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ON A FRACTIONAL NIRENBERG EQUATION: COMPACTNESS AND EXISTENCE RESULTS

AZEB ALGHANEMI AND RANDA BEN MAHMOUD

ABSTRACT. This paper deals with a fractional Nirenberg equation of order $\sigma \in (0, n/2)$, $n \geq 2$. We study the compactness defect of the associated variational problem. We determine precise characterizations of critical points at infinity of the problem, through the construction of a suitable pseudo-gradient at infinity. Such a construction requires detailed asymptotic expansions of the associated energy functional and its gradient. This study will then be used to derive new existence results for the equation.

1. Introduction

Over the past decades fractional analysis has aroused the interest of many scientists. This is mainly due to its numerous applications in various scientific domains such as biology, medicine, engineering and mathematical analysis; see [16] and [23] and references therein. In this paper we are concerned with a fractional partial differential equation arising in a geometric context. Namely, the prescribed fractional Q-curvature problem on the standard sphere. Let S^n, $n \geq 2$, be the unit sphere of $\mathbb{R}^{n+1}$ equipped with its standard metric g_0. Let $g = u^{\frac{4}{n-2\sigma}} g_0$, $u \in C^\infty(S^n, \mathbb{R}^+)$, $\sigma \in (0, \frac{n}{2})$, be a metric of S^n conformably equivalent to g_0. The fractional Q-curvature Q_σ of order σ associated to the metric g is defined by

$$Q_\sigma = c(n,\sigma)^{-1} u^{-\frac{n+2\sigma}{n-2\sigma}} P_\sigma^{g_0}(u) \quad \text{on } S^n,$$

where $c(n,\sigma) = \Gamma\big(\frac{n}{2} + \sigma\big)/\Gamma\big(\frac{n}{2} - \sigma\big)$, with Γ the gamma function, and

$$P_\sigma^{g_0} = \Gamma\bigg(B + \frac{1}{2} + \sigma\bigg)\Big/\Gamma\bigg(B + \frac{1}{2} - \sigma\bigg), \quad B = \sqrt{-\Delta_{g_0} + \bigg(\frac{n-1}{2}\bigg)^2}$$

is the conformal fractional operator of order σ of (S^n, g_0). It can be seen, via the stereographic projection, as the pull back operator of the fractional Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$. The problem of finding conformal metrics g with a fractional

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Q-curvature $Q_\sigma = K$ on S^n is then reduced to the solvability of the following fractional Nirenberg equation:

$$(P_\sigma) : \begin{cases} P_\sigma^{g_0} u = c(n,\sigma) K u^{\frac{n+2\sigma}{n-2\sigma}}, \\ u > 0 \quad \text{on } S^n, \end{cases}$$

$\sigma \in (0, \frac{n}{2})$. When $\sigma = 1$, (P_σ) corresponds to the well-known scalar curvature problem (or Nirenberg problem). When $\sigma = 2$, it is the Paneitz curvature problem. When $\sigma \in \mathbb{N}$, $\sigma \geq 3$, it is the higher-order Nirenberg problem related to the so-called GJMS operators. For these topics, we refer to [7, 9, 10, 11, 17, 18, 20, 21, 28, 29, 35] and references therein.

For $\sigma \notin \mathbb{N}$, problem (P_σ) has been the subject of several works after the seminal papers [13], [14] and [22]. We refer to [2, 3, 15, 19, 24, 25] for $\sigma \in (0,1)$, [36, 26] for $\sigma \in (0, \frac{n}{2})$ and [37, 38] for $\sigma \geq \frac{n}{2}$. Regarding some recent results on related fractional problems, we refer to [5, 4, 12, 31, 34, 33, 27].

In [36] Abdullah Sharaf and Chtioui studied problem (P_σ), $\sigma \in (0, \frac{n}{2})$. Their main hypothesis is the so-called non-degeneracy condition. Namely,

(nd): K is a C^2-function on S^n having only non-degenerate critical points and satisfies

$$\Delta K(y) \neq 0 \quad \text{if } \nabla K(y) = 0.$$

Under the above hypothesis, Abdullah Sharaf and Chtioui studied the lack of compactness of (P_σ) by characterizing the critical points at infinity of the problem and proved existence results through Euler–Hopf-type formulas.

Convinced that the (nd)-condition would exclude interesting classes of functions K and aiming to include a larger class of prescribed functions in the study of problem (P_σ), we opted in the present work for the following β-flatness hypothesis. Let

$$\mathcal{K} = \{y \in S^n,\, \nabla K(y) = 0\}.$$

$(f)_\beta$: K is a C^1 function on S^n such that around any $y \in \mathcal{K}$, K is expanded as

$$K(x) = K(y) + \sum_{k=1}^{n} b_k(y) \big|(x-y)_k\big|^{\beta(y)} + o\big(\|x-y\|^{\beta(y)}\big)$$

in some geodesic normal coordinate system. Here, $\beta(y) = \beta > 1$, $b_k(y) = b_k \in \mathbb{R} \setminus \{0\}$ for all $k = 1, \dots, n$, $\sum_{k=1}^n b_k(y) \neq 0$ and

$$\frac{1}{\beta^*(y)} + \frac{1}{\beta^*(y')} > \frac{2}{n-2\sigma} \quad \forall y \neq y' \in \mathcal{K},$$

where $\beta^*(z) = \min(\beta(z), n)$.

It is easy to see that for $\sigma \in (0, \frac{n}{2} - 1)$, the (nd)-condition coincides with the $(f)_\beta$-condition with $\beta(y) = 2$ for any $y \in \mathcal{K}$. Let

$$\Sigma = \Big\{u \in H^\sigma(S^n),\, \|u\|^2 = \int_{S^n} P_\sigma^{g_0} u\, u\, dvg_0 = 1\Big\},$$

where $H^\sigma(S^n)$ is the fractional Sobolev space of order σ. It is straightforward to see that the solutions of problem (P_σ) correspond to the positive critical points of the functional

$$J(u) = \frac{1}{\left(\int_{S^n} K u^{\frac{2n}{n-2\sigma}} dv g_0 \right)^{\frac{n-2\sigma}{n}}}, \quad u \in \Sigma.$$

Due to the compactness defect of the fractional Sobolev embedding $H^\sigma(S^n) \hookrightarrow L^{\frac{2n}{n-2\sigma}}(S^n)$, J fails to satisfy the Palais–Smale condition. It is the occurrence of the critical points at infinity; that are the ends of the non-precompact flow lines of the gradient of J; see [6, Definition 09]. The characterization of the critical points at infinity leads to identifying the locations in the variational space where the lack of compactness of the problem occurs and plays a fundamental role in the existence and non-existence results of problem (P_σ).

Let $a \in S^n$ and $\lambda > 0$. We define

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{\frac{n-2\sigma}{2}}}{\left(1 + \frac{1}{2}(\lambda^2 - 1)\big(1 - \cos d(a,x)\big)\right)^{\frac{n-2\sigma}{2}}},$$

where c_0 is a fixed positive constant. Following [24], $\delta_{(a,\lambda)}$, $a \in S^n$, $\lambda > 0$, are the only solutions of

$$P_\sigma^{g_0} u = u^{\frac{n+2\sigma}{n-2\sigma}}, \quad u > 0 \text{ on } S^n.$$

We shall prove the following result prescribing the lack of compactness of the problem. Let

$$\Sigma^+ = \big\{ u \in \Sigma,\, u \geq 0 \text{ on } S^n \big\},$$

and

$$\mathcal{K}^+ = \big\{ y \in \mathcal{K},\, -\textstyle\sum_{k=1}^n b_k > 0 \big\}.$$

For any $y \in \mathcal{K}$, we set

$$\widetilde{i}(y) = \#\big\{ b_k(y),\, 1 \leq k \leq n, \text{ s.t. } b_k(y) < 0 \big\}.$$

Theorem 1.1. *Let $K : S^n \to \mathbb{R}$ be a positive function satisfying the $(f)_\beta$-condition. Assume that J has no critical point in Σ^+. There exists a positive constant α_0 such that if $1 < \beta < n + \alpha_0$, the critical points at infinity of J in Σ^+ are*

$$(y_1, \dots, y_p)_\infty := \sum_{i=1}^p \frac{1}{K(y_i)^{\frac{n-2\sigma}{n}}} \delta_{(y_i,\infty)},$$

where $y_i \in \mathcal{K}^+$ $\forall\, i = 1, \dots, p$ and $y_i \neq y_j$ $\forall\, 1 \leq i \neq j \leq p$. Moreover, the index of J at $(y_1, \dots, y_p)_\infty$ is equal to $i(y_1, \dots, y_p)_\infty = p - 1 + \sum_{j=1}^p n - \widetilde{i}(y_j)$.

