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GROUND STATE SOLUTIONS FOR SCHRÖDINGER EQUATIONS IN THE PRESENCE OF A MAGNETIC FIELD

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Abstract. In this paper, we are dedicated to studying the Schrödinger equations in the presence of a magnetic field. Based on variational methods, especially the mountain pass theorem, we obtain ground state solutions for the system under certain assumptions.

1. Introduction

The linear Schrödinger equation is a basic tool of quantum mechanics, and it provides a description of the dynamics of a particle in a non-relativistic setting. The nonlinear Schrödinger equation arises in different physical theories, e.g., the description of Bose–Einstein condensates and nonlinear optics, see [7] and the references cited there. Both the linear and the nonlinear Schrödinger equations have been widely considered in the literature, see [1, 15, 13, 16, 14]. The authors in [8] considered the following problem:

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}v & \text{in } \Omega, \\ -\Delta v = \mu |v|^{2^*-1} + |u|^{2^*-1} & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 4$, $2^* = \frac{2N}{N-2}$, $\lambda \in \mathbb{R}$ and $\mu \geq 0$. They obtained existence and nonexistence results, depending on the value of the parameters λ and μ.

In the nonlocal framework, only few and recent works deal with fractional magnetic Schrödinger equations like

$$(-\Delta)^s_A u + V(x)u = f(x, |u|^2)u \quad \text{in } \mathbb{R}^N.$$

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Up to normalization constants, $(-\Delta)^s_A$ can be defined on smooth complex valued functions $u \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ as

$$(-\Delta)^s_A u(x) = \lim_{\varepsilon \to 0^+} \int_{B^c_\varepsilon(x)} \frac{e^{-i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(x) - u(y)}{|x-y|^{N+2s}} \,\mathrm{d}y \quad \text{in } \mathbb{R}^N,$$

where $B_\varepsilon(x)$ denotes a ball in $\mathbb{R}^N$ with radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^N$.

To the best of our knowledge, the earliest existence result for the nonlinear magnetic Schrödinger equation is the paper by Esteban and Lions where the magnetic field is assumed to be constant. Their approach was generalized to the periodic magnetic fields by Arioli and Szulkin [3].

For instance, d'Avenia and Squassina in [9] studied the existence of ground state solutions to the above equation, when $\epsilon = 1$, V is constant and f is a subcritical or critical nonlinearity. Fiscella proved the multiplicity of nontrivial solutions for a fractional magnetic problem with homogeneous boundary conditions in [10]. Then, in [6], Zhang obtained the existence of mountain pass solutions which tend to the trivial solution as $\epsilon \to 0$ for a fractional magnetic Schrödinger equation involving critical frequency and critical growth.

Moreover, Vincenzo Ambrosio [2] studied the following fractional Schrödinger–Poisson equation with magnetic field:

$$\epsilon^{2s}(-\Delta)^s_{A\backslash\epsilon} u + V(x)u + \epsilon^{-2t}(|x|^{2t-3} * |u|^2)u = f(|u|^2) + |u|^{2^*_s - 2}u \quad \text{in } \mathbb{R}^3,$$

where $\epsilon > 0$ is a small parameter, $s \in (\frac{3}{4}, 1)$, $t \in (0,1)$, $2^*_s = \frac{6}{3-2s}$ is the fractional critical exponent, $(-\Delta)^s_A$ is the fractional magnetic Laplacian, $V : \mathbb{R}^3 \to \mathbb{R}$ is a positive continuous potential, $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth magnetic potential and $f : \mathbb{R} \to \mathbb{R}$ is a subcritical nonlinearity. Under a local condition on the potential V, they studied the multiplicity and concentration of nontrivial solutions as $\epsilon \to 0$. In particular, they related the number of nontrivial solutions with the topology of the set where the potential V attains its minimum.

On the other hand, for quite a long time, some interesting papers (see [4, 5, 11] and the references therein) dealt with Schrödinger equations like

$$(-i\nabla + A)^2 u = \frac{|u|^{2^*(s)-2}u}{|x|^s}, \quad u \in D^{1,2}_A(\mathbb{R}^N),$$

and related systems thereof, viz.

$$\begin{cases} (-i\nabla + A)^2 u = \mu_1 \dfrac{|u|^{2^*(s)-2}u}{|x|^s} + \dfrac{\alpha\gamma}{2^*(s)} \dfrac{|u|^{\alpha-2}u|v|^\beta}{|x|^s}, \\ (-i\nabla + B)^2 v = \mu_2 \dfrac{|v|^{2^*(s)-2}v}{|x|^s} + \dfrac{\beta\gamma}{2^*(s)} \dfrac{|u|^\alpha |v|^{\beta-2}v}{|x|^s}, \\ \qquad u \in D^{1,2}_A(\mathbb{R}^N), \quad v \in D^{1,2}_B(\mathbb{R}^N), \end{cases}$$

where $u, v : \mathbb{R}^N \to \mathbb{C}$, $N \geq 3$, $A = (A_1, \ldots, A_N)$, $B = (B_1, \ldots, B_N) : \mathbb{R}^N \to \mathbb{R}^N$ are magnetic vector potentials, $0 \leq s < 2$, $\lambda, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma > 0$, $\alpha, \beta > 1$ with $\alpha + \beta = 2^*(s) := \frac{2(N-s)}{N-2}$, and Ω is a smooth bounded domain containing the origin as an interior point. Under proper conditions, the existence of ground state solutions to the above equation and systems was established.

However, the conclusion about Schrödinger equations in the presence of a magnetic field needs further study. Thus, by the above works, we consider the problem

$$\begin{cases}(-i\nabla + A)^2 u = \lambda u + |u|^{2^*-2}v, & x \in \Omega,\\ (-i\nabla + A)^2 v = \mu |v|^{2^*-2}v + |u|^{2^*-2}u, & x \in \Omega,\\ u = v = 0, & x \in \mathbb{R}^N \setminus \Omega,\end{cases} \tag{1.1}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ with Lipschitz boundary, $N \geq 4$, $2^* := \frac{2N}{N-2}$ is the Sobolev critical exponent, $\lambda > 0$ and $\mu \geq 0$. $A = (A_1, \dots, A_N) : \mathbb{R}^N \to \mathbb{R}^N$ is a vector (or magnetic) potential. Let $B := \operatorname{curl} A$. For $N = 3$, this is the usual curl operator and for general N, $B = (B_{jk})$, $1 \leq j, k \leq N$, where

$$B_{jk} := \partial_j A_k - \partial_k A_j.$$

A can also be thought of as the 1-form:

$$A = \sum_{j=1}^{N} A_j dx^j;$$

then $B = dA$, i.e.,

$$B = \sum_{j<k} B_{jk} dx^j \wedge dx^k,$$

where B_{jk} are as above. Here B represents an external magnetic field whose source is A.

Below we specify our assumptions. Suppose $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$. Write $\nabla_A u = (\nabla + iA)u$ and let

$$H^1_A(\Omega) := \{u \in L^2(\Omega) : \nabla_A u \in L^2(\Omega)\}$$

and

$$D^{1,2}_A(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N)\}.$$

Both $H^1_A(\Omega)$ and $D^{1,2}_A(\mathbb{R}^N)$ are Hilbert spaces with inner product

$$\operatorname{Re}\left(\int_\Omega \nabla_A u \cdot \overline{\nabla_A v}\, \mathrm{d}x\right) \quad \text{and} \quad \operatorname{Re}\left(\int_{\mathbb{R}^N} \nabla_A u \cdot \overline{\nabla_A v}\, \mathrm{d}x\right)$$

respectively, where the bar denotes complex conjugation. The norm in $L^p(\mathbb{R}^N)$ is denoted by

$$|u|_p^p = \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x.$$

The norm in $L^p(\Omega)$ is denoted by

$$|u|_{p,\Omega}^p = \int_\Omega |u|^p \,\mathrm{d}x.$$

Define

$$S_A := \inf_{(u,v) \in D_A(\mathbb{R}^N)\setminus\{(0,0)\}} \frac{\|(u,v)\|^2_{D_A}}{\left(\int_{\mathbb{R}^N} |u|^{2^*} + \mu |v|^{2^*} + |u|^{2^*-1} v \,\mathrm{d}x\right)^{\frac{2}{2^*}}},$$

$$S_{A,\lambda} := \inf_{(u,v)\in H_A(\mathbb{R}^N)\setminus\{(0,0)\}} \frac{\|(u,v)\|_{H_A}^2}{\left(\int_\Omega |u|^{2^*}+\mu|v|^{2^*}+|u|^{2^*-1}v\,\mathrm{d}x\right)^{\frac{2}{2^*}}},$$

where $D_A := D_A^{1,2}(\mathbb{R}^N)\times D_A^{1,2}(\mathbb{R}^N)$, endowed with norm

$$\|(u,v)\|_{D_A}^2 := |\nabla_A u|_2^2 + |\nabla_A v|_2^2$$

and $H_A := H_A^1(\Omega)\times H_A^1(\Omega)$, endowed with norm

$$\|(u,v)\|_{H_A}^2 := |\nabla_A u|_{2,\Omega}^2 - \lambda|u|_{2,\Omega}^2 + |\nabla_A v|_{2,\Omega}^2 - \lambda|v|_{2,\Omega}^2.$$

We look for solutions to the problem (1.1) as critical points of the C^1-functional J: $H_A^1(\Omega)\times H_A^1(\Omega)\to\mathbb{R}$ given by

$$\begin{aligned} J(u,v) = \frac{1}{2}\int_\Omega |\nabla_A u|^2\,\mathrm{d}x - \frac{1}{2}\lambda\int_\Omega |u|^2\,\mathrm{d}x + \frac{1}{2(2^*-1)}\int_\Omega |\nabla_A v|^2\,\mathrm{d}x \\ - \frac{\mu}{2^*(2^*-1)}\int_\Omega |v|^{2^*}\,\mathrm{d}x - \frac{1}{2^*-1}\int_\Omega |u|^{2^*-1}v\,\mathrm{d}x. \end{aligned}$$

We are concerned with ground state solutions to (1.1), namely solutions $(u,v)\in H_A^1(\Omega)\times H_A^1(\Omega)$ together with both $u\not\equiv 0$ and $v\not\equiv 0$. It is standard that the ground states are the solutions to (1.1) that minimize J on the Nehari manifold

$$\mathcal{N} = \Big\{(u,v)\in H_A^1(\Omega)\times H_A^1(\Omega)\setminus\{(0,0)\} : F(u,v) = (0,0)\Big\}, \tag{1.2}$$

where

$$\begin{aligned} F(u,v) = \Bigg(\int_\Omega |\nabla_A u|^2\,\mathrm{d}x - \lambda\int_\Omega |u|^2\,\mathrm{d}x - \int_\Omega |u|^{2^*-1}v\,\mathrm{d}x, \\ \int_\Omega |\nabla_A v|^2\,\mathrm{d}x - \mu\int_\Omega |v|^{2^*}\,\mathrm{d}x - \int_\Omega |u|^{2^*-1}v\,\mathrm{d}x\Bigg). \end{aligned}$$

The main results of this paper are as follows.

