrightarrow \Omega^s(L) \otimes (t+1)P_\varepsilon \stackrel{\text{Remark 4.2 and (3.8)}}{\simeq} 5s(t+1)P_L \oplus (s+1)(t+1)P_\varepsilon \longrightarrow \\ &\longrightarrow \Omega^s(L) \otimes \Omega^t(L) \stackrel{\text{IH}}{\simeq} \Omega^{s+t}(\varepsilon) \oplus stP_\varepsilon \oplus 2stP_L \longrightarrow 0. \end{aligned}$$

Then $\Omega^s(L) \otimes \Omega^{t+1}(L) \simeq \Omega^{s+t+1}(\varepsilon) \oplus s(3(t+1)+2)P_L$ as claimed. Finally, the fourth case which we must consider is with s even and t odd. We have

$$\begin{aligned} 0 &\longrightarrow \Omega^s(L) \otimes \Omega^{t+1}(L) \longrightarrow \\ &\longrightarrow \Omega^s(L) \otimes (t+1)P_\varepsilon \stackrel{\text{Remark 4.2 and (3.9)}}{\simeq} 5(s+1)(t+1)P_L \oplus s(t+1)P_\varepsilon \longrightarrow \\ &\longrightarrow \Omega^s(L) \otimes \Omega^t(L) \stackrel{\text{IH}}{\simeq} \Omega^{s+t}(\varepsilon) \oplus (3s+2)tP_L \longrightarrow 0. \end{aligned}$$

Then $\Omega^s(L) \otimes \Omega^{t+1}(L) \simeq \Omega^{s+t+1}(\varepsilon) \oplus s(t+1)P_\varepsilon \oplus 2(s+1)((t+1)+1)P_L$ as claimed. This concludes the double induction and hence (4.3) holds.

4.5. Tensor products between syzygies and cosyzygy modules. Let $s, t \in \mathbb{Z}_{\geq 0}$. The next isomorphisms can be demonstrated by a double induction procedure and following a strategy analogous to that used for showing (4.3).

$$\Omega^s(L) \otimes \Omega^{-t}(L) \simeq \Omega^{-t}(L) \otimes \Omega^s(L) \tag{4.6}$$

$$\simeq \Omega^{s-t}(\varepsilon) \oplus \begin{cases} (s+1)tP_\varepsilon \oplus 2s(t+1)P_L & \text{for } s \text{ odd } > t \text{ even,} \\ s(t+1)P_\varepsilon \oplus 2s(t+1)P_L & \text{for } s \text{ odd } < t \text{ even,} \\ (3s+1)tP_L & \text{for } s \text{ odd } \geq t \text{ odd,} \\ s(3t+1)P_L & \text{for } s \text{ odd } < t \text{ odd,} \\ (s+1)(3t+2)P_L & \text{for } s \text{ even } \geq t \text{ even,} \\ (3s+2)(t+1)P_L & \text{for } s \text{ even } < t \text{ even,} \\ (s+1)tP_\varepsilon \oplus 2(s+1)tP_L & \text{for } s \text{ even } > t \text{ odd,} \\ s(t+1)P_\varepsilon \oplus 2(s+1)tP_L & \text{for } s \text{ even } < t \text{ odd;} \end{cases}$$

$$\Omega^s(L) \otimes \Omega^{-t}(\varepsilon) \simeq \Omega^{-t}(L) \otimes \Omega^s(\varepsilon) \tag{4.7}$$

$$\simeq \Omega^{s-t}(L) \oplus \begin{cases} (t+1)(3s+2)P_L & \text{for } t \text{ odd } > s \text{ even,} \\ (3t+2)(s+1)P_L & \text{for } t \text{ odd } < s \text{ even,} \\ (t+1)sP_\varepsilon \oplus 2(t+1)sP_L & \text{for } t \text{ odd } \geq s \text{ odd,} \\ t(s+1)P_\varepsilon \oplus 2(t+1)sP_L & \text{for } t \text{ odd } < s \text{ odd,} \\ (t+1)sP_\varepsilon \oplus 2t(s+1)P_L & \text{for } t \text{ even } \geq s \text{ even,} \\ t(s+1)P_\varepsilon \oplus 2(t+1)sP_L & \text{for } t \text{ even } < s \text{ even,} \\ (3t+1)sP_L & \text{for } t \text{ even } > s \text{ odd,} \\ t(3s+1)P_L & \text{for } t \text{ even } < s \text{ odd;} \end{cases}$$

$$\Omega^s(\varepsilon) \otimes \Omega^{-t}(\varepsilon) \simeq \Omega^{-t}(\varepsilon) \otimes \Omega^s(\varepsilon) \tag{4.8}$$

$$\simeq \Omega^{s-t}(\varepsilon) \oplus \begin{cases} (s+1)tP_\varepsilon \oplus 2(s+1)tP_L & \text{for } s \text{ odd } > t \text{ even,} \\ s(t+1)P_\varepsilon \oplus 2(s+1)tP_L & \text{for } s \text{ odd } < t \text{ even,} \\ (s+1)(3t+2)P_L & \text{for } s \text{ odd } \geq t \text{ odd,} \\ (3s+2)(t+1)P_L & \text{for } s \text{ odd } < t \text{ odd,} \\ (s+1)(3t+1)P_L & \text{for } s \text{ even } \geq t \text{ even,} \\ (3s+1)(t+1)P_L & \text{for } s \text{ even } < t \text{ even,} \\ (s+1)tP_\varepsilon \oplus 2s(t+1)P_L & \text{for } s \text{ even } > t \text{ odd,} \\ s(t+1)P_\varepsilon \oplus 2s(t+1)P_L & \text{for } s \text{ even } < t \text{ odd.} \end{cases}$$

