REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 68, No. 1, 2025, Pages 103–129
Published online: March 31, 2025
https://doi.org/10.33044/revuma.3697

ON THE WELLPOSEDNESS OF A FUEL CELL PROBLEM

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Abstract. This paper investigates the existence of weak solutions to a fuel cell problem modeled by a boundary value problem (BVP) in the multiregion domain. The BVP consists of the coupled Stokes/Darcy-TEC (thermoelectrochemical) system of elliptic equations, with Beavers–Joseph–Saffman and regularized Butler–Volmer boundary conditions being prescribed on the interfaces, porous-fluid and membrane, respectively. The present model includes macrohomogeneous models for both hydrogen and methanol crossover. The novelty in the coupled Stokes/Darcy-TEC system lies in the presence of the Joule effect together with the quasilinear character given by (1) temperature dependence of the viscosities and the diffusion coefficients; (2) the concentration-temperature dependence of Dufour–Soret and Peltier–Seebeck cross-effect coefficients, and (3) the pressure dependence of the permeability. We derive quantitative estimates of the solutions to clarify smallness conditions on the data. We use fixed-point and compactness arguments based on the quantitative estimates of approximated solutions.

1. Introduction

In this paper, we study the so-called fuel cells. Our concern is on the mathematical analysis of thermoelectrochemical (TEC) models from devices that convert the chemical energy from a fuel into electricity. The chemical reaction produces charged ions, which move on a membrane. Positively charged ions are conducted by the proton exchange membrane at low operating temperature in polymer electrolyte membrane fuel cells (PEMFC) [1, 18, 26] and direct methanol fuel cells (DMFC) [27, 34], while negatively charged ions are conducted by adequate ionic condutors at high operating temperatures in solid oxide fuel cells (SOFC) [21, 17] and molten carbonate fuel cells (MCFC) [14]. Despite their differences, they all consist of two electrodes (one anode and one cathode), an electrolyte membrane separator between the two electrodes, and two or more channels. The domain consists of different pairwise disjoint Lipschitz subdomains (precisely, it is separated into five regions, and it has four interfaces of dimension $n-1$) as sagittally illustrated in Figure 1. It includes the membrane medium in contrast with the phase

2020 *Mathematics Subject Classification.* Primary 80A50; Secondary 35Q35, 35Q79, 35J57, 35D30, 76S05.

Key words and phrases. PEM fuel cell, multiregion domain, Stokes–Darcy system, porous media, heat-conducting fluids, Beavers–Joseph–Saffman boundary condition, electrochemistry.

change models, which consider the membrane as an interface separating the different conductive phases [8]. We refer to [13] for a numerical approach for the steady free boundary value problem, on which mixed Dirichlet–Neumann conditions are specified, motivated by the two-phase flow in fuel cell electrodes.

It is widely recognized that the behavior of the components in the interface boundaries plays an essential role in the cell performance, and it can determine its life. An exact solution of an electro-osmotic flow problem modeling polymer electrolyte membranes is derived in [4] under the assumption that Stokes flow is driven solely by an external field, rather than by a pressure gradient, in an infinite cylindrical pore. In [29, 32], the authors combine a PEM fuel cell and electrical circuits to potentiate the energy efficiency, to reduce the cost of FC technology, and to improve fuel usage. Computational fluid dynamics (CFD) based tools are developed for PEMFC (see [12, 24, 28], and the references therein).

Mathematically, the modeling tool is the coupled Stokes/Darcy system, completed by the thermoelectrochemical (TEC) system, in the multiregion domain. The Stokes/Darcy system consists of the Stokes equation on one part of the domain coupled to the Darcy equation, where the flow velocities are small and mainly driven by the pressure gradient in a porous medium. The TEC system consists of the energy equation and the mass transport associated with electrochemical reactions, where the fluxes are given by generalized Fourier, Fick and Ohm laws, by including the Dufour–Soret and Peltier–Seebeck cross effects.

The complexity of the present model is a true drawback by the presence of both the cross and Joule effects together with different types of interfaces: the fluid-porous interfaces that require stress boundary conditions such as the Beavers–Joseph–Saffman interface condition, and the membrane interface on which the electrochemical reactions occur. We refer to [6] for the existence of a weak solution of a 1D half-cell model.

The coupled system of elliptic equations (Stokes/Darcy-TEC) is quasilinear since the physical parameters such as the viscosity and the diffusion coefficients depend on the temperature while the cross-effect coefficients depend on the temperature and the concentrations. Moreover, the permeability depends on the pressure by the Klinkenberg equation. It is known that regularity results are available whenever the Navier–Stokes–Fourier system has constant coefficients [3]. We refer to [7] for the study of the Beavers–Joseph–Saffman–Stokes–Darcy–Fourier problem.

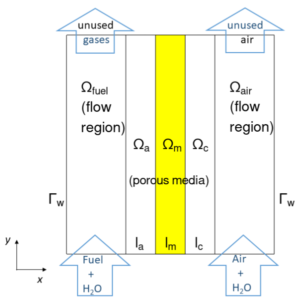
The existence of weak solutions is established by applying a fixed-point procedure under some assumptions on the nonlinear terms. The use of the Tychonoff fixed point theorem is somewhat standard. However, the presence of the dissipation term requires additional regularity, and some small coefficient conditions are enforced.

We confine ourselves to the study of the proton exchange membrane fuel cell. We focus our attention on H_2PEM fuel cells driven by gaseous hydrogen, but the present model may include other cells, such as direct methanol fuel cells operating on methanol in an aqueous solution.

The structure of the paper is as follows. We begin by introducing the concrete physical model under consideration. Next, the functional framework, the data under consideration and the main theorems are stated in Section 3. Some auxiliary results are proved in Section 4. In particular, the existence of an auxiliary velocity-pressure pair in Subsection 4.1, and an auxiliary partial density-temperature-potential triplet solution in Subsection 4.2. In Section 5, the fixed-point argument is applied to prove Theorem 3.5.

2. Statement of the fuel cell problem

Let Ω be a bounded multiregion domain of $\mathbb{R}^n$, $n \geq 2$, that is, $\Omega = \mathrm{int}\left(\overline{\Omega}_{\mathrm{f}} \cup \overline{\Omega}_{\mathrm{p}}\right)$ is a connected open set, with Ω_{f} and Ω_{p} being two disjoint open subsets of Ω. The multidomain Ω represents one single PEM fuel cell, whose 2D (two-dimensional) representations are schematically illustrated in Fig. 1.



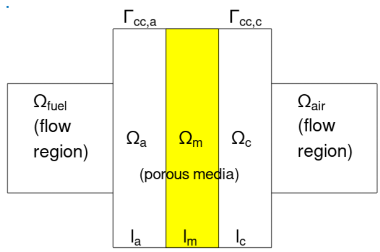


FIGURE 1. The flow region $\Omega_{\mathrm{f}} = \Omega_{\mathrm{fuel}} \cup \Omega_{\mathrm{air}}$ and the porous region $\Omega_{\mathrm{p}} = \Omega_{\mathrm{a}} \cup \overline{\Omega}_{\mathrm{m}} \cup \Omega_{\mathrm{c}}$ (not to scale), with length $l_{\mathrm{a}} + l_{\mathrm{m}} + l_{\mathrm{c}} \ll L$, where $L = 1\,\mathrm{cm}$ to $10\,\mathrm{cm}$ denotes each channel's length. Left: xy cross-section. Right: xz cross-section.

The fluid bidomain Ω_{f} consists of two channels, namely the anodic fuel channel Ω_{fuel} and the cathodic air channel Ω_{air}, constituted by mixtures (due to their noncontinuity) of the gas and liquid phases [1].

The membrane electrode assembly – what we call porous domain Ω_{p} – consists of the regions relative to the membrane separator Ω_{m} and the backing and catalyst layers of the two electrodes. The domain Ω_{m} stands for the proton conducting membrane (20 µm to 100 µm in thickness). It accounts for the transport of dissolved water (H_2O) and the hydronium (H_3O^+) ions, and it is electrically insulating, so that the electrons are forced to travel in an external circuit from the anode to the cathode. A usual catalyst layer, between the membrane separator and the backing layer, can be assumed to have negligible measure (the backing layers are approximately $l_a = l_c = 200\,\mu\mathrm{m}$ in thickness, while the catalyst layers are 5 µm to 10 µm [1, 2]), and it is denoted by Γ_{CL}. Another interface is the porous-fluid boundary $\Gamma = \partial\Omega_{\mathrm{p}} \cap \Omega$.

The fuel (for instance, pure hydrogen [25] or hydrocarbon type, which includes diesel, methanol [34] and chemical hydrides) is oxidized at the anode catalyst layer $\Gamma_{\rm a}$, generating positively charged ions and electrons. The positively charged ions travel through $\Omega_{\rm m}$, while the traveling of the free electrons produces the electric current in the backing layers, $\Omega_{\rm a}$ and $\Omega_{\rm c}$, through an external circuit that is attained by a current collector $\Gamma_{\rm cc}$. These two currents are interconnected through the electrochemical reactions. At the cathode catalyst layer $\Gamma_{\rm c}$, the oxygen reduction occurs: hydrogen ions, electrons, and oxygen react to form water. We set

$$\Omega_{\rm p} = \Omega_{\rm a} \cup \left(\overline{\Omega}_{\rm m} \cap \Omega\right) \cup \Omega_{\rm c} = \Omega_{\rm a} \cup \Gamma_{\rm a} \cup \Omega_{\rm m} \cup \Gamma_{\rm c} \cup \Omega_{\rm c}.$$

Hereafter, the subscripts 'a' and 'c' stand for anode and cathode, respectively.

The general phenomenological fluxes, $\mathbf{j}_i$ [$\mathrm{kg\,s^{-1}\,m^{-2}}$], $\mathbf{q}$ [$\mathrm{W\,m^{-2}}$] and $\mathbf{j}$ [$\mathrm{A\,m^{-2}}$], are explicitly driven by gradients of the temperature θ, the mass concentration vector $\boldsymbol{\rho}$, and the electric potential ϕ, in the form (up to some temperature- and concentration-dependent factors)

$$\mathbf{j}_i = -D_i(\theta)\nabla\rho_i - \sum_{\substack{j=1\\ j\neq i}}^{\rm I} D_{ij}(\theta)\nabla\rho_j - \rho_i S_i(c_i,\theta)\nabla\theta - u_i\rho_i\nabla\phi; \tag{2.1}$$

$$\mathbf{q} = -R\theta^2\sum_{j=1}^{\rm I} D'_j(c_j,\theta)\nabla c_j - k(\theta)\nabla\theta - \Pi(\theta)\sigma(\mathbf{c},\theta)\nabla\phi; \tag{2.2}$$

$$\mathbf{j} = -F\sum_{j=1}^{\rm I} z_j D_j(\theta)\nabla c_j - \alpha_{\rm S}(\theta)\sigma(\mathbf{c},\theta)\nabla\theta - \sigma(\mathbf{c},\theta)\nabla\phi,$$

with $i = 1,\dots,{\rm I}$ (see [9, 10] and the references therein). These include the Fick law (with the diffusion coefficient D_i [$\mathrm{m^2\,s^{-1}}$]), the Fourier law (with the thermal conductivity k [$\mathrm{W\,m^{-1}\,K^{-1}}$]), the Ohm law (with the electrical conductivity σ [$\mathrm{S\,m^{-1}}$]), the Dufour–Soret cross effect (with the Dufour coefficient D'_i [$\mathrm{m^2\,s^{-1}\,K^{-1}}$] and the Soret coefficient S_i [$\mathrm{m^2\,s^{-1}\,K^{-1}}$]), and the Peltier–Seebeck cross effect (with the Peltier coefficient Π [V] and the Seebeck coefficient $\alpha_{\rm S}$ [$\mathrm{V\,K^{-1}}$] being correlated by the first Kelvin relation).

In the fuel cell model, the main contribution for the electric potential is given at the membrane interface (cf. Subsection 2.6), and the electric flux is reduced to the Ohm law [1, 2]

$$\mathbf{j} = -\sigma(\mathbf{c},\theta)\nabla\phi \quad \text{in } \Omega_{\rm a}\cup\Omega_{\rm c}. \tag{2.3}$$

Each partial density is defined by

$$\rho_i = M_i c_i, \tag{2.4}$$

where M_i denotes the molar mass [$\mathrm{kg\,mol^{-1}}$] and c_i is the molar concentration [$\mathrm{mol\,m^{-3}}$] of the species i.

The mobility u_i [$\mathrm{m^2\,s^{-1}\,V^{-1}}$] satisfies the Nernst–Einstein relation $u_i = z_i F D_i/(R\theta)$, and according to Onsager's reciprocal theorem, the two coupling coefficients are equal. The universal constants are the Faraday constant $F = 9.6485 \times 10^4\,\mathrm{C\,mol^{-1}}$, and the gas constant $R = 8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$.

