REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 68, No. 1, 2025, Pages 243–249
Published online: May 21, 2025
https://doi.org/10.33044/revuma.3826

COUNTEREXAMPLES FOR SOME RESULTS IN "ON THE MODULE INTERSECTION GRAPH OF IDEALS OF RINGS"

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ABSTRACT. Let R be a commutative ring and M be an R-module, and let $I(R)^*$ be the set of all nontrivial ideals of R. The M-intersection graph of ideals of R, denoted by $G_M(R)$, is a graph with the vertex set $I(R)^*$, and two distinct vertices I and J are adjacent if and only if $IM \cap JM \neq 0$. In this note, we provide counterexamples for some results proved in a paper by Asir, Kumar, and Mehdi [Rev. Un. Mat. Argentina 63 (2022), no. 1, 93–107]. Also, we determine the girth of $G_M(R)$ and derive a necessary and sufficient condition for $G_M(R)$ to be weakly triangulated.

1. Introduction

The intersection graphs of some algebraic structures such as lattices, posets, groups, rings and modules have been studied by several authors. Let R be a commutative ring and M be an R-module, and $I(R)^*$ be the set of all non-zero proper ideals of R. In [2], the intersection graph of ideals of R, denoted by $G(R)$, was introduced as the graph with vertices $I(R)^*$ and two distinct vertices are adjacent if and only if they have non-zero intersection. In [6], the M-intersection graph of ideals of R, denoted by $G_M(R)$, is defined to be the graph with the vertex set $I(R)^*$, and two distinct vertices I and J are adjacent if and only if $IM \cap JM \neq 0$. Clearly, $G_R(R) = G(R)$, so $G_M(R)$ is in fact a generalization of $G(R)$. Also, the $\mathbb{Z}_n$-intersection graph of $\mathbb{Z}_m$, was studied in [7]. Recently, Asir et al. studied the M-intersection graph of ideals of R in [1]. In this note, we provide counterexamples for some results proved in [1]. Moreover, we determine the girth of $G_M(R)$ and derive a necessary and sufficient condition for $G_M(R)$ to be weakly triangulated. Throughout the paper, all rings are commutative with non-zero identity and all modules are unitary. The *annihilator* of an R-module M is denoted by $\operatorname{ann}(M)$. If $\operatorname{ann}(M) = 0$, then M is said to be a *faithful* R-module. An R-module M is a *multiplication* module if for each submodule N of M there is an ideal I of R such that $IM = N$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the set of integers and the set of integers modulo n, respectively.

2020 *Mathematics Subject Classification.* Primary 16P20; Secondary 05C25, 05C17.
Key words and phrases. intersection graph, perfect graph, weakly perfect graph, planar graph, girth.

Now, we recall some definitions and notations on graphs. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Suppose that $x, y \in V(G)$. If x and y are adjacent, then we write $x — y$. A graph G is *complete* if each pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. If a graph G has a cycle, then the *girth* of G (notated $\mathrm{gr}(G)$) is defined as the length of a shortest cycle of G; otherwise $\mathrm{gr}(G) = \infty$. A *clique* of a graph is a complete subgraph and the number of vertices in a largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. By $\chi(G)$, we denote the *chromatic number* of G, i.e., the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A graph is *perfect* if the clique number and the chromatic number of its induced subgraphs are equal. Also, it is *weakly perfect* if $\chi(G) = \omega(G)$. Recall that a graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends.

2. Connectedness

Recall that an ideal which is minimal in $I(R)^*$ with respect to inclusion is said to be a *minimal ideal* of R. The following theorem was proved in [1]:

Theorem 2.1 ([1, Theorem 2.5]). *Let R be a commutative ring and M an R-module. Then $G_M(R)$ is complete if and only if M is faithful and R is Artinian with a unique minimal ideal.*

Let $M = R = \mathbb{Z}$. Since any two nontrivial ideals of $\mathbb{Z}$ have non-zero intersection, $G_{\mathbb{Z}}(\mathbb{Z}) = G(\mathbb{Z})$ is a complete graph, and hence $G_{\mathbb{Z}}(\mathbb{Z})$ is a counterexample for Theorem 2.1.

