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A NOTE ON BERNSTEIN–SATO IDEALS

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Abstract. We define the Bernstein–Sato ideal associated to a tuple of ideals and we relate it to the jumping points of the corresponding mixed multiplier ideals.

1. Introduction

Let R be either the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ over the complex numbers or the ring $\mathbb{C}\{x_1, \dots, x_n\}$ of convergent power series in the neighbourhood of the origin, or any other point. The *multiplier ideals* of an element f or an ideal $\mathfrak{a}$ in R are a family of nested ideals that play a prominent role in birational geometry (see Lazarsfeld's book [10]). Associated to these ideals we have a set of invariants, the *jumping numbers*, which are intimately related to other invariants of singularities. For instance, Ein, Lazarsfeld, Smith and Varolin [8], and independently Budur and Saito [6], proved that the negatives of the jumping numbers of f in the interval $(0, 1)$ are roots of the *Bernstein–Sato polynomial* of f. Budur, Mustaţă and Saito [5] extended the classical theory of Bernstein–Sato polynomials to the case of ideals and also proved that the jumping numbers of an ideal $\mathfrak{a}$ in the interval $(0, 1)$ are roots of the Bernstein–Sato polynomial of $\mathfrak{a}$.

There is a natural extension of the theory of multiplier ideals to the context of tuples of germs $F := f_1, \dots, f_\ell$ or tuples of ideals $\boldsymbol{\mathfrak{a}} := \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ in R. The main difference that we encounter in this setting is that, whereas the multiplier ideals come with the set of associated jumping numbers, the mixed multiplier ideals come with a set of *jumping walls*. On the other hand, we have the notion of *Bernstein–Sato ideal* associated to a tuple of germs F given by Sabbah [14]. In the case of a tuple of plane curves, Cassou-Noguès and Libgober [7] related the Bernstein–Sato ideal with the so-called *faces of quasi-adjunction*, which are a set of invariants equivalent to the jumping walls.

The aim of this short note is to fill out a gap in the theory by introducing the notion of Bernstein–Sato ideal associated to a tuple of ideals $\boldsymbol{\mathfrak{a}} := \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$. To such purpose we are going to follow the approach given by Mustaţă [12], in which he relates the Bernstein–Sato polynomial of a single ideal $\mathfrak{a} = \big(f_1, \dots, f_r\big)$ to the reduced Bernstein–Sato polynomial of $g = f_1 y_1 + \cdots + f_r y_r$, where the y_j's are

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new variables. Finally, we show in Theorem 3.11 that the negative of the jumping points of the mixed multiplier ideals of the tuple $\mathbf{\mathfrak{a}}$ that are in the open ball of radius one centered at the origin belong to the zero locus of the Bernstein–Sato ideal of $\mathbf{\mathfrak{a}}$.

The theory of Bernstein–Sato polynomials and its relations with other invariants such as the multiplier ideals is vast and rich. In this note we tried to introduce only the essential concepts that we needed, so we recommend those who are not that familiar with these topics to take a look at the surveys of Budur [4], Granger [9], or Jeffries, Núñez-Betancourt and the author [1] for further insight.

2. Bernstein–Sato ideal of a tuple of ideals

Let R be either $\mathbb{C}[x_1,\ldots,x_n]$ or $\mathbb{C}\{x_1,\ldots,x_n\}$ and denote by $\mathfrak{m}=(x_1,\ldots,x_n)$ the (homogeneous) maximal ideal. Let $\mathbf{\mathfrak{a}}:=\mathfrak{a}_1,\ldots,\mathfrak{a}_\ell$ be a tuple of ideals in R. For each ideal described by a set of generators $\mathfrak{a}_i=\left(f_{i,1},\ldots,f_{i,r_i}\right)$, we consider $g_i=f_{i,1}y_{i,1}+\cdots+f_{i,r_i}y_{i,r_i}$, where the $y_{i,j}$'s are new variables. In particular, we get a tuple $G:=g_1,\ldots,g_\ell$ in the ring A that will be either $\mathbb{C}[x_1,\ldots,x_n,y_{1,1},\ldots,y_{\ell,r_\ell}]$ or $\mathbb{C}\{x_1,\ldots,x_n,y_{1,1},\ldots,y_{\ell,r_\ell}\}$. In what follows, $d:=n+r_1+\cdots+r_\ell$ will denote the number of variables in A.