4.6. Tensor products between (k, k) -types modules and syzygies. The following isomorphisms hold for all $s \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$ and $\mathbf{t} \in \mathfrak{A}$:

$$\begin{aligned}
 M_k(L, \mathbf{t}) \otimes \Omega^s(L) &\simeq \Omega^s(L) \otimes M_k(L, \mathbf{t}) & (4.9) \\
 &\simeq \begin{cases} M_k(\varepsilon, \mathbf{t}) \oplus (3s + 2)kP_L & \text{for } s \text{ even,} \\ M_k(L, \mathbf{t}) \oplus skP_\varepsilon \oplus 2skP_L & \text{for } s \text{ odd;} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 M_k(L, \mathbf{t}) \otimes \Omega^s(\varepsilon) &\simeq \Omega^s(\varepsilon) \otimes M_k(L, \mathbf{t}) & (4.10) \\
 &\simeq \begin{cases} M_k(L, \mathbf{t}) \oplus skP_\varepsilon \oplus 2skP_L & \text{for } s \text{ even,} \\ M_k(\varepsilon, \mathbf{t}) \oplus (3s + 2)kP_L & \text{for } s \text{ odd;} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 M_k(\varepsilon, \mathbf{t}) \otimes \Omega^s(L) &\simeq \Omega^s(L) \otimes M_k(\varepsilon, \mathbf{t}) & (4.11) \\
 &\simeq \begin{cases} M_k(L, \mathbf{t}) \oplus skP_\varepsilon \oplus 2(s + 1)kP_L & \text{for } s \text{ even,} \\ M_k(\varepsilon, \mathbf{t}) \oplus 3skP_L & \text{for } s \text{ odd;} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 M_k(\varepsilon, \mathbf{t}) \otimes \Omega^s(\varepsilon) &\simeq \Omega^s(\varepsilon) \otimes M_k(\varepsilon, \mathbf{t}) & (4.12) \\
 &\simeq \begin{cases} M_k(\varepsilon, \mathbf{t}) \oplus 3skP_L & \text{for } s \text{ even,} \\ M_k(\varepsilon, \mathbf{t}) \oplus skP_\varepsilon \oplus 2(s + 1)kP_L & \text{for } s \text{ odd.} \end{cases}
 \end{aligned}$$

We will prove (4.9) and (4.11). The proof of (4.10) and (4.12) is analogous. We begin by computing the indecomposable summands of $M_k(L, \mathbf{t}) \otimes L$ and $M_k(\varepsilon, \mathbf{t}) \otimes L$ by induction on k ; recall that $\Omega^0(L) = L$. For $k = 1$, we claim that the submodule $N = \mathcal{A} \cdot w \subset M_1(L, \mathbf{t}) \otimes L$, where

$$w = \bar{w}_\varepsilon \otimes v_{(132)} - \frac{t_{13}}{a + 2} \bar{w}_{(23)} \otimes v_{(12)} + \frac{t_{12}}{a - 1} \bar{w}_{(12)} \otimes v_{(13)},$$

is isomorphic to $M_1(\varepsilon, \mathbf{t})$. Indeed, the morphism $f : P_L \rightarrow M_1(L, \mathbf{t}) \otimes L$, induced by $\delta_L \mapsto w$, satisfies that $\dim \text{Im}(f) = 6$ and $x_{L, \mathbf{t}} \delta_L \in \text{Ker}(f)$; we verify this through a computation in GAP. Hence $N \simeq M_1(\varepsilon, \mathbf{t})$ as a consequence of Proposition 3.7. On the other hand, by applying $-\otimes L$ to (3.14), we deduce that $2P_L$ is a direct summand of $M_1(L, \mathbf{t}) \otimes L$. Since $\text{soc}(2P_L) = 2L$ and $\text{soc}(N) = \varepsilon$, we conclude that

$$M_1(L, \mathbf{t}) \otimes L \simeq M_1(\varepsilon, \mathbf{t}) \oplus 2P_L$$

by a dimensional argument. Now, if we apply $-\otimes L$ to this isomorphism, we obtain

$$\begin{aligned}
 (M_1(L, \mathbf{t}) \otimes L) \otimes L &\simeq M_1(\varepsilon, \mathbf{t}) \otimes L \oplus 2P_L \otimes L \stackrel{(4.2)}{\simeq} M_1(\varepsilon, \mathbf{t}) \otimes L \oplus 8P_L \oplus 2P_\varepsilon \\
 &\simeq M_1(L, \mathbf{t}) \otimes (L \otimes L) \stackrel{(4.1)}{\simeq} M_1(L, \mathbf{t}) \oplus 2M_1(L, \mathbf{t}) \otimes P_L \\
 &\stackrel{(4.2)}{\simeq} M_1(L, \mathbf{t}) \oplus 10P_L \oplus 2P_\varepsilon
 \end{aligned}$$

and hence the Krull-Schmidt theorem implies that

$$M_1(\varepsilon, \mathbf{t}) \otimes L \simeq M_1(L, \mathbf{t}) \oplus 2P_L.$$

We have proved (4.9) and (4.11) for $k = 1$ and $s = 0$.

