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RICCI–BOURGUIGNON SOLITONS ON REAL HYPERSURFACES IN THE COMPLEX PROJECTIVE SPACE

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ABSTRACT. We give a complete classification of Ricci–Bourguignon solitons on real hypersurfaces in the complex projective space $\mathbb{C}P^n = SU_{n+1}/S(U_1 \cdot U_n)$. Next, as an application, we give some non-existence properties for gradient Ricci–Bourguignon solitons on real hypersurfaces with isometric Reeb flow and contact real hypersurfaces in the complex projective space $\mathbb{C}P^n$.

1. Introduction

Among the class of Hermitian symmetric spaces with rank 1 of compact type, we have complex projective space $\mathbb{C}P^n = SU_{n+1}/S(U_1 \cdot U_n)$, which is geometrically quite different from the case of rank 2. It has a Kähler structure and Fubini–Study metric g of constant holomorphic sectional curvature 4 (see Romero [27, 26], and Smyth [28]). The complex projective space $\mathbb{C}P^n$ is considered as a kind of real Grassmann manifold of compact type with rank 1 (see Kobayashi and Nomizu [20]). In Hermitian symmetric space of rank 2, Jeong and Suh [18] gave a classification of Ricci soliton real hypersurfaces in the complex two-plane Grassmannian $G_2(\mathbb{C}^{n+2})$, and the other geometric properties of $G_2(\mathbb{C}^{n+2})$ are investigated in Suh [29, 30, 31].

Recently, Yamabe solitons and Ricci solitons on almost co-Kähler manifolds and 3-dimensional $N(k)$-contact manifolds have been investigated by De, Chaubey, and Suh [12, 13]. Moreover, the study of the Yamabe flow was initiated in the work of Hamilton [16], Morgan and Tian [21] and Perelman [24] as a geometric method to construct Yamabe metrics on Riemannian manifolds.

A time-dependent metric $g(t)$ on a Riemannian manifold M is said to be evolved by the Yamabe flow if the metric g satisfies

$$\frac{\partial}{\partial t} g(t) = -\gamma g(t), \quad g(0) = g_0$$

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on M, where γ denotes the scalar curvature on M. From such a viewpoint, in this paper we want to give a complete classification of Yamabe solitons and gradient Yamabe solitons on Hopf real hypersurfaces in the complex projective space $\mathbb{C}P^n$.

On the other hand, it is well known that there exist two focal submanifolds of real hypersurfaces in Hermitian symmetric spaces of compact type and only one focal submanifold in Hermitian symmetric spaces of non-compact type (see Berndt and Suh [1] and Helgason [17]). Since the complex projective space $\mathbb{C}P^n$ is an Hermitian symmetric space of compact type, any real hypersurface has two focal submanifolds (see Djorić and Okumura [14], Pérez [25]). Among them we consider two kinds of real hypersurfaces in $\mathbb{C}P^n$ with isometric Reeb flow or contact hypersurfaces. In $\mathbb{C}P^n$, Cecil and Ryan [9] and Okumura [22] gave a classification of real hypersurfaces with isometric Reeb flow as follows:

Theorem A. *Let M be a real hypersurface of the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube of radius $0 < r < \frac{\pi}{2}$ around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^n$ for some $k \in \{0, \ldots, n-1\}$ or a tube with radius $\frac{\pi}{2} - r$ over $\mathbb{C}P^\ell$, where $k + \ell = n - 1$.*

When a real hypersurface M in the complex projective space $\mathbb{C}P^n$ satisfies the formula $A\phi + \phi A = k\phi$, $k \neq 0$ and constant, we say that M is a *contact* real hypersurface in $\mathbb{C}P^n$. In the papers by Blair [3], Okumura [22], and Yano and Kon [35], these authors introduce the classification of contact real hypersurfaces in $\mathbb{C}P^n$ as follows:

Theorem B. *Let M be a connected orientable real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then M is a contact real hypersurface if and only if M is congruent to an open part of a tube of radius $0 < r < \frac{\pi}{4}$ around an n-dimensional real projective space $\mathbb{R}P^n$ or a tube of radius $\frac{\pi}{4} - r$ over Q^{n-1}, where $0 < r < \frac{\pi}{4}$.*

Motivated by these results, in this paper we give some characterizations of real hypersurfaces in the complex projective space $\mathbb{C}P^n$ regarding a family of geometric flows introduced by J. P. Bourguignon. We call this the *Ricci–Bourguignon flow*, which generalizes the Ricci flow. It is an intrinsic geometric flow on Riemannian manifolds whose fixed points are solitons. Indeed, we know that a solution of the Ricci flow equation $\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t))$ is given by

$$\frac{1}{2}(\mathcal{L}_V g)(X,Y) + \mathrm{Ric}(X,Y) = \Omega g(X,Y),$$

where the function Ω is constant and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Chaubey, De, and Suh [11], Morgan and Tian [21], Perelman [24], and Wang [33, 34]). Then this solution (M, V, Ω, g) is said to be a *Ricci soliton* with potential vector field V and Ricci soliton constant Ω. In the complex two-plane Grassmannian $G_2(\mathbb{C}^{n+2})$, Jeong and Suh [18] gave a classification of Ricci soliton for real hypersurfaces.

As a generalization of the notion of Ricci flow, the Ricci–Bourguignon flow (see Bourguignon [4, 5], Catino, Cremaschi, Djadli, Mantegazza, and Mazzieri [6]) is given by

$$\frac{\partial}{\partial t}g(t) = -2(\mathrm{Ric}(g(t)) - \theta\gamma g(t)), \quad g(0) = g_0.$$

This family of geometric flows with $\theta = 0$ reduces to the Ricci flow $\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}(g(t))$, $g(0) = g_0$. If the constant $\theta = \frac{1}{2}$, it is said to be the *Einstein flow*. The critical point of the Einstein flow

$$\frac{\partial}{\partial t}g(t) = -2(\mathrm{Ric}(g(t)) - \frac{1}{2}\gamma g(t)), \quad g(0) = g_0,$$

implies that the Einstein gravitational tensor $\mathrm{Ric}(g(t)) - \frac{1}{2}\gamma g(t)$ vanishes. For a 4-dimensional spacetime M^4, this is equivalent to the vanishing Ricci tensor by virtue of $d\gamma = 2\,\mathrm{div}(\mathrm{Ric})$. In this case M^4 becomes vacuum. That is, $g(t) = g(0)$, the metric is constant along the time (see O'Neill [23]). For $\theta = \frac{1}{n}$, the tensor $\mathrm{Ric} - \frac{\gamma}{n}g$ is said to be the traceless Ricci tensor, and for $\theta = \frac{1}{2(n-1)}$, it is said to be the Schouten tensor.