Hereafter the subscript i stands for the correspondence to the ionic component $i = 1,\dots,\mathrm{I}$ intervened in the reaction process, with $\mathrm{I} \in \mathbb{N}$ being either $\mathrm{I_p}$ or $\mathrm{I_f}$ whenever we consider Ω_p or Ω_f, respectively. To avoid confusion, in the present work we never show the components of vectors of $\mathbb{R}^n$, namely the velocity vector or the gradient.

2.1. **In the fluid bidomain $\Omega_\mathrm{f} = \Omega_\mathrm{fuel} \cup \Omega_\mathrm{air}$.** By the characteristics of the channels, the convection for fluid and heat flows may be neglected.

The governing equations are the conservation of mass, momentum, species and energy, a.e. in Ω_f,

$$\nabla \cdot (\rho \mathbf{u}) = 0; \tag{2.5}$$

$$-\nabla \cdot \tau = -\nabla p; \tag{2.6}$$

$$\nabla \cdot (\mathbf{u}\rho_i) + \nabla \cdot \mathbf{j}_i = 0; \tag{2.7}$$

$$\nabla \cdot \mathbf{q} = 0 \tag{2.8}$$

for the uncharged species $i = 1,\dots,\mathrm{I}$. The unknown functions are the density ρ, the velocity $\mathbf{u} = (u_x, u_y, u_z)$, the mass concentration vector $\boldsymbol{\rho} = (\rho_1,\dots,\rho_\mathrm{I})$ and the temperature θ.

We assume that the anode and cathode gas mixtures with water vapor act as ideal gases [26], that is, the pressure p obeys the Boyle–Marriotte law

$$p = R_\mathrm{specific}\rho\theta, \tag{2.9}$$

where $R_\mathrm{specific} = R/M$, with M denoting the molar mass [$\mathrm{kg\,mol^{-1}}$]. Moreover, the deviatoric stress tensor $\tau = p\mathsf{I} + \sigma$, where σ represents the Cauchy stress tensor and I denotes the identity $n \times n$ matrix. The stress tensor τ, which is temperature dependent, obeys the constitutive law

$$\tau = \mu(\theta) D\mathbf{u} + \lambda(\theta)\mathrm{tr}(D\mathbf{u})\mathsf{I}, \qquad \mathrm{tr}(D\mathbf{u}) = \mathsf{I} : D\mathbf{u} = \nabla \cdot \mathbf{u}, \tag{2.10}$$

where $D = (\nabla + \nabla^T)/2$ denotes the symmetric gradient, and μ and λ are the viscosity coefficients in accordance with the second law of thermodynamics

$$\mu(\theta) > 0, \quad \nu(\theta) := \lambda(\theta) + \mu(\theta)/n \geq 0, \tag{2.11}$$

with ν denoting the bulk (or volume) viscosity and $\mu/2$ being the shear (or dynamic) viscosity. Here we use the notation $\zeta : \varsigma = \zeta_{ij}\varsigma_{ij}$, taking into account the convention on implicit summation over repeated indices.

Finally, we emphasize that there is no electric current in the fluid bidomain.

2.2. **Number of species** I. The number of species $\mathrm{I} \in \mathbb{N}$ may indeed represent different numbers $\mathrm{I_a}$, $\mathrm{I_m}$ and $\mathrm{I_c}$ corresponding to the domains $\Omega_\mathrm{fuel} \cup \Omega_\mathrm{a}$, Ω_m and $\Omega_\mathrm{air} \cup \Omega_\mathrm{c}$, respectively.

The flow domain Ω_f accounts for the reactant gases (oxygen, nitrogen and water vapor on the cathodic channel) and liquid water. In the $\mathrm{H_2}$PEMFC, the dry hydrogen gas is humidified before being introduced into the fuel channel Ω_fuel. In the DMFC, the chemical reaction in the anode catalyst layer is the methanol ($\mathrm{CH_3OH}$) oxidation.

2.2.1. *The anodic fuel compartment* $\Omega_\mathrm{fuel}\cup\Omega_\mathrm{a}$. In the anodic region, different kinds of humidified fuel may be considered, for instance:

- $\mathrm{H_2}$, to produce the hydrogen oxidation reaction [1, 2, 18]: $\mathrm{H_2} \to 2\mathrm{H}^+ + 2\mathrm{e}^-$ on the membrane interface Γ_a, which means

$$\mathrm{H_2}(g) + 2\mathrm{H_2O}(l) \to 2\mathrm{H_3O}^+(aq) + 2\mathrm{e}^-, \qquad E^0 = 0.00\,\mathrm{V}.$$

  The gas composition obeys

$$\begin{aligned} \rho &= M(\mathrm{H_2})c_{\mathrm{H_2}} + M(\mathrm{H_2O})c_{\mathrm{H_2O}} && \text{in } \Omega_\mathrm{fuel}\cup\Omega_\mathrm{a}; \\ \rho &= Mc_{\mathrm{H_3O^+}} + M(\mathrm{H_2O})c_{\mathrm{H_2O}} && \text{in } \Omega_\mathrm{m}, \end{aligned}$$

  with $M(\mathrm{H_2}) = 2\,\mathrm{g\,mol^{-1}}$, $M(\mathrm{H_2O}) = 18\,\mathrm{g\,mol^{-1}}$ and $M = 1\,\mathrm{g\,mol^{-1}}$.
- methanol, to produce the oxidation reaction [34]: $\mathrm{CH_3OH}{+}\mathrm{H_2O} \to \mathrm{CO_2} + 6\mathrm{H}^+ + 6\mathrm{e}^-$ on the membrane interface Γ_a.

Then, we take $i =$ fuel, $\mathrm{H_2O}$.

2.2.2. *The cathodic air compartment* $\Omega_\mathrm{air}\cup\Omega_\mathrm{c}$. In the cathodic region, the air undergoes the oxygen reduction reaction [18, 25]: $\mathrm{O_2} + 4\mathrm{H}^+ + 4\mathrm{e}^- \to 2\mathrm{H_2O}$ on the membrane interface Γ_c, which means

$$\mathrm{O_2}(g) + 4\mathrm{H_3O}^+(aq) + 4\mathrm{e}^- \to 6\mathrm{H_2O}(l), \qquad E^0 = 1.23\,\mathrm{V}.$$

Then, we take $i = \mathrm{O_2}$, $\mathrm{H_2O}$. The liquid water byproduct drains away for a proper operating of the fuel cell. The gas composition obeys

$$\rho = M(\mathrm{H_2O})c_{\mathrm{H_2O}} + M(\mathrm{O_2})c_{\mathrm{O_2}},$$

with $M(\mathrm{H_2O}) = 18\,\mathrm{g\,mol^{-1}}$ and $M(\mathrm{O_2}) = 32\,\mathrm{g\,mol^{-1}}$. The mass density is assumed to be

$$\rho = \sum_{i=1}^{\mathrm{I}} \rho_i \quad \text{in } \Omega_\mathrm{f}. \tag{2.12}$$

Therefore, the overall balanced cell reactions are

$$\begin{aligned} &\mathbf{H_2PEMFC:} \quad 2\mathrm{H_2} + \mathrm{O_2} \to 2\mathrm{H_2O}, \quad E^0_\mathrm{cell} = 1.23\,\mathrm{V}. \\ &\mathbf{DMFC:} \quad 2\mathrm{CH_4O} + 3\mathrm{O_2} \to 2\mathrm{CO_2} + 4\mathrm{H_2O}. \end{aligned}$$

For the sake of simplicity, we consider the number of species (cf. Table 1)

$$\mathrm{I} = \mathrm{I_a} = \mathrm{I_m} = \mathrm{I_c} = 2.$$

The water is present in fluid and vapor states, and in both cases it can be modeled as a Newtonian (i.e., linearly viscous) fluid.

TABLE 1. Correspondence between each component and each region

i	$\Omega_{\text{fuel}} \cup \Omega_{\text{a}}$	Ω_{m}	$\Omega_{\text{air}} \cup \Omega_{\text{c}}$
1	fuel	H_3O^+	O_2
2	H_2O	H_2O	H_2O

2.3. **In the porous domain $\Omega_{\text{p}} = \Omega_{\text{a}} \cup \Omega_{\text{m}} \cup \Omega_{\text{c}}$.** The governing equations, after a volume averaging procedure [5], are

$$\nabla \cdot \mathbf{u}_{\text{D}} = 0; \tag{2.13}$$

$$\nabla \cdot \mathbf{j}_i = 0; \tag{2.14}$$

$$\nabla \cdot \mathbf{q} = Q \quad \text{a.e. in } \Omega_{\text{p}}, \tag{2.15}$$

for i according to Table 1, that is, $\text{I}_{\text{p}} = 2$. Here, we omit the bracket $\langle \cdot \rangle$, which usually represents the volume averaged. Thus, the temperature θ is the spatially averaged (over a representative elementary volume) microscopic quantity, and the Darcy velocity $\mathbf{u}_{\text{D}}$ $[\mathrm{m\,s^{-1}}]$ is the superficial average quantity. The volume averaged density ρ of the fluid is piecewise constant, $\rho_{\text{water}} = 970\,\mathrm{kg\,m^{-3}}$ and $\rho_{\text{air}} = 0.995\,\mathrm{kg\,m^{-3}}$, due to $\rho_{\text{air}} = p_{\text{atm}} M_{\text{air}}/(R\theta_r)$ at the typical operating temperature of $\theta_r = 357.15\,\mathrm{K}$ $(= 84\,^{\circ}\mathrm{C})$, $p_{\text{atm}} = 101.325\,\mathrm{kPa}$ and $M_{\text{air}} = 28.97\,\mathrm{g\,mol^{-1}}$.

The Darcy velocity $\mathbf{u}_{\text{D}}$ obeys

$$\mu \mathbf{u}_{\text{D}} = -K_g \nabla p, \tag{2.16}$$

where p is the intrinsic average pressure [Pa], $\mu = \mu(\theta)$ denotes the viscosity [Pa s] and K_g represents the gas permeability $[\mathrm{m^2}]$ that is given by the Klinkenberg equation

$$K_g = K_l \left(1 + \frac{b}{p} \right), \tag{2.17}$$

with $b \geq 0$ being a constant, $b > 0$ in $\Omega_{\text{a}} \cup \Omega_{\text{c}}$ and $b = 0$ in Ω_{m}, and with $K_l > 0$ being the liquid permeability of the porous media, that only depends on the porosity ϵ and therefore is constant.

The molar flux $\mathbf{J}_i$ of the water $i = H_2O$ obeys (2.1), where the second term means the electro-osmosis ($j \neq i$), with $D_{ij} = n_{\text{d}}$ representing the electro-osmostic drag coefficient [24]. The proton flux $\mathbf{J}_i$ of the ionic component $i = H_3O^+$ obeys (2.1), where in the first term $D_i = \kappa/(z_i F)$, with the proton ionic conductivity κ being nonconstant in accordance with the membrane not being fully hydrated.

In the energy equation (2.15), the Joule effect

$$Q = \chi_{\Omega_{\text{a}} \cup \Omega_{\text{c}}} \sigma |\nabla \phi|^2 \tag{2.18}$$

takes into account that the effect of flow velocity is negligible when compared to the electrical current that exists in $\Omega_{\text{a}} \cup \Omega_{\text{c}}$.

The electric current density $\mathbf{j}$ satisfies

$$\nabla \cdot \mathbf{j} = 0 \quad \text{a.e. in } \Omega_{\text{a}} \cup \Omega_{\text{c}}. \tag{2.19}$$

Notice that there is no electric current density in Ω_{m}, i.e., there is the ionic current density $\mathbf{j}_{\mathrm{m}}$ that satisfies $\mathbf{j}_{\mathrm{m}} = z_{\mathrm{H}^+} F \mathbf{J}_{\mathrm{H}^+}$, where the valence of species $z_{\mathrm{H}^+} = 1$. Also, $\sigma_m = 8.3\,\mathrm{S\,m^{-1}}$ is known for the ionomer Nafion.

2.4. **On the outer boundary $\partial\Omega$.** The boundary of Ω is constituted by three pairwise disjoint open $(n-1)$-dimensional sets, namely Γ_{in}, Γ_{out} and Γ_{w}, which represent the inlet, outlet and wall boundaries, respectively:

$$\partial\Omega = \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} \cup \overline{\Gamma}_{\mathrm{w}}.$$

The wall boundary has a subpart $\Gamma_{\mathrm{cc}} \subset \partial\Omega_{\mathrm{p}}$ that stands for the current collector, meaning that the remaining wall boundary is electrically insulated. The inlet and outlet sets are the union of two disjoint connected open $(n-1)$-dimensional sets, namely,

$$\Gamma_{\mathrm{in}} = \Gamma_{\mathrm{in,a}} \cup \Gamma_{\mathrm{in,c}};$$
$$\Gamma_{\mathrm{out}} = \Gamma_{\mathrm{out,a}} \cup \Gamma_{\mathrm{out,c}},$$

corresponding to the anodic and cathodic channels, Ω_{fuel} and Ω_{air}.