Associated to R or A we have the corresponding ring of differential operators

$$D_R=R\langle\partial_1,\ldots,\partial_n\rangle,\qquad D_A=A\langle\partial_1,\ldots,\partial_n,\partial_{1,1},\ldots,\partial_{\ell,r_\ell}\rangle,$$

where ∂_i (resp. $\partial_{i,j}$) is the partial derivative with respect to x_i (resp. $y_{i,j}$). That is, D_R (resp. D_A) is the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(R)$ (resp. $\mathrm{End}_{\mathbb{C}}(A)$) generated by the ring and the partial derivatives.

Definition 2.1. The *Bernstein–Sato ideal* of the tuple G is the ideal $B_G\subseteq\mathbb{C}[s_1,\ldots,s_\ell]$ generated by all the polynomials $b(s_1,\ldots,s_\ell)$ satisfying the *Bernstein–Sato functional equation*

$$\delta(s_1,\ldots,s_\ell)g_1^{s_1+1}\cdots g_\ell^{s_\ell+1}=b(s_1,\ldots,s_\ell)g_1^{s_1}\cdots g_\ell^{s_\ell},$$

where $\delta(s_1,\ldots,s_\ell)\in D_A[s_1,\ldots,s_\ell]$ and $b(s_1,\ldots,s_\ell)\in\mathbb{C}[s_1,\ldots,s_\ell]$.

Sabbah [14] proved that $B_G\neq 0$ in the convergent power series case. The proof of $B_G\neq 0$ in the polynomial ring case is completely analogous to the classical case of a single element. Indeed, it is enough to consider the local case.

Remark 2.2. Briançon and Maisonobe showed [2] that

$$B_G^{\mathbb{C}[x]}=\bigcap_{p\in\mathbb{C}^d}B_G^{\mathbb{C}\{x-p\}},$$

where $B_G^{\mathbb{C}[x]}$ denotes the Bernstein–Sato ideal of a tuple G over the polynomial ring and $B_G^{\mathbb{C}\{x-p\}}$ is the Bernstein–Sato ideal of G in the convergent power series around a point $p\in\mathbb{C}^d$.

Now, since the g_i are pairwise without common factors, we have

$$B_G\subseteq\big((s_1+1)\cdots(s_\ell+1)\big)$$

(see [11, 3] for details).

Definition 2.3. The *reduced Bernstein–Sato ideal* of the tuple G is the ideal $\widetilde{B}_G \subseteq \mathbb{C}[s_1, \dots, s_\ell]$ generated by the polynomials

$$\frac{b(s_1, \dots, s_\ell)}{(s_1+1)\cdots(s_\ell+1)},$$

with $b(s_1, \dots, s_\ell) \in B_G$.

Following the approach given by Mustaţă [12] for the case of a single ideal, we consider the following:

Definition 2.4. Let $\mathfrak{a} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals in $\mathcal{O}_{X,O}$ and let $G := g_1, \dots, g_\ell$ be its associated tuple of hypersurfaces. We define the *Bernstein–Sato ideal* of $\mathfrak{a}$ as

$$B_{\mathfrak{a}} := \widetilde{B}_G \subseteq \mathbb{C}[s_1, \dots, s_\ell].$$

Our next result shows that $B_{\mathfrak{a}}$ does not depend on the generators of the ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ and thus it is an invariant of the tuple $\mathfrak{a}$.

Theorem 2.5. *Let $\mathfrak{a} := \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals and, for each ideal, consider two different sets of generators, $\mathfrak{a}_i = \left(f_{i,1}, \dots, f_{i,r_i}\right)$ and $\mathfrak{a}_i = \left(f'_{i,1}, \dots, f'_{i,s_i}\right)$. Consider the tuple $G = g_1, \dots, g_\ell$ with $g_i = f_{i,1}y_{i,1} + \cdots + f_{i,r_i}y_{i,r_i}$ and the tuple $G' = g'_1, \dots, g'_\ell$ with $g'_i = f'_{i,1}y'_{i,1} + \cdots + f'_{i,s_i}y'_{i,s_i}$. Then $\widetilde{B}_G = \widetilde{B}_{G'}$.*