In what follows, we denote by C^∞ the set of all the critical points at infinity of problem (P_σ). Under the assumptions of Theorem 1.1,

$$C^\infty = \Big\{(y_1,\dots,y_p)_\infty := \sum_{i=1}^{p} \frac{1}{K(y_i)^{\frac{n-2\sigma}{n}}}\delta_{(y_i,\infty)},\ y_i \in \mathcal{K}^+\ \forall\, i=1,\dots,p \\ \text{and } y_i \neq y_j\ \forall\, 1\le i\neq j\le p\Big\}.$$

If $(y_1,\dots,y_p)_\infty \in C^\infty$, let $W_u^\infty(y_1,\dots,y_p)_\infty$ and $W_s^\infty(y_1,\dots,y_p)_\infty$ designate its unstable and stable manifolds respectively. According to [7], we have

$$\dim W_u^\infty(y_1,\dots,y_p)_\infty = \operatorname{codim} W_s^\infty(y_1,\dots,y_p)_\infty = i(y_1,\dots,y_p)_\infty.$$

In order to state our first existence result, we need to introduce the following notations. Let $k_0 \in \mathbb{N}$ and let $N_{k_0}^\infty$ be a subset of

$$C_{\le k_0}^\infty = \big\{(y_1,\dots,y_p)_\infty \in C^\infty \text{ s.t. } i(y_1,\dots,y_p)_\infty \le k_0\big\}.$$

Define

$$W_u^\infty(N_{k_0}^\infty) = \bigcup_{(y_1,\dots,y_p)_\infty \in N_{k_0}^\infty} W_u^\infty(y_1,\dots,y_p)_\infty.$$

$W_u^\infty(N_{k_0}^\infty)$ defines a stratified set of top dimension less than or equal to k_0. To simplify, we assume that it is equal to k_0. Since Σ^+ is a contractible space and since $W_u^\infty(N_{k_0}^\infty) \subset \Sigma^+$, there exists at least a contraction $\theta\big(W_u^\infty(N_{k_0}^\infty)\big)$ of $W_u^\infty(N_{k_0}^\infty)$ in Σ^+. We then have

Theorem 1.2. *Let $K : S^n \to \mathbb{R}$ be a positive function satisfying the $(f)_\beta$-condition, $1 < \beta < n + \alpha_0$. If there exist an integer $k_0 \in \mathbb{N}$ and a subset $N_{k_0}^\infty \subset C_{\le k_0}^\infty$ such that*

(a) $\displaystyle\sum_{(y_1,\dots,y_p)_\infty \in N_{k_0}^\infty} (-1)^{i(y_1,\dots,y_p)_\infty} \neq 1,$

(b) $\theta\big(W_u^\infty(N_{k_0}^\infty)\big) \bigcap W_s^\infty(y_1,\dots,y_p)_\infty = \emptyset\ \forall (y_1,\dots,y_p)_\infty \in C_{\le k_0+1}^\infty \backslash N_{k_0}^\infty,$

then problem (P_σ) has at least a solution.

As an application of the above theorem, we state the following existence result. Let y_0 and z_0 be two points in S^n such that $K(y_0) = \max_{S^n} K(x)$ and $K(z_0) = \min_{S^n} K(x)$. It is easy to see that, under the $(f)_\beta$-condition, $y_0 \in \mathcal{K}^+$ and $z_0 \in \mathcal{K}\backslash\mathcal{K}^+$.

Theorem 1.3. *Let $K : S^n \to \mathbb{R}$ be a positive function satisfying the $(f)_\beta$-condition, $\beta \in (1, n+\alpha_0)$, such that $\frac{K(y_0)}{K(z_0)} < 2^{\frac{1}{n-2\sigma}}$. If there exists $k_0 \in \mathbb{N}$ such that*

(a') $\displaystyle\sum_{y\in\mathcal{K}^+,\, n-\widetilde{i}(y)\le k_0} (-1)^{n-\widetilde{i}(y)} \neq 1;$

(b') *for any $y \in \mathcal{K}^+$ we have $n - \widetilde{i}(y) \neq k_0 + 1$,*

then (P_σ) admits a solution.

It is easy to see that any integer $k_0 \ge n$ satisfies condition (b'). Therefore the following existence result is an immediate consequence of Theorem 1.3.

Theorem 1.4. *Assume that K satisfies the $(f)_\beta$-condition, $1 < \beta < n + \alpha_0$. If*

$$\sum_{y \in \mathcal{K}^+} (-1)^{n - \widetilde{i(y)}} \neq 1,$$

then (P_σ) has a solution provided $\frac{K(y_0)}{K(z_0)} < 2^{\frac{1}{n-2\sigma}}$.

It should be noted that Theorem 1.4 covers the perturbation theorem of T. Jin, Y. Li and J. Xiong [24] in two ways. First, the flatness order of the prescribed function exceeds the dimension n of its domain. Second, the closeness rate to a positive constant is estimated in our theorem.

Our aim in what follows is to remove condition (b') in the existence result of Theorem 1.3. This leads to another kind of existence result. Nevertheless an additional condition concerning the closeness of $K(y_0)$ with respect to $K(z_0)$ will be imposed.

Theorem 1.5. *Let $K : S^n \to \mathbb{R}$ be a positive function satisfying the $(f)_\beta$-condition, $\beta \in (1, n + \alpha_0)$, such that $\frac{K(y_0)}{K(z_0)} < \left(\frac{3}{2}\right)^{\frac{1}{n-2\sigma}}$. If*

$$\mathcal{K}^+ \backslash \{y_0\} \neq \emptyset,$$

then (P_σ) has a solution.

Our method is based on the critical points at infinity theory of A. Bahri [6]. We follow closely the ideas developed in [11] and [32] where the prescribed scalar curvature problem was studied using some topological tools.

In Section 2 we recall some preliminaries related to the associated variational structure. In Section 3 we perform an asymptotic analysis on the gradient field of J under condition $(f)_\beta$, $\beta \in (1, \infty)$, and we construct a suitable pseudo-gradient allowing us to prove Theorem 1.1. The proof of the existence results will be performed in Section 4.

2. Preliminaries

We start this section by characterizing the sequences of Σ^+ which violate the Palais–Smale condition for the functional J. For $p \in \mathbb{N}$, and $\varepsilon > 0$ small enough, we set

$$\begin{aligned} V(p,\varepsilon) = \Big\{ u \in \Sigma,\ \exists\, \alpha_1, \dots, \alpha_p > 0,\ \exists\, \lambda_1, \dots, \lambda_p > \varepsilon^{-1} \text{ and } a_1, \dots, a_p \in S^n \text{ s.t.} \\ \Big\| u - \textstyle\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} \Big\| < \varepsilon, \\ \Big| J(u)^{\frac{n}{n-2\sigma}} \alpha_i^{\frac{4}{n-2\sigma}} K(a_i) J(u)^{\frac{n}{n-2\sigma}} - 1 \Big| < \varepsilon \ \forall\, 1 \le i \le p, \\ \text{and } \varepsilon_{ij} < \varepsilon \ \forall\, i \neq j \Big\}, \end{aligned}$$

where

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{-\frac{n-2\sigma}{2}}.$$

Following [8] and [30], we have

Proposition 2.1. *Let (u_k) be a non-precompact sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ tends to zero. There exist $p \in \mathbb{N}$, a positive sequence (ε_k) tending to zero and a subsequence of (u_k), denoted again by (u_k), such that $u_k \in V(p, \varepsilon_k)\ \forall k \geq 1$.*

A parametrization of $V(p, \varepsilon)$ is given in the following proposition.

Proposition 2.2 ([7])**.** *For any $u \in V(p, \varepsilon)$, the minimization problem*

$$\min_{\alpha_i > 0,\, \lambda_i > 0,\, a_i \in S^n} \Big\| u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} \Big\|$$

has a unique solution $(\bar{\alpha}, \bar{\lambda}, \bar{a})$. Moreover, $v := u - \sum_{i=1}^{p} \bar{\alpha}_i \delta_{(\bar{a}_i, \bar{\lambda}_i)}$ satisfies the orthogonality condition

$$(V_0):\ \langle v, \varphi \rangle = 0 \quad \forall \varphi \in \left\{ \delta_{(a_i, \lambda_i)}, \frac{\partial \delta_{(a_i, \lambda_i)}}{\partial \lambda_i}, \frac{\partial \delta_{(a_i, \lambda_i)}}{\partial a_i},\ i = 1, \ldots, p \right\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product of $H^\sigma(S^n)$.

Next we deal with the v-part of u. Following [2] and [8] we have

Proposition 2.3. *For any $\sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$, the minimization problem*

$$\min_{v \text{ satisfies } (V_0)} J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + v \Big)$$

has a unique solution $\overline{v} = \overline{v}(\alpha, a, \lambda)$. In addition, there exists a change of variables $V = v - \overline{v}$ such that the following expansion holds:

$$J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + v \Big) = J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + \overline{v} \Big) + \|V\|^2.$$

Moreover, under the $(f)_\beta$-condition we have following estimate:

$$\|\overline{v}\| \leq c \sum_{i=1}^{p} \left[\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{\frac{n+2\sigma}{2n}}}{\lambda_i^{\frac{n+2\sigma}{2}}} \right] + c \begin{cases} \displaystyle\sum_{i \neq j} \varepsilon_{i\,j}^{\frac{n+2\sigma}{2(n-2\sigma)}} \Big(\log \varepsilon_{i\,j}^{-1} \Big)^{\frac{n+2\sigma}{2n}}, & n \geq 3, \\ \displaystyle\sum_{i \neq j} \varepsilon_{i\,j} \Big(\log \varepsilon_{i\,j}^{-1} \Big)^{\frac{n-2\sigma}{n}}, & n = 2. \end{cases}$$

We now define a critical point at infinity [6].

Definition 2.4. A *critical point at infinity* of the functional J is an end of a non-precompact flow line $u(s)$ of the gradient vector field $(-\partial J)$. According to Propositions 2.1 and 2.2, $u(s)$ can be described at infinity in the form

$$u(s) = \sum_{i=1}^{p} \alpha_i(s)\, \delta_{(a_i(s), \lambda_i(s))} + v(s),$$

where $\|v(s)\| \to 0$ and $\lambda_i(s) \to \infty \; \forall \, i = 1, \dots, p$. If we set $y_i := \lim_{s \to +\infty} a_i(s)$ and $\alpha_i = \lim_{s \to +\infty} \alpha_i(s) \; \forall \, i = 1, \dots, p$, then

$$(y_1, \dots, y_p)_\infty = \sum_{i=1}^{p} \alpha_i \delta_{(y_i, \infty)}$$

denotes a critical point at infinity.

3. Proof of Theorem 1.1

In this section we characterize the critical points at infinity of problem (P_σ), $\sigma \in (0, \frac{n}{2})$, under condition $(f)_\beta$. We construct in $V(p, \varepsilon)$, $p \geq 1$, a suitable pseudo-gradient W for which J decreases and the Palais–Smale condition is satisfied along its flow lines as long as these flow lines do not enter the neighborhood of the critical points $y_1, \dots, y_p \in \mathcal{K}^+$ such that $y_i \neq y_j \; \forall \, 1 \leq i \neq j \leq p$. We shall prove the following result.