Now let us introduce the Ricci–Bourguignon soliton $(M, V, \Omega, \theta, \gamma, g)$, which is a solution of the Ricci–Bourguignon flow as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) = (\Omega + \theta\gamma)g(X, Y)$$

for any tangent vector fields X and Y on M, where Ω is a soliton constant, θ any constant and γ the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Morgan and Tian [21]). Then (M, g) is said to be a *Ricci–Bourguignon soliton* with potential vector field V and Ricci–Bourguignon soliton constant Ω. In recent years, many authors studied the Ricci–Bourguignon soliton on Riemannian manifolds. In [8], Catino, Mazzieri, and Mongodi classified noncompact gradient Ricci–Bourguignon solitons $(M, \nabla f, \Omega, \theta, \gamma, g)$ with bounded non-negative sectional curvature. Moreover, Blaga and Tastan [2] and Dwivedi [15] obtained some results on Ricci–Bourguignon solitons and almost Ricci–Bourguignon solitons on Riemannian manifolds.

On the other hand, when the Reeb vector field ξ satisfies $A\xi = \alpha\xi$ for the shape operator A on a real hypersurface M in the complex projective space $\mathbb{C}P^n$, M is said to be *Hopf*. By using this notion, in Section 4 it can be easily seen that a Hopf Ricci–Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$ in the complex projective space $\mathbb{C}P^n$ also satisfies the generalized pseudo-anti-commuting property mentioned above.

We now introduce some background on the study of real hypersurfaces in the complex projective space $\mathbb{C}P^n$, which serves as a key foundation for our theorems.

If the Ricci operator Ric of a real hypersurface M in $\mathbb{C}P^n$ satisfies

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi$$

for smooth functions a, b on M, then M is said to be *pseudo-Einstein*. Then by virtue of the classification for pseudo-Einstein Hopf real hypersurfaces in the complex projective space $\mathbb{C}P^n$ due to Cecil and Ryan [9] and Propositions 3.3 and 3.4 below, we introduce the following theorem.

Theorem C. *Let M be a pseudo-Einstein real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then M is locally congruent to one of the following:*

(i) *a geodesic hypersphere, $a = 2n + 2(n-1)\cot^2(r)$, and $b = -2n$;*

(ii) *a tube of radius r around a totally geodesic CP^k, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$, $\cot^2 r = \frac{k}{n-k-1}$, $a = 2n$, and $b = -2$;*

(iii) *a tube of radius r around a complex quadric Q^{n-1} where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = n-2$, $a = 2n$, and $b = -2n-1$.*

Now let us consider an Einstein hypersurface in the complex quadric $\mathbb{C}P^n$. Then the Ricci tensor of M becomes $\mathrm{Ric} = \lambda g$. In case (i) in above Theorem C, there do not exist any Einstein hypersurfaces in $\mathbb{C}P^n$, because the case (i) has two distinct constant principal curvatures, the function $b = -2n$ is non-vanishing. Moreover, the tubes in (ii) and (iii) are known to be three distinct constant principal curvatures. So it can be easily checked that the smooth functions $b = -2$ for (ii), and $b = -2(n-1)$ for (iii) in Theorem C are non-vanishing (see Cecil and Ryan [9] and Takagi [32]). From such a viewpoint we can conclude the following result.

Theorem D. *There does not exist an Einstein real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$.*

In Section 4, we show that every Hopf Ricci–Bourguignon soliton real hypersurface in the complex projective space $\mathbb{C}P^n$ satisfies the generalized pseudo-anti-commuting property. Then by virtue of the classification due to Ki and Suh [19], we introduce a classification given in Proposition 4.1. After doing this, we should check whether a geodesic hypersphere of type A_1, a pseudo-Einstein real hypersurface in type A_2 or of type B could admit a Ricci–Bourguignon soliton or not. Then we can assert the following theorem.

Main Theorem 1. *There does not exist a Hopf Ricci–Bourguignon soliton $(M, \xi, \Omega, \theta, \gamma, g)$ in the complex projective space $\mathbb{C}P^n$, $n \geq 3$.*

By virtue of Theorem 1, we can assert the following corollary.

Corollary 1.1. *There does not exist a Hopf Ricci soliton (M, ξ, Ω, g) in the complex projective space $\mathbb{C}P^n$, $n \geq 3$.*

Now let us denote by Df the gradient vector field of the function f on M defined by $g(Df, X) = g(\mathrm{grad}\, f, X) = X(f)$ for any tangent vector field X on M. We consider the *gradient Ricci–Bourguignon soliton* $(M, Df, \Omega, \theta, \gamma, g)$ (see Catino and Mazzieri [7], Cernea and Guan [10]) defined by

$$\mathrm{Hess}(f) + \mathrm{Ric} = (\Omega + \theta\gamma)g,$$

where $\mathrm{Hess}(f)$ is defined by $\mathrm{Hess}(f) = \nabla Df$ for any tangent vector fields X and Y on M in such a way that

$$\mathrm{Hess}(f)(X, Y) = g(\nabla_X Df, Y).$$

Then the gradient Ricci–Bourguignon soliton can be given by

$$\nabla_X Df + \mathrm{Ric}(X) = (\Omega + \theta\gamma)X$$

for any vector field X tangent to M in $\mathbb{C}P^n$. Then first by virtue of Theorem A we can give a non-existence theorem for the gradient Ricci–Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$ as follows:

Main Theorem 2. *There does not exist a real hypersurface with isomeric Reeb flow in the complex projective space $\mathbb{C}P^n$, $n \geq 3$, admitting the gradient Ricci–Bourguignon soliton.*

Next by Theorem B for a contact real hypersurface in the complex projective space $\mathbb{C}P^n$, we can assert the following for the gradient Ricci–Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$.