Theorem 1.1. *For $N\geq 4$, if one of the conditions*

(a_1) $A\in L^N_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)$, $\operatorname{curl} A\equiv 0$, or

(a_2) $A\in L^2_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)$, A is continuous at $\overline{x}$, and $\sigma(-\Delta_A-\lambda)\subset(0,+\infty)$,

and $\mu\geq 0$, $\lambda\in(0,\lambda_1(\Omega))$ are satisfied, then problem (1.1) has a ground state solution.

Theorem 1.2. *If $N>4$, $\mu\geq 0$ and (a_1) or (a_2) are satisfied, then for every $\epsilon>0$, $\left(m_0^{\frac{1}{2^*-2}}u_\epsilon, m_0^{\frac{3-2^*}{2^*-2}}u_\epsilon\right)$ is a ground state solution of (2.12) and*

$$J_0\left(m_0^{\frac{1}{2^*-2}}u_\epsilon, m_0^{\frac{3-2^*}{2^*-2}}u_\epsilon\right) = M = M' = \frac{1}{N}\left(k_0^2+\frac{l_0^2}{2^*-1}\right)S_A^{N/2},$$

where (k_0,l_0) is a solution to (3.2). The definition of u_ϵ and problem (2.12) are given in Section 3 and Section 2, respectively.

Theorem 1.3. *If $N = 4$, $\mu \geq 0$ and (a_1) or (a_2) are satisfied, then for every $\epsilon > 0$, $\left(\sqrt{m_0}u_\epsilon, \frac{1}{\sqrt{m_0}}u_\epsilon\right)$ is a ground state solution of* (2.12) *and*

$$J_0\left(\sqrt{m_0}u_\epsilon, \frac{1}{\sqrt{m_0}}u_\epsilon\right) = M = M' = \frac{1}{4}\left(\tilde{k}^2 + \frac{1}{3}\tilde{l}^2\right) S_A^2,$$

where $\tilde{k}, \tilde{l}$ is the unique solution to (3.7).

2. Preliminaries

Define

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{|\nabla u|_2^2}{|u|_{2^*}^2},$$

where

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}.$$

Then S is attained by functions of the form

$$(U_\epsilon(x), V_\epsilon(x)) := \left(\epsilon^{-\frac{N-2}{2}} U\left(\frac{x}{\epsilon}\right), \epsilon^{-\frac{N-2}{2}} V\left(\frac{x}{\epsilon}\right)\right) \tag{2.1}$$

and

$$(u_\epsilon(x), v_\epsilon(x)) := (\phi(x)U_\epsilon(x), \phi(x)V_\epsilon(x)),$$

where $\phi \in C_0^1(B_{2r})$ is a cut-off function satisfying $\phi \equiv 1$ in B_r. B_r is the ball centered at 0 with radius $r > 0$ and $\epsilon > 0$. We have

$$\int_\Omega |\nabla u_\epsilon|^2 \,\mathrm{d}x = \int_\Omega |\nabla(\phi U_\epsilon)|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x + O(\epsilon^{N-2}), \tag{2.2}$$

$$\int_\Omega |\nabla v_\epsilon|^2 \,\mathrm{d}x = \int_\Omega |\nabla(\phi V_\epsilon)|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^N} |\nabla V|^2 \,\mathrm{d}x + O(\epsilon^{N-2}), \tag{2.3}$$

$$\int_\Omega |u_\epsilon|^{2^*} \,\mathrm{d}x = \int_\Omega U_\epsilon^{2^*} \,\mathrm{d}x - \int_\Omega (1 - \phi^{2^*}) U_\epsilon^{2^*} \,\mathrm{d}x \geq \int_{\mathbb{R}^N} U^{2^*} \,\mathrm{d}x + O(\epsilon^N), \tag{2.4}$$

$$\int_\Omega |v_\epsilon|^{2^*} \,\mathrm{d}x = \int_\Omega V_\epsilon^{2^*} \,\mathrm{d}x - \int_\Omega (1 - \phi^{2^*}) V_\epsilon^{2^*} \,\mathrm{d}x \geq \int_{\mathbb{R}^N} V^{2^*} \,\mathrm{d}x + O(\epsilon^N), \tag{2.5}$$

$$\int_\Omega |u_\epsilon|^{2^*-1} v_\epsilon \,\mathrm{d}x \geq \int_{\mathbb{R}^N} |U|^{2^*-1} V \,\mathrm{d}x + O(\epsilon^N). \tag{2.6}$$

Define

$$S_0 := \inf_{(u,v) \in D(\mathbb{R}^N)\setminus\{(0,0)\}} \frac{\|(u,v)\|_D^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} + \mu |v|^{2^*} + |u|^{2^*-1} v \,\mathrm{d}x\right)^{\frac{2}{2^*}}},$$

where $D := D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, endowed with norm

$$\|(u,v)\|_D^2 := |\nabla u|_2^2 + |\nabla v|_2^2.$$

Then S_0 is attained by (U, V), where U, V are positive and satisfy the decay conditions

$$U(x) + V(x) \leq C(1 + |x|)^{2-N}, \quad |\nabla U(x)| + |\nabla V(x)| \leq C(1 + |x|)^{1-N}. \tag{2.7}$$

Lemma 2.1. *If $A \in L^N_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N)$, then $S_A = S_0$.*

This equality is generally known, but for the readers' convenience, we will briefly outline the proof.

Proof. For any $(u,v) \in D_A \setminus \{(0,0)\}$, by the Sobolev and the diamagnetic inequalities, we have

$$\begin{aligned} S_A &\geq \frac{|\nabla_A u|_2^2 + |\nabla_A v|_2^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} + \mu|v|^{2^*} + |u|^{2^*-1}v \,\mathrm{d}x\right)^{\frac{2}{2^*}}} \\ &\geq \frac{\big|\nabla|u|\big|_2^2 + \big|\nabla|v|\big|_2^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} + \mu|v|^{2^*} + |u|^{2^*-1}v \,\mathrm{d}x\right)^{\frac{2}{2^*}}} \geq S_0, \end{aligned}$$

By (2.1)–(2.6), we deduce that $\{u_\epsilon\}$ is bounded in $L^{2^*}(\mathbb{R}^N)$ and $u_\epsilon \to 0$ a.e. in $\mathbb{R}^N$ as $\epsilon \to 0$. Thus for any $\phi \in L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N} u_\epsilon \phi \,\mathrm{d}x\right| \leq \left(\int_{\mathbb{R}^N} u_\epsilon^{2^*} \,\mathrm{d}x\right)^{\frac{1}{2^*}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{2^*}{2^*-1}} \,\mathrm{d}x\right)^{\frac{2^*-1}{2^*}} \to 0,$$

that is, $u_\epsilon \rightharpoonup 0$ weakly in $L^{2^*}(\mathbb{R}^N)$. Hence, $u_\epsilon^2 \rightharpoonup 0$ weakly in $L^{\frac{2^*}{2}}(\mathbb{R}^N)$, where the duality product is taken with respect to $L^{\frac{N}{2}}(\mathbb{R}^N)$ and $L^{\frac{2^*}{2}}(\mathbb{R}^N)$. By the same argument,

$$\int_{\mathbb{R}^N} |Au_\epsilon|^2 \,\mathrm{d}x = \left\langle |A|^2, |u_\epsilon|^2 \right\rangle \to 0 \quad \text{as } \epsilon \to 0.$$

Let $\delta > 0$. Choosing ϵ small enough, we have

$$\begin{aligned} S_A &\leq \frac{|\nabla_A u_\epsilon|_2^2 + |\nabla_A v_\epsilon|_2^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} + \mu|v|^{2^*} + |u|^{2^*-1}v \,\mathrm{d}x\right)^{\frac{2}{2^*}}} \\ &= \frac{|\nabla u_\epsilon|_2^2 + |\nabla v_\epsilon|_2^2 + |Au_\epsilon|_2^2 + |Av_\epsilon|_2^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} + \mu|v|^{2^*} + |u|^{2^*-1}v \,\mathrm{d}x\right)^{\frac{2}{2^*}}} \\ &\leq \frac{|\nabla U|_2^2 + |\nabla V|_2^2 + |Au_\epsilon|_2^2 + |Av_\epsilon|_2^2 + O(\epsilon^{N-2})}{\left(\int_{\mathbb{R}^N} |U|^{2^*} + \mu|V|^{2^*} + |U|^{2^*-1}V \,\mathrm{d}x + O(\epsilon^N)\right)^{\frac{2}{2^*}}} = S_0 + \delta, \end{aligned}$$

that is, $S_A \leq S_0$. Thus $S_A = S_0$. □

Lemma 2.2. *The embedding $H^1_A(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for $1 \leq p \leq 2^*$ and it is compact for $1 \leq p < 2^*$. The embedding $D^{1,2}_A(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous.*

Proof. Let $\{u_n\}$ be a bounded sequence in $H^1_A(\Omega)$; then there exists a subsequence of $\{u_n\}$, still denoted by u_n. Thus $u_n \rightharpoonup u$ weakly in $H^1_A(\Omega)$, and then $u_n \rightharpoonup u$ weakly in $L^{2^*}(\Omega)$ and $|u_n - u|$ is bounded in $H^1_0(\Omega)$. Hence, up to a subsequence, $|u_n - u| \rightharpoonup 0$ weakly in $H^1_0(\Omega)$ and $u_n \to u$ a.e. on Ω.