Theorem 3.1. *Let $K : S^n \to \mathbb{R}$ be a positive function satisfying the $(f)_\beta$-condition. There exists $\alpha_0 > 0$ such that if $\beta \in (1, n + \alpha_0)$, the following holds: For any $p \geq 1$ there exists a bounded pseudo-gradient W in $V(p, \varepsilon)$, $\varepsilon > 0$ small enough, such that*

(i) $\Big\langle \partial J(u), W(u) \Big\rangle \leq -c\Bigg(\sum_{i=1}^{p} \bigg(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \bigg) + \sum_{i \neq j} \varepsilon_{ij} \Bigg)$,

(ii) $\Big\langle \partial J(u + \overline{v}), W(u) + \frac{\partial \overline{v}}{\partial(\alpha, a, \lambda)} W(u) \Big\rangle \leq -c\Bigg(\sum_{i=1}^{p} \bigg(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \bigg) + \sum_{i \neq j} \varepsilon_{ij} \Bigg)$, *for any $u \in V(p, \varepsilon)$.*

Moreover,

(iii) *For any $i = 1, \dots, p$, $\max_{s \geq 0} \lambda_i(s)$ is bounded unless $a_i(s) \to y_i \in \mathcal{K}^+ \; \forall \, i = 1, \dots, p$ with $y_i \neq y_j \; \forall \, 1 \leq i \neq j \leq p$. In this case all the parameters $\lambda_i(s)$ increase and tend to ∞.*

The first step in the construction of the required pseudo-gradient W is to describe the variation of the energy functional J with respect to the parameters λ_i and a_i, $i = 1, \dots, p$, of $V(p, \varepsilon)$.

Proposition 3.2. *Let K be a positive function satisfying the $(f)_\beta$-condition. There exists $\alpha_0 > 0$ such that if $\beta(y) \in (1, n + \alpha_0)$ for any $y \in \mathcal{K}$, the following holds:*

For any $u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in V(p,\varepsilon)$ *such that* $a_i \in B(y_i,\rho)$, $y_i \in \mathcal{K}$, *we have*

$$\begin{aligned}
&\Big\langle \partial J(u), \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \Big\rangle \\
&\quad = J(u) \tilde{c}_i \frac{\alpha_i^2 (n-2\sigma)}{n K(a_i)} \sum_{k=1}^{n} b_k(y_i) \begin{cases} \dfrac{1}{\lambda_i^{\beta^*(y_i)}} (1+o(1)) & \textit{if } \beta(y_i) \neq n, \\ \dfrac{\log \lambda_i}{\lambda_i^{\beta(y_i)}} (1+o(1)) & \textit{if } \beta(y_i) = n \end{cases} \\
&\qquad - c_1 J(u) \sum_{j \neq i} \alpha_i \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\Big(|a_i - y_i|^{\beta(y_i)} \Big) + O\Big(\frac{|a_i - y_i|^{\beta(y_i)-1}}{\lambda_i} \Big) \\
&\qquad + \sum_{j \neq i} o\big(\varepsilon_{ij}\big),
\end{aligned}$$

where

$$c_1 = \int_{\mathbb{R}^n} \frac{dz}{(1+|z|^2)^{\frac{n+2\sigma}{2}}}$$

and

$$\tilde{c}_i = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \frac{|t_1|^{\beta(y_i)} (|t|^2 - 1)}{(1+|t|^2)^{n+1}} \, dt & \textit{if } \beta(y_i) < n, \\ 1 & \textit{if } \beta(y_i) = n, \\ \dfrac{\rho^{\beta(y_i)-n} w_{n-1}}{\beta(y_i) - n} & \textit{if } \beta(y_i) > n. \end{cases}$$

Proof. Using a computation like the one in [1, Proposition 3.2], we have

$$\begin{aligned}
\Big\langle \partial J(u), \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \Big\rangle = {} & -2 J(u)^{\frac{2n-2\sigma}{n-2\sigma}} \alpha_i^{\frac{2n}{n-2\sigma}} \int_{S^n} K(x) \delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \, dv_{g_0} \\
& - 2 c_1 J(u) \sum_{j \neq i} \alpha_i \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}. \qquad (3.1)
\end{aligned}$$

Let π_N be the stereographic projection with respect to the north pole N of S^n. To simplify, we will identify any x of S^n with its projection in $\mathbb{R}^n$. Also, we identify the functional K with its composition with π_N. By an elementary calculation we have

$$\delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} = \frac{n-2\sigma}{2} \frac{\big(1 - \lambda_i^2 |x-a_i|^2\big) \lambda_i^n}{\big(1+\lambda_i^2 |x-a_i|^2\big)^{n+1}}.$$

It follows that

$$\begin{aligned}
I &:= \int_{\mathbb{R}^n} K(x) \delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \, dx \\
&= \int_{\mathbb{R}^n} \big(K(x) - K(y_i)\big) \delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \, dx,
\end{aligned}$$

since

$$\int_{\mathbb{R}^n} \delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}\, dx = 0.$$

Let $\rho > 0$ be small enough. In $B(a_i, \rho_0)^c$ we have

$$\begin{aligned}
\int_{B(a_i,\rho)^c} \big(K(x) - K(y_i)\big) \delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}\, dx \\
&\leq c(K) \int_{B(a_i,\rho)^c} \frac{\big(1 - \lambda_i^2 |x-a_i|^2\big) \lambda_i^n \, dx}{\big(1 + \lambda_i^2 |x-a_i|^2\big)^{n+1}} \\
&\leq c(K) \int_{\lambda_i \rho}^{\infty} \frac{r^{n-1} |1 - r^2|\, dr}{(1+r^2)^{n+1}} \leq O\Big(\frac{1}{\lambda_i^n}\Big).
\end{aligned} \tag{3.2}$$

In $B(a_i, \rho)$, we use $(f)_\beta$ expansion. We obtain

$$\begin{aligned}
\int_{B(a_i,\rho_0)} & \big(K(x) - K(y_i)\big) \delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}\, dx \\
&= \frac{n-2\sigma}{2} \sum_{k=1}^{n} b_k(y_i) \int_{B(a_i,\rho)} \big|(x-y_i)_k\big|^{\beta(y_i)} \frac{\big(1 - \lambda_i^2|x-a_i|^2\big)\lambda_i^n \, dx}{\big(1+\lambda_i^2|x-a_i|^2\big)^{n+1}} \\
&\quad + o\bigg(\int_{B(a_i,\rho)} \big|(x-y_i)\big|^{\beta(y_i)} \frac{\big|1 - \lambda_i^2|x-a_i|^2\big| \lambda_i^n \, dx}{\big(1+\lambda_i^2|x-a_i|^2\big)^{n+1}} \bigg) \\
&= \frac{n-2\sigma}{2} \sum_{k=1}^{n} b_k(y_i) \int_{B(0,\lambda_i\rho)} \frac{\big|t + \lambda_i(a_i - y_i)_k\big|^{\beta(y_i)}}{\lambda_i^{\beta(y_i)}} \frac{1 - |t|^2}{\big(1+|t|^2\big)^{n+1}}\, dt \\
&\quad + o\bigg(\int_{B(0,\lambda_i\rho)} \frac{\big|t + \lambda_i(a_i - y_i)_k\big|^{\beta(y_i)}}{\lambda_i^{\beta(y_i)}} \frac{|1 - |t|^2|}{\big(1+|t|^2\big)^{n+1}}\, dt \bigg),
\end{aligned}$$

by setting $t = \lambda_i(a_i - y_i)$. Observe that

$$\begin{aligned}
\int_{B(0,\lambda_i\rho)} & \frac{\big|t + \lambda_i(a_i - y_i)_k\big|^{\beta(y_i)}}{\lambda_i^{\beta(y_i)}} \frac{1 - |t|^2}{\big(1+|t|^2\big)^{n+1}}\, dt \\
&= \int_{B(0,\lambda_i\rho)} \frac{\big|t_k\big|^{\beta(y_i)} \big(1 - |t|^2\big)}{\lambda_i^{\beta(y_i)} \big(1+|t|^2\big)^{n+1}}\, dt \\
&\quad + O\bigg(\int_{B(0,\lambda_i\rho)} \frac{|a_i - y_i| |t|^{\beta(y_i)-1} |1 - |t|^2|\, dt}{\lambda_i^{\beta(y_i)-1} \big(1+|t|^2\big)^{n+1}} \bigg) + O\Big(\big|a_i - y_i\big|^{\beta(y_i)} \Big). \\
&= -\tilde{c}_i (1 + o(1)) \begin{cases} \dfrac{1}{\lambda_i^{\beta^*(y_i)}} & \text{if } \beta(y_i) \neq n \\[2ex] \dfrac{\log \lambda_i}{\lambda_i^{\beta(y_i)}} & \text{if } \beta(y_i) = n \end{cases} + O\Big(|a_i - y_i|^{\beta(y_i)} \Big),
\end{aligned}$$

where

$$\tilde{c}_i = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \frac{|t_1|^{\beta(y_i)}(|t|^2-1)}{(1+|t|^2)^{n+1}}\,dt & \text{if } \beta(y_i) < n, \\ 1 & \text{if } \beta(y_i) = n, \\ \dfrac{\rho^{\beta(y_i)-n} w_{n-1}}{\beta(y_i)-n} & \text{if } \beta(y_i) > n. \end{cases}$$

It follows from the above estimate and (3.2) that

$$I = -\frac{n-2\sigma}{2}\,\tilde{c}_i\,(1+o(1))\Big(\sum_{k=1}^{n} b_k(y_i)\Big) \begin{cases} \dfrac{1}{\lambda_i^{\beta^*(y_i)}} & \text{if } \beta(y_i) \neq n \\ \dfrac{\log \lambda_i}{\lambda_i^{\beta^*(y_i)}} & \text{if } \beta(y_i) = n \end{cases}$$
$$+\, O\Big(|a_i - y_i|^{\beta(y_i)}\Big) + O\Big(\frac{1}{\lambda_i^n}\Big).$$