Proof. Without loss of generality we may assume that, for each ideal $\mathfrak{a}_i$, the set of generators $f'_{i,1}, \dots, f'_{i,s_i}$ is just $f_{i,1}, \dots, f_{i,r_i}, h_i$ for a given $h_i \in \mathfrak{a}_i$. Let $z_1, \dots, z_{r_i}$ be such that $h_i = z_1 f_{i,1} + \cdots + z_{r_i} f_{i,r_i}$. Then we have

$$\begin{aligned} g'_i &= f_{i,1}y'_{i,1} + \cdots + f_{i,r_i}y'_{i,r_i} + h_i y'_{i,r_i+1} \\ &= f_{i,1}y'_{i,1} + \cdots + f_{i,r_i}y'_{i,r_i} + (z_1 f_{i,1} + \cdots + z_{r_i} f_{i,r_i})y'_{i,r_i+1} \\ &= f_1(y'_{i,1} + z_1 y'_{i,r_i+1}) + \cdots + f_\ell(y'_{i,r_i} + z_{r_i} y'_{i,r_i+1}). \end{aligned}$$

After a change of variables $y_{i,j} \mapsto y'_{i,j} + z_j y'_{i,r_i+1}$, this polynomial becomes g_i. Since Bernstein–Sato ideals do not change by a change of variables, we conclude that $B_G = B_{G'}$ and the result follows. ☐

3. Mixed multiplier ideals

Let $\pi\colon X' \to X$ be a common log-resolution of a tuple of ideals $\mathfrak{a} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ in R. Namely, π is a birational morphism such that

- X' is smooth;
- $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ for some effective Cartier divisor F_i, $i = 1, \dots, \ell$;
- $\sum_{i=1}^{\ell} F_i + E$ is a divisor with simple normal crossings, where $E = \mathrm{Exc}(\pi)$ is the exceptional locus.

The divisors $F_i = \sum_j e_{i,j} E_j$ are integral divisors in X' which can be decomposed into their *exceptional* and *affine* parts according to the support, i.e., $F_i = F_i^{\mathrm{exc}} + F_i^{\mathrm{aff}}$, where

$$F_i^{\mathrm{exc}} = \sum_{j=1}^{s} e_{i,j} E_j \qquad \text{and} \qquad F_i^{\mathrm{aff}} = \sum_{j=s+1}^{t} e_{i,j} E_j.$$

Whenever $\mathfrak{a}_i$ is an $\mathfrak{m}$-primary ideal, the divisor F_i is only supported on the exceptional locus, i.e., $F_i = F_i^{\mathrm{exc}}$. We will also consider the *relative canonical divisor*

$$K_\pi = \sum_{i=1}^{s} k_j E_j,$$

which is a divisor in X' supported on the exceptional locus E defined by the Jacobian determinant of the morphism π.

Definition 3.1. The *mixed multiplier ideal* associated to a tuple $\boldsymbol{\mathfrak{a}} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ of ideals in R and a point $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{R}^\ell_{\geqslant 0}$ is defined as

$$\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\lambda}) := \mathcal{J}(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) = \pi_* \mathcal{O}_{X'} \left(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_\ell F_\ell \rceil \right).$$

In the classical case of a single ideal we have the notion of *jumping numbers* associated to the sequence of multiplier ideals. The corresponding notion in the context of mixed multiplier ideals is more involved.

Definition 3.2. Let $\boldsymbol{\mathfrak{a}} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals in R. Then, for each $\boldsymbol{\lambda} \in \mathbb{R}^\ell_{\geqslant 0}$, we define:

- The *region* of $\boldsymbol{\lambda}$: $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) = \left\{ \boldsymbol{\lambda'} \in \mathbb{R}^\ell_{\geqslant 0} \mid \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda'}}) \supseteq \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}) \right\}$.
- The *constancy region* of $\boldsymbol{\lambda}$: $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) = \left\{ \boldsymbol{\lambda'} \in \mathbb{R}^\ell_{\geqslant 0} \mid \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda'}}) = \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}) \right\}$.

The boundaries of these regions is where we have a strict inclusion of ideals. Therefore we may define:

Definition 3.3. Let $\boldsymbol{\mathfrak{a}} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals in R. The *jumping wall* associated to $\boldsymbol{\lambda} \in \mathbb{R}^\ell_{\geqslant 0}$ is the boundary of the region $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$.