By the Rellich–Kondrachov theorem, we see that

$$u_n \to u \text{ strongly in } L^q(\Omega), \quad \text{where } 1 \leq q < 2^*.$$

Since $H^1_A(\Omega) \hookrightarrow L^{2^*}(\Omega)$, there exists a constant C such that $|u_n - u|^{2^*}_{2^*,\Omega} \le C$. For any $\epsilon > 0$, let

$$\Omega_\epsilon := \Omega \cap B_\epsilon \quad \text{and} \quad \Omega^c_\epsilon := \Omega \setminus \Omega_\epsilon,$$

where B_ϵ is the ball centered at 0 with radius ϵ. We have

$$\begin{aligned}\int_{\Omega_\epsilon} |u_n - u|^p \,\mathrm{d}x &\le \left(\int_{\Omega_\epsilon} |u_n - u|^{2^*} \,\mathrm{d}x\right)^{\frac{p}{2^*}} \left(\int_{\Omega_\epsilon} 1^{\frac{2^*}{2^*-p}} \,\mathrm{d}x\right)^{\frac{2^*-p}{2^*}} \\ &\le C\left(\epsilon^N\right)^{\frac{2^*-p}{2^*}} \\ &= O\left(\epsilon^{\frac{(N-2)(2^*-p)}{2}}\right).\end{aligned}$$

Moreover, it follows from the Rellich–Kondrachov compactness theorem that

$$\int_{\Omega^c_\epsilon} |u_n - u|^p \,\mathrm{d}x = o(1).$$

Thus

$$\lim_{n\to\infty} |u_n - u|^p_{p,\Omega} = 0.$$

So the embedding $H^1_A(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for $1 \le p \le 2^*$ and compact for $1 \le p < 2^*$. □

Lemma 2.3. *If* $A \in L^N_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$*, then* S_A *is attained by a* $u \in D^{1,2}_A(\mathbb{R}^N) \setminus \{0\}$ *if and only if* $\operatorname{curl} A \equiv 0$.

Proof. (Necessary condition) Assume that S_A is attained at (u, v), which satisfies

$$\int_{\mathbb{R}^N} |u|^{2^*} + \mu|v|^{2^*} + |u|^{2^*-1}v \,\mathrm{d}x = 1.$$

Using the diamagnetic inequality and Lemma 2.1, we obtain

$$S_A = |\nabla_A u|^2_2 + |\nabla_A v|^2_2 \ge \big|\nabla|u|\big|^2_2 + \big|\nabla|v|\big|^2_2 \ge S_0 = S_A.$$

Therefore

$$\begin{aligned}|\nabla_A u| &= \big|\nabla|u|\big| = \left|\mathrm{Re}\left(\nabla u \frac{\overline{u}}{|u|}\right)\right| = \left|\mathrm{Re}(\nabla u + iAu)\frac{\overline{u}}{|u|}\right|,\\ |\nabla_A v| &= \big|\nabla|v|\big| = \left|\mathrm{Re}\left(\nabla v \frac{\overline{v}}{|v|}\right)\right| = \left|\mathrm{Re}(\nabla v + iAv)\frac{\overline{v}}{|v|}\right|.\end{aligned}$$

Then, we have $\mathrm{Im}\left(\nabla u \frac{\overline{u}}{|u|}\right) = 0$ and $\mathrm{Im}\left(\nabla v \frac{\overline{v}}{|v|}\right) = 0$, which are equivalent to $A = -\mathrm{Im}\left(\frac{\nabla u}{u}\right) = -\mathrm{Im}\left(\frac{\nabla v}{v}\right)$. Since $\operatorname{curl}\left(\frac{\nabla u}{u}\right) = 0$ and $\operatorname{curl}\left(\frac{\nabla v}{v}\right) = 0$, we infer that $\operatorname{curl} A = 0$.

(Sufficient condition) Assume $\operatorname{curl} A = 0$. By [12], there exists $\vartheta \in W^{1,N}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R})$ such that $\nabla\vartheta = A$. Let

$$(u_\epsilon(x), v_\epsilon(x)) = \left(U_\epsilon(x)e^{-i\vartheta(x)}, V_\epsilon(x)e^{-i\vartheta(x)}\right),$$

where $\epsilon > 0$ and (U_ϵ, V_ϵ) is defined in (2.1). It follows from Lemma 2.1 that (u_ϵ, v_ϵ) is a minimizer for S_A. □

Lemma 2.4. *Assume that* $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$*,* $N > 4$ *and* $\sigma(-\Delta_A - \lambda) \subset (0, +\infty)$*, where* $\sigma(\cdot)$ *is the spectrum in* $L^2(\mathbb{R}^N)$*. If there exists* $\overline{x} \in \mathbb{R}^N$ *such that* $\lambda > 0$ *and* A *is continuous at* $\overline{x}$*, then* $S_{A,\lambda}$ *is attained by some* $(u, v) \in H_A$ *such that* $u \not\equiv 0$*,* $v \not\equiv 0$*.*

Proof. Without loss of generality, assume $\overline{x} = 0$. Setting

$$\theta(x) := -\sum_{j=1}^{N} A_j(0) x_j,$$

we have

$$\nabla\theta(x) = (-A_1(0), \ldots, -A_N(0)) = -A(0),$$

which implies that $(\nabla\theta + A)(0) = 0$. Then by continuity, there exists $\delta > 0$ satisfying

$$|(\nabla\theta + A)(x)|^2 \le \frac{\lambda}{2} \quad \text{for all } |x| < \delta. \tag{2.8}$$

There exists $\rho > 0$ such that $B_\rho \subset \Omega$. Let $2r := \min\{\delta, \rho\}$ and

$$(u_\epsilon(x), v_\epsilon(x)) = \left(\phi(x) U_\epsilon(x) e^{i\theta(x)}, \phi(x) V_\epsilon(x) e^{i\theta(x)}\right).$$

From (2.2), (2.3) and (2.8), we deduce that

$$\begin{aligned}
&\int_\Omega \left(|\nabla_A u_\epsilon|^2 - \lambda |u_\epsilon|^2 + |\nabla_A v_\epsilon|^2 - \lambda |v_\epsilon|^2\right) \mathrm{d}x \\
&\qquad = \int_\Omega \left(|\nabla(\phi U_\epsilon)|^2 + \phi^2 U_\epsilon^2 |\nabla\theta + A|^2 - \lambda \phi^2 U_\epsilon^2\right) \mathrm{d}x \\
&\qquad\quad + \int_\Omega \left(|\nabla(\phi V_\epsilon)|^2 + \phi^2 V_\epsilon^2 |\nabla\theta + A|^2 - \lambda \phi^2 V_\epsilon^2\right) \mathrm{d}x \\
&\qquad \le \int_{\mathbb{R}^N} (|\nabla U|^2 + |\nabla V|^2)\, \mathrm{d}x + O(\epsilon^{N-2}) + \frac{\lambda}{2} \int_{B_{2r}} \phi^2 U_\epsilon^2 \,\mathrm{d}x \\
&\qquad\quad - \lambda \int_{B_{2r}} \phi^2 U_\epsilon^2 \,\mathrm{d}x + \frac{\lambda}{2} \int_{B_{2r}} \phi^2 V_\epsilon^2 \,\mathrm{d}x - \lambda \int_{B_{2r}} \phi^2 V_\epsilon^2 \,\mathrm{d}x \\
&\qquad \le \int_{\mathbb{R}^N} (|\nabla U|^2 + |\nabla V|^2)\, \mathrm{d}x + O(\epsilon^{N-2}) - \frac{\lambda}{2} \int_{B_r} (U_\epsilon^2 + V_\epsilon^2)\, \mathrm{d}x.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{B_r} |U_\epsilon|^2 \,\mathrm{d}x &\ge \int_{|x| \le r} \epsilon^{2-N} \left|U\left(\frac{x}{\epsilon}\right)\right|^2 \mathrm{d}x = \epsilon^2 \int_{\mathbb{R}^N} |U(y)|^2 \,\mathrm{d}y - \epsilon^2 \int_{|y| \ge \frac{r}{\epsilon}} |U(y)|^2 \,\mathrm{d}y \\
&\ge C\epsilon^2 - C\epsilon^2 \int_{|y| \ge \frac{r}{\epsilon}} |y|^{4-2N} \,\mathrm{d}y = C\epsilon^2 - C\epsilon^2 \int_{|y| \ge \frac{r}{\epsilon}} \int_{\Sigma_r} r^{3-N} \,\mathrm{d}r \\
&= C\epsilon^2 + O(\epsilon^{N-2}),
\end{aligned}$$

and

$$\int_{B_r} |V_\epsilon|^2 \,\mathrm{d}x \ge C\epsilon^2 + O(\epsilon^{N-2}),$$

where Σ_r is the area of a sphere with radius r. By (2.4)–(2.6), we have

$$\begin{aligned} S_{A,\lambda} &\leq \frac{\|(u_\epsilon, v_\epsilon)\|_{H_A}^2}{\left(|u_\epsilon|_{2^*,\Omega}^{2^*} + \mu|v_\epsilon|_{2^*,\Omega}^{2^*} + \int_\Omega |u_\epsilon|^{2^*-1} v_\epsilon \,\mathrm{d}x\right)^{\frac{2}{2^*}}} \\ &\leq \frac{\int_{\mathbb{R}^N} (|\nabla U|^2 + |\nabla V|^2)\,\mathrm{d}x - C\epsilon^2 + O(\epsilon^{N-2})}{\left(|U|_{2^*}^{2^*} + \mu|V|_{2^*}^{2^*} + \int_{\mathbb{R}^N} |U|^{2^*-1} V \,\mathrm{d}x\right)^{\frac{2}{2^*}}} \\ &< S_\lambda. \end{aligned} \tag{2.9}$$

Let $\{(u_n, v_n)\}$ be a minimizing sequence for $S_{A,\lambda}$ normalized as

$$|u_n|_{2^*,\Omega}^{2^*} + \mu|v_n|_{2^*,\Omega}^{2^*} + \int_\Omega |u_n|^{2^*-1} v_n \,\mathrm{d}x = 1,$$

that is,

$$|\nabla_A u_n|_{2,\Omega}^2 - \lambda|u_n|_{2,\Omega}^2 + |\nabla_A v_n|_{2,\Omega}^2 - \lambda|v_n|_{2,\Omega}^2 = S_{A,\lambda} + o(1). \tag{2.10}$$