3. Perfectness

The next theorem was proved in [1]:

Theorem 3.1 ([1, Theorem 3.3]). *Let $R \cong R_1 \times \cdots \times R_n$, where each R_i, $1 \leq i \leq n$, is a Noetherian ring with unique minimal ideal, and let M be a faithful R-module. Then $G_M(R)$ is perfect if and only if $n \leq 4$.*

Note that even if a ring has a unique non-zero minimal ideal, there might be non-zero ideals not containing it, unless the ring is Artinian. Let $n = 1$, $R \cong R_1 = \mathbb{Z}_4 \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and $M = R$. Clearly, R is a Noetherian ring with a unique minimal ideal $J = 2\mathbb{Z}_4 \times 0 \times 0 \times 0 \times 0$. If $I_1 = \mathbb{Z}_4 \times \mathbb{Z} \times 0 \times 0 \times 0$, $I_2 = 0 \times \mathbb{Z} \times \mathbb{Z} \times 0 \times 0$, $I_3 = 0 \times 0 \times \mathbb{Z} \times \mathbb{Z} \times 0$, $I_4 = 0 \times 0 \times 0 \times \mathbb{Z} \times \mathbb{Z}$, and $I_5 = \mathbb{Z}_4 \times 0 \times 0 \times 0 \times \mathbb{Z}$, then the subgraph induced by the set $\{I_1, \ldots, I_5\}$ in $G_R(R) = G(R)$ is an induced cycle of length 5. Thus $G_R(R)$ is not perfect, and so $G_R(R)$ is a counterexample for Theorem 3.1. We show that if each R_i is an Artinian ring with a unique minimal ideal, and M is a faithful multiplication R-module, then the proof is correct.

A graph G is called *weakly triangulated* if neither G nor its complement $\overline{G}$ contains a chordless cycle of length more than 4. In [5], it is proved that all weakly triangulated graphs are perfect. Also, Chudnovsky et al. [3] provided a characterization of perfect graphs.

Theorem A (The Strong Perfect Graph Theorem [3]). *A finite graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least* 5.

Theorem 3.2. *Let $R \cong R_1 \times \cdots \times R_n$, where each R_i, $1 \leq i \leq n$, is an Artinian ring with a unique minimal ideal, and let M be a faithful multiplication R-module. Then $G_M(R)$ is weakly triangulated if and only if $n \leq 4$.*

Proof. ($\Rightarrow$): Suppose $n \geq 5$. Let $I_j = 0 \times \cdots \times 0 \times R_j \times R_{j+1} \times 0 \times \cdots \times 0$ for $j = 1, \ldots, 4$ and $I_5 = R_1 \times 0 \times 0 \times 0 \times R_5 \times 0 \times \cdots \times 0$. Since M is a faithful multiplication R-module, by [4, Theorem 1.6], we find that $I_iM \cap I_jM = (I_i \cap I_j)M$. Hence the subgraph induced by the set $\{I_1, \ldots, I_5\}$ in $G_M(R)$ is an induced cycle of length 5, and so $G_M(R)$ is not weakly triangulated.

($\Leftarrow$): Assume $n \leq 4$. Note that any ideal I_k of R is of the form $I_{k_1} \times \cdots \times I_{k_n}$, where I_{k_i} is an ideal of R_i for all $i = 1, \ldots, n$. If two vertices I_k and I_l are non-adjacent in $G_M(R)$, then $I_kM \cap I_lM = 0$. The fact that M is faithful leads to $I_k \cap I_l = 0$. Note that R_i is Artinian with a unique minimal ideal for all $i = 1, \ldots, n$. Therefore if I_k is not adjacent to I_l in $G_M(R)$, then either $I_{k_j} = 0$ or $I_{l_j} = 0$ for each $j = 1, \ldots, n$. First, let us consider the best possible choice, $n = 4$. We claim that every cycle of length more than 4 in $G_M(R)$ must have diagonals. In order to prove the claim, suppose $I_1 — I_2 — I_3 — \cdots — I_m — I_1$ is a cycle of length $m \geq 5$ in $G_M(R)$. If any three ideals from $\{I_{1_1}, I_{1_2}, I_{1_3}, I_{1_4}\}$ are the zero ideal, say $I_{1_1} = I_{1_2} = I_{1_3} = 0$, then $I_{2_4} \neq 0$ and $I_{m_4} \neq 0$. So I_2 and I_m form a diagonal edge. If exactly one ideal from $\{I_{1_1}, I_{1_2}, I_{1_3}, I_{1_4}\}$ is a zero ideal, say $I_{1_1} = 0$, then $I_{3_2} = I_{3_3} = I_{3_4} = 0$. This implies that $I_{2_1}, I_{4_1} \neq 0$. Therefore I_2 and I_4 form a diagonal edge. Thus every ideal of I_1, I_2, I_3 and I_4 can be decomposed into two zero ideals and two non-zero ideals. Let $I_{1_1} = I_{1_2} = 0$ and $I_{1_3}, I_{1_4} \neq 0$. Then $I_{3_3} = I_{3_4} = 0$ and $I_{4_3} = I_{4_4} = 0$. Hence $I_{3_1}, I_{3_2} \neq 0$ and $I_{4_1}, I_{4_2} \neq 0$. Since $I_2 — I_3$, either $I_{2_1} \neq 0$ or $I_{2_2} \neq 0$. So I_2 and I_4 form a diagonal edge. Therefore, the claim holds true for $n = 4$.