Main Theorem 3. *There does not exist a contact real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$, admitting the gradient Ricci–Bourguignon soliton.*

2. The complex projective space

This section is due to Berndt and Suh [1]. Let $(\bar{M}, g, J)$ be a Kähler manifold and $\bar{R}$ the Riemannian curvature tensor of $(\bar{M}, g)$. Since $\bar{\nabla} J = 0$, we immediately see that

$$\bar{R}(X, Y)JZ = J\bar{R}(X, Y)Z$$

holds for all $X, Y, Z \in T_x(\bar{M})$, $x \in \bar{M}$. From the curvature identities in Kobayashi and Nomizu [20] we also get

$$g(\bar{R}(X, Y)Z, W) = g(\bar{R}(JX, JY)Z, W) = g(\bar{R}(X, Y)JZ, JW).$$

Let $G_2^J(T\bar{M})$ be the Grassmann bundle over $\bar{M}$ consisting of all 2-dimensional J-invariant linear subspaces V of $T_p\bar{M}$, $p \in M$. Thus every $V \in G_2^J(T\bar{M})$ is a complex line in the corresponding tangent space of $\bar{M}$. The restriction of the sectional curvature function K to $G_2^J(T\bar{M})$ is called the *holomorphic sectional curvature function* on $\bar{M}$, and $K(V)$ is called the *holomorphic sectional curvature* of $\bar{M}$ with respect to $V \in G_2^J(T\bar{M})$.

A Kähler manifold M is said to have *constant holomorphic sectional curvature* if the holomorphic sectional curvature function is constant. Now we want to introduce the following.

Theorem 2.1. *A Kähler manifold $(\bar{M}, g, J)$ has constant holomorphic sectional curvature $c \in \mathbb{R}$ if and only if its Riemannian curvature tensor $\bar{R}$ is of the form*

$$\bar{R}(X, Y)Z = \frac{c}{4}\Big\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ\Big\}$$

for any vector fields X, Y, and Z on $\bar{M}$.

The complex vector space $\mathbb{C}^n$ ($n \in \mathbb{N}$) is in a canonical way an n-dimensional complex manifold. For $p \in \mathbb{C}^n$ denote by $\pi_p : T_p\mathbb{C}^n \to \mathbb{C}^n$ the canonical isomorphism. We define a Riemannian metric g on $\mathbb{C}^n$ by

$$g_p(u, v) = \langle \pi_p(u), \pi_p(v) \rangle$$

for all $u, v \in T_p\mathbb{C}^n$ and $p \in \mathbb{C}^n$, where $\langle\cdot,\cdot\rangle$ is the real part of the standard Hermitian inner product on $\mathbb{C}^n$, that is,

$$\langle a, b\rangle = \mathrm{Re}\left(\sum_{\nu=1}^{n} a_\nu \bar{b}_\nu\right) \qquad (a, b \in \mathbb{C}^n).$$

The metric g is called the *canonical Riemannian metric on* $\mathbb{C}^n$. The complex structure J on $\mathbb{C}^n$ is given by the equation $\pi_p(Ju) = i\pi_p(u)$. It is easy to verify that $(\mathbb{C}^n, g, J)$ is a Kähler manifold. In fact, $(\mathbb{C}^n, g, J)$ is a complex Euclidean space with vanishing constant holomorphic sectional curvature. The Kähler manifold $(\mathbb{C}^n, g, J)$ is known to be the *n-dimensional complex Euclidean space*.

We define an equivalence relation $\sim$ on $\mathbb{C}^{n+1}\setminus\{0\}$ by $z_1 \sim z_2$ if and only if there exists $\lambda \in \mathbb{C}\setminus\{0\}$ such that $z_2 = z_1\lambda$. We denote by $\mathbb{C}P^n$ the quotient space $(\mathbb{C}^{n+1}\setminus\{0\})/\sim$. By construction, the points in $\mathbb{C}P^n$ are in one-to-one correspondence with the complex lines through $0 \in \mathbb{C}^{n+1}$. We equip $\mathbb{C}P^n$ with the quotient topology with respect to the canonical projection $\tau : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{C}P^n$. Then $\mathbb{C}P^n$ is a compact Hausdorff space and τ is a continuous map. There exists a unique complex manifold structure on $\mathbb{C}P^n$ so that τ is a holomorphic submersion. In this way $\mathbb{C}P^n$ becomes an n-dimensional complex manifold $(\mathbb{C}P^n, J)$. For $z \in \mathbb{C}^{n+1}\setminus\{0\}$ we also write $[z] = \tau(z) \in \mathbb{C}P^n$.

Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$ and denote by π the restriction of τ to S^{2n+1}. We consider S^{2n+1} with the Riemannian metric induced from $\mathbb{C}^{n+1}$, which is the standard metric on S^{2n+1} turning it into a real space form with constant sectional curvature 1. The map $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a surjective submersion whose fibers are 1-dimensional circles. There exists a unique Riemannian metric g on $\mathbb{C}P^n$ so that π becomes a Riemannian submersion. In such a way, the map $\pi : S^{2n+1} \to \mathbb{C}P^n$ is known as the *Hopf map* from S^{2n+1} onto $\mathbb{C}P^n$ and the Riemannian metric g is known as the *Fubini–Study metric* on $\mathbb{C}P^n$. The manifold $(\mathbb{C}P^n, J, g)$ is a Kähler manifold and called the n-dimensional *complex projective space*. The complex projective space $(\mathbb{C}P^n, J, g)$ is a complex space form with constant holomorphic sectional curvature 4.