Observing that $\{u_n\}$, $\{v_n\}$ are bounded in $H_A^1(\Omega)$, by Lemma 2.2, we have

$$\begin{aligned} &u_n \rightharpoonup u, \quad v_n \rightharpoonup v \text{ in } H_A^1(\Omega), \\ &u_n \to u, \quad v_n \to v \text{ in } L^2(\Omega), \\ &u_n \to u, \quad v_n \to v \text{ a.e. on } \Omega, \end{aligned}$$

and

$$|u_n|_{2^*,\Omega}^{2^*} + \mu|v_n|_{2^*,\Omega}^{2^*} + \int_\Omega |u_n|^{2^*-1} v_n \,\mathrm{d}x \leq 1.$$

Setting

$$w_n := u_n - u, \quad z_n := v_n - v,$$

then $w_n \rightharpoonup 0$, $z_n \rightharpoonup 0$ weakly in $H_A^1(\Omega)$ and $w_n \to 0$, $z_n \to 0$ a.e. on Ω. By the diamagnetic inequality and (2.10), we have

$$|\nabla_A u_n|_{2,\Omega}^2 + |\nabla_A v_n|_{2,\Omega}^2 \geq \big|\nabla|u_n|\big|_{2,\Omega}^2 + \big|\nabla|v_n|\big|_{2,\Omega}^2 \geq S_\lambda,$$

$$S_{A,\lambda} + \lambda\left(|u_n|_{2,\Omega}^2 + |v_n|_{2,\Omega}^2\right) + o(1) \geq S_\lambda.$$

From (2.9), we see that

$$\lambda\left(|u_n|_{2,\Omega}^2 + |v_n|_{2,\Omega}^2\right) \geq S_\lambda - S_{A,\lambda} > 0,$$

which means $(u, v) \not\equiv (0, 0)$. Since $w_n \rightharpoonup 0$, $z_n \rightharpoonup 0$ weakly in $H_A^1(\Omega)$, we obtain

$$\begin{aligned} |\nabla_A u_n|_{2,\Omega}^2 &= \int_\Omega |\nabla_A w_n|^2 \,\mathrm{d}x + \int_\Omega |\nabla_A u|^2 \,\mathrm{d}x + 2\int_\Omega \nabla_A w_n \cdot \overline{\nabla_A u} \,\mathrm{d}x \\ &= |\nabla_A w_n|_{2,\Omega}^2 + |\nabla_A u|_{2,\Omega}^2 + o(1), \end{aligned}$$

and

$$|\nabla_A v_n|_{2,\Omega}^2 = |\nabla_A z_n|_{2,\Omega}^2 + |\nabla_A v|_{2,\Omega}^2 + o(1).$$

Then (2.10) yields

$$S_{A,\lambda} = |\nabla_A w_n|_{2,\Omega}^2 + |\nabla_A u|_{2,\Omega}^2 - \lambda|u|_{2,\Omega}^2 + |\nabla_A z_n|_{2,\Omega}^2 + |\nabla_A v|_{2,\Omega}^2 - \lambda|v|_{2,\Omega}^2 + o(1). \tag{2.11}$$

From the Brezis–Lieb Lemma, one has

$$
\begin{aligned}
1 &= |u+w_n|_{2^*,\Omega}^{2^*} + \mu|v+z_n|_{2^*,\Omega}^{2^*} + \int_\Omega |u+w_n|^{2^*-1}|v+z_n|\,\mathrm{d}x \\
&= |u|_{2^*,\Omega}^{2^*} + \mu|v|_{2^*,\Omega}^{2^*} + \int_\Omega |u|^{2^*-1}v\,\mathrm{d}x + |w_n|_{2^*,\Omega}^{2^*} + \mu|z_n|_{2^*,\Omega}^{2^*} \\
&\quad + \int_\Omega |w_n|^{2^*-1}|z_n|\,\mathrm{d}x + o(1).
\end{aligned}
$$

Noting

$$
|u|_{2^*,\Omega}^{2^*} + \mu|v|_{2^*,\Omega}^{2^*} + \int_\Omega |u|^{2^*-1}v\,\mathrm{d}x \le 1
$$

and

$$
|w_n|_{2^*,\Omega}^{2^*} + \mu|z_n|_{2^*,\Omega}^{2^*} + \int_\Omega |w_n|^{2^*-1}|z_n|\,\mathrm{d}x \le 1,
$$

we have

$$
\begin{aligned}
1 &\le \left(|u|_{2^*,\Omega}^{2^*} + \mu|v|_{2^*,\Omega}^{2^*} + \int_\Omega |u|^{2^*-1}|v\,\mathrm{d}x\right)^{\frac{2}{2^*}} \\
&\quad + \left(|w_n|_{2^*,\Omega}^{2^*} + \mu|z_n|_{2^*,\Omega}^{2^*} + \int_\Omega |w_n|^{2^*-1}|z_n|\,\mathrm{d}x\right)^{\frac{2}{2^*}} + o(1) \\
&\le \left(|u|_{2^*,\Omega}^{2^*} + \mu|v|_{2^*,\Omega}^{2^*} + \int_\Omega |u|^{2^*-1}|v\,\mathrm{d}x\right)^{\frac{2}{2^*}} + \frac{1}{S_\lambda}\left(\big|\nabla|w_n|\big|_{2,\Omega}^2 + \big|\nabla|z_n|\big|_{2,\Omega}^2\right) + o(1) \\
&\le \left(|u|_{2^*,\Omega}^{2^*} + \mu|v|_{2^*,\Omega}^{2^*} + \int_\Omega |u|^{2^*-1}|v\,\mathrm{d}x\right)^{\frac{2}{2^*}} + \frac{1}{S_\lambda}\left(|\nabla_A w_n|_{2,\Omega}^2 + |\nabla_A z_n|_{2,\Omega}^2\right) + o(1).
\end{aligned}
$$

Using (2.9), (2.11) and the fact that $S_{A,\lambda} > 0$, we obtain

$$
\begin{aligned}
&|\nabla_A u|_{2,\Omega}^2 - \lambda|u|_{2,\Omega}^2 + |\nabla_A v|_{2,\Omega}^2 - \lambda|v|_{2,\Omega}^2 \\
&\quad \le S_{A,\lambda}\left(|u|_{2^*,\Omega}^{2^*} + \mu|v|_{2^*,\Omega}^{2^*} + \int_\Omega |u|^{2^*-1}|v\,\mathrm{d}x\right)^{\frac{2}{2^*}} \\
&\qquad + \left(\frac{S_{A,\lambda}}{S_\lambda} - 1\right)\left(|\nabla_A w_n|_{2,\Omega}^2 + |\nabla_A z_n|_{2,\Omega}^2\right) + o(1) \\
&\quad < S_{A,\lambda}\left(|u|_{2^*,\Omega}^{2^*} + \mu|v|_{2^*,\Omega}^{2^*} + \int_\Omega |u|^{2^*-1}|v\,\mathrm{d}x\right)^{\frac{2}{2^*}} + o(1).
\end{aligned}
$$

Combining with $(u,v) \not\equiv (0,0)$, we have

$$
\frac{|\nabla_A u|_{2,\Omega}^2 - \lambda|u|_{2,\Omega}^2 + |\nabla_A v|_{2,\Omega}^2 - \lambda|v|_{2,\Omega}^2}{\left(|u|_{2^*,\Omega}^{2^*} + \mu|v|_{2^*,\Omega}^{2^*} + \int_\Omega |u|^{2^*-1}|v\,\mathrm{d}x\right)^{\frac{2}{2^*}}} \le S_{A,\lambda}.
$$

In summary, $S_{A,\lambda}$ is attained by (u,v). □

In order to obtain Theorem 1.1, we consider the limit case ($\Omega = \mathbb{R}^N$ and $\lambda = 0$)

$$\begin{cases} (-i\nabla + A)^2 u = |u|^{2^*-2} v, & x \in \mathbb{R}^N, \\ (-i\nabla + A)^2 v = \mu |v|^{2^*-2} v + |u|^{2^*-2} u, & x \in \mathbb{R}^N, \\ u, v \in D_A^{1,2}(\mathbb{R}^N). \end{cases} \tag{2.12}$$

Thus, we search nontrivial solutions to (2.12) as critical points of the functional

$$\begin{aligned} J_0(u,v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_A u|^2 \,\mathrm{d}x + \frac{1}{2(2^*-1)} \int_{\mathbb{R}^N} |\nabla_A v|^2 \,\mathrm{d}x \\ - \frac{1}{2^*(2^*-1)} \mu \int_{\mathbb{R}^N} |v|^{2^*} \,\mathrm{d}x - \frac{1}{2^*-1} \int_{\mathbb{R}^N} |u|^{2^*-1} v \,\mathrm{d}x \end{aligned}$$

defined in $D_A^{1,2}(\mathbb{R}^N) \times D_A^{1,2}(\mathbb{R}^N)$. In particular, we investigate ground state solutions of (2.12) of the form $(ku_\epsilon, lu_\epsilon)$ with $k, l > 0$. So we consider

$$\mathcal{N}_0 := \Big\{ (u,v) \in \big(D_A^{1,2}(\mathbb{R}^N) \times D_A^{1,2}(\mathbb{R}^N) \big) \setminus \{(0,0)\} : F_0(u,v) = (0,0) \Big\}, \tag{2.13}$$

where

$$\begin{aligned} F_0(u,v) = \Bigg(\int_{\mathbb{R}^N} |\nabla_A u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} |u|^{2^*-1} v \,\mathrm{d}x, \\ \int_{\mathbb{R}^N} |\nabla_A v|^2 \,\mathrm{d}x - \mu \int_{\mathbb{R}^N} |v|^{2^*} \,\mathrm{d}x - \int_{\mathbb{R}^N} |u|^{2^*-1} v \,\mathrm{d}x \Bigg), \end{aligned}$$

$$\mathcal{N}_0' := \Big\{ (u,v) \in \big(D_A^{1,2}(\mathbb{R}^N) \times D_A^{1,2}(\mathbb{R}^N) \big) \setminus \{(0,0)\} : H_0(u,v) = 0 \Big\}, \tag{2.14}$$

and

$$\begin{aligned} H_0(u,v) = \int_{\mathbb{R}^N} |\nabla_A u|^2 \,\mathrm{d}x + \frac{1}{2^*-1} \int_{\mathbb{R}^N} |\nabla_A v|^2 \,\mathrm{d}x \\ - \frac{2^*}{2^*-1} \int_{\mathbb{R}^N} |u|^{2^*-1} v \,\mathrm{d}x - \frac{\mu}{2^*-1} \int_{\mathbb{R}^N} |v|^{2^*} \,\mathrm{d}x. \end{aligned}$$

For the limit case, define $M := \inf_{(u,v) \in \mathcal{N}_0} J_0(u,v)$ and $M' := \inf_{(u,v) \in \mathcal{N}_0'} J_0(u,v)$. Then we obtain the following proof.