Observe that in the above expansion, the remainder term $O\Big(\frac{1}{\lambda_i^n}\Big)$ is very small with respect to $\frac{1}{\lambda_i^{\beta(y_i)}}$ if $\beta(y_i) < n$ and $\frac{\log \lambda_i}{\lambda_i^{\beta(y_i)}}$ if $\beta(y_i) = n$. However, if $\beta(y_i) > n$, $O\Big(\frac{1}{\lambda_i^n}\Big)$ is of the same size than $\frac{1}{\lambda_i^{\beta^*(y_i)}}$. This presents the difficulty of studying the problem for any flatness order $\beta(y) > n$, since the sign of the leading term in the above expansion is unknown. Nevertheless, for $n < \beta(y_i) < n + \alpha_0$, where α_0 is a small positive constant, the remainder term $O\Big(\frac{1}{\lambda_i^n}\Big)$ is very small with respect to $\frac{\rho^{n-1} w_{n-1}}{\beta(y_i)-n}\frac{1}{\lambda_i^{\beta^*(y_i)}}$. In this case the latest expansion will be reduced to

$$I = -\frac{n-2\sigma}{2}\,\tilde{c}_i\,(1+o(1))\Big(\sum_{k=1}^{n} b_k(y_i)\Big) \begin{cases} \dfrac{1}{\lambda_i^{\beta^*(y_i)}} & \text{if } \beta(y_i) \neq n \\ \dfrac{\log \lambda_i}{\lambda_i^{\beta^*(y_i)}} & \text{if } \beta(y_i) = n \end{cases}$$
$$+\, O\Big(|a_i - y_i|^{\beta(y_i)}\Big). \tag{3.3}$$

The expansion of Proposition 3.2 follows from (3.1), (3.3) and the relation

$$\alpha_i^{\frac{4\sigma}{n-2\sigma}} K(a_i) J(u)^{\frac{n}{n-2\sigma}} = 1 + o(1) \quad \forall\, i = 1, \ldots, p. \qquad \square$$

Proposition 3.3. *Assume that K is positive on S^n and satisfies the $(f)_\beta$-condition. For any $u = \sum_{i=1}^{p} \alpha_i \widehat{\delta}_{(a_i,\lambda_i)} \in V(p,\varepsilon)$ such that $a_i \in B(y_i,\rho)$, $y_i \in \mathcal{K}$, we have*

$$\Big\langle \partial J(u), \frac{\alpha_i}{\lambda_i}\frac{\partial \widehat{\delta}_{(a_i,\lambda_i)}}{\partial (a_i)_k} \Big\rangle$$
$$= -c_2 \alpha_i^{\frac{2n}{n-2\sigma}} J(u)^{\frac{2n-2\sigma}{n-2\sigma}} \beta(y_i) b_k(y_i)\,\mathrm{sign}\big[(a_i-y_i)_k\big]\frac{|a_i-y_i|^{\beta(y_i)-1}}{\lambda_i} + R,$$

where $(a_i)_k$, $k = 1, \dots, n$, *denotes the* k*-th component of* a_i *in the geodesic coordinate system,*

$$c_2 = \frac{n - 2\sigma}{n} \int_{\mathbb{R}^n} \frac{|t|^2\,dt}{(1 + |t|^2)^{n+1}},$$

and

$$R = o\Big(\frac{|a_i - y_i|^{\beta(y_i)-1}}{\lambda_i}\Big) + \sum_{l=2}^{E(\beta^*(y_i))} O\Big(\frac{|a_i - y_i|^{\beta(y_i)-l}}{\lambda_i^l}\Big) + \sum_{j\neq i} O(\varepsilon_{ij})$$
$$+ \begin{cases} O\Big(\dfrac{1}{\lambda_i^{\beta(y_i)}}\Big) & \textit{if } \beta(y_i) \le n, \\ o\Big(\dfrac{1}{\lambda_i^{\gamma}}\Big), \gamma \in \big(n, \min(n+1, \beta(y_i))\big) & \textit{if } \beta(y_i) > n. \end{cases}$$

Proof. Following [1, Proposition 3.3], we have

$$\Big\langle \partial J(u), \frac{\alpha_i}{\lambda_i} \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial (a_i)_k} \Big\rangle = -2J(u)^{\frac{2n-2\sigma}{n-2\sigma}} \alpha_i^{\frac{2n}{n-2\sigma}} \int_{S^n} K(x) \delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \frac{1}{\lambda_i} \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial (a_i)_k}\, dv_{g_0}$$
$$+ \sum_{j\neq i} O(\varepsilon_{ij}).$$

Performing a stereographic projection, for any $x \in \mathbb{R}^n$ we have

$$\delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \frac{1}{\lambda_i} \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial (a_i)_k} = (n - 2\sigma) \frac{\lambda_i^{n+1} (x - a_i)_k}{\big(1 + \lambda_i^2 |x - a_i|^2\big)^{n+1}}.$$

It follows that

$$I := \int_{\mathbb{R}^n} K(x) \delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial (a_i)_k}\, dx$$
$$\int_{B(a_i,\rho)} \big(K(x) - K(a_i)\big) \delta_{(a_i,\lambda_i)}^{\frac{n+2\sigma}{n-2\sigma}} \frac{1}{\lambda_i} \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial (a_i)_k} + O\Big(\frac{1}{\lambda_i^{n+1}}\Big).$$

By a Taylor expansion up to order $E\big(\beta^*(y_i)\big)$, we have

$$K(x) - K(a_i) = \sum_{l=1}^{E(\beta^*(y_i))} \frac{D^l K(a_i)(x - a_i)^l}{l!}$$
$$+ \begin{cases} O\big(|x - a_i|^{\beta(y_i)}\big) & \text{if } \beta(y_i) \le n, \\ o\big(|x - a_i|^{\gamma}\big),\ \gamma \in \big(n, \beta(y_i)\big) & \text{if } \beta(y_i) > n. \end{cases}$$

Therefore, by setting $t = \lambda_i(x - a_i)$, we get

$$I = (n-2\sigma)\sum_{l=1}^{E(\beta^*(y_i))}\int_{B(0,\lambda_i\rho)}\frac{D^lK(a_i)(t)^l}{l!\lambda_i^l}\frac{t_k}{(1+|t|^2)^{n+1}}\,dt + \begin{cases} O\Big(\dfrac{1}{\lambda_i^{\beta(y_i)}}\Big) & \text{if } \beta(y_i) \le n \\ o\Big(\dfrac{1}{\lambda_i^{\gamma}}\Big) & \text{if } \beta(y_i) > n \end{cases} + O\Big(\frac{1}{\lambda_i^{n+1}}\Big)$$

$$= (n-2\sigma)\int_{B(0,\lambda_i\rho)}\frac{DK(a_i)(t)}{\lambda_i}\frac{t_k}{(1+|t|^2)^{n+1}}\,dt + \sum_{l=2}^{E(\beta^*(y_i))} O\Big(\frac{|x-a_i|^{\beta(y_i)-l}}{\lambda_i^l}\Big) + \begin{cases} O\Big(\dfrac{1}{\lambda_i^{\beta(y_i)}}\Big) & \text{if } \beta(y_i) \le n \\ o\Big(\dfrac{1}{\lambda_i^{\gamma}}\Big) & \text{if } \beta(y_i) > n \end{cases} + O\Big(\frac{1}{\lambda_i^{n+1}}\Big).$$

Observe that

$$\begin{aligned}\int_{B(0,\lambda_i\rho)}\frac{t_kDK(a_i)(t)}{\lambda_i(1+|t|^2)^{n+1}}\,dt &= \sum_{j=1}^{n}\frac{\frac{\partial K}{\partial x_j}(a_i)}{\lambda_i}\int_{B(0,\lambda_i\rho)}\frac{t_kt_j\,dt}{(1+|t|^2)^{n+1}} \\ &= \frac{\frac{\partial K}{\partial x_k}(a_i)}{\lambda_i}\int_{B(0,\lambda_i\rho)}\frac{t_k^2\,dt}{(1+|t|^2)^{n+1}} \\ &= \frac{1}{n}\frac{\frac{\partial K}{\partial x_k}(a_i)}{\lambda_i}\bigg(\int_{\mathbb{R}^n}\frac{|t|^2}{(1+|t|^2)^{n+1}}\,dt + O\Big(\frac{1}{\lambda_i^n}\Big)\bigg).\end{aligned}$$

Moreover, by $(f)_\beta$ expansion we have

$$\frac{\partial K}{\partial x_k}(a_i) = \beta(y_i)b_k(y_i)\big[\operatorname{sign}(a_i-y_i)_k\big]\big|(a_i-y_i)_k\big|^{\beta(y_i)-1} + o\Big(\big|a_i-y_i\big|^{\beta(y_i)-1}\Big).$$

It follows that

$$I = c_2\beta(y_i)b_k(y_i)\operatorname{sign}\big[(a_i-y_i)_k\big]\frac{|a_i-y_i|^{\beta(y_i)-1}}{\lambda_i} + o\Big(\big|a_i-y_i\big|^{\beta(y_i)-1}\Big) + \sum_{l=2}^{E(\beta^*(y_i))} O\Big(\frac{|a_i-y_i|^{\beta(y_i)-l}}{\lambda_i^l}\Big) + \begin{cases} O\Big(\dfrac{1}{\lambda_i^{\beta(y_i)}}\Big) & \text{if } \beta(y_i) \le n \\ o\Big(\dfrac{1}{\lambda_i^{\gamma}}\Big) & \text{if } \beta(y_i) > n \end{cases} + O\Big(\frac{1}{\lambda_i^{n+1}}\Big).$$

This concludes the proof of Proposition 3.3. □

In order to prove Theorem 3.1 we introduce the following subsets of $V(p,\varepsilon)$, $p \geq 1$. Let

$$\widetilde{V}(p,\varepsilon) = \Big\{ u = \sum_{i=1}^{p} \alpha_i \, \delta_{(a_i,\lambda_i)} \in V(p,\varepsilon) \text{ s.t. } a_i \in B(y_i,\rho), \\ y_i \in \mathcal{K} \; \forall\, i = 1, \dots, p \text{ and } y_i \neq y_j \; \forall\, 1 \leq i \neq j \leq p \Big\}.$$

In [2] it is proved that there is no critical point at infinity in $V(p,\varepsilon) \setminus \widetilde{V}(p,\varepsilon)$. More precisely:

Proposition 3.4 ([2, Section 3])**.** *There exists a bounded pseudo-gradient $W \in V(p,\varepsilon) \setminus \widetilde{V}(p,\varepsilon)$ satisfying inequalities* (i) *and* (ii) *of Theorem* 3.1 *for any $u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in V(p,\varepsilon) \setminus \widetilde{V}(p,\varepsilon)$. Moreover,*

(iii') $\max_{1 \leq i \leq p} \lambda_i(s)$ *remains bounded as long as the associated flow line $u(s) = \sum_{i=1}^{p} \alpha_i(s) \delta_{(a_i(s),\lambda_i(s))}$ remains in $V(p,\varepsilon) \setminus \widetilde{V}(p,\varepsilon)$.*

The next proposition describes the concentration phenomenon in $\widetilde{V}(p,\varepsilon)$.