Let us continue with the inductive step. We assume that the decompositions of $M_k(L, \mathbf{t}) \otimes L$ and $M_k(\varepsilon, \mathbf{t}) \otimes L$ in (4.9) and (4.11) hold for k . Then, tensoring (3.15) with L , we get the exact sequence

$$0 \longrightarrow M_1(\varepsilon, \mathbf{t}) \oplus 2P_L \longrightarrow M_{k+1}(L, \mathbf{t}) \otimes L \longrightarrow M_k(\varepsilon, \mathbf{t}) \oplus 2kP_L \longrightarrow 0.$$

Hence $M_{k+1}(L, \mathbf{t}) \otimes L \simeq N \oplus 2(k+1)P_L$ with N fitting in the exact sequence

$$0 \longrightarrow M_1(\varepsilon, \mathbf{t}) \longrightarrow N \longrightarrow M_k(\varepsilon, \mathbf{t}) \longrightarrow 0.$$

By Proposition 3.5, $N \simeq M_{k+1}(\varepsilon, \mathbf{t})$ or $N \simeq M_1(\varepsilon, \mathbf{t}) \oplus M_k(\varepsilon, \mathbf{t})$. Suppose that the second isomorphism holds. We compute $M_{k+1}(L, \mathbf{t}) \otimes L \otimes L$ in two ways:

$$\begin{aligned} (M_{k+1}(L, \mathbf{t}) \otimes L) \otimes L &\stackrel{\text{IH}}{\simeq} M_1(L, \mathbf{t}) \oplus M_k(L, \mathbf{t}) \oplus P_1 \\ &\simeq M_{k+1}(L, \mathbf{t}) \otimes (L \otimes L) \stackrel{(4.1)}{\simeq} M_{k+1}(L, \mathbf{t}) \oplus P_2; \end{aligned}$$

here P_1 and P_2 denote certain projective modules. By the Krull–Schmidt theorem, the above isomorphisms cannot be possible. Therefore $N \simeq M_{k+1}(\varepsilon, \mathbf{t})$ and hence $M_{k+1}(L, \mathbf{t}) \otimes L \simeq M_{k+1}(\varepsilon, \mathbf{t}) \oplus 2(k+1)P_L$. Now, by applying $-\otimes L$ to this isomorphism, we can deduce that $M_{k+1}(\varepsilon, \mathbf{t}) \otimes L \simeq M_{k+1}(L, \mathbf{t}) \oplus 2(k+1)P_L$ by arguing as in the case $k = 1$. This completes the proof of the inductive step.

Summarizing, we have computed the indecomposable summands of the tensor products $M_k(L, \mathbf{t}) \otimes \Omega^s(L)$ and $M_k(\varepsilon, \mathbf{t}) \otimes \Omega^s(L)$ for all $k \in \mathbb{N}$ and $s = 0$. The indecomposable summands for $s > 0$ can be found by induction on s , in a similar way to the proof of (4.3). We leave it for the reader.

In order to finish the proof of (4.9) and (4.11) we must calculate the indecomposable summand of $\Omega^s(L) \otimes M_k(L, \mathbf{t})$ and $\Omega^s(L) \otimes M_k(\varepsilon, \mathbf{t})$. This can be made as we have proceed above. We leave it for the reader. We only observe that

$$v_{(132)} \otimes \bar{w}_\varepsilon - \frac{t_{13}}{a+2} v_{(12)} \otimes \bar{w}_{(13)} + \frac{t_{12}}{a-1} v_{(13)} \otimes \bar{w}_{(23)}$$

generates a direct summand of $\Omega^0(L) \otimes M_1(L, \mathbf{t})$ isomorphic to $M_1(\varepsilon, \mathbf{t})$.

4.7. Tensor products between (k, k) -types modules. The following isomorphisms hold for all $k, j \in \mathbb{N}$ and $\mathbf{t}, \mathbf{s} \in \mathfrak{A}$. Let us set $i := \min\{j, k\}$:

$$\begin{aligned} M_k(L, \mathbf{t}) \otimes M_j(L, \mathbf{s}) &\simeq M_j(L, \mathbf{s}) \otimes M_k(L, \mathbf{t}) && (4.13) \\ &\simeq \begin{cases} 2jkP_L \oplus jkP_\varepsilon & \text{if } \mathbf{t} \not\sim \mathbf{s}, \\ M_i(\varepsilon, \mathbf{t}) \oplus M_i(L, \mathbf{t}) \oplus 2jkP_L \oplus (jk-i)P_\varepsilon & \text{if } \mathbf{t} \sim \mathbf{s}; \end{cases} \end{aligned}$$