Now, let $I_1 — I_2 — I_3 — \cdots — I_m — I_1$ be a cycle C of length $m \geq 5$ in $\overline{G_M(R)}$. We show that C has a diagonal. If any three ideals from $\{I_{1_1}, I_{1_2}, I_{1_3}, I_{1_4}\}$ are the zero ideal, say $I_{1_1} = I_{1_2} = I_{1_3} = 0$, then $I_{3_4}, I_{4_4} \neq 0$, which yields a contradiction. If exactly one ideal from $\{I_{1_1}, I_{1_2}, I_{1_3}, I_{1_4}\}$ is a zero ideal, say $I_{1_1} = 0$, then $I_{2_2} = I_{2_3} = I_{2_4} = 0$. This implies that $I_{4_1}, I_{5_1} \neq 0$, a contradiction. Thus every ideal of I_1, I_2, I_3 and I_4 can be decomposed into two zero ideals and two non-zero ideals. Assume that $I_{1_1} = I_{1_2} = 0$ and $I_{1_3}, I_{1_4} \neq 0$. Then $I_{2_3} = I_{2_4} = 0$, and hence $I_{2_1}, I_{2_2} \neq 0$. This yields that $I_{3_1} = I_{3_2} = 0$, and so $I_{3_3}, I_{3_4} \neq 0$. Again, we deduce that $I_{4_3} = I_{4_4} = 0$, and then $I_{4_1}, I_{4_2} \neq 0$. Therefore, I_1 and I_4 form a diagonal edge.

Similar arguments to those above lead us to the cases $n = 3$ and $n = 2$. So let $n = 1$. Thus R is an Artinian ring with a unique minimal ideal, say J. Since J is a non-zero ideal and M is a faithful R-module, we have $JM \neq 0$. On the other hand, since R is an Artinian ring, $J \subseteq I$ for each non-zero ideal I of R. Thus we conclude that $G_M(R)$ is a complete graph, and hence $G_M(R)$ is weakly triangulated. □

Example 3.3 ([7, Example 1]). Let $R = \mathbb{Z}_{p_1 p_2^3}$, where p_1 and p_2 are distinct primes. It is not hard to see that $\mathbb{Z}_{p_1 p_2^2}$ is an R-module. Then we have the graph in Figure 1.

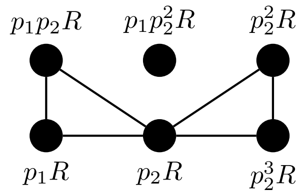

FIGURE 1. The graph $G_{\mathbb{Z}_{p_1 p_2^2}}(R)$.

The next theorem was proved in [1]:

Theorem 3.4 ([1, Theorem 3.4]). *The graph $G_M(R)$ is weakly perfect for any R-module M.*

The proof is not correct. Let $A = \{I \in I^*(R) \mid IM = 0\}$ and $A' = I^*(R) \setminus A$. In line 6 of the proof, the authors claimed that if $\omega(G_M(R)) = n$ and $S = \{I_1, \ldots, I_n\}$ is a clique of $G_M(R)$ such that $S \subset A'$, then the vertices $J + I_1, \ldots, J + I_n$ are the same as $I_1, \ldots, I_n$ in different order, where $J \in A' \setminus S$. But this claim does not hold. See Example 3.3. Let $I_1 = p_1R$, $I_2 = p_2R$, and $I_3 = p_1p_2R$. Clearly, $S = \{I_1, I_2, I_3\}$ is a clique of $G_{\mathbb{Z}_{p_1 p_2^2}}(R)$, and $A = \{p_1p_2^2R\}$. Consider $J = p_2^2R$. Then $\{J + I_1, J + I_2, J + I_3\} = \{I_2, R\}$. Because $p_2^2R + p_1R = R$, $p_2^2R + p_2R = p_2R$ and $p_2^2R + p_1p_2R = p_2R$. This contradicts the claim. (Also, the open neighborhood of $J \in A' \setminus S$ is not in S. This contradicts the sentence in line 9 of the proof.)