In particular, we will be interested in the points of these jumping walls. In what follows, $B_\varepsilon(\boldsymbol{\lambda})$ stands for the open ball of radius ε centered at a point $\boldsymbol{\lambda} \in \mathbb{R}^\ell$.

Definition 3.4. Let $\boldsymbol{\mathfrak{a}} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals in R. We say that $\boldsymbol{\lambda} \in \mathbb{R}^\ell_{\geqslant 0}$ is a *jumping point* of $\boldsymbol{\mathfrak{a}}$ if $\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda'}}) \supsetneq \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}})$ for all $\boldsymbol{\lambda'} \in \{\boldsymbol{\lambda} - \mathbb{R}^\ell_{\geqslant 0}\} \cap B_\varepsilon(\boldsymbol{\lambda})$ and $\varepsilon > 0$ small enough.

From the definition of mixed multiplier ideals we have that the jumping points $\boldsymbol{\lambda} \in \mathbb{R}^\ell_{\geqslant 0}$ must lie on hyperplanes of the form $H_j \colon e_{1,j} z_1 + \cdots + e_{\ell,j} z_\ell = k_j + \nu_j$ for $j = 1, \dots, s$ and $\nu_j \in \mathbb{Z}_{>0}$.

For $\lambda \in (0,1)$, we have $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(g_\alpha^\lambda)$, where $\mathfrak{a} = \left(f_1, \dots, f_r\right)$ is a single ideal in R, and $g_\alpha = \alpha_1 f_1 + \cdots + \alpha_r f_r \in R$ with $\alpha_i \in \mathbb{C}$ is a general element (see [10, Proposition 9.2.28]). As a consequence of a more general result of Mustaţă and Popa [13, Theorem 2.5] we also have a relation between $\mathcal{J}(\mathfrak{a}^\lambda)$ and the multiplier ideal of the associated hypersurface $g = f_1 y_1 + \cdots + f_r y_r$ in A.

Definition 3.5. Let $\mathcal{J} = \left(Q_1(y), \dots, Q_s(y)\right)$ be an ideal in A. Then, $\operatorname{Coeff}(\mathcal{J}) \subseteq R$ is the ideal generated by

$$\{Q_1(\alpha), \dots, Q_s(\alpha) \mid \alpha \in \mathbb{C}^r\}.$$

The result of Mustaţă and Popa in the form that we need is the following:

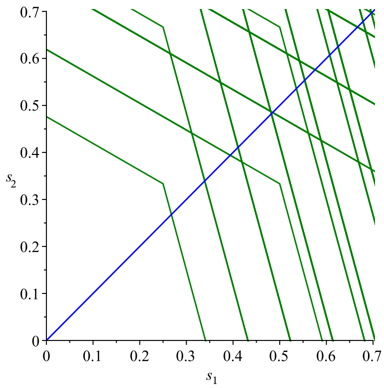
Proposition 3.6. *Let $\mathfrak{a} = \big(f_1, \dots, f_r\big)$ be an ideal in R and let $g = f_1y_1 + \cdots + f_ry_r$ be the associated hypersurface in A. Then, for any $\lambda \in \mathbb{Q} \cap (0, 1)$, we have*

$$\mathcal{J}(\mathfrak{a}^\lambda) = \operatorname{Coeff}\big(\mathcal{J}(g^\lambda)\big).$$

In particular, the set of jumping numbers in the interval $(0, 1)$ of $\mathfrak{a}$ and g coincide.

The mixed multiplier ideals version of this result follows immediately from the following observation.

Remark 3.7. Consider a ray through the origin $L\colon (0, \dots, 0) + \mu(\alpha_1, \dots, \alpha_\ell)$, where the α_i's are positive integers.



Then, the jumping points of a tuple $\boldsymbol{\mathfrak{a}} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ lying on L are the jumping numbers of the ideal $\mathfrak{a}_1^{\alpha_1} \cdots \mathfrak{a}_\ell^{\alpha_\ell}$.