On the wall boundary Γ_{w}, the no-outflow boundary conditions are imposed on the velocity and species,

$$\mathbf{u}\cdot\mathbf{n} = (\rho_i\mathbf{u} + \mathbf{j}_i)\cdot\mathbf{n} = 0 \quad (i = 1, \dots, \mathrm{I}). \tag{2.20}$$

Hereafter, $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$.

On the inlet and outlet boundaries $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}$, the velocity, partial densities and temperature are specified. Due to the characteristics of the domain, the velocity is specified as constant on the y-direction. Since the general case of prescribed partial densities and temperature can be handled by subtracting a background profile that fits the specified functions, we assume a homogeneous Dirichlet condition. Then, we assume:

- for a.e. $(x, 0, z) \in \Gamma_{\mathrm{in}}$:
$$\mathbf{u}(x,0,z) = u_{\mathrm{in}}\mathbf{e}_y \equiv (0, u_{\mathrm{in}}, 0);$$
$$\rho_i(x,0,z) = \theta(x,0,z) = 0.$$
- for a.e. $(x, L, z) \in \Gamma_{\mathrm{out}}$:
$$\mathbf{u}(x,L,z) = u_{\mathrm{out}}\mathbf{e}_y \equiv (0, u_{\mathrm{out}}, 0);$$
$$\rho_i(x,L,z) = \theta(x,L,z) = 0.$$

On the current collector wall boundary Γ_{cc}, the electric potential is prescribed through the cell voltage $E_{\mathrm{cell}} = \phi|_{\Gamma_{\mathrm{cc,c}}} - \phi|_{\Gamma_{\mathrm{cc,a}}}$; this means

$$\phi = E_{\mathrm{cell}} \text{ on } \Gamma_{\mathrm{cc,c}} \quad \text{and} \quad \phi = 0 \text{ on } \Gamma_{\mathrm{cc,a}}. \tag{2.21}$$

On the remaining wall boundary $\Gamma_{\mathrm{w}} \setminus \Gamma_{\mathrm{cc}}$, the no-outflow condition $\mathbf{j}\cdot\mathbf{n} = 0$ is considered.

Finally, the Newton law of cooling, which is mathematically known as a Robin-type boundary condition, is considered:

$$\mathbf{q}\cdot\mathbf{n} = h_c(\theta - \theta_e) \text{ on } \Gamma_{\mathrm{w}}, \tag{2.22}$$

where h_c denotes the conductive heat transfer coefficient, which may depend both on the spatial variable and the temperature function θ, and θ_e denotes the external coolant stream temperature at the wall.

2.5. **On the fluid-porous interface Γ.** The unit outward normal to the interface boundary Γ pointing from the fluid region to the porous medium is $\mathbf{e}_x$ on $\operatorname{int}(\partial\Omega_{\text{fuel}} \cap \partial\Omega_{\text{a}})$ and $-\mathbf{e}_x$ on $\operatorname{int}(\partial\Omega_{\text{air}} \cap \partial\Omega_{\text{c}})$.

We consider the continuity of mass flux, a constant interface temperature, and the balance of normal Cauchy stress vectors (namely, $\sigma_{fN} + \sigma_{pN} = 0$):

$$\mathbf{u} \cdot \mathbf{e}_x = \mathbf{u}_{\text{D}} \cdot \mathbf{e}_x; \tag{2.23}$$

$$\theta_{\text{f}} = \theta_{\text{p}}; \tag{2.24}$$

$$(\tau \cdot \mathbf{e}_x) \cdot \mathbf{e}_x = [p] := p_{\text{f}} - p_{\text{p}}, \tag{2.25}$$

where $[\cdot]$ denotes the jump of a quantity across the interface in the direction from the porous medium to the fluid medium. Condition (2.23) ensures that the exchange of fluid between the two domains is conservative.

The heat transfer transmission is completed by the continuous heat flux condition

$$\mathbf{q}_{\text{f}} \cdot \mathbf{e}_x = -\mathbf{q}_{\text{p}} \cdot \mathbf{e}_x. \tag{2.26}$$

Finally, we assume the fluid flow is almost parallel to the interface and the Darcy velocity is much smaller than the slip velocity. Thus, the Beavers–Joseph–Saffman (BJS) interface boundary condition may be considered [15]:

$$(\tau \cdot \mathbf{n}) \cdot \mathbf{e}_j = -\beta \mathbf{u} \cdot \mathbf{e}_j \qquad (j = y, z), \tag{2.27}$$

where the coefficient $\beta = \alpha_{BJ} K^{-1/2} > 0$ denotes the Beavers–Joseph slip coefficient, with α_{BJ} being dimensionless and characterizing the nature of the porous surface.

2.6. **On the membrane interface $\Gamma_{\text{CL}} = \Gamma_{\text{a}} \cup \Gamma_{\text{c}}$.** In what follows, we focus on the H_2PEMFC. In both half cell reactions, the number of electrons that participate in each half cell reaction n is equal to 4 (see Subsections 2.2.1 and 2.2.2).

On $\Gamma_{\text{a}} = \partial\Omega_{\text{a}} \cap \overline{\Omega}_{\text{m}}$, it occurs the oxidation reaction of the fuel, that is,

$$\mathbf{j}_1 \cdot \mathbf{e}_x = -\frac{sM(\mathrm{H}_2)}{nF} j_a \quad \text{a.e. on } \Gamma_{\text{a}},$$

with the anodic stoichiometry number $s = 2$.

On $\Gamma_{\text{c}} = \partial\Omega_{\text{c}} \cap \overline{\Omega}_{\text{m}}$, it occurs the oxygen reduction reaction, that is,

$$\mathbf{j}_1 \cdot \mathbf{e}_x = -\frac{sM(\mathrm{O}_2)}{nF} j_c \quad \text{a.e. on } \Gamma_{\text{c}},$$

with the cathodic stoichiometry number $s = 1$.

The reaction rates j_ℓ [$\mathrm{A\,m^{-2}}$] are given by the Butler–Volmer equation:

$$j_{\mathrm{a}} = j_{\mathrm{a},0}\left(\frac{c_{\mathrm{fuel}}}{c_{\mathrm{fuel},0}}\right)^{\nu}\left(\exp\left[\frac{F\eta_a}{R\theta_a}\right] - \exp\left[-\frac{F\eta_a}{R\theta_a}\right]\right);$$
$$j_{\mathrm{c}} = j_{\mathrm{c},0}\frac{c_{\mathrm{O}_2}}{c_{\mathrm{O}_2,0}}\left(\exp\left[\frac{F\eta_c}{R\theta_c}\right] - \exp\left[-\frac{F\eta_c}{R\theta_c}\right]\right)$$

for some $j_{\ell,0} > 0$ only spatial dependent and such that $j_{\mathrm{a},0} > j_{\mathrm{c},0}$ [26]. Here, it is considered the charge transfer coefficient equal to $1/2$, θ_a and θ_c are some reference temperatures, $\nu = 1/2$ for H_2 fuel, and $\eta_\ell = \phi_\ell - \phi_m - \phi_r$ stands for the overpotential ($\ell = a, c$), for some reference potential ϕ_r.

Thus, the electric current may be modeled by the Butler–Volmer boundary condition

$$-\mathbf{j}\cdot\mathbf{e}_x = j_\ell \quad \text{a.e. on } \Gamma_\ell \quad (\ell = \mathrm{a}, \mathrm{c}). \tag{2.28}$$

Notice that the reaction rates are affected by the transport of species near the electrode, and may be represented as a current in terms of the limiting current

$$j_\ell = j_{\ell,L}\left(1 - \left(\frac{c}{c_0}\right)^{\nu_\ell}\right),$$

with $\nu_\ell = \nu$ if $\ell = a$, and $\nu = 1$ if $\ell = c$. Then, we may consider

$$j_\ell(\eta) = j_{\ell,L}\frac{2j_{\ell,0}\sinh[\eta/B_\ell]}{j_{\ell,L} + 2j_{\ell,0}\sinh[\eta/B_\ell]} \quad \text{for } \eta \geq 0, \tag{2.29}$$

with $B_\ell = R\theta_\ell/F$ being the Tafel slope at $\ell = a, c$. For a mathematical analysis, we assume that

$$j_\ell(\eta) = -j_\ell(-\eta) \quad \text{if } \eta < 0. \tag{2.30}$$

We emphasize that this assumption prevents the existence of infinitely many nontrivial solutions, which occur in boundary value problems under the Butler–Volmer boundary condition [23]. Similarly, j_ℓ, when representing the dual-pathway kinetic equation based on the Tafel–Heyrovsky–Volmer mechanism [31, 33], may be treated in the same way.

3. Variational formulation and main result

In the framework of Sobolev and Lebesgue functional spaces, for $r > 1$, we introduce the following spaces of test functions:

$$\mathbf{V}(\Omega_f) = \{\mathbf{v} \in \mathbf{H}^1(\Omega_{\mathrm{f}}) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}};\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{w}}\};$$
$$V_r(\Omega_p) = \{v \in W^{1,r}(\Omega_{\mathrm{p}}) : v = 0 \text{ on } \Gamma_{\mathrm{cc}}\};$$
$$V(\Omega) = \{v \in H(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}\};$$
$$H(\Omega_p) = \{v \in H^1(\Omega_{\mathrm{p}}) : v_a := v|_{\Omega_{\mathrm{a}}},\ v_c := v|_{\Omega_{\mathrm{c}}},\ v_m := v|_{\Omega_{\mathrm{m}}},\ v_a = v_m \text{ on } \Gamma_{\mathrm{a}},\ v_c = v_m \text{ on } \Gamma_{\mathrm{c}}\};$$
$$H(\Omega) = \{v \in H^1(\Omega) : v_f := v|_{\Omega_{\mathrm{f}}},\ v_p := v|_{\Omega_{\mathrm{p}}},\ v_f = v_p \text{ on } \Gamma\},$$

with their usual norms. Considering that the Poincaré inequality occurs whenever the trace of the function vanishes on a part with positive measure of the boundary $\partial\Omega$, then the Hilbert spaces $\mathbf{V}(\Omega_f)$, $V_2(\Omega_p)$ and $V(\Omega)$ are endowed with the standard seminorms (cf. (4.4)).

We write $V(\Omega_p)=V_2(\Omega_p)$, for the sake of simplicity.

Set the $(\boldsymbol{\rho},\theta)$-dependent $(\mathrm{I}+2)^2$-matrix

$$\mathsf{A}(\boldsymbol{\rho},\theta)=\begin{bmatrix} D_1(\theta) & \cdots & D_{1\mathrm{I}}(\theta) & a_{1,\mathrm{I}+1}(\rho_1,\theta) & a_{1,\mathrm{I}+2}(\rho_1,\theta) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ D_{\mathrm{I}1}(\theta) & \cdots & D_{\mathrm{I}}(\theta) & a_{\mathrm{I},\mathrm{I}+1}(\rho_{\mathrm{I}},\theta) & a_{\mathrm{I},\mathrm{I}+2}(\rho_{\mathrm{I}},\theta) \\ a_{\mathrm{I}+1,1}(\rho_1,\theta) & \cdots & a_{\mathrm{I}+1,\mathrm{I}}(\rho_{\mathrm{I}},\theta) & k(\theta) & a_{\mathrm{I}+1,\mathrm{I}+2}(\boldsymbol{\rho},\theta) \\ a_{\mathrm{I}+2,1}(\rho_1,\theta) & \cdots & a_{\mathrm{I}+2,\mathrm{I}}(\rho_{\mathrm{I}},\theta) & a_{\mathrm{I}+2,\mathrm{I}+1}(\boldsymbol{\rho},\theta) & \sigma(\boldsymbol{\rho},\theta) \end{bmatrix},$$

where the leading coefficients are kept denoted according to the Fick, Fourier and Ohm laws, for the reader's convenience.

The fuel cell problem, whose strong formulation is stated in Section 2, is equivalent to the following variational formulation.