It is noteworthy that Nikandish and Nikmerh [8] conjectured that, for every ring R, $G(R)$ is a weakly perfect graph. The conjecture will be true if Theorem 3.4 is proved. Also, see the problem posed by Heydari [6].

4. CYCLIC SUBGRAPH AND PLANARITY

The following theorem was proved in [1]:

Theorem 4.1 ([1, Theorem 4.1]). *Let M be an R-module. If $G_M(R)$ contains a cycle, then* $\mathrm{gr}(G_M(R)) = 3$. *That is,* $\mathrm{gr}(G_M(R)) \in \{3, \infty\}$.

The proof is not correct. Let $I_1 — I_2 — I_3 — I_4$ be a path in $G_M(R)$. In line 5 of the proof, the authors claimed that if I_k and I_l $(1 \leq k \neq l \leq 4)$ are two vertices that are incomparable, then $I_k — I_k + I_l — I_k + I_m — I_k$ is a cycle, where $m \in \{1, 2, 3, 4\} \setminus \{k, l\}$. This claim does not hold. We note that maybe $I_k + I_l = R$ and then $I_k + I_l$ cannot be a vertex. We prove the theorem as follows.

Theorem 4.2. *Let M be an R-module. Then* $\mathrm{gr}(G_M(R)) \in \{3, \infty\}$.

Proof. Suppose that $I_1 — I_2 — \cdots — I_n — I_1$ is a cycle of length n in $G_M(R)$. If $n = 3$, we are done. Thus assume that $n \geq 4$.

First, assume that M is a faithful R-module. Suppose that I_1 and I_2 are not comparable. Let $\mathfrak{m}_1, \mathfrak{m}_2$ be two maximal ideals of R such that $I_1 \subseteq \mathfrak{m}_1$ and $I_2 \subseteq \mathfrak{m}_2$. If $I_1 \neq \mathfrak{m}_1$ (resp. $I_2 \neq \mathfrak{m}_2$), then $I_1 — I_2 — \mathfrak{m}_1 — I_1$ (resp. $I_1 — I_2 — \mathfrak{m}_2 — I_1$) is a cycle of length 3. So let I_1 and I_2 be two maximal ideals of R. If $I_1 \cap I_2 = 0$, then R is a direct sum of two fields which implies that $|I(R)^*| = 2$, a contradiction. Thus $I_1 \cap I_2 \neq 0$, and hence $I_1 — I_2 — I_1 \cap I_2 — I_1$ is a triangle. Now, assume that I_1 and I_2 are comparable. Similarly, we can assume that I_i and I_{i+1} are comparable, for every i, $1 < i < n$. Hence we can compile into two cases. If $I_1 \subseteq I_2, I_3 \subseteq I_2$ and $I_3 \subseteq I_4$, then $I_3 \subseteq I_2 \cap I_4$. So $(I_2 \cap I_4)M \neq 0$. Thus $I_2 — I_3 — I_4 — I_2$ is a cycle of length 3. If $I_2 \subseteq I_1$ and $I_2 \subseteq I_3$, then $I_2 \subseteq I_1 \cap I_3$ and so $I_1 — I_2 — I_3 — I_1$ is a cycle of length 3. Therefore, $\mathrm{gr}(G_M(R)) = 3$.