3. Proof of the main results

3.1. The limit problem for $N > 4$.

Lemma 3.1. *Define a function $f_N : (0, +\infty) \to \mathbb{R}$,*

$$f_N(m) = m^{2^*-1} - m^{2^*-3} + \mu.$$

Then the function f_N is strictly increasing and satisfies

$$\lim_{m \to 0^+} f_N(m) = -\infty \quad \text{and} \quad \lim_{m \to +\infty} f_N(m) = +\infty,$$

namely, f_N has at least one zero point. Let $k, l > 0$ satisfy

$$k^2 + \frac{1}{2^*-1} l^2 \leq \frac{2^*}{2^*-1} k^{2^*-1} l + \frac{1}{2^*-1} \mu l^{2^*}. \tag{3.1}$$

Considering the system

$$\begin{cases} k^{2^*-3}l = 1, \\ \mu l^{2^*-1} + k^{2^*-1} = l, \\ k, l > 0, \end{cases} \tag{3.2}$$

we have

$$k_0^2 + \frac{1}{2^*-1}l_0^2 = \min_{i=1,2,\dots,n}\left\{k_i^2 + \frac{1}{2^*-1}l_i^2\right\} \le k^2 + \frac{1}{2^*-1}l^2, \tag{3.3}$$

where (k_i, l_i) *are solutions of system* (3.2)*, and* (k_0, l_0) *is a particular solution of system* (3.2)*.*

Proof. By a simple calculation of system (3.2), we obtain

$$\left(\frac{k}{l}\right)^{2^*-1} - \left(\frac{k}{l}\right)^{2^*-3} + \mu = 0.$$

Clearly, f_N has at most finite multiple solutions and system (3.2) has some solutions correspondingly.

(i) If f_N has a unique zero point m_1, then system (3.2) has a unique solution denoted as (k_1, l_1).

(ii) If f_N has n zero points, which are denoted as m_i $(i = 1, 2, \dots, n)$, we can assume $m_1 < m_2 < \cdots < m_n$, then system (3.2) has n solutions correspondingly denoted as $(k_i, l_i) = \left(m_i^{\frac{1}{2^*-2}}, m_i^{\frac{3-2^*}{2^*-2}}\right)$. Thus, there exists a minimum one, which is denoted as $(k_0, l_0) := \left(m_0^{\frac{1}{2^*-2}}, m_0^{\frac{3-2^*}{2^*-2}}\right)$. Then

$$k_0^2 + \frac{1}{2^*-1}l_0^2 := \min_{i=1,2,\dots,n}\left\{k_i^2 + \frac{1}{2^*-1}l_i^2\right\},$$

where (k_0, l_0) is one of the solutions of system (3.2).

Fix $k, l > 0$ satisfying (3.2) and let $h := \frac{(2^*-1)k^2+l^2}{l(2^*k^{2^*-1}+\mu l^{2^*-1})}$. Then

$$k_i = kh^{\frac{1}{2^*-2}}, \quad l_i = lh^{\frac{1}{2^*-2}},$$

which implies that $\frac{k_i}{l_i} = \frac{k}{l}$, so (k_i, l_i) are solutions of system (3.2). Since

$$0 < k_i \le k, \quad 0 < l_i \le l, \tag{3.4}$$

we have

$$k_i^2 + \frac{1}{2^*-1}l_i^2 \le k^2 + \frac{1}{2^*-1}l^2.$$

Thus

$$k_0^2 + \frac{1}{2^*-1}l_0^2 = \min_{i=1,2,\dots,n}\left\{k_i^2 + \frac{1}{2^*-1}l_i^2\right\} \le k^2 + \frac{1}{2^*-1}l^2.$$

□

Proof of Theorem 1.2. Assume $(au_\epsilon, bu_\epsilon) \in \mathcal{N}_0$. Then $F_0(au_\epsilon, bu_\epsilon) = (0,0)$, that is,

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla_A(au_\epsilon)|^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} |au_\epsilon|^{2^*-1} bu_\epsilon \, \mathrm{d}x = 0, \\ \int_{\mathbb{R}^N} |\nabla_A(bu_\epsilon)|^2 \, \mathrm{d}x - \mu \int_{\mathbb{R}^N} |bu_\epsilon|^{2^*} \, \mathrm{d}x - \int_{\mathbb{R}^N} |au_\epsilon|^{2^*-1} bu_\epsilon \, \mathrm{d}x = 0. \end{cases}$$

Then the above can be rewritten as

$$\begin{cases} \dfrac{\int_{\mathbb{R}^N} |\nabla_A u_\epsilon|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^N} |u_\epsilon|^{2^*} \, \mathrm{d}x} = a^{2^*-3} b, \\ \dfrac{1}{b^{2^*-2}} \cdot \dfrac{\int_{\mathbb{R}^N} |\nabla_A u_\epsilon|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^N} |u_\epsilon|^{2^*} \, \mathrm{d}x} - \mu - \dfrac{a^{2^*-1}}{b^{2^*-1}} = 0; \end{cases}$$

therefore

$$\left(\frac{a}{b}\right)^{2^*-1} - \left(\frac{a}{b}\right)^{2^*-3} + \mu = 0.$$

Since f_N admits a minimum nontrivial zero point m_0, we assume $m_0 = \frac{a}{b}$. Then we obtain

$$a = \left[m_0 \int_{\mathbb{R}^N} |\nabla_A u_\epsilon|^2 \, \mathrm{d}x \left(\int_{\mathbb{R}^N} |u_\epsilon|^{2^*} \, \mathrm{d}x \right)^{-1} \right]^{\frac{1}{2^*-2}},$$

$$b = \left[m_0^{3-2^*} \int_{\mathbb{R}^N} |\nabla_A u_\epsilon|^2 \, \mathrm{d}x \left(\int_{\mathbb{R}^N} |u_\epsilon|^{2^*} \, \mathrm{d}x \right)^{-1} \right]^{\frac{1}{2^*-2}}.$$

Thus

$$\Bigg(\left[m_0 \int_{\mathbb{R}^N} |\nabla_A u_\epsilon|^2 \, \mathrm{d}x \left(\int_{\mathbb{R}^N} |u_\epsilon|^{2^*} \, \mathrm{d}x \right)^{-1} \right]^{\frac{1}{2^*-2}} u_\epsilon, \\ \left[m_0^{3-2^*} \int_{\mathbb{R}^N} |\nabla_A u_\epsilon|^2 \, \mathrm{d}x \left(\int_{\mathbb{R}^N} |u_\epsilon|^{2^*} \, \mathrm{d}x \right)^{-1} \right]^{\frac{1}{2^*-2}} u_\epsilon \Bigg) \in \mathcal{N}_0$$

and system (3.2) has a solution $(k_0, l_0) = \left(m_0^{\frac{1}{2^*-2}}, m_0^{\frac{3-2^*}{2^*-2}} \right)$. Since $\mathcal{N}_0 \subset \mathcal{N}_0'$, we have $M' \leq M$. Also by $J_0'(k_0 u_\epsilon, l_0 u_\epsilon) = 0$ and $(k_0 u_\epsilon, l_0 u_\epsilon) \in \mathcal{N}_0 \subset \mathcal{N}_0'$, we have

$$M' \leq M \leq J_0\left(m_0^{\frac{1}{2^*-2}} u_\epsilon, m_0^{\frac{3-2^*}{2^*-2}} u_\epsilon \right) = \frac{1}{N} \left(k_0^2 + \frac{1}{2^*-1} l_0^2 \right) S_A^{N/2}.$$

Let$\{(u_n,v_n)\}\subset\mathcal{N}_0'$ be a minimizing sequence, such that $J_0(u_n,v_n)\to M'$. By the Sobolev embedding theorem and Hölder's inequality, we have

$$\begin{aligned}
&S_A\left(|u_n|_{2^*}^2+\frac{1}{2^*-1}|v_n|_{2^*}^2\right)\\
&\quad\le|\nabla_A u_n|_2^2+\frac{1}{2^*-1}|\nabla_A v_n|_2^2\\
&\quad=\frac{2^*}{2^*-1}\int_{\mathbb{R}^N}|u_n|^{2^*-1}v_n\,\mathrm{d}x+\frac{\mu}{2^*-1}\int_{\mathbb{R}^N}|v_n|^{2^*}\,\mathrm{d}x\\
&\quad\le\frac{2^*}{2^*-1}\int_{\mathbb{R}^N}|u_n|^{\frac{2^*-1}{2^*}}\,\mathrm{d}x\int_{\mathbb{R}^N}|v_n|^{2^*}\,\mathrm{d}x+\frac{\mu}{2^*-1}\int_{\mathbb{R}^N}|v_n|^{2^*}\,\mathrm{d}x.
\end{aligned}\tag{3.5}$$

By direct computation, we obtain

$$\begin{aligned}
&\left(S_A^{\frac{2-N}{4}}|u_n|_{2^*}\right)^2+\frac{1}{2^*-1}\left(S_A^{\frac{2-N}{4}}|v_n|_{2^*}\right)^2\\
&\le\frac{2^*}{2^*-1}\left(S_A^{\frac{2-N}{4}}|u_n|_{2^*}\right)^{2^*-1}\left(\frac{1}{2^*-1}S_A^{\frac{2-N}{4}}|v_n|_{2^*}\right)+\frac{\mu}{2^*-1}\left(S_A^{\frac{2-N}{4}}|u_n|_{2^*}\right)^{2^*}.
\end{aligned}$$

Then it follows from Lemma 3.1 that

$$k_0^2+\frac{1}{2^*-1}l_0^2\le S_A^{1-\frac{N}{2}}\left(|u_n|_{2^*}^2+\frac{1}{2^*-1}|v_n|_{2^*}^2\right).$$