$$\begin{aligned} M_k(L, \mathbf{s}) \otimes M_j(\varepsilon, \mathbf{t}) &\simeq M_j(\varepsilon, \mathbf{t}) \otimes M_k(L, \mathbf{s}) && (4.14) \\ &\simeq \begin{cases} 3jkP_L & \text{if } \mathbf{t} \not\sim \mathbf{s}, \\ M_i(\varepsilon, \mathbf{t}) \oplus M_i(L, \mathbf{t}) \oplus (3jk-i)P_L & \text{if } \mathbf{t} \sim \mathbf{s}; \end{cases} \end{aligned}$$

$$\begin{aligned} M_k(\varepsilon, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{s}) &\simeq M_j(\varepsilon, \mathbf{s}) \otimes M_k(\varepsilon, \mathbf{t}) && (4.15) \\ &\simeq \begin{cases} 2jkP_L \oplus jkP_\varepsilon & \text{if } \mathbf{t} \not\sim \mathbf{s}, \\ M_i(\varepsilon, \mathbf{t}) \oplus M_i(L, \mathbf{t}) \oplus 2jkP_L \oplus (jk-i)P_\varepsilon & \text{if } \mathbf{t} \sim \mathbf{s}. \end{cases} \end{aligned}$$

4.7.1. *Proof for the case $\mathbf{t} \not\sim \mathbf{s}$.* We demonstrate first (4.13) by double induction. For $k = 1 = j$, we see that $M_1(L, \mathbf{t}) \otimes M_1(L, \mathbf{s})$ has $2P_L$ as a direct summand, by tensoring (3.14) with $M_1(L, \mathbf{s})$ and using (4.9) for $s = 0$. On the other hand, a computation in GAP allows us to verify that

$$x_{\text{top}} \cdot (\bar{w}_e \otimes \bar{w}_e) = (t_{(12)}s_{(13)} - t_{(13)}s_{(12)})(\bar{w}_{(123)} \otimes \bar{w}_{(132)} - \bar{w}_{(132)} \otimes \bar{w}_{(123)}) \\ (2a + 1)\bar{w}_{(23)} \otimes \bar{w}_{(23)} - (a + 2)\bar{w}_{(12)} \otimes \bar{w}_{(12)} - (a - 1)\bar{w}_{(13)} \otimes \bar{w}_{(13)}),$$

where $\mathbf{t} = (t_{(23)}, t_{(12)}, t_{(13)})$ and $\mathbf{s} = (s_{(23)}, s_{(12)}, s_{(13)})$. Thus $x_{\text{top}} \cdot (\bar{w}_e \otimes \bar{w}_e) \neq 0$ as $\mathbf{t} \not\sim \mathbf{s}$. Then P_ε is a direct summand of $M_1(L, \mathbf{t}) \otimes M_1(L, \mathbf{s})$ by Lemma 3.1. Putting all together and by a dimensional argument, we get (4.13) for $k = 1 = j$. Explicitly,

$$M_1(L, \mathbf{t}) \otimes M_1(L, \mathbf{s}) \simeq 2P_L \oplus P_\varepsilon.$$

We continue by proving (4.13) for $k + 1$ and $j = 1$ assuming that it holds for k and $j = 1$. Tensoring the exact sequence (3.15) by $M_1(L, \mathbf{s})$, we get

$$0 \longrightarrow 2P_L \oplus P_\varepsilon \longrightarrow M_{k+1}(L, \mathbf{t}) \otimes M_1(L, \mathbf{s}) \longrightarrow 2kP_L \oplus kP_\varepsilon \longrightarrow 0$$

thanks to the inductive hypothesis. Therefore the middle term must obey (4.13).

Finally, we prove (4.13) for k and $j + 1$ assuming that it holds for k and j . We apply $M_k(L, \mathbf{t}) \otimes -$ to the exact sequence (3.15), with j instead of k , and obtain

$$0 \longrightarrow 2kP_L \oplus kP_\varepsilon \longrightarrow M_k(L, \mathbf{t}) \otimes M_{j+1}(L, \mathbf{s}) \longrightarrow 2kjP_L \oplus kjP_\varepsilon \longrightarrow 0.$$

As above, the middle term must obey (4.13). This completes the double induction proof for (4.13); we notice that $M_k(L, \mathbf{t}) \otimes M_j(L, \mathbf{s}) \simeq M_j(L, \mathbf{s}) \otimes M_k(L, \mathbf{t})$ because our arguments hold for all $k, j \in \mathbb{N}$ and $\mathbf{t}, \mathbf{s} \in \mathfrak{A}$.

Now, we demonstrate (4.14) for $\mathbf{t} \not\sim \mathbf{s}$. We apply $-\otimes M_k(L, \mathbf{s})$ to the exact sequence (3.17) and get

$$0 \longrightarrow M_j(\varepsilon, \mathbf{t}) \otimes M_k(L, \mathbf{s}) \longrightarrow 5jkP_L \oplus jkP_\varepsilon \longrightarrow 2jkP_L \oplus jkP_\varepsilon \longrightarrow 0;$$

the middle term is due to Remark 4.2 and the term on the right-hand side is due to (4.13). Therefore $M_j(\varepsilon, \mathbf{t}) \otimes M_k(L, \mathbf{s})$ decomposes as in (4.14) and the same holds for $M_k(L, \mathbf{s}) \otimes M_j(\varepsilon, \mathbf{t})$ because the tensor products in Remark 4.2 and (4.13) are commutative.