Definition 3.1. We say that the function $(\mathbf{u},p,\boldsymbol{\rho},\theta,\phi)$ is a *weak solution to the fuel cell problem* if it satisfies the following variational formulations to

- the momentum conservation (Beavers–Joseph–Saffman/Stokes–Darcy problem):

$$\int_{\Omega_\mathrm{f}}\mu(\theta)D\mathbf{u}:D\mathbf{v}\,\mathrm{d}x+\int_{\Omega_\mathrm{f}}\lambda(\theta)\nabla\cdot\mathbf{u}\nabla\cdot\mathbf{v}\,\mathrm{d}x+\int_{\Omega_\mathrm{p}}\frac{K_g(p)}{\mu(\theta)}\nabla p\cdot\nabla v\,\mathrm{d}x \\ +\int_\Gamma\beta(\theta)\mathbf{u}_T\cdot\mathbf{v}_T\,\mathrm{d}s+\int_\Gamma p\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s-\int_\Gamma\mathbf{u}\cdot\mathbf{n}v\,\mathrm{d}s \\ =R_{\mathrm{specific}}\int_{\Omega_\mathrm{f}}\rho\theta\nabla\cdot\mathbf{v}\,\mathrm{d}x \quad (3.1)$$

  holds for all $(\mathbf{v},v)\in\mathbf{V}(\Omega_f)\times H(\Omega_p)$;
- the species conservation:

$$\int_{\Omega_\mathrm{f}}\rho_i\mathbf{u}\cdot\nabla v\,\mathrm{d}x+\int_\Omega D_i(\theta)\nabla\rho_i\cdot\nabla v\,\mathrm{d}x \\ +\sum_{\substack{j=1\\ j\neq i}}^{\mathrm{I}}\int_{\Omega_\mathrm{m}}D_{ij}(\theta)\nabla\rho_j\cdot\nabla v\,\mathrm{d}x+\int_\Omega a_{i,\mathrm{I}+1}(\rho_i,\theta)\nabla\theta\cdot\nabla v\,\mathrm{d}x \\ +\int_{\Omega_\mathrm{p}}a_{i,\mathrm{I}+2}(\rho_i,\theta)\nabla\phi\cdot\nabla v\,\mathrm{d}x=0 \quad (3.2)$$

  holds for all $v\in V(\Omega)$ and $i=1,2,\dots,\mathrm{I}$;

- the energy conservation:

$$\int_\Omega k(\theta)\nabla\theta\cdot\nabla v\,\mathrm{d}x+\int_{\Gamma_\mathrm{w}} h_c(\theta)\theta v\,\mathrm{d}s$$
$$+\sum_{j=1}^{\mathrm{I}}\int_\Omega a_{\mathrm{I}+1,j}(\rho_j,\theta)\nabla\rho_j\cdot\nabla v\,\mathrm{d}x+\int_{\Omega_\mathrm{p}} a_{\mathrm{I}+1,\mathrm{I}+2}(\boldsymbol{\rho},\theta)\nabla\phi\cdot\nabla v\,\mathrm{d}x$$
$$=\int_{\Gamma_\mathrm{w}} h_c(\theta)\theta_e v\,\mathrm{d}s+\int_{\Omega_\mathrm{a}\cup\Omega_\mathrm{c}}\sigma(\boldsymbol{\rho},\theta)|\nabla\phi|^2 v\,\mathrm{d}x \quad (3.3)$$

holds for all $v\in V(\Omega)$;
- the electricity conservation:

$$\int_{\Omega_\mathrm{p}}\sigma(\boldsymbol{\rho},\theta)\nabla\phi\cdot\nabla w\,\mathrm{d}x+\sum_{j=1}^{\mathrm{I}}\int_{\Omega_\mathrm{m}} a_{\mathrm{I}+2,j}(\rho_j,\theta)\nabla\rho_j\cdot\nabla w\,\mathrm{d}x$$
$$+\int_{\Omega_\mathrm{m}} a_{\mathrm{I}+2,\mathrm{I}+1}(\boldsymbol{\rho},\theta)\nabla\theta\cdot\nabla w\,\mathrm{d}x+\int_{\Gamma_\mathrm{a}} j_a([\phi])[w]\,\mathrm{d}s$$
$$=\int_{\Gamma_\mathrm{c}} j_c(\phi_c-\phi_m-E_{\mathrm{cell}})[w]\,\mathrm{d}s \quad (3.4)$$

holds for all $w\in V(\Omega_p)$,
- and ρ obeying (2.12).

Hereafter, we use the notation $\mathrm{d}s$ for the surface element in the integrals on the boundary as well as any subpart of the boundary $\partial\Omega$. Although in Section 2.5 the notation $[\cdot]$ was used for the jump of a quantity across the interface in the direction of the fluid media, for the sake of clearness, in (3.4) it means $[w]=w_\ell-w_m$, where the subscripts denote the restriction to Ω_ℓ, $\ell=\mathrm{a},\mathrm{c}$, or Ω_m.

The equivalence between the strong and variational formulations use standard arguments [30]. Indeed, the variational formulation (3.1) follows from the strong formulations (2.6), (2.13) and (2.16), via the Green formula,

$$-\int_{\Omega_\mathrm{f}}\tau : D\mathbf{v}\,\mathrm{d}x+\langle\tau_T+\tau_N\mathbf{n},\mathbf{v}\rangle_\Gamma=\int_{\Omega_\mathrm{f}} p\nabla\cdot\mathbf{v}\,\mathrm{d}x$$
$$-\langle p_\mathrm{f},\mathbf{v}\cdot\mathbf{n}\rangle_\Gamma\quad\forall\mathbf{v}\in\mathbf{V}(\Omega_f);$$
$$\int_{\Omega_\mathrm{p}}\frac{K_g(p)}{\mu(\theta)}\nabla p\cdot\nabla v\,\mathrm{d}x=\int_\Gamma\mathbf{u}_\mathrm{D}\cdot\mathbf{n}v\,\mathrm{d}s\quad\forall v\in H(\Omega_p),$$

by considering (2.23), (2.25) and (2.27).

The variational formulations (3.2), (3.3) and (3.4) follow from the respective strong formulations, namely, from (2.7), (2.14) with boundary conditions (2.20), (2.28)–(2.30); from (2.8), (2.15), (2.18) with boundary conditions (2.22), (2.24) and (2.26); and from (2.19) with boundary conditions (2.20)–(2.21).

Remark 3.2. All terms are meaningful in the integral identities (3.1)–(3.4). In particular, the Joule effect $Q=\sigma|\nabla\phi|^2$ belonging to $L^t(\Omega_\mathrm{a}\cup\Omega_\mathrm{c})$ is meaningful for any $t>1$ if $n=2$ or for any $t\geq 2n/(n+2)$ if $n>2$.

The set of hypothesis is as follows.

(H1) The viscosities μ and λ are assumed to be Carathéodory functions from $\Omega_\mathrm{f} \times \mathbb{R}$ into $\mathbb{R}$ such that

$$\exists \mu_\#, \mu^\# > 0 : \mu_\# \leq \mu(x,e) \leq \mu^\#; \tag{3.5}$$

$$\exists \lambda^\# > 0 : \, -\mu/n \leq \lambda(x,e) \leq \lambda^\# \tag{3.6}$$

for a.e. $x \in \Omega_\mathrm{f}$ and for all $e \in \mathbb{R}$, while K_g is assumed to be a Carathéodory function from $\Omega_\mathrm{p} \times \mathbb{R}$ into $\mathbb{R}$ such that

$$\exists K_l, b > 0 : K_l \leq K_g(x,e) \leq K_l + b \tag{3.7}$$

for a.e. $x \in \Omega_\mathrm{p}$ and for all $e \in \mathbb{R}$.

(H2) The matrix of coefficients A has components that are Carathéodory functions from $\Omega \times \mathbb{R}^{\mathrm{I}+1}$ to $\mathbb{R}$, except the leading coefficients D_i, k that are Carathéodory functions from $\Omega \times \mathbb{R}$ to $\mathbb{R}$. While the leading coefficients D_i, k and σ satisfy

$$\exists D_i^\#, D_{i,\#}, D_{i,p} > 0 : \quad D_{i,\#} \leq D_i(x,e) \leq D_i^\# \quad \text{for a.e. } x \in \Omega_\mathrm{f}; \tag{3.8}$$

$$D_{i,p} \leq D_i(x,e) \leq D_i^\# \quad \text{for a.e. } x \in \Omega_\mathrm{p}; \tag{3.9}$$

$$\exists k^\#, k_\# > 0 : \quad k_\# \leq k(x,e) \leq k^\# \quad \text{for a.e. } x \in \Omega; \tag{3.10}$$

$$\exists \sigma^\#, \sigma_\#, \sigma_m > 0 : \quad \sigma_\# \leq \sigma(x,\mathbf{e}) \leq \sigma^\# \quad \text{for a.e. } x \in \Omega_\mathrm{a} \cup \Omega_\mathrm{c}; \tag{3.11}$$

$$\sigma_m \leq \sigma(x,\mathbf{e}) \leq \sigma^\# \quad \text{for a.e. } x \in \Omega_\mathrm{m} \tag{3.12}$$

for all $e \in \mathbb{R}$ and $\mathbf{e} \in \mathbb{R}^{\mathrm{I}+1}$, the remaining coefficients satisfy

$$\exists a^\#_{\mathrm{I}+2,j} > 0 : \quad |a_{\mathrm{I}+2,j}(\cdot,\mathbf{e})| \leq a^\#_{\mathrm{I}+2,j} \text{ a.e. in } \Omega_\mathrm{m}; \tag{3.13}$$

$$\exists a^\#_{i,j} > 0 : \quad |a_{i,j}(\cdot,\mathbf{e})| \leq a^\#_{i,j} \quad \text{a.e. in } \Omega \ \forall \mathbf{e} \in \mathbb{R}^{\mathrm{I}+1} \tag{3.14}$$

for all $i \in \{1,\dots,\mathrm{I}+1\}$ and $j \in \{1,\dots,\mathrm{I}+2\}$ such that $i \neq j$. Moreover, we assume the following:

$$a_{i,\#} = \min\{D_{i,\#}, D_{i,p}\} - 2^\mathrm{I} \frac{\left(a^\#_{\mathrm{I}+1,i}\right)^2}{k_\#} - 2^\mathrm{I} \frac{\left(a^\#_{\mathrm{I}+2,i}\right)^2}{\sigma_m} - 2^{\mathrm{I}+1} \sum_{\substack{j=1\\ j\neq i}}^{\mathrm{I}} \frac{\left(a^\#_{j,i}\right)^2}{D_{j,\#}} > 0; \tag{3.15}$$

$$a_{\mathrm{I}+1,\#} = k_\# - 4 \sum_{j=1}^{\mathrm{I}} \frac{\left(a^\#_{j,\mathrm{I}+1}\right)^2}{D_{j,\#}} - 2 \frac{\left(a^\#_{\mathrm{I}+2,\mathrm{I}+1}\right)^2}{\sigma_m} > 0; \tag{3.16}$$

$$a_{\mathrm{I}+2,\#} = \min\{\sigma_\#, \sigma_m\} - 2 \frac{\left(a^\#_{\mathrm{I}+1,\mathrm{I}+2}\right)^2}{k_\#} - 2 \sum_{j=1}^{\mathrm{I}} \frac{\left(a^\#_{j,\mathrm{I}+2}\right)^2}{D_{j,\#}} > 0 \tag{3.17}$$

for each $i = 1,\dots,\mathrm{I}$. We observe that these assumptions are required for the Legendre–Hadamard ellipticity condition.

(H3) The boundary coefficient β is assumed to be a Carathéodory function from $\Gamma \times \mathbb{R}$ into $\mathbb{R}$. Moreover, there exist $\beta_\#, \beta^\# > 0$ such that
$$\beta_\# \leq \beta(\cdot, e) \leq \beta^\# \tag{3.18}$$
a.e. in Γ and for all $e \in \mathbb{R}$.

(H4) The boundary coefficient h_c is assumed to be a Carathéodory function from $\Gamma_{\rm w} \times \mathbb{R}$ into $\mathbb{R}$. Moreover, there exist $h_\#, h^\# > 0$ such that
$$h_\# \leq h_c(\cdot, e) \leq h^\# \tag{3.19}$$
a.e. in $\Gamma_{\rm w}$ and for all $e \in \mathbb{R}$.

(H5) The boundary functions j_ℓ, $\ell = \mathrm{a}, \mathrm{c}$, are assumed to be the odd continuous functions from $\mathbb{R}$ into $\mathbb{R}$ defined in (2.29)–(2.30).

(H6) There exists $u_0 \in H^1(\Omega_{\rm f})$ such that $u_0 = u_{\rm in}$ on $\Gamma_{\rm in}$ and $u_0 = u_{\rm out}$ on $\Gamma_{\rm out}$. Indeed, due to the characteristics of the problem, u_0 has explicit expression
$$u_0(x, y, z) = u_{\rm in} + (u_{\rm out} - u_{\rm in})y/L.$$

Remark 3.3. The assumption (3.5) is a sufficient condition for the validity of the second law of thermodynamics (2.11). The physical meaning of this sufficient condition is consistent with known values of viscosities in the operating temperature range 320 K to 390 K, for instance $\mu_\# \approx 4.2 \times 10^{-5}$ Pa s and $\mu^\# \approx 4.8 \times 10^{-5}$ Pa s for the air. In (3.6), the upper bound $\lambda^\#$ may be assumed $\lambda^\# \geq \mu^\#/n$, for the sake of simplicity, to ensure that $|\lambda| \leq \lambda^\#$. Indeed, the bulk viscosity can be decomposed, at a first approach [22], into a vibrational contribution $\nu_{\rm vib} \gg \mu$ and a rotational contribution $\nu_{\rm rot} = c(\theta)\mu$, where c stands for a temperature-dependent function such that $0 < c(\theta) < 1/3$ for three-dimensional space. While the presence of the vibrational contribution means that $\lambda > 0$, if there exists only a rotational contribution then $\lambda = -(1/3 - c(\theta))\mu < 0$.