Corollary 3.8. *Let $\boldsymbol{\mathfrak{a}} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals in R, let $G := g_1, \dots, g_\ell$ be its associated tuple of hypersurfaces, and consider $\boldsymbol{\lambda} \in \mathbb{Q}^\ell_{\geqslant 0}$ with Euclidean norm $\|\boldsymbol{\lambda}\| < 1$. Then, $\boldsymbol{\lambda}$ is a jumping point of $\boldsymbol{\mathfrak{a}}$ if and only if it is a jumping point of G.*

Proof. After Remark 3.7 we may assume that we have a single ideal $\mathfrak{a} = \big(f_1, \dots, f_r\big)$ so its associated hypersurface is $g = f_1y_1 + \cdots + f_ry_r$. The result then follows from 3.6. $\square$

In order to prove the main result of this section we will need the analytic definition of mixed multiplier ideal associated to a tuple $G = g_1, \dots, g_\ell$.

Definition 3.9. Let $G = g_1, \dots, g_\ell$ be a tuple in A. Let $\overline{B}_\varepsilon(O)$ be a closed ball of radius ε and center the origin $O \in \mathbb{C}^d$. The *mixed multiplier ideal* (at the origin O) of G associated with $\boldsymbol{\lambda} \in \mathbb{Q}^\ell_{>0}$ is

$$\mathcal{J}(g_1^{\lambda_1} \cdots g_\ell^{\lambda_\ell})_O = \left\{ h \in A \mid \exists\, \varepsilon \ll 1 \text{ such that } \int_{\overline{B}_\varepsilon(O)} \frac{|h|^2}{|g_1|^{2\lambda_1} \cdots |g_\ell|^{2\lambda_\ell}}\, dx\, dy\, d\bar{x}\, d\bar{y} < \infty \right\}.$$

Remark 3.10. As in the case of Bernstein–Sato ideals, it is enough to consider this local case since we have

$$\mathcal{J}(g_1^{\lambda_1}\cdots g_\ell^{\lambda_\ell}) = \bigcap_{p\in\mathbb{C}^d} \mathcal{J}(g_1^{\lambda_1}\cdots g_\ell^{\lambda_\ell})_p.$$

If it is clear from the context we will omit the subscript referring to the point.

Theorem 3.11. *Let $\mathfrak{a} = \mathfrak{a}_1, \ldots, \mathfrak{a}_\ell$ be a tuple of ideals in R. Let $\boldsymbol{\lambda} \in \mathbb{Q}^\ell_{\geqslant 0}$ be a jumping point of $\mathfrak{a}$ with Euclidean norm $\|\boldsymbol{\lambda}\| < 1$. Then $-\boldsymbol{\lambda} \in Z(B_{\mathfrak{a}})$.*

Proof. Let $\boldsymbol{\lambda} \in \mathbb{Q}^\ell_{\geqslant 0}$ be a jumping point of the tuple $G = g_1, \ldots, g_\ell$ associated to $\mathfrak{a}$ with $\|\boldsymbol{\lambda}\| < 1$ and take $h \in \mathcal{J}(f^{\boldsymbol{\lambda}'}) \setminus \mathcal{J}(f^{\boldsymbol{\lambda}})$ with $\boldsymbol{\lambda}' \in \{\boldsymbol{\lambda} - \mathbb{R}^\ell_{\geqslant 0}\} \cap B_\varepsilon(\boldsymbol{\lambda})$ for $\varepsilon > 0$ small enough. Therefore

$$\frac{|h|^2}{|g_1|^{2\lambda'_1}\cdots|g_\ell|^{2\lambda'_\ell}}$$

is integrable, but when we take the limit $\varepsilon \to 0$ we end up with

$$\frac{|h|^2}{|g_1|^{2\lambda_1}\cdots|g_\ell|^{2\lambda_\ell}},$$

which is not integrable. Set $d = n + r_1 + \cdots + r_\ell$ and consider the complex zeta function

$$\int_{\mathbb{C}^d} |g_1|^{2s_1}\cdots|g_\ell|^{2s_\ell}\varphi(x,y,\bar{x},\bar{y})\,dx\,dy\,d\bar{x}\,d\bar{y},$$

where $s_1, \ldots, s_\ell$ are indeterminate variables and $\varphi(x,\bar{x}) \in C_c^\infty(\mathbb{C}^d)$ is a *test function*, i.e., an infinitely many times differentiable function with compact support. Moreover, φ has holomorphic and antiholomorphic parts. For any $b(s_1,\ldots,s_\ell) \in B_G$, we have a Bernstein–Sato functional equation

$$\delta(s_1,\ldots,s_\ell)g_1^{s_1+1}\cdots g_\ell^{s_\ell+1} = b(s_1,\ldots,s_\ell)g_1^{s_1}\cdots g_\ell^{s_\ell}.$$