Finally, we note that the isomorphism (4.15) for $\mathbf{t} \not\sim \mathbf{s}$ follows by dualizing (4.13). □

4.7.2. *Proof for the case $\mathbf{t} \sim \mathbf{s}$.* We can assume $\mathbf{t} = \mathbf{s}$. We prove first the decomposition of $M_k(L, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t})$ for $j \geq k$ given in (4.14).

By the Morita equivalence and [9, Lemma 3.28], there exists an exact sequence

$$0 \longrightarrow M_k(L, \mathbf{t}) \longrightarrow N \longrightarrow \varepsilon \longrightarrow 0$$

with $N \simeq \Omega^k(L)$ for k odd, $N \simeq \Omega^k(\varepsilon)$ otherwise. Tensoring by $M_j(\varepsilon, \mathbf{t})$, we get

$$0 \longrightarrow M_k(L, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t}) \xrightarrow{\iota} \\ \xrightarrow{\iota} N \otimes M_j(\varepsilon, \mathbf{t}) \stackrel{(4.11)-(4.12)}{\simeq} M_j(\varepsilon, \mathbf{t}) \oplus 3jkP_L \xrightarrow{\pi} M_j(\varepsilon, \mathbf{t}) \longrightarrow 0.$$

Let \overline{N} be a submodule of $M_k(L, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t})$ isomorphic to $M_j(\varepsilon, \mathbf{t})$, which exists by Lemma 4.3, see below. As ι is a monomorphism, $\iota(\overline{N}) \simeq \overline{N} \simeq M_j(\varepsilon, \mathbf{t})$. By observing the socles, we conclude that $\iota(\overline{N}) \cap (3jkP_L) = 0$ and hence the above exact sequence looks as follows:

$$0 \longrightarrow M_k(L, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t}) \xrightarrow{\iota} \iota(\overline{N}) \oplus 3jkP_L \xrightarrow{\pi} M_j(\varepsilon, \mathbf{t}) \longrightarrow 0.$$

Since $\pi \circ \iota = 0$, the restriction $\pi|_{3jkP_L} : 3jkP_L \longrightarrow M_j(\varepsilon, \mathbf{t})$ is an epimorphism. By Lemma 4.1 and Proposition 3.7, we have that $\ker(\pi|_{3jkP_L}) \simeq M_k(L, \mathbf{t}) \oplus (3j-1)kP_L$. Therefore

$$M_k(L, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t}) \simeq M_j(\varepsilon, \mathbf{t}) \oplus M_j(\varepsilon, \mathbf{t}) \oplus (3j-1)kP_L$$

as we wanted.

The proof for $k \geq j$ is analogous but it starts with an exact sequence involving $M_j(\varepsilon, \mathbf{t})$ to which we apply $M_k(L, \mathbf{t}) \otimes -$.

Now, we can deduce (4.15) from (4.14) by applying $L \otimes -$. In fact, by the formulas of the above subsections we have

$$L \otimes (M_k(L, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t})) \simeq M_i(\varepsilon, \mathbf{t}) \oplus M_i(L, \mathbf{t}) \oplus 12jkP_L \oplus (3jk-i)P_\varepsilon$$

and, on the other hand, we get

$$(L \otimes M_k(L, \mathbf{t})) \otimes M_j(\varepsilon, \mathbf{t}) \simeq M_k(\varepsilon, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t}) \oplus 10jkP_L \oplus 2jkP_\varepsilon.$$

By the associative property, the right-hand side of both isomorphisms are isomorphic and hence (4.15) holds due to the Krull-Schmidt theorem; notice that k and j do not play any role in our reasoning and then $M_j(\varepsilon, \mathbf{t}) \otimes M_k(\varepsilon, \mathbf{t}) \simeq M_k(\varepsilon, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t})$.

The isomorphism in (4.14) follows by applying $-\otimes L$ to (4.15) and arguing as in the above paragraph.

Finally, the isomorphism (4.13) for $\mathbf{t} \sim \mathbf{s}$ follows by dualizing (4.15). □

Lemma 4.3. *If $l \leq j, k$, then $M_k(L, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t})$ has a submodule isomorphic to $M_l(\varepsilon, \mathbf{t})$.*

Proof. We introduce the following elements in $M_k(L, \mathbf{t}) \otimes M_j(\varepsilon, \mathbf{t})$:

$$\begin{aligned} \mathbf{a}_{i,\ell} &:= \overline{w}_e^i \otimes w_e^\ell + \overline{w}_{(23)}^i \otimes w_{(23)}^\ell + \overline{w}_{(12)}^i \otimes w_{(12)}^\ell + \overline{w}_{(13)}^i \otimes w_{(13)}^\ell \\ &\quad - \overline{w}_{(123)}^i \otimes w_{(132)}^\ell - \overline{w}_{(132)}^i \otimes w_{(123)}^\ell, \\ \mathbf{b}_{i,\ell} &:= \overline{w}_e^i \otimes w_{(132)}^\ell - \frac{t_{(13)}}{a+2} \overline{w}_{(23)}^i \otimes w_{(12)}^\ell - \frac{t_{(12)}}{a-1} \overline{w}_{(12)}^i \otimes w_{(13)}^\ell, \\ \mathbf{c}_{i,\ell} &:= -\frac{\widehat{t}_{(13)}}{a+2} \overline{w}_{(23)}^i \otimes w_{(12)}^\ell - \frac{\widehat{t}_{(12)}}{a-1} \overline{w}_{(12)}^i \otimes w_{(13)}^\ell \end{aligned}$$