Remark 3.4. The choice of (3.15)–(3.17) depends on the application of the relation $(a_1 + \cdots + a_N)^2 \leq 2^{N-1}(a_1^2 + a_2^2) + 2^{N-2}a_3^2 + \cdots + 2a_N^2$, $N > 2$ in inequality (4.22). The meaningfulness of (3.15)–(3.17) follows from the fact that the cross-effect parameters are some orders of magnitude smaller in comparison to the leading coefficients (see [1, 2] and the references therein).

Using a fixed-point argument, we establish the following existence result under a smallness condition on the data.

Theorem 3.5. *Let Ω be a bounded multiregion domain of $\mathbb{R}^n$, $n = 2, 3$. Under the assumptions* (H1)–(H6)*, the fuel cell problem admits, at least, one solution according to Definition* 3.1 *such that*

- *the velocity $\mathbf{u} \in \mathbf{u}_0 + \mathbf{V}(\Omega_f)$, with $\mathbf{u}_0 = u_0\mathbf{e}_y$;*
- *the pressure $p \in H(\Omega_p)$;*
- *the partial densities $\boldsymbol{\rho} \in [V(\Omega)]^{\rm I}$;*
- *the temperature $\theta \in V(\Omega)$;*
- *the potential $\phi \in E_{\rm cell}\chi_{\Omega_{\rm c}} + V_r(\Omega_p)$, for $r > 2$,*

provided one of the smallness conditions (5.5) *or* (5.6) *holds.*

The existence of a weak solution to the fuel cell problem relies on the fixed-point argument

$$\begin{aligned}(\pi, \boldsymbol{\varrho}, \xi, \varphi, \Phi) \in E := H(\Omega_p) \times [H^1(\Omega)]^{\mathrm{I}+1} \times V(\Omega_p) \times L^t(\Omega_{\mathrm{a}} \cup \Omega_{\mathrm{c}})\\ \mapsto (\mathbf{U}, p) \in \mathbf{V}(\Omega_f) \times (H(\Omega_p)/\mathbb{R})\\ \mapsto (\boldsymbol{\rho}, \theta, \phi_{cc}) \in [V(\Omega)]^{\mathrm{I}+1} \times V(\Omega_p)\\ \mapsto (p, \boldsymbol{\rho}, \theta, \phi_{cc}, |\nabla\phi|_{\Omega_{\mathrm{a}}\cup\Omega_{\mathrm{c}}}|^2),\end{aligned} \tag{3.20}$$

where

- $(\mathbf{U}, p) = (\mathbf{U}, p)(\pi, \boldsymbol{\varrho}, \xi)$ stands for the auxiliary velocity-pressure pair given in Section 4.1;
- $(\rho_1, \dots, \rho_{\mathrm{I}}, \theta, \phi_{cc}) = (\boldsymbol{\rho}, \theta, \phi)(\mathbf{w}, \boldsymbol{\varrho}, \xi, \varphi, \Phi)$ stands for the auxiliary partial densities, temperature and potential given in Section 4.2, for $t \geq 2n/(n+2)$ if $n > 2$ or $t > 1$ if $n = 2$, with $\mathbf{w} = \mathbf{u}(\pi, \boldsymbol{\varrho}, \xi)$ being the auxiliary velocity field given in Section 4.1;
- $\phi = \phi_{cc} + E_{\mathrm{cell}}\chi_{\Omega_{\mathrm{c}}}$, with $\chi_{\Omega_{\mathrm{c}}}$ denoting the characteristic function.

4. Auxiliary results

In this section, although our result is only valid for $n = 2, 3$, we keep the space dimension n as general whenever possible. Thus, the reader is able to see where the dimension becomes an obstacle and may reflect on it.

We begin by naming some known constants that are used in this work (see, for instance, [16]).

Notation. We denote by

- S^* the continuity constant of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, i.e., it obeys the Sobolev inequality
$$\|v\|_{2^*,\Omega} \leq S^*\|v\|_{1,2,\Omega} \quad \forall v \in H^1(\Omega), \tag{4.1}$$
with $2^* = 2n/(n-2)$ being the critical Sobolev exponent if $n > 2$. If $n = 2$, the Sobolev inequality holds for any $1 \leq p < \infty$. For the sake of simplicity, we also denote by 2^* any arbitrary real number greater than one, if $n = 2$.
- S_* the continuity constant of the trace embedding $H^1(\Omega) \hookrightarrow L^{2_*}(\partial\Omega)$, i.e., it obeys the trace inequality
$$\|v\|_{2_*,\partial\Omega} \leq S_*\|v\|_{1,2,\Omega} \quad \forall v \in H^1(\Omega), \tag{4.2}$$
with $2_* = 2(n-1)/(n-2)$ being the critical trace exponent if $n > 2$. If $n = 2$, we denote by 2_* an arbitrary real number greater than one.

Remark 4.1. The Rellich–Kondrachov compact embeddings $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\partial\Omega)$ stand for any exponent q lower than the critical Sobolev exponent p^* and the critical trace exponent p_*, respectively. If $q = p^*$, only continuous embedding holds. Similarly with the embedding of $W^{1,p}(\Omega)$ into $L^q(\partial\Omega)$, if $q = p_*$ it is only continuous.

The Poincaré constant C_Ω can have different forms, i.e., it obeys one of the Poincaré-type inequalities

$$\inf_{\alpha\in\mathbb{R}} \|v-\alpha\|_{2,\Omega_\mathrm{p}} \leq C_{\Omega_\mathrm{p}} \|\nabla v\|_{2,\Omega_\mathrm{p}} \quad \forall v \in H^1(\Omega_\mathrm{p}); \tag{4.3}$$

$$\|v\|_{2,\Omega_\ell} \leq C_{\Omega_\ell} \|\nabla v\|_{2,\Omega_\ell} \quad \forall v \in V(\Omega_\ell), \tag{4.4}$$

where $V(\Omega_\ell)$ stands for $\mathbf{V}(\Omega_f)$, $V(\Omega_p)$ or simply $V(\Omega)$. The inequality (4.3) is known as the Deny–Lions lemma [11].

We recall the classical Korn inequality.

Lemma 4.2 (Korn inequality)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain. Then, there exists a constant $C_K > 0$ such that*

$$\|\nabla \mathbf{v}\|_{2,\Omega}^2 \leq C_K \|D\mathbf{v}\|_{2,\Omega}^2$$

for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$.

Next, we state the properties of the transport term for some exponent q.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For each $\mathbf{w} \in \mathbf{L}^q(\Omega)$, $q = n > 2$ or $q > n = 2$, the following functional is well defined and continuous: $e \in H^1(\Omega) \mapsto \int_\Omega \mathbf{w}\cdot\nabla e v \,\mathrm{d}x$ for all $v \in H^1(\Omega)$. In particular, the relation*

$$\left|\int_\Omega \mathbf{w}\cdot\nabla e v \,\mathrm{d}x\right| \leq \|\mathbf{w}\|_{q,\Omega} \|\nabla e\|_{2,\Omega} \|v\|_{2^*,\Omega} \tag{4.5}$$

holds for any $e, v \in H^1(\Omega)$.

Proof. The wellposedness of the functional is a consequence of the Hölder inequality for $1/q + 1/2^* = 1/2$, i.e., $2q/(q-2) = 2^*$. ☐

Notice that the Rellich–Kondrachov embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ is valid with exponents q, p and 2 such that (cf. Remark 4.1)

$$\frac{1}{2^*} < \frac{1}{p} = \frac{1}{2} - \frac{1}{q} \Leftrightarrow q > n.$$

4.1. **Auxiliary velocity-pressure pair.** For $\pi \in L^2(\Omega_\mathrm{p})$, $\boldsymbol{\varrho} \in [L^4(\Omega_\mathrm{f})]^\mathrm{I}$ and $\xi \in H^1(\Omega)$, we define the Dirichlet–BJS/Stokes–Darcy problem

$$\begin{aligned}
&\int_{\Omega_\mathrm{f}} \mu(\xi) D\mathbf{U} : D\mathbf{v}\,\mathrm{d}x + \int_{\Omega_\mathrm{f}} \lambda(\xi)\nabla\cdot\mathbf{U}\nabla\cdot\mathbf{v}\,\mathrm{d}x \\
&\quad + \int_\Gamma \beta(\xi)\mathbf{U}_T\cdot\mathbf{v}_T\,\mathrm{d}s + \int_{\Omega_\mathrm{p}} \frac{K_g(\pi)}{\mu(\xi)}\nabla p\cdot\nabla v\,\mathrm{d}x + \int_\Gamma p\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s - \int_\Gamma \mathbf{U}\cdot\mathbf{n} v\,\mathrm{d}s \\
&\quad\quad = R_\mathrm{specific}\int_{\Omega_\mathrm{f}} \varrho\xi\nabla\cdot\mathbf{v}\,\mathrm{d}x - G(\xi,\mathbf{u}_0,\mathbf{v},v) \quad \forall(\mathbf{v},v)\in\mathbf{V}(\Omega_f)\times H(\Omega_p),
\end{aligned} \tag{4.6}$$

where

$$\varrho = \sum_{i=1}^{\mathrm{I}} \varrho_i, \tag{4.7}$$

$$\begin{aligned} G(\xi, \mathbf{z}, \mathbf{v}, v) = & \int_{\Omega_\mathrm{f}} \mu(\xi) D\mathbf{z} : D\mathbf{v} \,\mathrm{d}x + \int_{\Omega_\mathrm{f}} \lambda(\xi) \nabla \cdot \mathbf{z} \nabla \cdot \mathbf{v} \,\mathrm{d}x \\ & + \int_\Gamma \beta(\xi) \mathbf{z}_T \cdot \mathbf{v}_T \,\mathrm{d}s - \int_\Gamma \mathbf{z} \cdot \mathbf{n} v \,\mathrm{d}s. \end{aligned} \tag{4.8}$$

The existence of a unique weak solution $(\mathbf{U}, p) = (\mathbf{U}, p)(\pi, \boldsymbol{\varrho}, \xi)$ to the variational equality (4.6) can be stated as follows.

Proposition 4.4 (Auxiliary velocity-pressure pair)**.** *Let* $\pi \in L^2(\Omega_\mathrm{p})$*,* $\boldsymbol{\varrho} \in [L^4(\Omega_\mathrm{f})]^\mathrm{I}$ *and* $\xi \in H^1(\Omega)$*,* $n = 2, 3$*. Under the assumptions* (H1)*,* (H3) *and* (H6)*, the Dirichlet–BJS/Stokes–Darcy problem* (4.6) *admits a unique weak solution* $(\mathbf{U}, p) \in \mathbf{V}(\Omega_f) \times (H(\Omega_p)/\mathbb{R})$*. Moreover, if* $\mathbf{u} = \mathbf{U} + \mathbf{u}_0$ *the following quantitative estimate holds:*

$$\begin{aligned} & \frac{\mu_\#}{2C_K} \|\nabla \mathbf{u}\|^2_{2,\Omega_\mathrm{f}} + \beta_\# \|\mathbf{u}_T\|^2_{2,\Gamma} + \frac{K_l}{\mu^\#} \|\nabla p\|^2_{2,\Omega_\mathrm{p}} \\ & \quad \leq \left(\frac{R_\mathrm{specific}}{\sqrt{\mu_\#}} \|\varrho\|_{4,\Omega_\mathrm{f}} \|\xi\|_{4,\Omega_\mathrm{f}} + \sqrt{\mu^\#} \|D\mathbf{u}_0\|_{2,\Omega_\mathrm{f}} + \frac{\lambda^\#}{\sqrt{\mu_\#}} \|\nabla \cdot \mathbf{u}_0\|_{2,\Omega_\mathrm{f}} \right)^2 \\ & \qquad + \max\left\{ \beta^\#, \frac{\mu^\#}{K_l} \right\} \|\mathbf{u}_0\|^2_{2,\Gamma}. \end{aligned} \tag{4.9}$$

Proof. The existence of a unique weak solution $(\mathbf{U}, p) \in \mathbf{V}(\Omega_f) \times (H(\Omega_p)/\mathbb{R})$ to the variational equality (4.6) can be obtained by the Lax–Milgram lemma, due to the assumptions (H1), (H3) and (H6). Indeed, the uniqueness of p in $H(\Omega_p)$ follows from the contradiction argument. Assuming that p_1 and $p_2 = p_1 + c$, $c \in \mathbb{R}$, satisfy (4.6), then subtracting the corresponding relations we obtain

$$c \int_\Gamma \mathbf{v} \cdot \mathbf{n} \,\mathrm{d}s = 0 \quad \forall \mathbf{v} \in \mathbf{V}(\Omega_f),$$

which implies $c = 0$.