Next, suppose that $\mathrm{ann}(M) \neq 0$. Let $S = R/\mathrm{ann}(M)$ and $J_i = (I_i + \mathrm{ann}(M))/\mathrm{ann}(M)$, for $i = 1, \ldots, n$. Note that $I_i + \mathrm{ann}(M) \neq \mathrm{ann}(M)$, otherwise $I_iM = 0$ which yields that I_i is an isolated vertex in $G_M(R)$, a contradiction. Also, if $I_i + \mathrm{ann}(M) = R$, then $I_iM = M$. This implies that I_i is adjacent to all other vertices of the cycle, and hence $\mathrm{gr}(G_M(R)) = 3$. On the other hand, if $i \neq k$ and $I_i + \mathrm{ann}(M) = I_k + \mathrm{ann}(M)$, then $I_iM = I_kM$. Consider $m \in \{1, \ldots, n\} \setminus \{i\}$ such that I_m is adjacent to I_k. Thus $I_i — I_k — I_m — I_i$ is a cycle of length 3 in $G_M(R)$. Therefore, we can assume that $J_1 — J_2 — \cdots — J_n — J_1$ is a cycle of length n in $G_M(S)$. Since M is a faithful S-module, as we saw above, $G_M(S)$ contains a triangle, say $L_1/\mathrm{ann}(M) — L_2/\mathrm{ann}(M) — L_3/\mathrm{ann}(M) — L_1/\mathrm{ann}(M)$, and so $L_1 — L_2 — L_3 — L_1$ is a cycle of length 3 in $G_M(R)$. Hence $\mathrm{gr}(G_M(R)) = 3$, as desired. □

Remark 4.3 ([1, Remark 4.4])**.** Let M be a faithful R-module and $|I(R)^*| \geq 3$.

(a) If R is an Artinian local ring or M is uniform, then $G_M(R)$ is complete and so it is Hamiltonian.

(b) If M is not a faithful R-module, then $\mathrm{ann}(M)$ is an isolated vertex in $G_M(R)$, so $G_M(R)$ is not Hamiltonian.

Let $R = F[x,y]/\overline{(x,y)^2}$, where F is a field. Clearly, R is an Artinian local ring with maximal ideal $\overline{(x,y)}$. But $G_R(R) = G(R)$ is not a complete graph, because $\overline{(x)}$ and $\overline{(y)}$ are two non-adjacent vertices. This contradicts the statement (a) of Remark 4.3.

Lemma 4.4. *Let R be the direct product of $n \geq 2$ local rings such that at least one of them is not a field, and let M be a faithful R-module. Then K_{2^n-1} is a subgraph of $G_M(R)$.*

Proof. Let $R = R_1 \times \cdots \times R_n$, where each R_i is local with maximal ideal $\mathfrak{m}_i$ for $i = 1, \ldots, n$. With no loss of generality, assume that R_1 is not a field. Let $A = \{I_1 \times \cdots \times I_n \mid I_i = R_i \text{ or } \mathfrak{m}_i \text{ for } i = 1, \ldots, n\} \setminus \{R_1 \times \cdots \times R_n\}$. Then

$A \subseteq I^*(R)$ and $|A| = 2^n - 1$. Since I_1 is non-zero, the subgraph induced by A is a complete subgraph of $G_M(R)$. Therefore K_{2^n-1} is a subgraph of $G_M(R)$. □

In [1, page 106, line 1], the authors claimed that the vertex $0 \times R_2 \times \cdots \times R_n$ is adjacent to all the vertices of A in $G_M(R)$. This is not correct. For example, consider $M = R = R_1 \times \cdots \times R_n$, with $R_2, \ldots, R_n$ fields. Then $R_1 \times 0 \times \cdots \times 0 \in A$. But $R_1 \times 0 \times \cdots \times 0$ and $0 \times R_2 \times \cdots \times R_n$ are not adjacent.

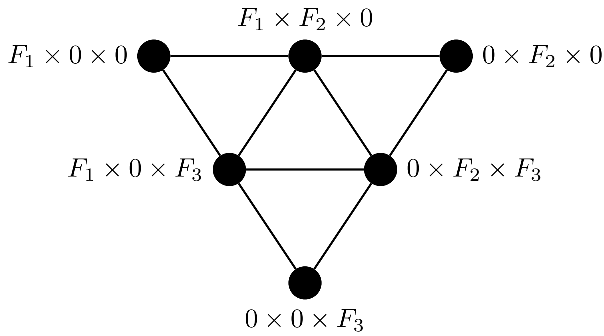

Figure 2. The graph $G(F_1 \times F_2 \times F_3)$.

Let F_1, F_2, F_3 be fields and let $M = R = F_1 \times F_2 \times F_3$. It is not hard to see that Figure 2 is a counterexample for [1, Theorem 4.10].

Acknowledgments

The authors would like to express their deep gratitude to the referee for comments and suggestions.

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Received: December 19, 2022
Accepted: July 18, 2023
Early view: August 24, 2024