for all $1 \leq i, \ell \leq j, k$. By a computation in GAP², we see that

$$x_\sigma \mathbf{a}_{i,\ell} = \widehat{t}(\overline{w}_\sigma^{i-1} \otimes w_e^\ell - \overline{w}_e^i \otimes w_\sigma^{\ell-1}),$$

²We compute in GAP the action of x_σ on $\mathbf{a}_{1,2}$, $\mathbf{a}_{2,3}$ and $\mathbf{a}_{2,1}$ in $M_3(L, \mathbf{t}) \otimes M_3(\varepsilon, \mathbf{t})$. This is enough to deduce the action for all i and ℓ because the action of x_σ on \overline{w}_g^ℓ and w_g^ℓ depends only on the elements \overline{w}_g^ℓ , $\overline{w}_g^{\ell-1}$, w_g^ℓ and $w_g^{\ell-1}$. We do the same for $\mathbf{b}_{i,\ell}$ and $\mathbf{c}_{i,\ell}$.

$$\begin{aligned}
 x_{(12)}x_{(23)}\mathbf{b}_{i,\ell} &= t_{(12)}\mathbf{a}_{i,\ell} + \widehat{t}_{(12)}\left(\overline{w}_e^i \otimes w_e^{\ell-1} + \overline{w}_{(12)}^{i-1} \otimes w_{(12)}^\ell\right) \\
 &\quad - \widehat{t}_{(23)}\left(\overline{w}_{(23)}^{i-1} \otimes w_{(23)}^\ell - \overline{w}_{(123)}^{i-1} \otimes w_{(132)}^\ell\right), \\
 x_{(13)}x_{(12)}\mathbf{b}_{i,\ell} &= t_{(13)}\mathbf{a}_{i,\ell} + \widehat{t}_{(13)}\left(\overline{w}_e^i \otimes w_e^{\ell-1} + \overline{w}_{(13)}^{i-1} \otimes w_{(13)}^\ell\right) \\
 &\quad - \widehat{t}_{(12)}\left(\overline{w}_{(12)}^{i-1} \otimes w_{(12)}^\ell - \overline{w}_{(123)}^{i-1} \otimes w_{(132)}^\ell\right), \\
 x_{(12)}x_{(23)}\mathbf{c}_{i,\ell} &= \widehat{t}_{(12)}\left(\overline{w}_{(13)}^i \otimes w_{(13)}^\ell - \overline{w}_{(132)}^i \otimes w_{(123)}^\ell\right) \\
 &\quad - \widehat{t}_{(13)}\left(\overline{w}_{(23)}^i \otimes w_{(23)}^\ell - \overline{w}_{(123)}^i \otimes w_{(132)}^\ell\right), \\
 x_{(13)}x_{(12)}\mathbf{c}_{i,\ell} &= \widehat{t}_{(13)}\left(\overline{w}_{(23)}^i \otimes w_{(23)}^\ell - \overline{w}_{(132)}^i \otimes w_{(123)}^\ell\right) \\
 &\quad - \widehat{t}_{(23)}\left(\overline{w}_{(12)}^i \otimes w_{(12)}^\ell - \overline{w}_{(123)}^i \otimes w_{(132)}^\ell\right)
 \end{aligned}$$

for all $1 \leq i, \ell \leq j, k$ and transposition σ ; the summands corresponding to $i - 1 = 0 = \ell - 1$ must be omitted.

Finally, for $\ell \leq l \leq j, k$, we set

$$\begin{aligned}
 \mathbf{w}_e^\ell &:= \sum_{i=1}^{\ell} \mathbf{a}_{i,\ell+1-i}, & \mathbf{w}_{(132)}^\ell &:= \sum_{i=1}^{\ell} \mathbf{b}_{i,\ell+1-i} + \sum_{i=1}^{\ell-1} \mathbf{c}_{i,\ell-i}, \\
 \mathbf{w}_{(23)}^\ell &:= x_{(13)}\mathbf{w}_{(132)}^\ell, & \mathbf{w}_{(12)}^\ell &:= x_{(23)}\mathbf{w}_{(132)}^\ell, & \mathbf{w}_{(13)}^\ell &:= x_{(12)}\mathbf{w}_{(132)}^\ell, \\
 \mathbf{w}_{(123)}^\ell &:= \frac{1}{1+2a}x_{(12)}x_{(13)}\mathbf{w}_{(132)}^\ell.
 \end{aligned}$$

Using the above formulas and the defining relations of \mathcal{A} , we can see that the action of x_σ on the elements \mathbf{w}_g^ℓ obeys the recipe (3.10). Therefore $\{\mathbf{w}_g^\ell \mid g \in \mathbb{S}_3, 1 \leq \ell \leq l\}$ spans a submodule isomorphic to $M_l(\varepsilon, \mathbf{t})$. □

4.8. Tensor products between cosyzygy modules. These tensor products can be calculated by dualizing the isomorphism of Subsection 4.4.