To prove the coercivity, we observe that

$$\begin{aligned} \mu(\xi)|D\mathbf{U}|^2 + \lambda(\xi)|\nabla \cdot \mathbf{U}|^2 & \geq \mu(\xi) \left(|D\mathbf{U}|^2 - \frac{1}{n} |\nabla \cdot \mathbf{U}|^2 \right) \geq \mu(\xi) \left(1 - \frac{1}{n} \right) |D\mathbf{U}|^2 \\ & \geq \frac{n-1}{n} \mu_\# |D\mathbf{U}|^2 \geq \frac{\mu_\#}{2} |D\mathbf{U}|^2, \end{aligned} \tag{4.10}$$

using (3.6), the fact that $(\nabla \cdot \mathbf{U})^2 \leq |D\mathbf{U}|^2$, (3.5) and $n \geq 2$. Then, coercivity follows, on the one hand, from the Poincaré-type inequality (4.3) for p in the quotient space $H(\Omega_p)/\mathbb{R}$, and on the other hand, from inequality (4.10), taking the Korn inequality into account (cf. Lemma 4.2), for $\mathbf{U}$ in $\mathbf{V}(\Omega_f)$.

The quantitative estimate (4.9) follows from taking $(\mathbf{v}, v) = (\mathbf{U}, p)$ as a test function in (4.6), and next taking the Hölder and Young inequalities into account, applying assumptions (3.5)–(3.7), and (3.18), and using inequality (4.10) and Lemma 4.2. $\square$

The continuous dependence is established as follows.

Proposition 4.5 (Continuous dependence). *Suppose that the assumptions of Proposition 4.4 are fulfilled. Let $\{\pi_m\}$, $\{\boldsymbol{\varrho}_m\}$ and $\{\xi_m\}$ be sequences such that $\pi_m \to \pi$ in $L^2(\Omega_{\mathrm{p}})$, $\boldsymbol{\varrho}_m \to \boldsymbol{\varrho}$ in $[L^4(\Omega)]^{\mathrm{I}}$, and $\xi_m \rightharpoonup \xi$ in $H^1(\Omega)$, respectively. If $(\mathbf{u}_m, p_m) = (\mathbf{U} + \mathbf{u}_0, p)(\pi_m, \boldsymbol{\varrho}_m, \xi_m)$ are the unique solutions to the corresponding variational equalities $(4.6)_m$ for $m \in \mathbb{N}$, then*

$$\mathbf{U}_m \rightharpoonup \mathbf{U} \text{ in } \mathbf{V}(\Omega_f); \tag{4.11}$$

$$p_m \rightharpoonup p \text{ in } H(\Omega_p), \tag{4.12}$$

with $(\mathbf{u}, p) = (\mathbf{U} + \mathbf{u}_0, p)(\pi, \boldsymbol{\varrho}, \xi)$ being the solution to (4.6).

Proof. Let $\{\pi_m\}$, $\{\boldsymbol{\varrho}_m\}$ and $\{\xi_m\}$ be sequences in the conditions of the proposition. The uniform estimate (4.9) allows us to find a subsequence of $\{(\mathbf{u}_m, p_m)\}$, still denoted by $\{(\mathbf{u}_m, p_m)\}$, such that the convergences (4.11)–(4.12) hold. It remains to prove that $(\mathbf{u}, p)$ solves the variational equality (4.6).

By appealing to the Rellich–Kondrashov compact embeddings $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$ and $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Gamma)$, $n = 2, 3$, we have

$$\xi_m \to \xi \text{ in } L^4(\Omega) \text{ and a.e. in } \Omega; \tag{4.13}$$

$$\xi_m \to \xi \text{ in } L^2(\Gamma) \text{ and a.e. on } \Gamma. \tag{4.14}$$

Applying Krasnolselskii's theorem to the Nemytskii operators K_g and μ, along with the Lebesgue dominated convergence theorem, we obtain

$$\frac{K_g(\pi_m)}{\mu(\xi_m)} \nabla v \to \frac{K_g(\pi)}{\mu(\xi)} \nabla v \quad \text{in } \mathbf{L}^2(\Omega_{\mathrm{p}}).$$

Analogously for the coefficients μ, λ and β.

Then, we pass to the limit the variational equality $(4.6)_m$ as m tends to infinity, concluding that $(\mathbf{u}, p)$ solves the variational equality (4.6). $\square$

4.2. **Auxiliary partial density-temperature-potential triplet solution.** In this section, we seek for the triplet solution $(\boldsymbol{\rho}, \theta, \phi)$. Let $\mathbf{w} \in \mathbf{L}^q(\Omega_{\mathrm{f}})$ for

$$q \geq n > 2 \quad \text{or} \quad q > n = 2. \tag{4.15}$$

For $\boldsymbol{\varrho} \in [H^1(\Omega)]^{\mathrm{I}}$, $\xi \in H^1(\Omega)$, $\varphi \in H^1(\Omega_{\mathrm{p}})$ and $\Phi \in L^t(\Omega_{\mathrm{a}} \cup \Omega_{\mathrm{c}})$, with

$$t \geq 2n/(n+2) \text{ if } n > 2 \quad \text{or} \quad t > 1 \text{ if } n = 2, \tag{4.16}$$

we define the coupled problem

$$\int_{\Omega_{\mathrm{f}}} \rho_i \mathbf{w}\cdot\nabla v \,\mathrm{d}x + \int_{\Omega} D_i(\xi)\nabla\rho_i\cdot\nabla v\,\mathrm{d}x \\ + \sum_{\substack{j=1\\ j\neq i}}^{\mathrm{I}} \int_{\Omega} D_{ij}(\varrho_j,\xi)\nabla\rho_j\cdot\nabla v\,\mathrm{d}x + \int_{\Omega} a_{i,\mathrm{I}+1}(\varrho_i,\xi)\nabla\theta\cdot\nabla v\,\mathrm{d}x \\ + \int_{\Omega_{\mathrm{p}}} a_{i,\mathrm{I}+2}(\varrho_i,\xi)\nabla\phi\cdot\nabla v\,\mathrm{d}x = 0; \tag{4.17}$$

$$\int_{\Omega} k(\xi)\nabla\theta\cdot\nabla v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{w}}} h_c(\xi)\theta v\,\mathrm{d}s \\ + \sum_{j=1}^{\mathrm{I}} \int_{\Omega} a_{\mathrm{I}+1,j}(\varrho_j,\xi)\nabla\rho_j\cdot\nabla v\,\mathrm{d}x + \int_{\Omega_{\mathrm{p}}} a_{\mathrm{I}+1,\mathrm{I}+2}(\boldsymbol{\varrho},\xi)\nabla\phi\cdot\nabla v\,\mathrm{d}x \\ = \int_{\Gamma_{\mathrm{w}}} h_c(\xi)\theta_e v\,\mathrm{d}s + \int_{\Omega_{\mathrm{a}}\cup\Omega_{\mathrm{c}}} \sigma(\boldsymbol{\varrho},\xi)\Phi v\,\mathrm{d}x; \tag{4.18}$$

$$\int_{\Omega_{\mathrm{p}}} \sigma(\boldsymbol{\varrho},\xi)\nabla\phi\cdot\nabla w\,\mathrm{d}x + \sum_{j=1}^{\mathrm{I}} \int_{\Omega_{\mathrm{m}}} a_{\mathrm{I}+2,j}(\varrho_j,\xi)\nabla\rho_j\cdot\nabla w\,\mathrm{d}x \\ + \int_{\Omega_{\mathrm{m}}} a_{\mathrm{I}+2,\mathrm{I}+1}(\boldsymbol{\varrho},\xi)\nabla\theta\cdot\nabla w\,\mathrm{d}x \\ + \int_{\Gamma_{\mathrm{a}}} j_a([\phi])[w]\,\mathrm{d}s = \int_{\Gamma_{\mathrm{c}}} j_c([\varphi])[w]\,\mathrm{d}s \tag{4.19}$$

for every $i = 1, 2, \dots, \mathrm{I}$, and for all $v \in V(\Omega)$ and $w \in V(\Omega_p)$.

The existence of a unique weak solution $(\boldsymbol{\rho}, \theta, \phi_{cc}) = (\boldsymbol{\rho}, \theta, \phi)(\mathbf{w}, \boldsymbol{\varrho}, \xi, \varphi, \Phi)$ to the variational equalities (4.17)–(4.19) can be stated as follows.

Proposition 4.6 (Auxiliary partial density-temperature-potential triplet)**.** *Let $\mathbf{w} \in \mathbf{L}^q(\Omega_{\mathrm{f}})$, $q \geq n > 2$ or $q > n = 2$, be such that*

$$\|\mathbf{w}\|_{q,\Omega_{\mathrm{f}}} < \min_i \frac{D_{i,\#}}{S^*}, \tag{4.20}$$

$\boldsymbol{\varrho} \in [H^1(\Omega)]^{\mathrm{I}}$, $\xi \in H^1(\Omega)$, $\varphi \in H^1(\Omega_{\mathrm{p}})$ and $\Phi \in L^t(\Omega_{\mathrm{a}} \cup \Omega_{\mathrm{c}})$, $t \geq 2n/(n+2)$ if $n > 2$ or $t > 1$ if $n = 2$, be given. Under the assumptions (H2)*,* (H4) *and* (H5)*, the variational problem* (4.17)*–*(4.19) *admits a unique solution $(\boldsymbol{\rho}, \theta, \phi_{cc}) \in$*

$[V(\Omega)]^{\mathrm{I}+1} \times V(\Omega_p)$. *Moreover, the quantitative estimate*

$$\begin{aligned}\sum_{i=1}^{\mathrm{I}} \left(a_{i,\#} - S^* \|\mathbf{w}\|_{q,\Omega_\mathrm{f}}\right) \|\nabla \rho_i\|_{2,\Omega_\mathrm{f}}^2 + \sum_{i=1}^{\mathrm{I}} a_{i,\#} \|\nabla \rho_i\|_{2,\Omega_\mathrm{p}}^2 + a_{\mathrm{I}+2,\#} \|\nabla \phi\|_{2,\Omega_p}^2 \\ + a_{\mathrm{I}+1,\#} \|\nabla \theta\|_{2,\Omega}^2 + h_\# \|\theta\|_{2,\Gamma_\mathrm{w}}^2 \\ \leq \frac{(S^* \sigma^\#)^2}{k_\#} \|\Phi\|_{t,\Omega_\mathrm{a}\cup\Omega_\mathrm{c}}^2 + \frac{j_L^2}{\min\{\sigma_\#, \sigma_m/2\}} + h^\# \|\theta_e\|_{2,\Gamma_\mathrm{w}}^2 \end{aligned} \tag{4.21}$$

holds for $\phi = \phi_{cc} + E_{\mathrm{cell}} \chi_{\Omega_\mathrm{c}}$, *i.e., for a solution such that* $\phi = 0$ *on* $\Gamma_{\mathrm{cc,a}}$ *and* $\phi = E_{\mathrm{cell}}$ *on* $\Gamma_{\mathrm{cc,c}}$.

Proof. Let $\mathbf{w} \in \mathbf{L}^q(\Omega_\mathrm{f})$, $q = n > 2$ or $q > n = 2$, $\boldsymbol{\varrho} \in [H^1(\Omega)]^{\mathrm{I}}$, $\xi \in H^1(\Omega)$, $\varphi \in H^1(\Omega_\mathrm{p})$ and $\Phi \in L^t(\Omega_\mathrm{a} \cup \Omega_\mathrm{c})$, $t = 2n/(n+2)$ if $n > 2$ or $t > 1$ if $n = 2$, be fixed.

The existence of a unique weak solution $(\boldsymbol{\rho}, \theta, \phi_{cc}) \in [V(\Omega)]^{\mathrm{I}+1} \times V(\Omega_p)$ to the variational equalities (4.17)–(4.19) can be obtained by the Browder–Minty theorem. Indeed, the operator $L : [V(\Omega)]^{\mathrm{I}+1} \times V(\Omega_p) \to \left([V(\Omega)]^{\mathrm{I}+1} \times V(\Omega_p)\right)'$, defined by

$$\begin{aligned}\langle L(\mathbf{Y}), \mathbf{v}\rangle = \int_\Omega \mathsf{A}(\boldsymbol{\varrho}, \xi) \nabla \mathbf{Y} \cdot \nabla \mathbf{v} \, \mathrm{d}x + \int_{\Gamma_\mathrm{w}} h_c(\xi) \theta v \, \mathrm{d}s \\ + \int_{\Gamma_\mathrm{a}} j_a([\phi])[w] \, \mathrm{d}s + \sum_{i=1}^{\mathrm{I}} \int_{\Omega_\mathrm{f}} \rho_i \mathbf{w} \cdot \nabla v \, \mathrm{d}x,\end{aligned}$$

where $\mathbf{Y} = (\boldsymbol{\rho}, \theta, \phi)$ and $\mathbf{v} = (v, \dots, v, w)$, is hemicontinuous, strictly monotone and coercive (see (4.22)) if (4.20) is satisfied.

Hereafter, the symbol $\langle \cdot, \cdot \rangle$ denotes the duality pairing $\langle \cdot, \cdot \rangle_{X' \times X}$, with X being a Banach space and X' denoting the dual space of X.