By virtue of Theorem 2.1, the Riemannian curvature tensor $\bar{R}$ of $\mathbb{C}P^n$ can be given for any vector fields X, Y and Z in $T_z(\mathbb{C}P^n)$, $z \in \mathbb{C}P^n$, as follows:

$$\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ.$$

3. Some general equations

Let M be a real hypersurface in the complex projective space $\mathbb{C}P^n$ and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. Then the vector field ξ is said to be the *Reeb* vector field on M in $\mathbb{C}P^n$. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

In another way, the complex projective space $\mathbb{C}P^n$ is defined by using the fibration

$$\tilde{\pi} : S^{2n+1}(1) \to \mathbb{C}P^n, \quad z \to [z],$$

which is said to be a Riemannian submersion. Then naturally we can consider the following diagram for a real hypersurface in the complex projective space $\mathbb{C}P^n$:

$$\begin{array}{ccc} M' = \tilde{\pi}^{-1}(M) & \xrightarrow{\tilde{i}} & S^{2n+1}(1) \subset \mathbb{C}^{n+1} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\tilde{\pi}}\big\downarrow \\ M & \xrightarrow{i} & \mathbb{C}P^n \end{array}$$

We now assume that M is a Hopf hypersurface. Then we have

$$A\xi = \alpha\xi,$$

where A denotes the shape operator of M in $\mathbb{C}P^n$ and the smooth function α is defined by $\alpha = g(A\xi, \xi)$ on M. When we consider the transformed vector field JX by the Kähler structure J on $\mathbb{C}P^n$ for any vector field X on M in $\mathbb{C}P^n$, we may write

$$JX = \phi X + \eta(X)N.$$

Then, by using Kähler structure $\nabla J = 0$, we get the following:

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \quad \text{and} \quad \nabla_X \xi = \phi AX.$$

Now we consider the equation of Codazzi

$$g((\nabla_X A)Y - (\nabla_Y A)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y).$$

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in $\mathbb{C}P^n$ induced from the curvature tensor $\bar{R}$ of $\mathbb{C}P^n$ can be described in terms of the almost contact structure tensor ϕ and the shape operator A of M in $\mathbb{C}P^n$ as follows:

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{3.1}$$

for any vector fields $X, Y, Z \in T_z M$, $z \in M$. From this, contracting Y and Z on M in $\mathbb{C}P^n$, we get the Ricci tensor of a real hypersurface M in $\mathbb{C}P^n$ as follows:

$$\mathrm{Ric}(X) = (2n + 1)X - 3\eta(X)\xi + (\mathrm{Tr}\, A)AX - A^2 X. \tag{3.2}$$

Then, by contracting the Ricci operator in (3.2), the scalar curvature γ of M in $\mathbb{C}P^n$ is given by

$$\gamma = \sum_{i=1}^{2n-1} g(\mathrm{Ric}(e_i), e_i) = 4(n^2 - 1) + h^2 - \mathrm{Tr}\, A^2, \tag{3.3}$$

where the function h denotes the trace of the shape operator A of M in $\mathbb{C}P^n$.

Putting $Z = \xi$ in the Codazzi equation, we get

$$g((\nabla_X A)Y - (\nabla_Y A)X, \xi) = -2g(\phi X, Y).$$

Since we have assumed that M is Hopf in $\mathbb{C}P^n$, differentiating $A\xi = \alpha\xi$ gives

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX.$$

From this, the left side of the above equation becomes

$$\begin{aligned} g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned}$$

Putting $X = \xi$ in the above two equations and using the almost contact structure of (M, g), we have

$$Y\alpha = (\xi\alpha)\eta(Y).$$

Inserting this formula into the two previous equations implies that

$$0 = 2g(A\phi AX, Y) - \alpha g((\phi A + A\phi)X, Y) - 2g(\phi X, Y).$$

By virtue of this equation, we can assert the following lemma.

Lemma 3.1. *Let M be a Hopf real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Then we obtain*

$$2A\phi AX = \alpha(A\phi + \phi A)X + 2\phi X$$

for any tangent vector field X on M.

In the proof of our Theorems 1 and 2, we want to give more information about Hopf real hypersurfaces in the complex projective space. By using the formulas given in Section 3 we introduce an important lemma due to Okumura [22] and Yano and Kon [35].

Lemma 3.2. *Let M be a Hopf real hypersurface in $\mathbb{C}P^n$. Then the Reeb function α is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$ and ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$ on M.*

Now, by using (3.2) and (3.3), we introduce an important proposition due to Cecil and Ryan [9] and Djorić and Okumura [14].

Proposition 3.3. *Let M be the tube of radius $0 < r < \frac{\pi}{2}$ around the totally geodesic $\mathbb{C}P^k$, $k \in \{1, \ldots, n-2\}$ in $\mathbb{C}P^n$. Then the following statements hold:*

(1) *M is a Hopf hypersurface.*
(2) *The principal curvatures and corresponding principal curvature spaces of M are given by*

principal curvature	*eigenspace*	*multiplicity*
$\lambda = \cot(r)$	T_λ	2ℓ
$\mu = -\tan(r)$	T_μ	$2k$
$\alpha = 2\cot(2r)$	$T_\alpha = \mathbb{R}JN$	1

where $\ell = n - k - 1$.

(3) *The shape operator A commutes with the structure tensor field ϕ:*

$$A\phi = \phi A.$$

(4) *The trace h of the shape operator A and its square h^2 become the following, respectively:*

$$h = (2\ell+1)\cot(r) - (2k+1)\tan(r),$$
$$h^2 = (2\ell+1)^2\cot^2(r) + (2k+1)^2\tan^2(r) - 2(2\ell+1)(2k+1).$$

(5) *The trace of the matrix A^2 is given by*

$$\operatorname{Tr} A^2 = (2\ell+1)\cot^2(r) + (2k+1)\tan^2(r) - 2.$$

(6) *The scalar curvature γ of the tube M is given by*

$$\gamma = 4(n-1)n - 8k\ell + 2(2\ell+1)\ell\cot^2(r) + 2(2k+1)k\tan^2(r).$$

(7) *M is pseudo-Einstein, that is,*

$$\operatorname{Ric}(X) = 2nX - 2\eta(X)\xi$$

for $\cot^2(r) = \frac{k}{n-k-1}$.

The tube of radius r around totally geodesic and totally real projective space $\mathbb{R}P^n$ has therefore three distinct constant principal curvatures: $2\tan(2r)$, $-\cot(r)$, and $\tan(r)$. It also can be regarded as a tube of radius $\frac{\pi}{4} - r$ over a totally geodesic complex quadric Q^{n-1}. Then, by (3.2) and (3.3), we want to give the following important proposition due to Berndt and Suh [1], Cecil and Ryan [9], and Takagi [32].