4.9. Tensor products between (k, k) -types modules and cosyzygies. These tensor products can be calculated by dualizing the isomorphisms of Subsection 4.6.

4.10. The Green ring of \mathcal{A} . By definition, this ring is generated by the isomorphism classes of modules with operations $[M] + [N] = [M \oplus N]$ and $[M] \cdot [N] = [M \otimes N]$, where $[M]$ and $[N]$ denote the respective isomorphism classes of the modules M and N . For $k \in \mathbb{N}$ and $\mathbf{t} \in \mathfrak{A}/\sim$, we recall that

$$1 := [\varepsilon], \quad \lambda := [L], \quad \rho := [P_L], \quad \omega := [\Omega(\varepsilon)], \quad \overline{\omega} := [\Omega^{-1}(\varepsilon)], \quad \mu_{k,\mathbf{t}} := [M_k(\varepsilon, \mathbf{t})].$$

Proof of Theorem 1.2. In the previous subsections we have seen that the tensor products between indecomposable modules satisfy $M \otimes N \simeq N \otimes M$. Then the Green ring of \mathcal{A} is commutative. Let $\mathbb{Z}[\lambda, \rho]$ denote the \mathbb{Z} -subalgebra generated by λ and ρ . The following facts are also deduced from those subsections:

$$\begin{aligned}
 [P_\varepsilon] &= \lambda\rho - 4\rho \quad \text{by (4.2),} \\
 [\Omega^s(\varepsilon)] &\in \omega^s \oplus_{i=0}^{s-1} \mathbb{Z}[\lambda, \rho]\omega^i \quad \text{by (4.5) and induction in } s, \\
 [\Omega^s(L)] &\in \lambda\omega^s \oplus_{i=0}^{s-1} \mathbb{Z}[\lambda, \rho]\omega^i \quad \text{by (4.4) and the above one,} \\
 [\Omega^{-s}(\varepsilon)] &\in \bar{\omega}^s \oplus_{i=0}^{s-1} \mathbb{Z}[\lambda, \rho]\bar{\omega}^i \quad \text{by dual arguments to the above ones,} \\
 [\Omega^{-s}(L)] &\in \lambda\bar{\omega}^s \oplus_{i=0}^{s-1} \mathbb{Z}[\lambda, \rho]\bar{\omega}^i \quad \text{by dual arguments to the above ones,} \\
 [M_k(L, \mathbf{t})] &= -2k\rho + \lambda\mu_{k, \mathbf{t}} \quad \text{by (4.11)}
 \end{aligned}$$

for all $s, k \in \mathbb{N}$ and $\mathbf{t} \in \mathfrak{A}$. These imply that \mathcal{B} forms a \mathbb{Z} -basis of the Green ring of \mathcal{A} since the classes of the indecomposable modules do so. In particular, the elements $\lambda, \rho, \omega, \bar{\omega}$, and $\mu_{k, \mathbf{t}}$ generate the Green ring of \mathcal{A} as a \mathbb{Z} -algebra.

Let us prove that the relations in Table 1 hold. Relations (Rel. 1) and (Rel. 5) follow from (4.1) and (4.8). By (4.2) and Remark 4.2, we know that $[P_\varepsilon] = \lambda\rho - 4\rho$ and $\rho^2 = 2[P_\varepsilon] + 10\rho$. These imply (Rel. 2); (Rel. 3) and (Rel. 4) are deduced similarly. (Rel. 6)–(Rel. 10) follow from the isomorphisms in Subsections 4.6–4.7.

Now, we have an epimorphism from the commutative \mathbb{Z} -algebra R presented by generators and relations as in the statement over the Green ring of \mathcal{A} . In order to show the isomorphism it is enough to prove that R is the \mathbb{Z} -span of the corresponding elements of \mathcal{B} ; notice that this set is linearly independent over \mathbb{Z} because it projects over a \mathbb{Z} -basis. We leave it for the reader to verify that every element

$$\lambda^{n_1} \rho^{n_2} \omega^{n_3} \bar{\omega}^{n_4} \mu_{k_5, \mathbf{t}_5}^{n_5} \cdots \mu_{k_\ell, \mathbf{t}_\ell}^{n_\ell} \in R$$

can be expressed as a linear combination of elements in \mathcal{B} using the relations. □

4.11. Semisimplification of $\text{Rep } \mathcal{A}$. The element $\chi = \sum_{g \in \mathbb{S}_3} \text{sgn}(g)\delta_g$ is an involutory pivot of \mathcal{A} [4, Proposition 30]. Therefore \mathcal{A} is a spherical Hopf algebra and we can form the quotient category $\underline{\text{Rep}} \mathcal{A}$ which is a semisimple tensor category. We refer to [1] for details on this matter.