Let us establish the quantitative estimate (4.21). We take $v = \rho_i$, $v = \theta$ and $w = \phi_{cc}$ as test functions in (4.17), (4.18) and (4.19), respectively. Next, applying the limiting current bound j_L $(= j_{c,L})$, the assumptions (3.8)–(3.14), (3.19), the Hölder and Young inequalities, inequality (4.5), and summing the obtained expressions, we get

$$\begin{aligned}\sum_{i=1}^{\mathrm{I}} \frac{D_{i,\#}}{2} \|\nabla \rho_i\|_{2,\Omega_\mathrm{f}}^2 + \sum_{i=1}^{\mathrm{I}} \frac{D_{i,p}}{2} \|\nabla \rho_i\|_{2,\Omega_\mathrm{p}}^2 + \frac{k_\#}{2} \|\nabla \theta\|_{2,\Omega}^2 + \frac{h_\#}{2} \|\theta\|_{2,\Gamma_\mathrm{w}}^2 \\ + \sigma_\# \|\nabla \phi\|_{2,\Omega_\mathrm{a}\cup\Omega_\mathrm{c}}^2 + \frac{\sigma_m}{2} \|\nabla \phi\|_{2,\Omega_\mathrm{m}}^2 \\ \leq \sum_{i=1}^{\mathrm{I}} \frac{1}{2D_{i,\#}} \left(\sum_{j=1,\, j\neq i}^{\mathrm{I}} a_{i,j}^\# \|\nabla \rho_j\|_{2,\Omega} + a_{i,\mathrm{I}+1}^\# \|\nabla \theta\|_{2,\Omega} + a_{i,\mathrm{I}+2}^\# \|\nabla \phi\|_{2,\Omega_\mathrm{p}} \right)^2 \\ + \frac{1}{2k_\#} \left(\sum_{j=1}^{\mathrm{I}} a_{\mathrm{I}+1,j}^\# \|\nabla \rho_j\|_{2,\Omega} + a_{\mathrm{I}+1,\mathrm{I}+2}^\# \|\nabla \phi\|_{2,\Omega_\mathrm{p}} \right)^2 + \frac{h^\#}{2} \|\theta_e\|_{2,\Gamma_\mathrm{w}}^2\end{aligned}$$

$$+ S^*\|\mathbf{w}\|_{q,\Omega_\mathrm{f}} \sum_{i=1}^{\mathrm{I}} \|\nabla\rho_i\|_{2,\Omega_\mathrm{f}}\|\rho_i\|_{1,2,\Omega_\mathrm{f}} + \sigma^{\#}\|\Phi\|_{t,\Omega_\mathrm{a}\cup\Omega_\mathrm{c}}\|\theta\|_{t',\Omega_\mathrm{a}\cup\Omega_\mathrm{c}}$$

$$+ \frac{1}{2\sigma_m}\left(\sum_{j=1}^{\mathrm{I}} a^{\#}_{\mathrm{I}+2,j}\|\nabla\rho_j\|_{2,\Omega_\mathrm{m}} + a^{\#}_{\mathrm{I}+2,\mathrm{I}+1}\|\nabla\theta\|_{2,\Omega_\mathrm{m}}\right)^2 + j_L\|[\phi]\|_{1,\Gamma_\mathrm{c}}. \qquad (4.22)$$

Then, we consider the Sobolev embedding $V(\Omega) \hookrightarrow L^{t'}(\Omega)$ for $t' = 2^*$, with the corresponding optimal Sobolev constant S^*. We emphasize that the fundamental theorem of calculus may be applied to our special domain, and taking $t' = 4$ into account, explicit constants may be derived. Instead using the trace-Poincaré inequality (namely, S_* and C_Ω) on the last term in the right-hand side, we apply the fundamental theorem of calculus to the domain Ω_c with $v = 0$ a.e. on Γ_c, i.e., at $x = x_c + l_c$, and next the Schwarz inequality,

$$|v(x_c)|^2 = \left|\int_{x_c}^{x_c+l_c} \partial_x v \,\mathrm{d}t\right|^2 \le l_c \int_{x_c}^{x_c+l_c} |\partial_x v|^2 \,\mathrm{d}t.$$

Hence, we have

$$\int_{\Gamma_\mathrm{c}} |v|^2 \,\mathrm{d}s \le l_c \int_{\Omega_\mathrm{c}} |\nabla v|^2 \,\mathrm{d}x. \qquad (4.23)$$

Recall that the notation $\mathrm{d}x$ refers to the 2D $\mathrm{d}x\,\mathrm{d}y$ and the 3D $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. An analogous inequality is valid for $v = \phi|_{\Omega_\mathrm{a}\cup\Omega_\mathrm{m}}$. Therefore, using Remark 3.4 and (3.15)–(3.17), the claimed quantitative estimate (4.21) arises. ☐

The continuous dependence is established as follows.

Proposition 4.7 (Continuous dependence)**.** *Suppose that the assumptions of Proposition* 4.6 *are fulfilled. Let* $\{\mathbf{w}_m\}$, $\{\boldsymbol{\varrho}_m\}$, $\{\xi_m\}$, $\{\varphi_m\}$ *and* $\{\Phi_m\}$ *be sequences such that* $\mathbf{w}_m \to \mathbf{w}$ *in* $\mathbf{L}^q(\Omega_\mathrm{f})$, $\boldsymbol{\varrho}_m \rightharpoonup \boldsymbol{\varrho}$ *in* $[H^1(\Omega)]^\mathrm{I}$, $\xi_m \rightharpoonup \xi$ *in* $H^1(\Omega)$, $\varphi_m \rightharpoonup \varphi$ *in* $H^1(\Omega_\mathrm{p})$, *and* $\Phi_m \rightharpoonup \Phi$ *in* $L^t(\Omega_\mathrm{a}\cup\Omega_\mathrm{c})$*. If*

$$(\boldsymbol{\rho}_m, \theta_m, (\phi_{cc})_m) = (\boldsymbol{\rho}, \theta, \phi)(\mathbf{w}_m, \boldsymbol{\varrho}_m, \xi_m, \varphi_m, \Phi_m)$$

are the unique solutions to the corresponding variational systems $(4.17)_m$–$(4.19)_m$ *for* $m \in \mathbb{N}$*, then*

$$\boldsymbol{\rho}_m \rightharpoonup \boldsymbol{\rho} \text{ in } [H^1(\Omega)]^\mathrm{I}; \qquad (4.24)$$

$$\theta_m \rightharpoonup \theta \text{ in } H^1(\Omega); \qquad (4.25)$$

$$\phi_m \rightharpoonup \phi \text{ in } H^1(\Omega_\mathrm{p}), \qquad (4.26)$$

with $(\boldsymbol{\rho}, \theta, \phi_{cc}) = (\boldsymbol{\rho}, \theta, \phi)(\mathbf{w}, \boldsymbol{\varrho}, \xi, \varphi, \Phi)$ *being the solution to* (4.17)–(4.19)*.*

Proof. Let $\{\mathbf{w}_m\}$, $\{\boldsymbol{\varrho}_m\}$, $\{\xi_m\}$, $\{\varphi_m\}$ and $\{\Phi_m\}$ be sequences in the conditions of the proposition, and let $(\boldsymbol{\rho}_m, \theta_m, (\phi_{cc})_m)$ solve the corresponding variational system $(4.17)_m$–$(4.19)_m$. Thanks to the estimate (4.21), we can extract a (not relabeled) subsequence $\{(\boldsymbol{\rho}_m, \theta_m, (\phi_{cc})_m)\}$ such that the convergences (4.24)–(4.26) hold.

Let us prove that $(\boldsymbol{\rho}, \theta, \phi_{cc})$ solves the variational equalities (4.17)–(4.19). By appealing to the Rellich–Kondrashov compact embeddings $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Gamma_{\rm w})$ and $H^1(\Omega_{\rm p}) \hookrightarrow\hookrightarrow L^2(\Gamma_{\rm CL})$ for $n \geq 2$, we have

$$(\boldsymbol{\varrho}_m, \xi_m) \to (\boldsymbol{\varrho}, \xi) \text{ in } [L^2(\Omega)]^{\rm I+1} \text{ and a.e. in } \Omega; \tag{4.27}$$

$$\xi_m \to \xi \text{ in } L^2(\Gamma_{\rm w}); \tag{4.28}$$

$$\varphi_m \to \varphi \text{ in } L^2(\Gamma_{\rm c}) \text{ and a.e. on } \Gamma_{\rm c}; \tag{4.29}$$

$$\phi_m \to \phi \text{ in } L^2(\Gamma_{\rm a}) \text{ and a.e. on } \Gamma_{\rm a}. \tag{4.30}$$

Thanks to the continuity of the Nemytskii operators $a_{l,j}$ for $l, j = 1, 2, \ldots, {\rm I}+2$, where $a_{i,i} = D_i$, $k = a_{\rm I+1,I+1}$, and $\sigma = a_{\rm I+2,I+2}$, using (4.27) together with the Lebesgue dominated convergence theorem, we obtain

$$a_{l,j}(\boldsymbol{\varrho}_m, \xi_m)\nabla v \to a_{l,j}(\boldsymbol{\varrho}, \xi)\nabla v \text{ in } \mathbf{L}^2(\Omega); \tag{4.31}$$

$$a_{{\rm I}+2,j}(\boldsymbol{\varrho}_m, \xi_m)\nabla w \to a_{{\rm I}+2,j}(\boldsymbol{\varrho}, \xi)\nabla w \text{ in } \mathbf{L}^2(\Omega_{\rm p}) \tag{4.32}$$

for all $l = 1, 2, \ldots, {\rm I}+1$. Analogously, for the Nemytskii operators h_c, j_a, and j_c, using (4.28)–(4.30) and the Lebesgue dominated convergence theorem, we obtain

$$h_c(\xi_m)v \to h_c(\xi)v \text{ in } L^2(\Gamma_{\rm w});$$

$$j_a([\phi_m]) \to j_a([\phi]) \text{ in } L^2(\Gamma_{\rm a}); \tag{4.33}$$

$$j_c([\varphi_m]) \to j_c([\varphi]) \text{ in } L^2(\Gamma_{\rm c}). \tag{4.34}$$

Hence, we may pass to the limit the variational equalities $(4.17)_m$–$(4.19)_m$ as m tends to infinity. Therefore, we conclude that $(\boldsymbol{\rho}, \theta, \phi_{cc})$ solves the variational equalities (4.17)–(4.19).

By the uniqueness of the limit, the weak limit of the initial sequence is the claimed solution. □

Next, we establish the convergence of the gradient of the auxiliary potential solutions a.e. in $\Omega_{\rm a} \cup \Omega_{\rm c}$.

Proposition 4.8 (Compactness). *If $\{(\boldsymbol{\varrho}_m, \xi_m, \varphi_m)\}_{m\in\mathbb{N}}$ is a sequence in $[H^1(\Omega_{\rm p})]^{\rm I+2}$ that converges weakly to $(\boldsymbol{\varrho}, \xi, \varphi)$ in $[H^1(\Omega_{\rm p})]^{\rm I+2}$, and $(\phi_{cc})_m$ solves the corresponding variational equality $(4.19)_m$ for each $m \in \mathbb{N}$, then $\nabla\phi_m \to \nabla\phi$ in $L^2(\Omega_{\rm a} \cup \Omega_{\rm c})$, where $\phi_{cc} = \phi - E_{\rm cell}\chi_{\Omega_{\rm c}}$ solves the variational equality (4.19). In particular, $\nabla\phi_m \to \nabla\phi$ almost everywhere, up to a subsequence, in $\Omega_{\rm a} \cup \Omega_{\rm c}$.*

Proof. Let $(\phi_{cc})_m$ and ϕ_{cc} solve $(4.19)_m$ and (4.19), respectively. Let us take $w = (\phi_m - \phi)\chi_{\Omega_{\rm a}\cup\Omega_{\rm c}}$ as a test function in $(4.19)_m$ and (4.19); subtracting the expressions, we obtain

$$\begin{aligned}\sigma_{\#}\int_{\Omega_{\mathrm{a}}\cup\Omega_{\mathrm{c}}}|\nabla(\phi_m-\phi)|^2\,\mathrm{d}x &\le \int_{\Omega_{\mathrm{a}}\cup\Omega_{\mathrm{c}}}\sigma(\boldsymbol{\varrho}_m,\xi_m)|\nabla(\phi_m-\phi)|^2\,\mathrm{d}x\\ &= I_1+I_2\\ &:= \int_{\Omega_{\mathrm{a}}\cup\Omega_{\mathrm{c}}}(\sigma(\boldsymbol{\varrho},\xi)-\sigma(\boldsymbol{\varrho}_m,\xi_m))\nabla\phi\cdot\nabla(\phi_m-\phi)\,\mathrm{d}x\\ &\qquad+\int_{\Gamma_{\mathrm{CL}}}\big(j(\phi_m)-j(\phi)\big)\,(\phi_m-\phi)\,\mathrm{d}s,\end{aligned}$$

with j being defined by

$$j(\phi)=\begin{cases} j_a(\phi_a-\phi) & \text{a.e. on } \Gamma_{\mathrm{a}},\\ -j_c(\varphi_c-\varphi) & \text{a.e. on } \Gamma_{\mathrm{c}},\end{cases} \tag{4.35}$$

where v_ℓ denotes the trace of $v|_{\Omega_\ell}$ on Γ_ℓ, $\ell=\mathrm{a},\mathrm{c}$, while v stands for the trace of a function defined in Ω_{m}. Thus, considering (4.27)–(4.30) and (4.32), we conclude that each integral, I_1 and I_2, converges to zero, and consequently the proof of Proposition 4.8 is finished. □

Finally, some higher integrability can be obtained for the gradient of the auxiliary potential solution (cf. [9] and the references therein).