Thus

$$\begin{aligned}
M'+o_n(1)&=J_0(u_n,v_n)\\
&=\frac{1}{2}|\nabla_A u_n|_2^2+\frac{1}{2(2^*-1)}|\nabla_A v_n|_2^2-\frac{\mu}{2^*(2^*-1)}\int_{\mathbb{R}^N}|v_n|^{2^*}\,\mathrm{d}x\\
&\quad-\frac{1}{2^*-1}\int_{\mathbb{R}^N}|u_n|^{2^*-2}u_nv_n\,\mathrm{d}x\\
&=\frac{1}{N}\left(|\nabla_A u_n|_2^2+\frac{1}{2^*-1}|\nabla_A v_n|_2^2\right)\\
&\ge\frac{S_A}{N}\left[\left(\int_{\mathbb{R}^N}|u_n|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}+\frac{1}{2^*-1}\left(\int_{\mathbb{R}^N}|v_n|^{2^*}\right)^{\frac{2}{2^*}}\,\mathrm{d}x\right]\\
&\ge\frac{1}{N}\left(k_0^2+\frac{l_0^2}{2^*-1}\right)S_A^{N/2}.
\end{aligned}$$

So $M'=\frac{1}{N}\left(k_0^2+\frac{l_0^2}{2^*-1}\right)S_A^{N/2}$ and we have that $\left(m_0^{\frac{1}{2^*-2}}u_\epsilon,m_0^{\frac{3-2^*}{2^*-2}}u_\epsilon\right)$ is a ground state solution of (2.12). ☐

3.2. **The limit problem for** $N=4$**.** In this subsection, we consider the limit problem for general $N=4$. We notice that in the previous subsection the key points consist in the existence of a zero point of the function f_N and the solution of the system (3.2). Similarly to the proof of Theorem 1.1, we have

$$f(m)=m^3-m+\mu,\quad m>0,\tag{3.6}$$

$$\begin{cases} kl = 1, \\ \mu l^3 + k^3 = l, \\ k, l > 0. \end{cases} \tag{3.7}$$

To prove Theorem 1.2, we give the following properties.

Proposition 3.2. *Let* $\mu \in [0, \frac{1}{4})$.

(i) *If* $\mu = 0$, $\mathcal{N}_0'$ *does not contain semitrivial couples.*

(ii) *If* $\mu \in (0, \frac{1}{4})$, $\mathcal{N}_0'$ *does not contain semitrivial couples* $(u, 0)$ *and*

$$M' < \inf_{(0,v)\in\mathcal{N}_0'} J_0(0, v).$$

Proof. (i) If $\mu = 0$, then

$$H_0(u, v) = |\nabla_A u|_2^2 + \frac{1}{3}|\nabla_A v|_2^2 - \frac{4}{3}\int_{\mathbb{R}^4} |u|^3 v \, \mathrm{d}x.$$

Assume $(u, v) \in \mathcal{N}_0'$. If $u = 0$, $v \neq 0$, then $H_0(u, v) = H_0(0, v) = |\nabla_A v|_2^2$, which is in contradiction with the definition of $\mathcal{N}_0'$. Likewise, if $v = 0$, $u \neq 0$, we also get a contradiction.

(ii) It is obvious that if $\mu \in (0, \frac{1}{4})$, $\mathcal{N}_0'$ does not contain semitrivial couples $(u, 0)$. Next we prove the second part of (ii). Let $(0, v) \in \mathcal{N}_0'$; we have

$$H_0(0, v) = \frac{1}{3}\left(|\nabla_A v|_2^2 - \mu \int_{\mathbb{R}^4} |v|^4 \, \mathrm{d}x\right) = 0$$

and

$$J_0(0, v) = \frac{1}{6}|\nabla_A v|_2^2 - \frac{1}{12}\mu \int_{\mathbb{R}^4} |v|^4 \, \mathrm{d}x = \frac{1}{6}|\nabla_A v|_2^2 - \frac{1}{12}|\nabla_A v|_2^2 = \frac{1}{12}|\nabla_A v|_2^2.$$

For every $r > 0$, $\big(t(r)rv, t(r)v\big) \in \mathcal{N}_0'$ with $t(r) = \left(\frac{(3r^2+1)^2\mu}{4r^3+\mu}\right)^{\frac{1}{2}}$ and then

$$M' \leq J_0\Big(t(r)rv, t(r)v\Big) = \frac{(3r^2+1)^2\mu}{12(4r^3+\mu)}|\nabla_A v|_2^2.$$

So, according to the definition of infimum, we obtain

$$M' \leq \frac{(3r^2+1)^2\mu}{4r^3+\mu} \inf_{(0,v)\in\mathcal{N}_0'} J_0(0, v).$$

As $\mu \in (0, \frac{1}{4})$ and $r = 4\mu$, we have $\frac{(3r^2+1)^2\mu}{4r^3+\mu} < 1$. Thus $M' < \inf_{(0,v)\in\mathcal{N}_0'} J_0(0, v)$. □

Proof of Theorem 1.3. As we said before, assume $(au_\epsilon, bu_\epsilon) \in \mathcal{N}_0$. Then we have $F_0(au_\epsilon, bu_\epsilon) = (0, 0)$, i.e.,

$$\begin{cases} \int_{\mathbb{R}^4} |\nabla_A(au_\epsilon)|^2 \, \mathrm{d}x - \int_{\mathbb{R}^4} |au_\epsilon|^3 bu_\epsilon \, \mathrm{d}x = 0, \\ \int_{\mathbb{R}^4} |\nabla_A(au_\epsilon)|^2 \, \mathrm{d}x - \mu \int_{\mathbb{R}^4} |bu_\epsilon|^4 \, \mathrm{d}x - \int_{\mathbb{R}^4} |au_\epsilon|^3 bu_\epsilon \, \mathrm{d}x = 0, \end{cases}$$

similar to the proof of Theorem 1.2, we have

$$a=\left[m_0\int_{\mathbb{R}^4}|\nabla_A u_\epsilon|^2\,\mathrm{d}x\left(\int_{\mathbb{R}^4}|u_\epsilon|^4\,\mathrm{d}x\right)^{-1}\right]^{\frac{1}{2}},$$

$$b=\left[\frac{1}{m_0}\int_{\mathbb{R}^4}|\nabla_A u_\epsilon|^2\,\mathrm{d}x\left(\int_{\mathbb{R}^4}|u_\epsilon|^4\,\mathrm{d}x\right)^{-1}\right]^{\frac{1}{2}}.$$

Then system (3.2) has a minimum solution $\left(\tilde{k},\tilde{l}\right)=\left(\sqrt{m_0},\frac{1}{\sqrt{m_0}}\right)$. Since $\mathcal{N}_0\subset\mathcal{N}_0'$, we have

$$M'\le M\le J_0\left(\sqrt{m_0}u_\epsilon,\frac{1}{\sqrt{m_0}}u_\epsilon\right)=\frac{1}{4}\left(\tilde{k}^2+\frac{1}{3}\tilde{l}^2\right)S_A^2=M',$$

that is, $\left(\sqrt{m_0}u_\epsilon,\frac{1}{\sqrt{m_0}}u_\epsilon\right)$ is a ground strate solution of (2.12).

If $\mu\in[0,\frac{1}{4})$, let $\{(u_n,v_n)\}\subset\mathcal{N}_0'$ be a minimizing sequence such that $J_0(u_n,v_n)\to M'$. By Proposition 3.2, we assume $u_n\neq0$ and $v_n\neq0$. Then

$$\begin{aligned}M'+o_n(1)=J_0(u_n,v_n)&=\frac{1}{4}\left(|\nabla_A u_n|_2^2+\frac{1}{3}|\nabla_A v_n|_2^2\right)\\&\ge\frac{1}{4}S_A\left(|u_n|_4^2+\frac{1}{3}|v_n|_4^2\right)\\&\ge\frac{1}{4}\left(\tilde{k}^2+\frac{1}{3}\tilde{l}^2\right)S_A^2,\end{aligned}$$

thus $M'=\frac{1}{4}\left(\tilde{k}^2+\frac{1}{3}\tilde{l}^2\right)S_A^2$. We have $\left(\sqrt{m_0}u_\epsilon,\frac{1}{\sqrt{m_0}}u_\epsilon\right)$ is a nontrivial ground state solution of (2.12). □

3.3. **Ground state solution for** (1.1)**.** In this section, we study the existence of ground state solutions of problem (1.1) and we will give the proof of Theorem 1.1. Before proving the main result, we show some lemmas. Since

$$\begin{aligned}F'(u,v)[u,v]&=\left((2-2^*)\left(\int_\Omega|\nabla_A u|^2\,\mathrm{d}x-\int_\Omega|u|^2\,\mathrm{d}x\right),(2-2^*)\int_\Omega|v|^{2^*}\,\mathrm{d}x\right)\\&\neq(0,0)\end{aligned}$$

for all $(u,v)\in\mathcal{N}$, we have that $\mathcal{N}$ is a C^1-manifold, where $\mathcal{N}$ is defined in (2.12).

Lemma 3.3. *If $\lambda\in(0,\lambda_1(\Omega))$ and $\mu\ge0$, then $\mathcal{N}\neq\emptyset$.*

Proof. Take $u\in H_A^1(\Omega)$, and

$$\theta=\frac{|\nabla_A u|_{2,\Omega}^2}{|\nabla_A u|_{2,\Omega}^2-\lambda|\nabla_A u|_{2,\Omega}^2},\quad\overline{\theta}:=\frac{|\nabla_A u|_{2,\Omega}^2-\lambda|\nabla_A u|_{2,\Omega}^2}{|u|_{2^*,\Omega}^{2^*}}.$$

Then m_0 is a strictly positive solution of

$$m^{2^*-1}-\theta m^{2^*-3}+\mu=0,$$

so we have

$$\Big((m_0\overline{\theta})^{\frac{1}{2^*-2}}u,(m_0^{3-2^*}\overline{\theta})^{\frac{1}{2^*-2}}u\Big)\in\mathcal{N}. \qquad \square$$

Now, let

$$\mathcal{B}:=\inf_{w\in\Gamma}\max_{t\in[0,1]}J(w(t)),$$

where $\Gamma:=\big\{w\in C\big([0,1],H_A^1(\Omega)\times H_A^1(\Omega)\big):w(0)=(0,0),J(w(1))<0\big\}$.