Corollary 4.4. *Let Γ denote the group $C_2 \times \mathbb{Z}$. Then $\underline{\text{Rep}} \mathcal{A}$ is monoidally equivalent to the category of Γ -graded finite-dimensional vector spaces.*

Proof. The simple objects in $\underline{\text{Rep}} \mathcal{A}$ are the classes of indecomposable modules with non-zero quantum dimension. The quantum dimension of a module is the trace of the action of the pivot χ . Therefore only the simple and the (co)syzygies modules survive in $\underline{\text{Rep}} \mathcal{A}$. Let c and z be generators of C_2 and \mathbb{Z} , respectively. The functor $F : \underline{\text{Rep}} \mathcal{A} \rightarrow \text{vect}^\Gamma$ induced by $F(L) = \mathbb{k}_c, F(\Omega(\varepsilon)) = \mathbb{k}_z, F(\Omega^{-1}(\varepsilon)) = \mathbb{k}_{z^{-1}}, F(\Omega(L)) = \mathbb{k}_{cz}$ and $F(\Omega^{-1}(L)) = \mathbb{k}_{c z^{-1}}$ gives the desired equivalence thanks to Theorem 1.2. □

APPENDIX A. ON THE COMPUTATIONS IN GAP

We set as underlying field $\mathbf{F} = \mathbb{Q}(\mathbf{a}, \mathbf{t12}, \mathbf{t13}, \mathbf{s12}, \mathbf{s13})$, the function field over the rationals in 5 indeterminates. The first indeterminate represents the scalar involved in the definition of $\mathcal{A} = \mathcal{A}_{[(1, a, -1, -a)]}$. Thus, the computations hold for

any scalar a except for $1, -\frac{1}{2}, -2$. The remaining indeterminates stand for the parameters defining the (k, k) -types modules.

Let M be an \mathcal{A} -module of dimension m with matrix representation $\varrho_M : \mathcal{A} \rightarrow \text{Mat}_{m \times m}(\mathbb{F})$. In order to calculate the action of $x \in \mathcal{A}$ over $m \in M$ in GAP, we construct the matrices $\varrho_M(x_{(ij)})$ and $\varrho_M(\delta_g)$ of all the generators of \mathcal{A} in a suitable basis of M . Then the image of ϱ_M is constructed in GAP using the command `AlgebraWithOne` and the above matrices. The elements of M are represented by m -uples and the command `LeftAlgebraModule` allows us to calculate $x \cdot m$. We next explain how we have constructed the matrix representation of the generators.

A.1. Matrix representation of L . We use the basis and formula in (3.1).

A.2. Matrix representation of tensor products. We use the formulas of the comultiplication given in Section 2 since $\varrho_{M \otimes N}(x) = \varrho_M(x_{(1)}) \otimes \varrho_N(x_{(2)})$. We recall that the command `KroneckerProduct` builds up the tensor product of vector spaces.

A.3. Matrix representation of P_L . By Subsection 4.2, we know that P_L is isomorphic to the submodule generated by $n_1 \in L \otimes L$. Thus, the command `SubAlgebraModule` allows us to work with P_L as a submodule of $L \otimes L$. In particular, we can consider the basis $\{x \cdot n_1 \mid x \in \mathbb{B}\}$ of P_L . Then, we construct the matrix representation of the generators using the command `MatrixOfAction`. Of course, one can construct these matrices by hand using the formulas [4, (19)–(52)] but it is more tedious.

A.4. Matrix representation of P_ε . We proceed as for P_L . We use that P_ε is a submodule of $L \otimes P_L$ by Subsection 4.3. A computation in GAP shows that $v_{(123)} \otimes \delta_L \in L \otimes P_L$ generates such a submodule.

A.5. Matrix representation of the (k, k) -types modules. We use the bases and formulas in (3.10) and (3.13). The elements $(t_{(23)}, t_{(12)}, t_{(13)})$ and $(\widehat{t}_{(23)}, \widehat{t}_{(12)}, \widehat{t}_{(13)}) \in \mathfrak{A}$ are represented by $(-\mathbf{t}12 - \mathbf{t}13, \mathbf{t}12, \mathbf{t}13)$ and $(-\mathbf{s}12 - \mathbf{s}13, \mathbf{s}12, \mathbf{s}13)$, respectively. For our purposes, it is enough to make calculations for $k \in \{1, 2, 3\}$ because the action of $x_{(ij)}$ on the basis elements \overline{w}_g^ℓ and w_g^ℓ depends only on the elements $\overline{w}_g^\ell, \overline{w}_g^{\ell-1}, w_g^\ell$ and $w_g^{\ell-1}$.

In Subsection 4.7.1, we carry out computations in $M_1(L, \mathbf{t}) \otimes M_1(L, \mathbf{s})$ with $\mathbf{t} \not\sim \mathbf{s}$. Then we use $(-\mathbf{t}12 - \mathbf{t}13, \mathbf{t}12, \mathbf{t}13)$ in the representation of $M_1(L, \mathbf{t})$ and $(-\mathbf{s}12 - \mathbf{s}13, \mathbf{s}12, \mathbf{s}13)$ in the representation of $M_1(L, \mathbf{s})$ since, for $k = 1$, the element $(\widehat{t}_{(23)}, \widehat{t}_{(12)}, \widehat{t}_{(13)})$ does not play any role in the definitions of these modules.

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