Proposition 4.9 (Regularity)**.** *Let $\phi_{cc}\in V(\Omega_p)$ be the solution of the variational equality* (4.19)*. Then, $(\phi_{cc})|_{\Omega_{\mathrm{a}}\cup\Omega_{\mathrm{c}}}$ belongs to the Sobolev space $W^{1,2+\varepsilon}(\Omega_{\mathrm{a}}\cup\Omega_{\mathrm{c}})$, for some $\varepsilon>0$ depending exclusively on the boundary, and the quantitative estimate*

$$\|\nabla\phi\|_{r,\Omega_a\cup\Omega_c}\le\frac{\sigma_{\#}M_r}{\sigma^{\#}\left(\sigma^{\#}-M_r\sqrt{(\sigma^{\#})^2-\sigma_{\#}^2}\right)}j_L|\Gamma_{CL}|=:R_3 \tag{4.36}$$

holds. Moreover, in the conditions of Proposition 4.8*, we have the strong convergence $\sigma(\boldsymbol{\varrho}_m,\xi_m)|\nabla\phi_m|^2\to\sigma(\boldsymbol{\varrho},\xi)|\nabla\phi|^2$ in $L^{1+\varepsilon/2}(\Omega_{\mathrm{a}}\cup\Omega_{\mathrm{c}})$.*

Proof. Let $\mathbf{Y}=(\boldsymbol{\varrho},\xi)\in[H^1(\Omega_{\mathrm{p}})]^{\mathrm{I}+1}$and $\varphi\in H^1(\Omega_{\mathrm{p}})$ be fixed, and let $\phi_{cc}\in V(\Omega_p)$ be the solution of the variational equality (4.19). Let us define the operator A : $V(\Omega_a\cup\Omega_c)\to\big(V(\Omega_a\cup\Omega_c)\big)'$ by

$$\langle A(\mathbf{Y};\phi),w\rangle=\int_{\Omega_{\mathrm{a}}\cup\Omega_{\mathrm{c}}}\sigma(\mathbf{Y})\nabla\phi\cdot\nabla w\,\mathrm{d}x.$$

By the uniqueness of the solution, the solution ϕ_{cc} satisfies

$$\langle A(\mathbf{Y};\phi),w\rangle=\int_{\Gamma_{\mathrm{CL}}}j(\phi)w\,\mathrm{d}s\quad\forall w\in V(\Omega_a\cup\Omega_c),$$

where j is defined in (4.35).

Denoting by A_r the restriction of A to $V_r(\Omega_a\cup\Omega_c)$, and $f=j(\phi)\in L^\infty(\Gamma_{\mathrm{CL}})\subset(V_r(\Omega_a\cup\Omega_c))'$, for any $r>2$, the regularity established in the celebrated paper by Gröger and Rehberg [20] guarantees that there exists a $r_0>2$ such that A_r is bijective from $V_r(\Omega_a\cup\Omega_c)$ onto $(V_r(\Omega_a\cup\Omega_c))'$ for every $r\in[2,r_0]$. The existence of

$r_0 > 2$ is determined by a class of the domain, which is formulated in [19]. Indeed, the domains Ω_a and Ω_c are regular in the sense formulated in [19] for every $r_0 \geq 2$. In particular, it is proved that $M_r < \sigma^\# / \sqrt{(\sigma^\#)^2 - \sigma_\#^2}$, with

$$M_r := \sup\{\|v\|_{1,r,\Omega_a\cup\Omega_c} : v \in V_r(\Omega_a \cup \Omega_c), \|Jv\|_{(V_r(\Omega_a\cup\Omega_c))'} \leq 1\},$$

where

$$\langle J\phi, w\rangle = \int_{\Omega_\mathrm{a}\cup\Omega_\mathrm{c}} \nabla\phi \cdot \nabla w \,\mathrm{d}x.$$

We remind that the dual space X' is equipped with the usual induced norm $\|f\|_{X'} = \sup\{\langle f, w\rangle,\, w \in X : \|w\|_X \leq 1\}$.

Moreover, the estimate

$$\|A^{-1}\|_{\mathcal{L}((V_r(\Omega_a\cup\Omega_c))';V_r(\Omega_a\cup\Omega_c))} \leq \frac{\sigma_\# M_r}{\sigma^\#\left(\sigma^\# - M_r\sqrt{(\sigma^\#)^2 - \sigma_\#^2}\right)}$$

holds true, where $\|A^{-1}\|_{\mathcal{L}(Y;X)} = \sup\{\|A^{-1}y\|_X : \|y\|_Y \leq 1\}$. Indeed, the estimate

$$\|\nabla\phi_1 - \nabla\phi_2\|_{r,\Omega_a\cup\Omega_c} \leq \frac{\sigma_\# M_r}{\sigma^\#\left(\sigma^\# - M_r\sqrt{(\sigma^\#)^2 - \sigma_\#^2}\right)}\|j(\phi_1) - j(\phi_2)\|_{(V_r(\Omega_a\cup\Omega_c))'}$$

holds true and plays an essential role. The quantitative estimate (4.36) holds true, due to the limiting current bound.

Considering the inequalities

$$|a^2 - b^2|^{r/2} \leq (a+b)^{r/2}|a-b|^{r/2} \leq \frac{1}{2}(a+b)^r + \frac{1}{2}|a-b|^r \quad \forall a, b > 0,$$

we conclude that

$$\begin{aligned}
&\|\sigma(\mathbf{Y}_m)|\nabla\phi_m|^2 - \sigma(\mathbf{Y})|\nabla\phi|^2\|_{r/2,\Omega_a\cup\Omega_c} \\
&\quad \leq \|\sigma(\mathbf{Y}_m)\left(|\nabla\phi_m|^2 - |\nabla\phi|^2\right)\|_{r/2,\Omega_a\cup\Omega_c} + \|(\sigma(\mathbf{Y}_m) - \sigma(\mathbf{Y}))|\nabla\phi|^2\|_{r/2,\Omega_a\cup\Omega_c} \\
&\quad \leq \frac{\sigma_\# M_r}{\sigma^\#\left(\sigma^\# - M_r\sqrt{(\sigma^\#)^2 - \sigma_\#^2}\right)}\|j(\phi_m) - j(\phi)\|_{(V_r(\Omega_a\cup\Omega_c))'}.
\end{aligned}$$

Therefore, the final claim is obtained by taking the strong convergences (4.29)–(4.30) into account. □

By considering Proposition 4.9, we may define $t = 1 + \varepsilon/2$ that obeys (4.16).

5. Fixed-point argument (proof of Theorem 3.5)

Our aim is to apply the Tychonoff fixed point theorem to the operator $\mathcal{T}$ defined in (3.20).

The closed ball $K \subset E = H(\Omega_p) \times [V(\Omega)]^{\mathrm{I}+1} \times V(\Omega_p) \times L^t(\Omega_\mathrm{a} \cup \Omega_\mathrm{c})$, $t > 1$, defined as

$$K = \left\{(\pi, \boldsymbol{\varrho}, \xi, \varphi, \Phi) : \|\pi\| \leq R_1, \left(\|\boldsymbol{\varrho}\|^2 + \|\xi\|^2 + \|\varphi\|^2\right)^{1/2} \leq R_2,\, \|\Phi\| \leq R_3\right\}$$

is compact when the topological vector space is endowed with the weak topology, or simply weakly compact, because E is reflexive. The radii R_1, R_2 and R_3 are the positive constants defined in (5.2), in the cases (1) and (2) below, and in (4.36), respectively.

The operator $\mathcal{T}$ is well defined for $n = 2, 3$, due to Proposition 4.4, considering that $V(\Omega) \hookrightarrow L^4(\Omega)$ for $n = 2, 3$, and due to Propositions 4.6 and 4.9, taking (4.16) and $\mathbf{w} = \mathbf{u} \in \mathbf{H}^1(\Omega_\mathrm{f}) \hookrightarrow \mathbf{L}^q(\Omega_\mathrm{f})$ into account, observing that $2n/(n-2) \geq q \geq n = 3, 4$ or any $q > n = 2$. Its continuity is due to Propositions 4.5, 4.7 and 4.8, if we consider $6 > q > n = 3$ or any $q > n = 2$.

It remains to prove that $\mathcal{T}$ maps K into itself. Let $(\pi, \boldsymbol{\varrho}, \xi, \varphi, \Phi) \in K$ be given, and $(p, \boldsymbol{\rho}, \theta, \phi, |\nabla \phi|_{\Omega_\mathrm{a} \cup \Omega_\mathrm{c}}|^2) = \mathcal{T}(\pi, \boldsymbol{\varrho}, \xi, \varphi, \Phi)$. In particular, there exists the auxiliary velocity field $\mathbf{u} = \mathbf{u}(\boldsymbol{\varrho}|_{\Omega_\mathrm{f}}, \xi)$ being in accordance with Section 4.1.

On the one hand, the estimate (4.9) may be rewritten as

$$\left(\|\nabla \mathbf{u}\|^2_{2,\Omega_\mathrm{f}} + \|\mathbf{u}_T\|^2_{2,\Gamma}\right)^{1/2} \leq aR_2^2 + \frac{1}{\sqrt{\min\{\mu_\#/(2C_K), \beta_\#\}}} C_0; \tag{5.1}$$

$$\|\nabla p\|_{2,\Omega_\mathrm{p}} \leq \sqrt{\frac{\mu^\#}{K_l}} \left(\frac{R_\mathrm{specific}}{\sqrt{\mu_\#}} R_2^2 + C_0 \right) := R_1, \tag{5.2}$$

where

$$a := \frac{1}{\sqrt{\min\{\mu_\#/(2C_K), \beta_\#\}}} \frac{R_\mathrm{specific}}{\sqrt{\mu_\#}};$$

$$C_0 := \sqrt{\mu^\#} \|D\mathbf{u}_0\|_{2,\Omega_\mathrm{f}} + \frac{\lambda^\#}{\sqrt{\mu_\#}} \|\nabla \cdot \mathbf{u}_0\|_{2,\Omega_\mathrm{f}} + \sqrt{\max\left\{\beta^\#, \frac{\mu^\#}{K_l}\right\}} \|\mathbf{u}_0\|_{2,\Gamma}.$$

On the other hand, the estimate (4.21) yields

$$\|\nabla \boldsymbol{\rho}\|^2_{2,\Omega} + \|\nabla \theta\|^2_{2,\Omega} + \|\nabla \phi\|^2_{2,\Omega_\mathrm{p}} \leq c/a_\# \qquad \text{if } b - aR_2^2 \geq a_\#; \tag{5.3}$$

$$\|\nabla \boldsymbol{\rho}\|^2_{2,\Omega} + \|\nabla \theta\|^2_{2,\Omega} + \|\nabla \phi\|^2_{2,\Omega_\mathrm{p}} \leq c/(b - aR_2^2) \qquad \text{if } b - aR_2^2 < a_\#, \tag{5.4}$$

where

$$a_\# := \min_{j=1,\dots,\mathrm{I}+2} a_{j,\#};$$

$$b := \min_{i=1,\dots,\mathrm{I}} \frac{D_{i,\#}}{S^*} - \frac{1}{\sqrt{\min\{\mu_\#/(2C_K), \beta_\#\}}} C_0;$$

$$c := \frac{(S^* \sigma^\#)^2}{k_\#} R_3^2 + \frac{j_L^2}{\min\{\sigma_\#, \sigma_m/2\}} + h^\# \|\theta_e\|^2_{2,\Gamma_\mathrm{w}}.$$

The existence of $R_2 > 0$ is guaranteed in both cases:

(1) The case $b - aR_2^2 \geq a_\#$ means

$$c/a_\# \leq x := R_2^2 \leq (b - a_\#)/a,$$

which is true if

$$b \geq a_\# + ac/a_\#. \tag{5.5}$$

(2) The case $b - aR_2^2 < a_\#$ means

$$(b-a_\#)/a < x := R_2^2 < b/a;$$
$$b - \sqrt{\Delta} \leq 2ax \leq b + \sqrt{\Delta} \quad \text{if } \Delta = b^2 - 4ac > 0,$$

which is true if

$$2\sqrt{ac} < b < 2a_\# + \sqrt{\Delta}. \tag{5.6}$$

Therefore, Theorem 3.5 is completely proved. □

Acknowledgments. The author is deeply grateful to the anonymous referee for the valuable comments.

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Received: November 5, 2022 · Accepted: May 16, 2023 · Early view: August 22, 2024