Proposition 3.5. *Under the assumption that K is positive on S^n and satisfies the $(f)_\beta$-condition, $\beta \in (1, n+\alpha_0)$, there exists a bounded pseudo-gradient W in $\widetilde{V}(p,\varepsilon)$ such that the assertions* (i), (ii) *and* (iii) *of Theorem* 3.1 *hold.*

Proof. Let γ_0 be a small positive constant. For any $y \in \mathcal{K}$ and for any $\lambda > \varepsilon^{-1}$, we define a neighborhood $V_\lambda(y)$ of y as follows:

$$V_\lambda(y) = \Big\{ a \in S^n, \; \big|a - y\big|^{\beta(y)} < \frac{\gamma_0}{\lambda^{\beta^*(y)}} \Big\}.$$

We divide $\widetilde{V}(p,\varepsilon)$ into the following three subsets:

$$\widetilde{V}_1(p,\varepsilon) = \Big\{ u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in \widetilde{V}(p,\varepsilon) \text{ s.t. } a_i \in V_{\lambda_i}(y_i), \, y_i \in \mathcal{K}^+ \; \forall\, i = 1, \dots, p \Big\},$$

$$\widetilde{V}_2(p,\varepsilon) = \Big\{ u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in \widetilde{V}(p,\varepsilon) \text{ s.t. } a_i \in V_{\lambda_i}(y_i), \, y_i \in \mathcal{K} \; \forall\, i = 1, \dots, p \\ \text{and } \exists \text{ at least an index } i \text{ s.t. } y_i \notin \mathcal{K}^+ \Big\},$$

$$\widetilde{V}_3(p,\varepsilon) = \Big\{ u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in \widetilde{V}(p,\varepsilon) \text{ s.t. } \exists \text{ at least an index } i \\ \text{s.t. } a_i \notin \bigcup_{y \in \mathcal{K}} V_{\lambda_i}(y) \Big\}.$$

We define on each subset $\widetilde{V}_j(p,\varepsilon)$, $j = 1,2,3$, an appropriate pseudo-gradient W_j. The required pseudo-gradient W of Proposition 3.4 will be defined by a convex combination of W_j, $j = 1,2,3$.

Pseudo-gradient in $\widetilde{V}_1(p,\varepsilon)$. Let $u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i,\lambda_i)}\in\widetilde{V}_1(p,\varepsilon)$. We define

$$W_1(u)=\sum_{i=1}^{p}\alpha_i\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}.$$

Along $W_1(u)$, all the parameters λ_i, $i=1,\dots,p$, increase according to the differential $\dot{\lambda}_i=\lambda_i$. Using the asymptotic expansion of Proposition 3.2, we have

$$\big\langle\partial J(u),W_1(u)\big\rangle\le -c\sum_{i=1}^{p}\begin{cases}\dfrac{1}{\lambda_i^{\beta^*(y_i)}} & \text{if } \beta(y_i)\neq n\\ \dfrac{\log\lambda_i}{\lambda_i^{\beta(y_i)}} & \text{if } \beta(y_i)=n\end{cases}+\sum_{j\neq i}O(\varepsilon_{ij}), \tag{3.4}$$

since $-\sum_{k=1}^{n}b_k(y_i)>0\ \forall\, i=1,\dots,p$.

Claim 1. For any $i\neq j$ we have

$$\varepsilon_{ij}=o\bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}}\bigg)+o\bigg(\frac{1}{\lambda_j^{\beta^*(y_j)}}\bigg)\qquad\text{as } \varepsilon\to 0.$$

Indeed, since $y_i\neq y_j\ \forall\, i\neq j$, we have

$$\varepsilon_{ij}\le\frac{c}{\big(\lambda_i\lambda_j\big)^{\frac{n-2\sigma}{2}}}.$$

Let γ_1 be a small positive constant. If $\lambda_j^{\frac{n-2\sigma}{2}}\ge\varepsilon^{-\gamma_1}\lambda_i^{\beta^*(y_i)-\frac{n-2\sigma}{2}}$, then

$$\varepsilon_{ij}\le\frac{c\,\varepsilon^{\gamma_1}}{\lambda_i^{\beta^*(y_i)}}.$$

It follows that $\varepsilon_{ij}=o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big)$ as $\varepsilon\to 0$.

If $\lambda_j^{\frac{n-2\sigma}{2}}\le\varepsilon^{-\gamma_1}\,\lambda_i^{\beta^*(y_i)-\frac{n-2\sigma}{2}}$, then for $\gamma_1<\frac{n-2\sigma}{2}$ the exponent $\beta^*(y_i)-\frac{n-2\sigma}{2}$ is positive (if not, the parameter λ_j will be less than ε^{-1}). It follows that

$$\frac{n-2\sigma}{\lambda_j^{2\beta^*(y_i)-(n-2\sigma)}}\le\frac{-2\gamma_1}{\varepsilon^{2\beta^*(y_i)-(n-2\sigma)}}\,\lambda_i.$$

Therefore,

$$\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}}\le\varepsilon^{-\frac{(n-2\sigma)\gamma_1}{2\beta^*(y_i)-(n-2\sigma)}}\,\lambda_j^{-\frac{(n-2\sigma)^2}{4\beta^*(y_i)-2(n-2\sigma)}}.$$

Thus,

$$\varepsilon_{ij}\le c\,\varepsilon^{-\frac{(n-2\sigma)\gamma_1}{2\beta^*(y_i)-(n-2\sigma)}}\,\lambda_j^{-\frac{n-2\sigma}{2}\left(1+\frac{n-2\sigma}{2\beta^*(y_i)-(n-2\sigma)}\right)}.$$

Using the fact that $\frac{1}{\beta^*(y_i)}+\frac{1}{\beta^*(y_j)}>\frac{2}{n-2\sigma}$, we get

$$\varepsilon_{ij}\le c\,\varepsilon^{-\frac{(n-2\sigma)\gamma_1}{2\beta^*(y_i)-(n-2\sigma)}}\,\lambda_j^{-\beta^*(y_j)-\frac{(n-2\sigma)}{2\beta^*(y_i)-(n-2\sigma)}\left(\beta^*(y_i)+\beta^*(y_j)-2\frac{\beta^*(y_i)\,\beta^*(y_j)}{n-2\sigma}\right)}.$$

Using also the fact that $\lambda_j > \varepsilon^{-1}$, we obtain

$$\varepsilon_{ij} \leq c\, \varepsilon^{\frac{n-2\sigma}{2\beta^*(y_i)-(n-2\sigma)}\left(\beta^*(y_i)+\beta^*(y_j)-\frac{2\beta^*(y_i)\,\beta^*(y_j)}{n-2\sigma}-\gamma_1\right)} \frac{1}{\lambda_j^{\beta^*(y_j)}}.$$

Thus for $\gamma_1 < \beta^*(y_i) + \beta^*(y_j) - 2\frac{\beta^*(y_i)\,\beta^*(y_j)}{n-2\sigma}$,

$$\varepsilon_{ij} = o\left(\frac{1}{\lambda_j^{\beta^*(y_j)}}\right) \quad \text{as } \varepsilon \to 0.$$

This concludes the justification of Claim 1.

Using now the result of Claim 1 and inequality (3.4), we get

$$\big\langle \partial J(u), W_1(u) \big\rangle \leq -c\left(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta^*(y_i)}} + \sum_{i\neq j} \varepsilon_{ij} \right). \tag{3.5}$$

Using $(f)_\beta$ expansion, we have

$$\big|\nabla K(a_i)\big| \sim \big|a_i - y_i\big|^{\beta(y_i)-1}. \tag{3.6}$$

Therefore, in $\widetilde{V}_1(p,\varepsilon)$ we have

$$\frac{\big|\nabla K(a_i)\big|}{\lambda_i} \leq \frac{c}{\lambda_i^{\beta^*(y_i)}}.$$

It follows from (3.5) that

$$\big\langle \partial J(u), W_1(u) \big\rangle \leq -c\left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\beta^*(y_i)}} + \frac{\big|\nabla K(a_i)\big|}{\lambda_i} \right) + \sum_{i\neq j} \varepsilon_{ij} \right).$$

Pseudo-gradient in $\widetilde{V}_2(p,\varepsilon)$. Let $u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in \widetilde{V}_2(p,\varepsilon)$. Set

$$I = \big\{ i,\ 1 \leq i \leq p,\ \text{ s.t. } y_i \notin \mathcal{K}^+ \big\}.$$

We define

$$Z_I(u) = -\sum_{i\in I} \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$