Lemma 3.4. *If $\lambda>0$ and $\mu\geq 0$, then $\mathcal{B}<M$.*

Proof. In order to prove $\mathcal{B}<M$, we may assume $0\in\Omega$ without loss of generality. Then by the definition of S, we have

$$\begin{aligned}|U_\epsilon|_{2^*,\Omega}^{2^*}&=S^{N/2}+O(\epsilon^N),\\ |U_\epsilon|_{2,\Omega}^2&\geq C\psi_N(\epsilon)+O(\epsilon^{N-2}),\\ |\nabla U_\epsilon|_{2,\Omega}^2&=S^{N/2}+O(\epsilon^{N-2})\end{aligned}$$

for some $C>0$, where

$$\psi_N(\epsilon)=\begin{cases}\epsilon^2 & \text{if } N>4,\\ \epsilon^2|\log\epsilon| & \text{if } N=4.\end{cases}$$

Define $(u_\epsilon,v_\epsilon):=(kU_\epsilon,lU_\epsilon)$, where $(k,l)\in\mathbb{R}^2$, $k,l>0$ and $(kU_\epsilon,lU_\epsilon)$ is a ground state solution of the limit problem (2.12). Then

$$\begin{aligned}|u_\epsilon|_{2,\Omega}^2&\geq C\psi(\epsilon)+O(\epsilon^{N-2}),\\ |v_\epsilon|_{2^*,\Omega}^{2^*}&=l^{2^*}S^{N/2}+O(\epsilon^N),\\ |\nabla u_\epsilon|_{2,\Omega}^2&=k^2S^{N/2}+O(\epsilon^{N-2}),\\ |\nabla v_\epsilon|_{2,\Omega}^2&=l^2S^{N/2}+O(\epsilon^{N-2}),\\ \int_\Omega u_\epsilon^{2^*-1}v_\epsilon\,\mathrm{d}x&=k^{2^*-1}lS^{N/2}+O(\epsilon^N).\end{aligned}$$

Noting that

$$k^2+\frac{1}{2^*-1}l^2=\frac{2^*}{2^*-1}k^{2^*-1}l+\frac{\mu}{2^*-1}l^{2^*},$$

we have

$$\begin{aligned}J(tu_\epsilon, tv_\epsilon) &= \frac{1}{2}|\nabla(tu)_\epsilon|_{2,\Omega}^2 - \frac{1}{2}\lambda\int_\Omega |tu_\epsilon|^2\,\mathrm{d}x + \frac{1}{2(2^*-1)}|\nabla(tv)_\epsilon|_{2,\Omega}^2 \\ &\quad - \frac{\mu}{2^*(2^*-1)}\int_\Omega |tv_\epsilon|^{2^*}\,\mathrm{d}x - \frac{1}{2^*-1}\int_\Omega |tu_\epsilon|^{2^*-1}tv_\epsilon\,\mathrm{d}x \\ &\le \frac{1}{2}t^2\left[\left(k^2+\frac{l^2}{2^*-1}\right)S^{N/2} - \lambda C\psi(\epsilon) + O(\epsilon^{N-2})\right] \\ &\quad - \frac{t^{2^*}}{2^*}\left[\left(k^2+\frac{l^2}{2^*-1}\right)S^{N/2} + O(\epsilon^N)\right] \\ &= \frac{1}{2}t^2\left(NA - \lambda C\psi(\epsilon) + O(\epsilon^2)\right) - \frac{1}{2_s^*}t^{2^*}\left(NA + O(\epsilon^N)\right).\end{aligned}$$

We consider

$$h(t) := \frac{t^2}{2}a_\epsilon - \frac{t^{2^*}}{2^*}b_\epsilon,$$

where

$$a_\epsilon = NA - \lambda C\psi(\epsilon) + O(\epsilon^2), \quad b_\epsilon = NA + O(\epsilon^N).$$

Obviously, for $\epsilon > 0$ and small enough,

$$\max_{t>0} h(t) = \frac{1}{N}\left(\frac{a_\epsilon}{b_\epsilon^{(N-2)/N}}\right)^{\frac{N}{2}} < M,$$

thus

$$\mathcal{B} \le \max_{t>0} J(tu_\epsilon, tv_\epsilon) < M. \qquad \square$$

Now we define some notions which will be useful in this paper.

$$\mathcal{N}' = \Big\{(u,v) \in \big(H_A^1(\Omega)\times H_A^1(\Omega)\big)\setminus\{(0,0)\} : H(u,v) = 0\Big\},$$

where

$$\begin{aligned}H(u,v) &= \int_\Omega |\nabla_A u|^2\,\mathrm{d}x - \lambda\int_\Omega |u|^2\,\mathrm{d}x + \frac{1}{2^*-1}\int_\Omega |\nabla_A v|^2\,\mathrm{d}x \\ &\quad - \frac{\mu}{2^*-1}\int_\Omega |v|^{2^*}\,\mathrm{d}x - \frac{2^*}{2^*-1}\int_\Omega |u|^{2^*-1}v\,\mathrm{d}x\end{aligned}$$

and

$$\mathcal{A} := \left\{(u,v) \in \big(H_A^1(\Omega)\times H_A^1(\Omega)\big) : \mu\int_\Omega |v|^{2^*}\,\mathrm{d}x + 2^*\int_\Omega |u|^{2^*-1}v\,\mathrm{d}x > 0\right\}$$

denotes the set of admissible pairs. Moreover, if $\lambda \in (0, \lambda_1(\Omega))$ for all $(u,v) \in \mathcal{N}'$, we have that $\mathcal{N}'$ is a C^1-manifold and

$$H'(u,v)[u,v] = (2-2^*)\left(\int_\Omega |\nabla_A u|^2\,\mathrm{d}x - \lambda\int_\Omega |u|^2\,\mathrm{d}x + \frac{1}{2^*-1}\int_\Omega |\nabla_A v|^2\,\mathrm{d}x\right) \ne 0.$$

Since $\mathcal{N} \subset \mathcal{N}' \subset \mathcal{A}$, we have

$$H(u,v) \ge \|(u,v)\|^2 - C\|(u,v)\|^{2^*}, \tag{3.8}$$

where $C > 0$ and

$$\|(u,v)\|^2 := \int_\Omega |\nabla_A u|^2\,\mathrm{d}x - \lambda \int_\Omega |u|^2\,\mathrm{d}x + \frac{1}{2^*-1}\int_\Omega |\nabla_A v|^2\,\mathrm{d}x.$$

Proposition 3.5. *If $\lambda \in (0, \lambda_1(\Omega))$ and $\mu \geq 0$, then*

$$\inf_{(u,v)\in\mathcal{N}'} J(u,v) = \inf_{(u,v)\in\mathcal{A}} \max_{t\geq 0} J(tu,tv) = \mathcal{B} > 0.$$

Proof. Let $(u,v) \in \mathcal{A}$ and

$$\tilde{t} = \left[\left(|\nabla_A u|_{2,\Omega}^2 - \lambda|u|_{2,\Omega}^2 + |\nabla_A v|_{2,\Omega}^2\right)\left(\tfrac{\mu}{2^*-1}|v|_{2,\Omega}^{2^*} + \tfrac{2^*}{2^*-1}\int_\Omega |u|^{2^*-1}v\,\mathrm{d}x\right)^{-1}\right]^{\frac{1}{2^*-2}}.$$

Then $\left(\tilde{t}u, \tilde{t}v\right) \in \mathcal{N}'$ and $J\left(\tilde{t}u, \tilde{t}v\right) \geq \inf_{(u,v)\in\mathcal{N}'} J(u,v)$. If $(u,v) \in \mathcal{A}$, then there exists $t > 0$ such that $J(tu,tv) < 0$ and

$$\inf_{(u,v)\in\mathcal{A}} \max_{t\geq 0} J(tu,tv) \geq \mathcal{B}. \tag{3.9}$$

Morever, if $(u,v) \in \mathcal{N}'$, then $\tilde{t} = 1$ and we have

$$\inf_{(u,v)\in\mathcal{N}'} J(u,v) \geq \inf_{(u,v)\in\mathcal{A}} \max_{t\geq 0} J(tu,tv). \tag{3.10}$$

Taking $w = (w_1, w_2) \in \Gamma$, then for a small t we have $H(w(t)) > 0$ and

$$H(w(1)) = 2J(w(1)) - \frac{2}{N(2^*-1)}\left(\mu\int_\Omega |w_2(1)|^{2^*}\,\mathrm{d}x + 2^*\int_\Omega |w_1(1)|^{2^*-1}w_2(1)\,\mathrm{d}x\right) < 0,$$

which implies that there exists $t' > 0$ such that $H(w(t')) = 0$, i.e., $w(t') \in \mathcal{N}$. Then

$$\mathcal{B} \geq \inf_{(u,v)\in\mathcal{N}'} J(u,v). \tag{3.11}$$

By (3.9), (3.10), (3.11), we have

$$\inf_{(u,v)\in\mathcal{N}'} J(u,v) = \inf_{(u,v)\in\mathcal{A}} \max_{t\geq 0} J(tu,tv) = \mathcal{B}.$$

Next, we prove $\mathcal{B} > 0$. If $J(u_n,v_n) \to 0$ and $(u_n,v_n) \in \mathcal{N}'$, then $\|(u_n,v_n)\| \to 0$, which is in contradiction with the inequality (3.8). So we have $\inf_{(u,v)\in\mathcal{N}'} J(u,v) = \mathcal{B} > 0$. □

Now we show a relative property before we prove the main result of this section.