Proposition 3.4. *Let M be the tube of radius $0 < r < \frac{\pi}{4}$, around the complex quadric Q^{n-1} which is a complex hypersurface in $\mathbb{C}P^n$. Then the following statements hold:*

(1) *M is a Hopf hypersurface.*
(2) *The principal curvatures and corresponding principal curvature spaces of M are*

principal curvature	*eigenspace*	*multiplicity*
$\lambda = -\cot(\frac{\pi}{4} - r)$	T_λ	$n-1$
$\mu = \tan(\frac{\pi}{4} - r)$	T_μ	$n-1$
$\alpha = 2\cot(2r)$	$\mathbb{R}JN$	1

(3) *The shape operator A and the structure tensor field ϕ satisfy*

$$A\phi + \phi A = k\phi, \quad k \neq 0 \text{ constant.}$$

(4) *The trace h of the shape operator A and its square h^2 become the following, respectively:*

$$h = \operatorname{Tr} A = 2\cot(2r) - 2(n-1)\tan(2r),$$
$$h^2 = 4\cot^2(2r) + 4(n-1)^2\tan^2(2r) - 8(n-1).$$

(5) *The trace of the matrix A^2 is given by*

$$\operatorname{Tr} A^2 = 4\cot^2(2r) + 4(n-1)\tan^2(2r).$$

(6) *The scalar curvature γ of the tube M is given by*

$$\gamma = 4(n-1)^2 + 4(n-1)(n-2)\tan^2(2r).$$

(7) *M is pseudo-Einstein, that is,*

$$\operatorname{Ric}(X) = 2nX - 2(n-1)\eta(X)\xi$$

for $\cot^2(2r) = n-2$.

4. Hopf Ricci–Bourguignon soliton real hypersurfaces

Now let us introduce the Ricci–Bourguignon soliton $(M, V, \Omega, \theta, \gamma, g)$, which is a solution of the Ricci–Bourguignon flow as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X,Y) + \operatorname{Ric}(X,Y) = (\Omega + \theta\gamma)g(X,Y)$$

for any tangent vector fields X and Y on M, where Ω is a Ricci–Bourguignon soliton constant, θ any constant and γ the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Morgan and Tian [21]). Then let us consider the Reeb vector field ξ as the Ricci–Bourguignon soliton vector field V as follows:

$$\frac{1}{2}(\mathcal{L}_\xi g)(X,Y) + \operatorname{Ric}(X,Y) = (\Omega + \theta\gamma)g(X,Y). \tag{4.1}$$

Then, by virtue of the Lie derivative $(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)$, the formula (4.1) can be given by

$$\operatorname{Ric}(X) = \frac{1}{2}(A\phi - \phi A)X + (\Omega + \theta\gamma)X. \tag{4.2}$$

From this, by applying the structure tensor ϕ to both sides, we get the following two formulas:

$$\begin{aligned}\operatorname{Ric}(\phi X) &= \frac{1}{2}(A\phi^2 - \phi A\phi)X + (\Omega + \theta\gamma)\phi X,\\ \phi\operatorname{Ric}(X) &= \frac{1}{2}(\phi A\phi - \phi^2 A)X + (\Omega + \theta\gamma)\phi X.\end{aligned}$$

By using the almost contact structure (ϕ, ξ, η, g) in the right side above, we know that the *generalized pseudo-anti-commuting property* holds as follows:

$$\operatorname{Ric}(\phi X) + \phi\operatorname{Ric}(X) = 2(\Omega + \theta\gamma)\phi X.$$

Now we want to introduce an important proposition due to Ki and Suh [19] and Yano and Kon [35], which will be used in the proof of our Theorem 1.

Proposition 4.1. *Let M be a connected complete Hopf real hypersurface in the complex projective space $\mathbb{C}P^n$. If M satisfies the generalized pseudo-anti-commuting property, then M is locally congruent to a geodesic hypersphere in type A_1, a pseudo-Einstein hypersurface in type A_2, or M is locally congruent to a real hypersurface of type B.*

Among real hypersurfaces of type A_2, only pseudo-Einstein real hypersurfaces satisfy the generalized pseudo-anti-commuting property (4.2). Then it is exactly the second case in Theorem C. That is, M is locally congruent to a tube of radius r around a totally geodesic CP^k, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$ and $\cot^2(r) = \frac{k}{n-k-1}$.

Now geodesic hyperspheres and pseudo-Einstein real hypersurfaces are included in the class of type A_1 and A_2, respectively. So by Theorem A, they are characterized by the commuting shape operator. That is, $A\phi = \phi A$. Accordingly, from the notion of Ricci–Bourguignon soliton $(M, \xi, \Omega, \theta, \gamma, g)$ of M, (4.1) becomes

$$\mathrm{Ric} = (\Omega + \theta\gamma)g.$$

This means that those hypersurfaces are Einstein. But, by Theorem D, there do not exist such hypersurfaces in the complex projective space $\mathbb{C}P^n$.

Next, in the remaining case let us check real hypersurfaces of type B in $\mathbb{C}P^n$. They are characterized by $A\phi + \phi A = k\phi$, where $k \neq 0$ is constant. Moreover, by Proposition 3.4, the principal curvatures are given by $\lambda = -\cot(\frac{\pi}{4} - r)$, $\mu = \tan(\frac{\pi}{4} - r)$ and $\alpha = 2\cot(2r)$. So $k = \lambda + \mu = -\frac{4}{\alpha}$. For any $X \in T_\lambda$ the vector field $\phi X \in T_\mu$. So the Ricci–Bourguignon soliton (4.1) gives the following:

$$\left(-\mu - \frac{2}{\alpha} \right) \phi X + \mathrm{Ric}\, X = (\Omega + \theta\gamma) X$$

for any $X \in T_\lambda$. Since the first term is skew-symmetric, by taking symmetric part the first term vanishes. So it becomes $\mathrm{Ric}(X) = (\Omega + \theta\gamma)X$ for any $X \in T_\lambda$.

Similarly, for any $Y \in T_\mu$ we get $\mathrm{Ric}(Y) = (\Omega + \theta\gamma)Y$. Of course, if we put $X = \xi$ in (4.2), we get $\mathrm{Ric}(\xi) = (\Omega + \theta\gamma)\xi$. Consequently, $\mathrm{Ric} = \lambda g$. That is, it is Einstein. But, by Theorem D, there does not exist such a type B on a hypersurface in complex projective space $\mathbb{C}P^n$. From this fact we can assert our Main Theorem 1 in the Introduction.