Observe that along Z_I, all the parameters λ_i, $i \in I$, decrease according to the differential equation $\dot{\lambda}_i = -\lambda_i$. Using the expansion of Proposition 3.2 and the fact that $-\sum_{k=1}^{n} b_k(y_i) < 0\ \forall\, i \in I$, we have

$$\big\langle \partial J(u), Z_I(u) \big\rangle \leq -c \sum_{i\in I} \begin{cases} \dfrac{1}{\lambda_i^{\beta^*(y_i)}} & \text{if } \beta(y_i) \neq n \\[2ex] \dfrac{\log \lambda_i}{\lambda_i^{\beta^*(y_i)}} & \text{if } \beta(y_i) = n \end{cases} + \sum_{i\neq j} O(\varepsilon_{ij}).$$

Using Claim 1, the above inequality can be improved as follows:

$$\big\langle \partial J(u), Z_I(u) \big\rangle \leq -c \sum_{i\in I} \frac{1}{\lambda_i^{\beta^*(y_i)}} + \sum_{i\notin I} o\left(\frac{1}{\lambda_i^{\beta^*(y_i)}} \right).$$

Let us denote by i_1 an index in I such that $\lambda_{i_1}^{\beta^*(y_{i_1})} = \min_{i\in I} \lambda_i^{\beta^*(y_i)}$. We set

$$\widetilde{I} = \Big\{i,\ 1 \le i \le p,\ \text{ s.t. } \lambda_i^{\beta^*(y_i)} \ge \frac{1}{2}\lambda_{i_1}^{\beta^*(y_{i_1})}\Big\}.$$

We have $I \subset \widetilde{I}$ and the preceding inequality is reduced to

$$\big\langle \partial J(u), Z_I(u)\big\rangle \le -c \sum_{i\in\widetilde{I}} \frac{1}{\lambda_i^{\beta^*(y_i)}} + \sum_{i\notin\widetilde{I}} o\bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}}\bigg).$$

To get inequality (i) of Proposition 3.2, we define

$$V_{\widetilde{I}}(u) = \sum_{i\notin\widetilde{I}} \alpha_i\lambda_i \frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}.$$

According to $V_{\widetilde{I}}(u)$, λ_i increases for all $i\notin\widetilde{I}$, but does not exceed $\Big(\frac{1}{2}\lambda_{i_1}^{\beta^*(y_{i_1})}\Big)^{\frac{1}{\beta^*(y_i)}}$.

Using the fact that $-\sum_{k=1}^n b_k(y_i) > 0\ \forall\, i \notin I$, we get by Proposition 3.2

$$\big\langle \partial J(u), V_{\widetilde{I}}(u)\big\rangle \le -c \sum_{i\notin\widetilde{I}} \frac{1}{\lambda_i^{\beta^*(y_i)}} + \sum_{i\in\widetilde{I}} o\bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}}\bigg).$$

Let

$$W_2(u) = Z_I(u) + V_{\widetilde{I}}(u).$$

Using the above two inequalities, relation (3.6) and Claim 1,

$$\big\langle \partial J(u), W_2(u)\big\rangle \le -c\bigg(\sum_{i=1}^p \bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\bigg) + \sum_{i\ne j}\varepsilon_{ij}\bigg).$$

Pseudo-gradient in $\widetilde{V}_3(p,\varepsilon)$. Let $u = \sum_{i=1}^p \alpha_i\delta_{(a_i,\lambda_i)} \in \widetilde{V}_3(p,\varepsilon)$. We set

$$I = \Big\{i,\ 1 \le i \le p,\ \text{ s.t. } |a_i - y_i|^{\beta(y_i)} > \frac{\gamma_0}{2\lambda_i^{\beta^*(y_i)}}\Big\}.$$

We introduce the following lemma that will be proved in the appendix of this paper.

Lemma 3.6. *For any $i \in I$ there exists a bounded vector field $X_i(u)$ which acts only on the parameter a_i of u and satisfies*

$$\big\langle \partial J(u), X_i(u)\big\rangle \le -c\bigg(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\bigg) + \sum_{i\ne j} O(\varepsilon_{ij}).$$

Using Lemma 3.6, we have

$$\Big\langle \partial J(u), \sum_{i\in I} X_i(u)\Big\rangle \le -c\sum_{i\in I}\bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\bigg) + \sum_{\substack{j\ne i\\ i\in I}} O(\varepsilon_{ij}).$$

Denote by i_1 an index of I such that

$$\lambda_{i_1}^{\beta^*(y_{i_1})} = \min_{i\in I} \lambda_i^{\beta^*(y_i)}.$$

Setting

$$\widetilde{I} = \Big\{ i,\, 1 \le i \le p, \text{ s.t. } \lambda_i^{\beta^*(y_i)} \ge \frac{1}{2}\lambda_{i_1}^{\beta^*(y_{i_1})} \Big\},$$

the above inequality can improved as follows:

$$\Big\langle \partial J(u), \sum_{i\in I} X_i(u) \Big\rangle \le -c \sum_{i\in \widetilde{I}} \bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \bigg) + \sum_{\substack{j\neq i\\ i\in I}} O(\varepsilon_{ij}).$$

Using the result of Claim 1, we obtain

$$\Big\langle \partial J(u), \sum_{i\in I} X_i(u) \Big\rangle \le -c \sum_{i\in \widetilde{I}} \bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \bigg) + \sum_{i\notin \widetilde{I}} o\bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}} \bigg).$$

Now set $\widehat{u} = \sum_{i\notin \widetilde{I}} \alpha_i \delta_{(a_i,\lambda_i)}$. Of course $\widehat{u}$ belongs to $\widetilde{V}_1(q,\varepsilon)$ or $\widetilde{V}_2(q,\varepsilon)$, where $q = \#\widetilde{I}^c$. Denote by $Y(\widehat{u})$ the corresponding vector field defined in the above two regions: $Y(\widehat{u}) = W_1(\widehat{u})$ or $Y(\widehat{u}) = W_2(\widehat{u})$. According to $Y(\widehat{u})$, the parameters λ_i, $i \notin \widetilde{I}$, can increase but they do not exceed $\Big(\frac{1}{2}\lambda_{i_1}^{\beta^*(y_{i_1})}\Big)^{\frac{1}{\beta^*(y_{i_1})}}$. Moreover, we have

$$\big\langle \partial J(u), Y(\widehat{u}) \big\rangle \le -c \sum_{i\notin \widetilde{I}} \bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \bigg) + \sum_{\substack{j\neq i\\ i\notin \widetilde{I}}} O(\varepsilon_{ij}).$$

Let

$$W_3(u) = \sum_{i\in I} X_i(u) + Y(\widehat{u}).$$

Using Claim 1 and the above two inequalities, we have

$$\big\langle \partial J(u), W_3(u) \big\rangle \le -c \Bigg(\sum_{i=1}^{p} \bigg(\frac{1}{\lambda_i^{\beta^*(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \bigg) + \sum_{i\neq j} \varepsilon_{ij} \Bigg).$$

The required pseudo-gradient W of Proposition 3.4 is defined by a convex combination of W_1, W_2 and W_3. By construction W satisfies the properties (i) and (iii) of Proposition 3.4. Concerning (ii), it follows from (i) and the estimate of $\|\overline{v}\|$ given in Proposition 2.1. This finishes the proof of Proposition 3.4. □

Proof of Theorem 3.1. It results from the construction of Proposition 3.4 and Proposition 3.3. □

Proof of Theorem 1.1. It follows from Proposition 2.1 that, under the assumption that J has no critical point in Σ^+, there exists a positive constant c such that

$$\big|\partial J(u)\big| \ge c \quad \forall u \in \bigcup_{p\ge 1} V\Big(p, \frac{\varepsilon}{2}\Big). \tag{3.7}$$

Let $\widetilde{W}$ be a vector field defined by a convex combination of $(-\partial J)$ in $\bigcup_{p\ge1} V(p, \frac{\varepsilon}{2})$ and W in $\bigcup_{p\ge1} V(p,\varepsilon)$. Here W is the pseudo-gradient defined in Theorem 3.1.

It results from (3.7) that for any $u_0 \in \Sigma^+$ there exists $p = p(u_0)$ and $\varepsilon(s) \searrow_0$ such that the motion $u(s, u_0)$ of $\widetilde{W}$ starting from u_0 ties in $V(p, \varepsilon(s))$ for any $s \geq s_0$. Therefore $u(s, u_0)$ can be expressed as

$$u(s, u_0) = \sum_{i=1}^{p} \alpha_i(s)\delta_{(a_i(s),\lambda_i(s))} + \overline{v}(s) \quad \forall s \geq s_0.$$

Using the properties of W given in Theorem 3.1, the flow line $u(s, u_0)$ stays in $\widetilde{V}_1(p, \varepsilon(s))$ for any sufficiently large s. Thus for any i, $1 \leq i \leq p$, there exists $y_i \in \mathcal{K}^+$ such that

$$\left|a_i(s) - y_i\right|^{\beta(y_i)} \leq \frac{\gamma_0}{\lambda_i(s)^{\beta^*(y_i)}}$$

with $y_i \neq y_j$ $\forall\, 1 \leq i \neq j \leq p$. Using the fact that $\lambda_i(s) > \varepsilon^{-1}(s)$, we get $a_i(s) \to y_i$ $\forall i = 1, \dots, p$. This ends the characterization of the critical points at infinity of J.

Where each critical point at infinity

$$\sum_{i=1}^{p} \frac{1}{K(y_i)^{\frac{n-2\sigma}{n}}} \delta_{(y_i,\infty)},$$

the functional J can be extended as

$$\begin{aligned} J&\left(\sum_{i=1}^{p} \alpha_i\delta_{(a_i,\lambda_i)} + \overline{v}\right) \\ &= \left(\sum_{i=1}^{p} \frac{S_n}{K(y_i)^{\frac{n-2\sigma}{2}}}\right)^{\frac{2\sigma}{n}} \\ &\quad \times \left(1 - \sum_{i=1}^{p}\left(\sum_{k=1}^{p} b_k(y_i)\left|(a_i - y_i)_k\right|^{\beta(y_i)}\right) - |H|^2 + \sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta^*(y_i)}}\right). \end{aligned} \tag{3.8}$$

Here $H \in \mathbb{R}^{p-1}$ and S_n is the best constant of Sobolev. Under the assumption that $b_k(y_i) \neq 0$ $\forall k, \dots, n$, the index of such a critical point at infinity is equal to $i(y_1, \dots, y_p)_\infty = p - 1 + \sum_{l=1}^{p}(n - i(y_l))$. The proof of Theorem 1.1 is thereby completed. ◻

4. Proof of the existence results

We shall prove the existence results of this paper by contradiction. Therefore throughout this section we assume that the variational functional J has no positive critical point.