Proposition 3.6. *Let $\lambda \in (0, \lambda_1(\Omega))$ and $\mu \geq 0$. If a ground state solution (u,v) of (1.1) exists, then (u,v) is nontrivial.*

Proof. Assume $(u,v) \in \mathcal{N}$ is such that $J(u,v) = \inf_{(u,v)\in\mathcal{N}} J$. If $v = 0$, then $\langle J'(u,0),(u,0)\rangle = 0$ implies $u = 0$. Now suppose that $u = 0$. If $\mu = 0$, then $v = 0$. So let $\mu > 0$ and let v be a nontrivial solution to

$$\begin{cases} (-i\nabla + A)^2 v = \mu|v|^{2^*-2}v, & x \in \Omega,\\ v = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Notice that

$$\inf\left\{J(0,w): w\in H_A^1(\Omega)\setminus\{0\}, |\nabla_A w|_{2,\Omega}^2=\mu\int_\Omega |w|^{2^*}\,\mathrm{d}x\right\}\le J(0,v)$$
$$=\inf_{\mathcal{N}} J=\inf\left\{J(0,w): w\in H_A^1(\Omega)\setminus\{0\}, |\nabla_A w|_{2,\Omega}^2=\mu\int_\Omega |w|^{2^*}\,\mathrm{d}x\right\}$$

and

$$\inf\left\{J(0,w): w\in H_A^1(\Omega)\setminus\{0\},\ |w|_{2,\Omega}^2=\mu\int_\Omega |w|^{2^*}\,\mathrm{d}x\right\}$$
$$=\frac{1}{N(2^*-1)}\inf\left\{|\nabla_A w|_{2,\Omega}^2: w\in H_A^1(\Omega)\setminus\{0\},\ |\nabla_A w|_{2,\Omega}^2=\mu\int_\Omega |w|^{2^*}\,\mathrm{d}x\right\}$$
$$=\frac{\mu^{\frac{2-N}{2}}}{N(2^*-1)}\inf\left\{|\nabla_A w|_{2,\Omega}^N: w\in H_A^1(\Omega),\ |w|_{2^*,\Omega}=1\right\}.$$

Then $\tilde v=\left(\frac{\mu}{|\nabla_A v|_{2,\Omega}^2}\right)^{\frac{1}{2^*}} v$ satisfies $|\tilde v|_{2^*,\Omega}=1$ and

$$|v|_{2,\Omega}^N=N(2^*-1)\mu^{(N-2)/2}J(0,v)=\inf\{|\nabla_A w|_{2,\Omega}^N: w\in H_A^1(\Omega),\ |w|_{2^*,\Omega}=1\},$$

which is a contradiction. Thus, if (u,v) is a ground state solution of (1.1), then (u,v) is nontrivial. ☐

Theorem 3.7. *If $\lambda\in(0,\lambda_1(\Omega))$, $\mu\ge 0$, then there exists a ground state (u,v) of J such that $J(u,v)=\inf_{\mathcal{N}} J=\inf_{\mathcal{N}'} J=\mathcal{B}$.*

Proof. The functional J satisfies the geometrical assumptions of the mountain pass theorem. By the Sobolev and Poincaré inequalities, we have

$$J(u,v)\ge C(|\nabla_A u|_{2,\Omega}^2+|\nabla_A v|_{2,\Omega}^2-|\nabla_A v|_{2,\Omega}^{2^*}-|\nabla_A u|_{2,\Omega}^{2^*-1}|\nabla_A v|_{2,\Omega})\ge d$$

for some $d>0$ and $\rho=\sqrt{|\nabla_A u|_{2,\Omega}^2+|\nabla_A v|_{2,\Omega}^2}$ sufficiently small.

If $(u,v)\in H_A^1(\Omega)\times H_A^1(\Omega)$ satisfies $\mu\int_\Omega |v|^{2^*}\,\mathrm{d}x+2^*\int_\Omega |u|^{2^*-1}v\,\mathrm{d}x>0$, then

$$J(tu,tv)=\frac{t^2}{2}\left(|\nabla_A u|_{2,\Omega}^2-\lambda\int_\Omega |u|^2\,\mathrm{d}x+\frac{1}{2^*-1}|\nabla_A v|_{2,\Omega}^2\right)$$
$$-\frac{t^{2^*}}{2^*-1}\left(\frac{\mu}{2^*}\int_\Omega |v|^{2^*}\,\mathrm{d}x+\int_\Omega |u|^{2^*-1}v\,\mathrm{d}x\right)\to-\infty\quad\text{as } t\to+\infty.$$

So there exists a $(PS)_{\mathcal{B}}$-sequence $\{(u_n,v_n)\}\in H_A^1(\Omega)\times H_A^1(\Omega)$ for J at level $\mathcal{B}$, namely, a sequence such that $J(u_n,v_n)\to\mathcal{B}$ and $J'(u_n,v_n)\to 0$. Since

$$C(|\nabla u_n|_{2,\Omega}^2+|\nabla_A v_n|_{2,\Omega}^2)\le J(u_n,v_n)-\frac{1}{2^*}\langle J'(u_n,v_n),(u_n,v_n)\rangle$$
$$\le(\mathcal{B}+1)+\sqrt{|\nabla_A u_n|_{2,\Omega}^2+|\nabla_A v_n|_{2,\Omega}^2}$$

for some constant $C>0$, we have that the sequence $\{(u_n,v_n)\}$ is bounded. Thus, up to a subsequence, according to the Sobolev embedding theorem, there exists

$(u,v) \in H^1_A(\Omega) \times H^1_A(\Omega)$ such that

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } H^1_A(\Omega), \qquad u_n \to u \text{ in } L^2(\Omega), \qquad u_n \to u \text{ a.e. on } \Omega,\\
&v_n \rightharpoonup v \text{ in } H^1_A(\Omega), \qquad v_n \to v \text{ a.e. on } \Omega,\\
&|u_n|^{2^*-1} \rightharpoonup |u|^{2^*-1} \qquad \text{in } L^{2^*/(2^*-1)}(\Omega),\\
&|v_n|^{2^*-1} \rightharpoonup |v|^{2^*-1} \qquad \text{in } L^{2^*/(2^*-1)}(\Omega),\\
&|u_n|^{2^*-3}u_n v_n \rightharpoonup |u|^{2^*-3}uv \quad \text{in } L^{2^*/(2^*-1)}(\Omega).
\end{aligned}$$

So, for every $(\xi,\eta) \in H^1_A(\Omega) \times H^1_A(\Omega)$, we have

$$\begin{aligned}
&|\langle J'(u_n,v_n),(\xi,\eta)\rangle - \langle J'(u,v),(\xi,\eta)\rangle|\\
&\quad = \Bigg| (|\nabla_A u_n|_{2,\Omega} - |\nabla_A u|_{2,\Omega})\,|\xi|_{2,\Omega} - \int_\Omega (|u_n|^{2^*-2}v_n - |u|^{2^*-2}v)\eta\,\mathrm{d}x\\
&\qquad + \frac{1}{2^*-1}\,(|\nabla_A v_n|_{2,\Omega} - |\nabla_A v|_{2,\Omega})\,|\nabla_A \eta|_{2,\Omega}\\
&\qquad - \lambda \int_\Omega (u_n - u)\,\xi\,\mathrm{d}x - \frac{\mu}{2^*-1}\int_\Omega \left(|v_n|^{2^*-1} - v^{2^*-1}\right)\eta\,\mathrm{d}x\\
&\qquad - \frac{1}{2^*-1}\int_\Omega \left(|u_n|^{2^*-1} - |u|^{2^*-1}\right)\eta\,\mathrm{d}x\Bigg| \to 0.
\end{aligned}$$

Thus $J'(u,v) = 0$. We claim that $(u,v) \neq (0,0)$. Otherwise,

$$u_n \to 0 \quad \text{in } L^2(\Omega). \tag{3.12}$$

Since J is continuous and $J(u_n,v_n) \to \mathcal{B} > 0$, (u_n,v_n) cannot converge to $(0,0)$ in $H^1_A(\Omega) \times H^1_A(\Omega)$. Thus, up to a subsequence, we may assume that $(u_n,v_n) \neq (0,0)$ and $\|(u_n,v_n)\| \geq C > 0$, $(u_n,v_n) \in \mathcal{A}$ for all $N \in \mathbb{N}$. Taking a subsequence $\{(u_{n_k},v_{n_k})\}$ of $\{(u_n,v_n)\}$ in $\left(H^1_A(\Omega) \times H^1_A(\Omega)\right) \cap \mathcal{A}^c$, we have

$$\langle J'(u_{n_k},v_{n_k}),(u_{n_k},v_{n_k})\rangle \geq \|(u_{n_k},v_{n_k})\|^2. \tag{3.13}$$

Since

$$\langle J'(u_{n_k},v_{n_k}),(u_{n_k},v_{n_k})\rangle \to 0 \quad \text{as } k \to +\infty,$$

we get a contradiction. If we take

$$t_n = \left[\left((2^*-1)|\nabla_A u_n|^2_{2,\Omega} + |\nabla_A v_n|^2_{2,\Omega}\right)\left(\mu|v_n|^{2^*}_{2^*,\Omega} + 2^*\int_\Omega |u_n|^{2^*-1}v_n\,\mathrm{d}x\right)^{-1}\right]^{\frac{1}{2^*-2}}$$

and we denote in the same way the functions in $H^1_A(\Omega)$ and their extensions in $\mathbb{R}^N$ putting the function equal to zero in $\mathbb{R}^N \setminus \Omega$, we have $(t_n u_n, t_n v_n) \in \mathcal{N}'_0$ and so

$$\langle J'_0(t_n u_n, t_n v_n),\ (t_n u_n, t_n v_n)\rangle = 0. \tag{3.14}$$

In addition, by (3.12),

$$\langle J'_0(u_n,v_n),\,(u_n,v_n)\rangle = \langle J'(u_n,v_n),\,(u_n,v_n)\rangle + o(1) = o(1). \tag{3.15}$$

Then, using (3.14) and (3.15) we have $t_n \to 1$. Therefore by Lemma 3.4 and Theorem 1.2, we have

$$\mathcal{B} < A = A' \leq \lim_n J(t_n u_n, t_n v_n) = \mathcal{B},$$

getting a contradiction. Thus $(u, v) \neq (0, 0)$ and $(u, v) \in \mathcal{N} \subset \mathcal{N}'$. Likewise, we find $t_n \to 1$ such that $(t_n u_n, t_n v_n) \in \mathcal{N}'$, and according to Proposition 3.5 we get

$$\inf_{\mathcal{N}} J \leq J(u, v) \leq \lim_{n\to\infty} J(t_n u_n, t_n v_n) = \mathcal{B} = \inf_{\mathcal{N}'} J \leq \inf_{\mathcal{N}} J,$$

so the proof is completed, that is, the problem (1.1) has a ground state solution. □

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