Therefore

$$\begin{aligned} b^2(s_1,\ldots,s_\ell)&\int_{\mathbb{C}^d}\varphi(x,y,\bar{x},\bar{y})|g_1|^{2s_1}\cdots|g_\ell|^{2s_\ell}\,dx\,dy\,d\bar{x}\,d\bar{y}\\ &= \int_{\mathbb{C}^d}\bar{\delta}^*\delta^*(s_1,\ldots,s_\ell)\big(\varphi(x,y,\bar{x},\bar{y})\big)|g_1|^{2(s_1+1)}\cdots|g_\ell|^{2(s_\ell+1)}\,dx\,dy\,d\bar{x}\,d\bar{y}, \end{aligned}$$

where $\bar{\delta}^*$ and δ^* denote, respectively, the *conjugate* and the *adjoint* differential operators associated to δ. Notice that $|h|^2\varphi(x,\bar{x})$ is still a test function, so

$$\begin{aligned} b^2(s_1,\ldots,s_\ell)&\int_{\mathbb{C}^d}|h|^2\varphi(x,y,\bar{x},\bar{y})|g_1|^{2s_1}\cdots|g_\ell|^{2s_\ell}\,dx\,dy\,d\bar{x}\,d\bar{y}\\ &= \int_{\mathbb{C}^d}\bar{\delta}^*\delta^*(s_1,\ldots,s_\ell)\big(|h|^2\varphi(x,y,\bar{x},\bar{y})\big)|g_1|^{2(s_1+1)}\cdots|g_\ell|^{2(s_\ell+1)}\,dx\,dy\,d\bar{x}\,d\bar{y}. \end{aligned}$$

Now we take a test function φ which is zero outside the ball $\overline{B}_\varepsilon(O)$ and identically one on a smaller ball $\overline{B}_{\varepsilon'}(O) \subseteq \overline{B}_\varepsilon(O)$, and thus we get

$$\begin{aligned}b^2(s_1,\ldots,s_\ell)&\int_{\overline{B}_{\varepsilon'}(O)} |h|^2|g_1|^{2s_1}\cdots|g_\ell|^{2s_\ell}\,dx\,dy\,d\bar{x}\,d\bar{y}\\ &= \int_{\overline{B}_{\varepsilon'}(p)} \bar{\delta}^*\delta^*(s_1,\ldots,s_\ell)\big(|h|^2\big)|g_1|^{2(s_1+1)}\cdots|g_\ell|^{2(s_\ell+1)}\,dx\,dy\,d\bar{x}\,d\bar{y}.\end{aligned}$$

Taking $s = -(\lambda'_1,\ldots,\lambda'_\ell)$ we get

$$\begin{aligned}b^2(-\lambda'_1,\ldots,-\lambda'_\ell)&\int_{\overline{B}_{\varepsilon'}(O)} \frac{|h|^2}{|g_1|^{2\lambda'_1}\cdots|g_\ell|^{2\lambda'_\ell}}\,dx\,dy\,d\bar{x}\,d\bar{y}\\ &= \int_{\overline{B}_{\varepsilon'}(O)} \bar{\delta}^*\delta^*(-\lambda'_1,\ldots,-\lambda'_\ell)\big(|h|^2\big)|g_1|^{2(1-\lambda'_1)}\cdots|g_\ell|^{2(1-\lambda'_\ell)}\,dx\,dy\,d\bar{x}\,d\bar{y},\end{aligned}$$

but the right-hand side is uniformly bounded for all $\varepsilon > 0$. Thus we have

$$b^2(-\lambda'_1,\ldots,-\lambda'_\ell)\int_{\overline{B}_{\varepsilon'}(O)} \frac{|h|^2}{|g_1|^{2\lambda'_1}\cdots|g_\ell|^{2\lambda'_\ell}}\,dx\,dy\,d\bar{x}\,d\bar{y} \le M < \infty$$

for some positive number M that depends on h. Then, by the monotone convergence theorem we have to have $b^2(-\lambda_1,\ldots,-\lambda_\ell) = 0$, and thus $-\boldsymbol{\lambda} \in Z(\tilde{B}_G) = Z(B_{\mathfrak{a}})$. □

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