5. Gradient Ricci–Bourguignon soliton on isometric Reeb flow in $\mathbb{C}P^n$

In this section, let M be a tube of radius r, $0 < r < \frac{\pi}{2}$, over a totally geodesic $\mathbb{C}P^k$, $k \in \{0, 1, \dots, n-2, n-1\}$ in $\mathbb{C}P^n$, which is said to be of type A_1 and of type A_2. In Theorem A, we have mentioned that the Reeb flow on M in $\mathbb{C}P^n$ is isometric if and only if M is locally congruent to a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$ for $k \in \{0, 1, \dots, n-1\}$. Then, for $k = 0$ or $k = n-1$, we say that M is a geodesic hypersphere which is said to be of type A_1 and it has two distinct principal curvatures. For $k \in \{1, \dots, n-2\}$, M is locally congruent to a tube over $\mathbb{C}P^k$ in $\mathbb{C}P^n$. Moreover, it is said to be of type A_2 and has three distinct constant principal curvatures.

Then the shape operator of M in the complex projective space $\mathbb{C}P^n$ with isometric Reeb flow can be expressed as

$$A = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cot(r) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cot(r) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\tan(r) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -\tan(r) \end{bmatrix}$$

for three constant principal curvatures $\alpha = 2\cot(2r)$, $\cot(r)$, and $-\tan(r)$ with multiplicities 1, 2ℓ, and $2k$, respectively, where $\ell = n-k-1$.

Then, by putting $X = \xi$ in (3.2), and using $A\xi = \alpha\xi$, we have the following:

$$\begin{aligned} \mathrm{Ric}(\xi) &= (2n+1)\xi - 3\xi + hA\xi - A^2\xi \\ &= 2(n-1)\xi + (h\alpha - \alpha^2)\xi \\ &= \kappa\xi, \end{aligned}$$

where we have put $\kappa = 2(n-1) + h\alpha - \alpha^2$. So, by Proposition 3.3, the constant κ is given by

$$\begin{aligned} \kappa &= 2(n-1) + (h\alpha - \alpha^2) \\ &= 2(n-1) + \{(2\ell+1)\cot(r) - (2k+1)\tan(r)\}2\cot(2r) - (2\cot(2r))^2 \\ &= 2(n-1) + 2\{\ell\cot^2(r) + k\tan^2(r) - (k+\ell)\} \\ &= 2\ell\cot^2(r) + 2k\tan^2(r). \end{aligned}$$

Then, by taking the covariant derivative, we get the following two formulas:

$$\begin{aligned} (\nabla_X \mathrm{Ric})\xi &= \kappa\phi AX - \mathrm{Ric}(\phi AX), \\ (\nabla_\xi \mathrm{Ric})X &= h(\nabla_\xi A)X - (\nabla_\xi A^2)X. \end{aligned}$$

Since M admits a gradient Ricci–Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$, we could consider the soliton vector field W as $W = Df$ for any smooth function on M. In the Introduction we have noted that $\mathrm{Hess}(f)$ is defined by $\mathrm{Hess}(f) = \nabla Df$ for any tangent vector fields X and Y on M in such a way that

$$\mathrm{Hess}(f)(X, Y) = g(\nabla_X Df, Y).$$

Then the gradient Ricci–Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$ can be given by

$$\nabla_X Df + \mathrm{Ric}(X) = (\Omega + \theta\gamma)X$$

for any tangent vector field X on M. Then, by taking the covariant derivative and using the fact that the scalar curvature γ is constant for an isometric Reeb flow, we obtain

$$\nabla_X \nabla_Y Df + (\nabla_X \mathrm{Ric})(Y) + \mathrm{Ric}(\nabla_X Y) = (\Omega + \theta\gamma)\nabla_X Y$$

for any vector field X and Y tangent to M in $\mathbb{C}P^n$. From this, together with the above two formulas, it follows that

$$\begin{aligned} R(\xi, Y)Df &= \nabla_\xi \nabla_Y Df - \nabla_Y \nabla_\xi Df - \nabla_{[\xi,Y]} Df \\ &= (\nabla_Y \operatorname{Ric})\xi - (\nabla_\xi \operatorname{Ric})Y \\ &= \kappa\phi AY - \operatorname{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y. \end{aligned} \tag{5.1}$$

Then, from (3.1), we have the following for a real hypersurface M in $\mathbb{C}P^n$ with isometric Reeb flow:

$$R(\xi, Y)Df = g(Y, Df)\xi - g(\xi, Df)Y + g(AY, Df)A\xi - g(A\xi, Df)AY. \tag{5.2}$$

From this, let us take a vector field $Y \in T_\lambda$, $\lambda = \cot(r)$. Moreover, we can decompose the tangent space $T\mathbb{C}P^n$ as

$$T\mathbb{C}P^n = T_\lambda \oplus T_\mu \oplus T_\alpha \oplus \mathbb{R}N,$$

where $\lambda = \cot(r)$, $\mu = -\tan(r)$, and $\alpha = 2\cot(2r)$. If M is of type A_1, that is, a geodesic hypersphere in $\mathbb{C}P^n$, it can be decomposed as

$$T\mathbb{C}P^n = T_\lambda \oplus T_\alpha \oplus \mathbb{R}N,$$

or otherwise

$$T\mathbb{C}P^n = T_\mu \oplus T_\alpha \oplus \mathbb{R}N.$$

Then, for $Y \in T_\lambda$, (5.2) gives

$$\begin{aligned} R(\xi, Y)Df &= g(Y, Df)\xi - g(\xi, Df)Y + \alpha\lambda g(Y, Df)\xi - \alpha\lambda g(\xi, Df)Y \\ &= (1 + \alpha\lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\}. \end{aligned} \tag{5.3}$$

Then, by taking the inner product of (5.3) with the Reeb vector field ξ and using (5.1), it follows that $(1 + \alpha\lambda)g(Y, Df) = \cot^2(r)g(Y, Df) = 0$. But $\cot^2(r) \neq 0$ for the radius $0 < r < \frac{\pi}{2}$ of isometric Reeb flow M in $\mathbb{C}P^n$. It implies the following, for any $Y \in T_\lambda$:

$$g(Y, Df) = 0.$$

Now let us check (5.2) for $Y \in T_\mu$, $\mu = -\tan(r)$. Then (5.2) gives

$$R(\xi, Y)Df = g(Y, Df)\xi - g(\xi, Df)Y + \alpha\mu g(Y, Df)\xi - \alpha\mu g(\xi, Df)Y. \tag{5.4}$$