Proof of Theorem 1.2. Under the assumptions of Theorem 1.2, $\theta(N_{k_0}^\infty)$ defines a contraction of $N_{k_0}^\infty$ of dimension k_0+1. Let $\widetilde{W}$ be the pseudo-gradient defined in the proof of Theorem 1.1. We use $\widetilde{W}$ to deform $\theta(N_{k_0}^\infty)$. Since we have supposed that J has no critical point in Σ^+, $\theta(N_{k_0}^\infty)$ retracts by deformation $(:\simeq)$ on a union of the unstable manifolds of critical points at infinity. By transversality and dimension arguments, we may suppose that the deformation avoids the unstable manifolds of the critical points at infinity of indices larger than or equal to k_0+2. Thus using the characterization of the critical points at infinity given by Theorem 1.1 we have

$$\begin{aligned}\theta(N^\infty_{k_0}) &\simeq \bigcup_{\substack{(y_1,\dots,y_p)_\infty\in C^\infty_{\leq k_0+1}\\ W^\infty_s(y_1,\dots,y_p)_\infty\cap\,\theta(N^\infty_{k_0})\neq\emptyset}} W^\infty_u(y_1,\dots,y_p)_\infty\\ &= N^\infty_{k_0}\cup\left(\bigcup_{\substack{(y_1,\dots,y_p)_\infty\in C^\infty_{\leq k_0+1}\setminus N^\infty_{k_0}\\ W^\infty_s(y_1,\dots,y_p)_\infty\cap\,\theta(N^\infty_{k_0})\neq\emptyset}} W^\infty_u(y_1,\dots,y_p)_\infty\right).\end{aligned}$$

Under the assumption (b) of Theorem 1.2, the above retract by deformation will be reduced to

$$\theta(N^\infty_{k_0})\simeq N^\infty_{k_0}.$$

Using the Euler characteristic of both sides of the above retracts, we get

$$1=\sum_{(y_1,\dots,y_p)_\infty\in N^\infty_{k_0}}(-1)^{i(y_1,\dots,y_p)_\infty}.$$

This contradicts assumption (a) of Theorem 1.2. □

Proof of Theorem 1.3. Just check the conditions of Theorem 1.2 under the assumptions of Theorem 1.3. Let k_0 be the integer satisfying conditions (a') and (b') of Theorem 1.3. We work with

$$N^\infty_{k_0}=\Big\{(y)_\infty=\frac{1}{K(y)^{\frac{n-2\sigma}{n}}}\delta_{(y,\infty)}\text{ s.t. } y\in\mathcal{K}^+,\text{ and } i(y)_\infty\leq k_0\Big\},$$

and

$$W^\infty_u(N^\infty_{k_0})=\bigcup_{\substack{y\in\mathcal{K}^+\\ i(y)_\infty\leq k_0}}W^\infty_u(y)_\infty.$$

It is easy to see that condition (a') implies condition (a) of Theorem 1.2. In order to complete the proof, it remains only to construct a contraction $\theta\big(W^\infty_u(N^\infty_{k_0})\big)$ of $W^\infty_u(N^\infty_{k_0})$ satisfying condition (b) of Theorem 1.2.

Recall that from expansion (3.8) the critical value at infinity $C_\infty(y_1,\dots,y_p)_\infty$ of a critical point at infinity $(y_1,\dots,y_p)_\infty$, $p\geq 1$, is

$$C_\infty(y_1,\dots,y_p)_\infty=S_n^{\frac{2}{n}}\left(\sum_{i=1}^p\frac{1}{K(y_i)^{\frac{n-2\sigma}{n}}}\right)^{\frac{2}{n}} \tag{4.1}$$

Let $K(z_0)=\min_{S^n}K(x)$ and $K(y_0)=\max_{S^n}K(x)$. It is easy to check that

$$C_\infty(y)<\frac{S_n^{\frac{2}{n}}}{K(z_0)^{\frac{n-2\sigma}{n}}}\quad\forall y\in\mathcal{K}^+; \tag{4.2}$$

moreover, for any $p \geq 2$, we have

$$C_\infty(y_1, \ldots, y_p)_\infty \geq \frac{(2S_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2\sigma}{n}}}. \tag{4.3}$$

Let

$$C_1 = \frac{S_n^{\frac{2}{n}}}{K(z_0)^{\frac{n-2\sigma}{n}}}.$$

Since J decreases along the flow lines of $(-\partial J)$, we obtain from (4.2)

$$J(u) \leq C_1 \quad \forall u \in W_u^\infty(N_{k_0}^\infty),$$

and therefore $W_u^\infty(N_{k_0}^\infty) \subset J_{C_1}$, where $J_c = \{u \in \Sigma^+,\ J(u) \leq c\}$.

Denote by J^1 the variation functional associated to the problem where the prescribed functional is equal to 1. An easy computation shows that

$$\frac{1}{K(y_0)^{\frac{n-2\sigma}{n}}} J^1(u) \leq J(u) \leq \frac{1}{K(z_0)^{\frac{n-2\sigma}{n}}} J^1(u) \quad \forall u \in \Sigma. \tag{4.4}$$

We obtain

$$J_{C_1} \subset J^1_{K(y_0)^{\frac{n-2\sigma}{n}} C_1} \subset J_{\left(\frac{K(y_0)}{K(z_0)}\right)^{\frac{n-2\sigma}{n}} C_1}. \tag{4.5}$$

Observe that, under the assumption $\frac{K(y_0)}{K(z_0)} < 2^{\frac{1}{n-2\sigma}}$, we have

$$\left(\frac{K(y_0)}{K(z_0)}\right)^{\frac{n-2\sigma}{n}} C_1 < \frac{(2S_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2\sigma}{n}}},$$

and therefore by (4.2) and (4.3), J has no critical point at infinity between the levels C_1 and $\left(\frac{K(y_0)}{K(z_0)}\right)^{\frac{n-2\sigma}{n}} C_1$. Using a retract by the deformation lemma, we have

$$J_{\left(\frac{K(y_0)}{K(z_0)}\right)^{\frac{n-2\sigma}{n}}} \simeq J_{C_1}.$$

It results from (4.5) that

$$J^1_{K(y_0)^{\frac{n-2\sigma}{n}} C_1} \simeq J_{C_1}.$$

Note that $J^1_{K(y_0)^{\frac{n-2\sigma}{n}} C_1}$ is a contractible set. It then follows from the above retract by deformation that J_{C_1} is a contractible set. Using the fact that $W_u^\infty(N_{k_0}^\infty)$ is included in J_{C_1}, there exists at least a contraction $\theta\big(W_u^\infty(N_{k_0}^\infty)\big)$ of $W_u^\infty(N_{k_0}^\infty)$ in J_{C_1}. Using (4.3) and assumption (b') of Theorem 1.3, the following holds:

$$\theta\big(W_u^\infty(N_{k_0}^\infty)\big) \cap W_s^\infty(y_1, \ldots, y_p)_\infty = \emptyset \quad \forall\, (y_1, \ldots, y_p)_\infty \in C_{k_0+1}^\infty \setminus N_{k_0}^\infty.$$

Condition (b) is valid and Theorem 1.2 applies. □

Proof of Theorem 1.5. Let

$$C_2 = \frac{(2S_n)^{\frac{2}{n}}}{K(z_0)^{\frac{n-2\sigma}{n}}}.$$

It follows from (4.1) that for any $y_i \neq y_j \in \mathcal{K}^+$, we have

$$C_\infty(y_i, y_j) < C_2. \tag{4.6}$$

Moreover, we derive from (4.4) that

$$J_{C_2} \subset J^1_{K(y_0)^{\frac{n-2\sigma}{n}} C_2} \subset J_{\left(\frac{K(y_0)}{K(z_0)}\right)^{\frac{n-2\sigma}{n}} C_2}. \tag{4.7}$$

Observe that under the assumption $\frac{K(y_0)}{K(z_0)} < \left(\frac{3}{2}\right)^{\frac{1}{n-2\sigma}}$, we have

$$\left(\frac{K(y_0)}{K(z_0)}\right)^{\frac{n-2\sigma}{n}} C_2 < \frac{(3S_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2\sigma}{n}}} \leq C_\infty(y_1, \dots, y_p)_\infty \quad \forall p \geq 3. \tag{4.8}$$

It follows from (4.6) and (4.8) that J has no critical points at infinity between the levels C_2 and $\left(\frac{K(y_0)}{K(z_0)}\right)^{\frac{n-2\sigma}{n}} C_2$, and therefore,

$$J_{\left(\frac{K(y_0)}{K(z_0)}\right)^{\frac{n-2\sigma}{n}} C_2} \simeq J_{C_2}. \tag{4.9}$$

We then derive from (4.7) that J_{C_2} is a contractible space, since, according to (4.9), J_{C_2} is a strong retract by deformation of $J^1_{K(y_0)^{\frac{n-2\sigma}{n}} C_2}$ and the latter is contractible.

Let χ be the Euler–Poincaré characteristic. It follows from (4.2), (4.3) and (4.8) that

$$1 = \chi(J_{C_2}) = \chi(J_{C_1}) + \sum_{y_i \neq y_j \in \mathcal{K}^+} (-1)^{1+2n-\left(\tilde{i}(y_i)+\tilde{i}(y_j)\right)}.$$

This implies that

$$\sum_{y_i \neq y_j \in \mathcal{K}^+} (-1)^{1+2n-\left(\tilde{i}(y_i)+\tilde{i}(y_j)\right)} = 0,$$

since $\chi(J_{C_1}) = 1$. Using the computation of [32, p. 16] we get

$$\#\mathcal{K}^+ = 1 \quad \text{and thus} \quad \mathcal{K}^+ \setminus \{y_0\} = \emptyset.$$

This yields a contradiction with the assumption of Theorem 1.5. ☐

Appendix A. Proof of Lemma 3.6

Let $i \in I$. We distinguish two cases.