Then, by taking the inner product (5.4) with the Reeb vector field ξ and $Y \in T_\mu$ respectively and using (5.1), we get

$$(1 + \alpha\mu)g(Y, Df) = 0 \quad \text{and} \quad (1 + \alpha\mu)g(\xi, Df) = 0, \tag{5.5}$$

where $g(R(\xi, Y)Df, \xi) = 0$ and the left side $g(R(\xi, Y)Df, Y) = 0$ is given by virtue of the following formulas:

$$g(\phi AY, Y) = \mu g(\phi Y, Y) = 0,$$

$$\operatorname{Ric}(\phi AY) = \mu\{(2n + 1) + \mu h - \mu^2\}\phi Y,$$

and

$$g((\nabla_\xi A)Y, Y) = -\mu g(\nabla_\xi Y, Y) = 0.$$

Since $1+\alpha\mu = 1+(\cot(r)-\tan(r))(-\tan(r)) = \tan^2(r) \neq 0$ for $0 < r < \frac{\pi}{2}$ for isometric Reeb flow M in $\mathbb{C}P^n$, (5.5) implies that

$$g(Y, Df) = 0 \quad \text{and} \quad g(\xi, Df) = 0 \tag{5.6}$$

for any $Y \in T_\mu$, $\mu = -\tan(r)$. For a geodesic hypersphere of type A_1 in $\mathbb{C}P^n$, it holds either $g(Y, Df) = 0$ for $Y \in T_\lambda = \mathcal{C}$ or for $Y \in T_\mu = \mathcal{C}$ from the above decomposition, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ in the tangent space TM of M in $\mathbb{C}P^n$. Of course, it also holds $g(\xi, Df) = 0$ for a geodesic hypersphere in $\mathbb{C}P^n$.

Summing up (5.3), (5.6), and the above discussion, the gradient of the smooth function f is identically vanishing, that is, $g(Y, Df) = 0$ for any tangent vector field $Y \in T_zM$, $z \in M$. Consequently, we can conclude that the gradient Ricci–Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$ is trivial. So it becomes Einstein. That is, $\mathrm{Ric}(X) = \lambda X$. Then, by Theorem D, we get a complete proof of our Main Theorem 2 in the Introduction.

6. Gradient Ricci–Bourguignon solitons on contact real hypersurfaces in $\mathbb{C}P^n$

In this section, we want to give a property for gradient Ricci–Bourguignon solitons on a contact real hypersurface M in the complex projective space $\mathbb{C}P^n$. Of course, by Theorem B we know that the scalar curvature γ is constant. The $(M, Df, \Omega, \theta, \gamma, g)$ gives the following for any tangent vector field X on M in $\mathbb{C}P^n$:

$$\nabla_X Df + \mathrm{Ric}(X) = (\Omega + \theta\gamma)X. \tag{6.1}$$

Then, by differentiating (6.1), the curvature tensor of $\operatorname{grad} f$ is given by

$$\begin{aligned} R(X,Y)Df &= \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df \\ &= -(\nabla_X \mathrm{Ric})Y - \mathrm{Ric}(\nabla_X Y) + (\Omega+\theta\gamma)\nabla_X Y \\ &\quad + (\nabla_Y \mathrm{Ric})X + \mathrm{Ric}(\nabla_Y X) - (\Omega+\theta\gamma)\nabla_Y X \\ &\quad + \mathrm{Ric}([X,Y]) - (\Omega+\theta\gamma)[X,Y] \\ &= (\nabla_Y \mathrm{Ric})X - (\nabla_X \mathrm{Ric})Y, \end{aligned} \tag{6.2}$$

where we have used the Ricci soliton constant ν and the gradient Ricci–Bourguignon soliton constant θ, and the scalar curvature γ is constant on a contact real hypersurface M in $\mathbb{C}P^n$ in Proposition 3.4.

Now let us assume that M is a contact real hypersurface in $\mathbb{C}P^n$, which is characterized by

$$A\phi + \phi A = \kappa\phi, \quad \text{where } \kappa \neq 0 \text{ is constant.}$$

Then it is Hopf and the Ricci operator is given by

$$\mathrm{Ric}(X) = (2n+1)X - 3\eta(X)\xi + hAX - A^2X$$

for any tangent vector field X on M. From this, let us put $X = \xi$. Then M being Hopf and $A\xi = \alpha\xi$ implies

$$\mathrm{Ric}(\xi) = \ell\xi,$$

where $\ell = 2(n-1) + h\alpha - \alpha^2$ is constant, and the mean curvature $h = \operatorname{Tr} A$ is constant for a contact hypersurface M in $\mathbb{C}P^n$. Then, by taking covariant derivative to the Ricci operator, we have

$$(\nabla_X \operatorname{Ric})\xi = \nabla_X(\operatorname{Ric}(\xi)) - \operatorname{Ric}(\nabla_X \xi) = \ell\phi AX - \operatorname{Ric}(\phi AX)$$

and

$$\begin{aligned}(\nabla_\xi \operatorname{Ric})X &= \nabla_\xi(\operatorname{Ric} X) - \operatorname{Ric}(\nabla_\xi X)\\ &= h(\nabla_\xi A)X - (\nabla_\xi A^2)X.\end{aligned}$$

From (6.2), together with the above formula, by putting $X = \xi$ we have the following for a contact hypersurface M in $\mathbb{C}P^n$:

$$\begin{aligned}R(\xi, Y)Df &= (\nabla_Y \operatorname{Ric})\xi - (\nabla_\xi \operatorname{Ric})Y\\ &= \ell\phi AY - \operatorname{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y.\end{aligned} \tag{6.3}$$

Then, the diagonalization of the shape operator A of the contact real hypersurface in the complex hyperbolic quadric $\mathbb{C}P^n$ is given by

$$A = \begin{bmatrix} 2\cot(2r) & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & -\cot(\frac{\pi}{4}-r) & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -\cot(\frac{\pi}{4}-r) & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & \tan(\frac{\pi}{4}-r) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & \tan(\frac{\pi}{4}-r)\end{bmatrix}.$$

Here, by Proposition 3.4, the principal curvatures are given by $\alpha = 2\cot(2r)$, $\lambda = -\cot(\frac{\pi}{4} - r)$, and $\mu = \tan(\frac{\pi}{4} - r)$, with multiplicities 1, $n-1$, and $n-1$, respectively. All of these principal curvatures satisfy

$$\kappa = \lambda + \mu = -\cot\left(\frac{\pi}{4} - r\right) + \tan\left(\frac{\pi}{4} - r\right) = -2\tan(2r) = -\frac{4}{\alpha}.$$

On the other hand, the curvature tensor $R(X,Y)Z$ of M induced from the curvature tensor $\bar{R}(X,Y)Z$ of the complex projective space $\mathbb{C}P^n$ gives

$$\begin{aligned}R(\xi, Y)Df &= g(Y, Df)\xi - g(\xi, Df)Y\\ &\quad + g(AY, Df)A\xi - g(A\xi, Df)AY\\ &= (1+\alpha\lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\}\end{aligned} \tag{6.4}$$

for any $Y \in T_\lambda$, $\lambda = -\cot(\frac{\pi}{4} - r)$ for a contact real hypersurface M in the complex projective space $\mathbb{C}P^n$. Consequently, (6.3) and (6.4) give

$$\ell\phi AY - \operatorname{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y = (1+\alpha\lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\}.$$

From this, by taking the inner product with the Reeb vector field ξ, we have

$$(1+\alpha\lambda)g(Y, Df) = 0. \tag{6.5}$$

Then, for any $Y \in T_\lambda$ in (6.5), it follows that

$$g(Y, Df) = 0, \tag{6.6}$$

where we have noted that $1+\alpha\lambda = 1+2\cot(2r)(-\cot(\frac{\pi}{4}-r)) \neq 0$. Because if we assume that $2\cot(2r)\cot(\frac{\pi}{4}-r) = 1$, then $\tan(2r) = 2\cot(\frac{\pi}{4}-r)$. Then it follows that

$$(\cos(r)-\sin(r))\sin(r)\cos(r) = (\cos(r)+\sin(r))^2(\cos(r)-\sin(r)),$$

which gives $\sin(r)\cos(r) = -1$. This implies a contradiction for $0 < r < \frac{\pi}{4}$. Accordingly, the gradient vector field Df is orthogonal to the eigenspace T_λ, that is, $g(Y, Df) = 0$ for any $Y \in T_\lambda$.

Next, we consider for $Y \in T_\mu$, $\mu = \tan(\frac{\pi}{4}-r)$ in Proposition 3.4. Then, using these properties in (6.3) and (6.4) implies the following:

$$\ell\phi AY - \mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y = (1+\alpha\mu)\{g(Y, Df)\xi - g(\xi, Df)Y\}.$$

From this, by taking the inner product with the Reeb vector field ξ, we get

$$g(Y, Df) = 0 \quad \text{for any } Y \in T_\mu, \tag{6.7}$$

where $1+\alpha\mu \neq 0$. If we assume that $1+\alpha\mu = 0$, then, by Proposition 3.4, we can write $1+2\cot(2r)\tan(\frac{\pi}{4}-r) = 0$. Then we obtain $-\tan(2r) = 2\frac{\cos(r)-\sin(r)}{\cos(r)+\sin(r)}$. Since $\tan(2r) = \frac{\sin(2r)}{\cos(2r)}$, we get the following:

$$\begin{aligned}(\cos(r)+\sin(r))\sin(r)\cos(r) &= -(\cos(r)-\sin(r))(\cos^2(r)-\sin^2(r))\\ &= -(\cos(r)-\sin(r))^2(\cos(r)+\sin(r)).\end{aligned}$$

Since $\cos(r)+\sin(r) \neq 0$ for $0 < r < \frac{\pi}{4}$, we get $\sin(r)\cos(r) = 1$, which gives also a contradiction.

Finally, let us take the inner product to the above formula with $Y \in T_\mu$, and use $AY = \mu Y$, $A\phi Y = \lambda\phi Y$ for a contact hypersurface in $\mathbb{C}P^n$, we have

$$\begin{aligned}-(1+\alpha\mu)g(\xi, Df) &= \ell g(\phi AY, Y) - g(\mathrm{Ric}(\phi AY), Y)\\ &\quad - hg((\nabla_\xi A)Y, Y) + g((\nabla_\xi A^2)Y, Y)\\ &= 0,\end{aligned} \tag{6.8}$$

where in the second equality we have used the following formulas:

$$\begin{aligned}\mathrm{Ric}(\phi AY) &= (2n+1)\phi AY + hA\phi AY - A^2\phi AY\\ &= \mu\{(2n+1) + \lambda h - \lambda^2\}\phi Y,\end{aligned}$$

$$\begin{aligned}g((\nabla_\xi A)Y, Y) &= g(\nabla_\xi(AY) - A\nabla_\xi Y, Y)\\ &= g(\mu\nabla_\xi Y - A\nabla_\xi Y, Y) = 0,\end{aligned}$$

and

$$\begin{aligned}g((\nabla_\xi A^2)Y, Y) &= g(\nabla_\xi(A^2Y) - A^2\nabla_\xi Y, Y)\\ &= g(\mu^2\nabla_\xi Y - A^2\nabla_\xi Y, Y)\\ &= \mu^2 g(\nabla_\xi Y, Y) - \mu^2 g(\nabla_\xi Y, Y) = 0.\end{aligned}$$

From this, together with (6.8), it follows that

$$g(\xi, Df) = 0. \tag{6.9}$$

Consequently, from (6.6), (6.7), and (6.9) it follows that the gradient vector field Df is identically vanishing. Then $Df = 0$ in (6.1) means that M is Einstein. But Theorem D in the Introduction gives that there does not exist an Einstein real hypersurface in the complex projective space $\mathbb{C}P^n$. From this, we give a complete proof of our Main Theorem 3 in the Introduction.

Remark 6.1. The metric g of a Riemannian manifold M of dimension $n \geq 3$ is said to be a *gradient Einstein soliton* [7] if there exists a smooth function f on M such that

$$\operatorname{Ric} + \nabla^2 f = \Big(\Omega + \frac{1}{2}\gamma\Big) g,$$

where γ denotes the scalar curvature of M and $\theta = \frac{1}{2}$ a constant on M. Here $\nabla^2 f$ denotes the Hessian operator of g and f the Einstein potential function of the gradient Einstein soliton. So this soliton is an example of gradient Ricci–Bourguignon soliton.

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