Case 1: $\beta(y_i) > n$. In this case we define

$$X_i(u) = \psi_{\gamma_0}\left(\lambda_i^{\beta^*(y_i)} |a_i - y_i|^{\beta(y_i)}\right) \times \sum_{k=1}^{n} \frac{b_k(y_i)}{\lambda_i} \operatorname{sign}(a_i - y_i)_k \frac{|(a_i - y_i)_k|^{\beta(y_i)-1}}{|a_i - y_i|^{\beta(y_i)-1}} \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial (a_i)_k},$$

where ψ_{γ_0} is a cutoff function such that

$$\psi_{\gamma_0}(t) = 1 \text{ if } t > \frac{\gamma_0}{4} \quad \text{and} \quad \psi_{\gamma_0}(t) = 0 \text{ if } t < \frac{\gamma_0}{8}.$$

Using the expansion of Proposition 3.3, we have

$$\big\langle \partial J(u), X_i(u)\big\rangle \leq -c\sum_{k=1}^{n}\frac{\big|(a_i-y_i)_k\big|^{\beta(y_i)-1}}{\lambda_i}+\sum_{l=2}^{[\beta^*(y_i)]}O\bigg(\frac{\big|a_i-y_i\big|^{\beta(y_i)-l}}{\lambda_i^l}\bigg)+o\bigg(\frac{1}{\lambda_i^{\gamma}}\bigg)+\sum_{j\neq i}O(\varepsilon_{ij}). \tag{A.1}$$

Here γ is any constant such that $n<\gamma<\min(n+1,\beta(y_i))$. We claim the following:
For any $l=2,\dots,n$, we have

$$\frac{\big|a_i-y_i\big|^{\beta(y_i)-l}}{\lambda_i^l}=o\bigg(\frac{\big|a_i-y_i\big|^{\beta(y_i)-1}}{\lambda_i}\bigg)\quad \text{as } \varepsilon\to 0. \tag{A.2}$$

Moreover,

$$\text{for } \gamma\in\bigg(\frac{n(\beta-1)}{\beta}+1,\min(\beta,n+1)\bigg),\ \text{ we have } \frac{1}{\lambda_i^{\gamma}}=o\bigg(\frac{\big|a_i-y_i\big|^{\beta(y_i)-1}}{\lambda_i}\bigg) \text{ as } \varepsilon\to 0. \tag{A.3}$$

Indeed,

$$\frac{\big|a_i-y_i\big|^{\beta(y_i)-l}}{\lambda_i^l}\frac{\lambda_i}{\big|a_i-y_i\big|^{\beta(y_i)-1}}=\frac{1}{(\lambda_i\big|a_i-y_i\big|)^{l-1}}.$$

Using the fact that $i\in I$, we get

$$\frac{1}{\big|a_i-y_i\big|^{\beta(y_i)}}\leq\frac{2}{\gamma_0}\lambda_i^{\beta^*(y_i)},$$

so

$$\frac{1}{\big|a_i-y_i\big|}\leq\bigg(\frac{2}{\gamma_0}\bigg)^{\frac{1}{\beta(y_i)}}\lambda_i^{\frac{\beta^*(y_i)}{\beta(y_i)}}.$$

Thus

$$\begin{aligned}\frac{1}{(\lambda_i\big|a_i-y_i\big|)^{l-1}}&\leq\frac{1}{\lambda_i^{l-1}}\bigg(\frac{2}{\gamma_0}\bigg)^{\frac{l-1}{\beta(y_i)}}\lambda_i^{\frac{\beta^*(y_i)}{\beta(y_i)}(l-1)}\\&\leq\bigg(\frac{2}{\gamma_0}\bigg)^{\frac{l-1}{\beta(y_i)}}\frac{1}{\lambda_i^{(l-1)\big(1-\frac{\beta^*(y_i)}{\beta(y_i)}\big)}}\to 0\quad \text{as } \varepsilon\to 0,\end{aligned}$$

since $\beta^*(y_i)=n$ and $\beta(y_i)>n$. Estimate (A.2) follows.

Now for $\gamma \in \Big(\frac{n(\beta(y_i)-1)}{\beta(y_i)}+1, \min(\beta(y_i), n+1)\Big)$, we have

$$\begin{aligned}\frac{1}{\lambda_i^\gamma}\frac{\lambda_i}{\big|a_i-y_i\big|^{\beta(y_i)-1}} &= \frac{1}{\lambda_i^{\gamma-1}}\frac{1}{\big|a_i-y_i\big|^{\beta(y_i)-1}}\\ &\leq \frac{1}{\lambda_i^{\gamma-1}}\left(\frac{2}{\gamma_0}\right)^{\frac{\beta(y_i)-1}{\beta(y_i)}}\lambda_i^{\frac{\beta^*(y_i)}{\beta(y_i)}(\beta(y_i)-1)}\\ &\leq \left(\frac{2}{\gamma_0}\right)^{\frac{\beta(y_i)-1}{\beta(y_i)}}\frac{1}{\lambda_i^{(\gamma-1)-(\beta(y_i)-1)\frac{\beta^*(y_i)}{\beta(y_i)}}} \to 0 \quad \text{as } \varepsilon\to 0,\end{aligned}$$

since $\gamma > \frac{n(\beta(y_i)-1)}{\beta(y_i)}+1$. Claim (A.3) follows. Using (A.2) and (A.3), we get from (A.1)

$$\big\langle \partial J(u), X_i(u)\big\rangle \leq -c\sum_{k=1}^{n}\frac{\big|(a_i-y_i)_k\big|^{\beta(y_i)-1}}{\lambda_i}+\sum_{j\neq i}O(\varepsilon_{ij}).$$

Using the fact that

$$\frac{1}{\lambda_i^{\beta(y_i)}} = o\left(\frac{1}{\lambda_i^\gamma}\right) = o\left(\frac{\big|a_i-y_i\big|^{\beta(y_i)-1}}{\lambda_i}\right) \quad \text{as } \varepsilon\to 0$$

and $|\nabla K(a_i)| \sim \big|a_i-y_i\big|^{\beta(y_i)-1}$, we obtain

$$\big\langle \partial J(u), X_i(u)\big\rangle \leq -c\left(\frac{1}{\lambda_i^{\beta(y_i)}}+\frac{\big|\nabla K(a_i)\big|}{\lambda_i}\right)+\sum_{j\neq i}O(\varepsilon_{ij}).$$

Lemma 3.6 follows in this case.

Case 2: $\beta(y_i)\leq n$. Let M be a sufficiently large positive constant.
If $|a_i-y_i|^{\beta(y_i)} \geq \frac{M}{\lambda_i^{\beta^*(y_i)}}$, we consider in this case the vector field

$$\begin{aligned}X_i(u) = {}&\psi_M\Big(\lambda_i^{\beta(y_i)}|a_i-y_i|^{\beta(y_i)}\Big)\\ &\times\sum_{k=1}^{n}\frac{b_k(y_i)}{\lambda_i}\,\mathrm{sign}(a_i-y_i)_k\frac{|(a_i-y_i)_k|^{\beta(y_i)-1}}{|a_i-y_i|^{\beta(y_i)-1}}\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial(a_i)_k},\end{aligned}$$

where ψ_M is a cutoff function such that

$$\psi_M(t) = \begin{cases}1 & \text{if } t\geq M,\\ 0 & \text{if } t\leq \frac{M}{2}.\end{cases}$$

Observe that $\langle\partial J(u), X_i(u)\rangle$ satisfies inequality (A.1). Moreover, the following holds: For any $l=2,\dots,p$, we have

$$\frac{\big|a_i-y_i\big|^{\beta(y_i)-l}}{\lambda_i^l} = o\left(\frac{\big|a_i-y_i\big|^{\beta(y_i)-1}}{\lambda_i}\right) \quad \text{as } M \text{ becomes large}$$

and

$$\frac{1}{\lambda_i^{\beta(y_i)}} = o\left(\frac{\left|a_i - y_i\right|^{\beta(y_i)-1}}{\lambda_i}\right) \quad \text{as } M \text{ becomes large.}$$

Thus

$$\begin{aligned}\left\langle \partial J(u), X_i(u)\right\rangle &\leq -c\sum_{k=1}^{n}\frac{\left|(a_i - y_i)_k\right|^{\beta(y_i)-1}}{\lambda_i} + \sum_{j\neq i} O(\varepsilon_{ij}) \\ &\leq -c\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{\left|\nabla K(a_i)\right|}{\lambda_i}\right) + \sum_{j\neq i} O(\varepsilon_{ij}).\end{aligned}$$

If $|a_i - y_i|^{\beta(y_i)} \leq \frac{M}{\lambda_i^{\beta^*(y_i)}}$, we use the vector field

$$X_i(u) = \sum_{k=1}^{n} b_k \int_{\mathbb{R}^n} \frac{\left|t_k + \lambda_i(a_i - y_i)_k\right|^{\beta(y_i)}}{\left(1+|t|^2\right)^{n+1}}\, dt \frac{\alpha_i}{\lambda_i}\frac{\partial \delta_{(a_i,\lambda_i)}}{\partial (a_i)_k}.$$

Note that X_i is used in [2, p. 1293]. Using the same computation of [2, pp. 1307–1308], we get

$$\left\langle \partial J(u), X_i(u)\right\rangle \leq -c\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{\left|\nabla K(a_i)\right|}{\lambda_i}\right) + \sum_{j\neq i} O(\varepsilon_{ij}).$$

The proof of Lemma 3.6 is thereby completed. □

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Azeb Alghanemi
Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
aalghanemi@kau.edu.sa

Randa Ben Mahmoud✉
Sfax University, Faculty of Sciences of Sfax, 3018 Sfax, Tunisia
randa_benmahmoud@yahoo.fr

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